EXHIBIT E

Chapter 2
Basic Navigational Mathematics, Reference Frames and the Earth's Geometry

Navigation algorithms involve various coordinate frames and the transformation of coordinates between them. For example, inertial sensors measure motion with respect to an inertial frame which is resolved in the host platform's body frame. This information is further transformed to a navigation frame. A GPS receiver initially estimates the position and velocity of the satellite in an inertial orbital frame. Since the user wants the navigational information with respect to the Earth, the satellite's position and velocity are transformed to an appropriate Earth-fixed frame. Since measured quantities are required to be transformed between various reference frames during the solution of navigation equations, it is important to know about the reference frames and the transformation of coordinates between them. But first we will review some of the basic mathematical techniques.

2.1 Basic Navigation Mathematical Techniques

This section will review some of the basic mathematical techniques encountered in navigational computations and derivations. However, the reader is referred to (Chatfield 1997; Rogers 2007 and Farrell 2008) for advanced mathematics and derivations. This section will also introduce the various notations used later in the book.

2.1.1 Vector Notation

In this text, a vector is depicted in bold lowercase letters with a superscript that indicates the coordinate frame in which the components of the vector are given. The vector components do not appear in bold, but they retain the superscript. For example, the three-dimensional vector r for a point in an arbitrary frame *k* is depicted as

$$\mathbf{r}^k = \begin{bmatrix} x^k \\ y^k \\ z^k \end{bmatrix} \quad (2.1)$$

In this notation, the superscript k represents the k-frame, and the elements (x^k, y^k, z^k) denote the coordinate components in the k-frame. For simplicity, the superscript is omitted from the elements of the vector where the frame is obvious from the context.

2.1.2 Vector Coordinate Transformation

Vector transformation from one reference frame to another is frequently needed in inertial navigation computations. This is achieved by a transformation matrix. A matrix is represented by a capital letter which is not written in bold. A vector of any coordinate frame can be represented into any other frame by making a suitable transformation. The transformation of a general k-frame vector $\mathbf{r}^k$ into frame m is given as

$$\mathbf{r}^m = R_k^m \mathbf{r}^k \quad (2.2)$$

where R_k^m represents the matrix that transforms vector $\mathbf{r}$ from the k-frame to the m-frame. For a valid transformation, the superscript of the vector that is to be transformed must match the subscript of the transformation matrix (in effect they cancel each other during the transformation).

The inverse of a transformation matrix R_k^m describes a transformation from the m-frame to the k-frame

$$\mathbf{r}^k = \left(R_k^m\right)^{-1} \mathbf{r}^m = R_m^k \mathbf{r}^m \quad (2.3)$$

If the two coordinate frames are mutually orthogonal, their transformation matrix will also be orthogonal and its inverse is equivalent to its transpose. As all the computational frames are orthogonal frames of references, the inverse and the transpose of their transformation matrices are equal. Hence for a transformation matrix R_k^m we see that

$$R_k^m = \left(R_m^k\right)^T = \left(R_m^k\right)^{-1} \quad (2.4)$$

A square matrix (like any transformation matrix) is orthogonal if all of its vectors are mutually orthogonal. This means that if

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \quad (2.5)$$

where

$$\mathbf{r}_1 = \begin{bmatrix} r_{11} \\ r_{21} \\ r_{31} \end{bmatrix}, \mathbf{r}_2 = \begin{bmatrix} r_{12} \\ r_{22} \\ r_{33} \end{bmatrix}, \mathbf{r}_3 = \begin{bmatrix} r_{13} \\ r_{23} \\ r_{33} \end{bmatrix} \tag{2.6}$$

then for matrix R to be orthogonal the following should be true

$$\mathbf{r}_1 \cdot \mathbf{r}_2 = 0, \mathbf{r}_1 \cdot \mathbf{r}_3 = 0, \mathbf{r}_2 \cdot \mathbf{r}_3 = 0 \tag{2.7}$$

2.1.3 Angular Velocity Vectors

The angular velocity of the rotation of one computational frame about another is represented by a three component vector $\boldsymbol{\omega}$. The angular velocity of the k-frame relative to the m-frame, as resolved in the p-frame, is represented by ω^p_{mk} as

$$\boldsymbol{\omega}^p_{mk} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \tag{2.8}$$

where the subscripts of $\boldsymbol{\omega}$ denote the direction of rotation (the k-frame with respect to the m-frame) and the superscripts denote the coordinate frame in which the components of the angular velocities $(\omega_x, \omega_y, \omega_z)$ are given.

The rotation between two coordinate frames can be performed in two steps and expressed as the sum of the rotations between two different coordinate frames, as shown in Eq. (2.9). The rotation of the k-frame with respect to the p-frame can be performed in two steps: firstly a rotation of the m-frame with respect to the p-frame and then a rotation of the k-frame with respect to the m-frame

$$\boldsymbol{\omega}^k_{pk} = \boldsymbol{\omega}^k_{pm} + \boldsymbol{\omega}^k_{mk} \tag{2.9}$$

For the above summation to be valid, the inner indices must be the same (to cancel each other) and the vectors to be added or subtracted must be in the same reference frame (i.e. their superscripts must be the same).

2.1.4 Skew-Symmetric Matrix

The angular rotation between two reference frames can also be expressed by a skew-symmetric matrix instead of a vector. In fact this is sometimes desired in order to change the cross product of two vectors into the simpler case of matrix multiplication. A vector and the corresponding skew-symmetric matrix forms of an angular velocity vector $\boldsymbol{\omega}^p_{mk}$ are denoted as

$$\underbrace{\boldsymbol{\omega}^p_{mk} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}}_{\text{Angular velocity vector}} \Rightarrow \underbrace{\Omega^p_{mk} = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}}_{\text{Skew-symmetric form of angular the velocity vector}} \quad (2.10)$$

Similarly, a velocity vector $\mathbf{v}^p$ can be represented in skew-symmetric form V^p as

$$\underbrace{\mathbf{v}^p = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}}_{\text{Velocity vector}} \Rightarrow \underbrace{\mathrm{V}^p = \begin{bmatrix} 0 & -v_z & v_y \\ v_z & 0 & -v_x \\ -v_y & v_x & 0 \end{bmatrix}}_{\text{Skew-symmetric form of the velocity vector}} \quad (2.11)$$

Note that the skew-symmetric matrix is denoted by a non-italicized capital letter of the corresponding vector.

2.1.5 Basic Operations with Skew-Symmetric Matrices

Since a vector can be expressed as a corresponding skew-symmetric matrix, the rules of matrix operations can be applied to most vector operations. If **a**, **b** and **c** are three-dimensional vectors with corresponding skew-symmetric matrices A, B and C, then following relationships hold

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{a}^T\mathbf{b} = \mathbf{b}^T\mathbf{a} \quad (2.12)$$

$$\mathbf{a} \times \mathbf{b} = \mathrm{A}\mathbf{b} = B^T\mathbf{a} = -\mathrm{B}\mathbf{a} \quad (2.13)$$

$$[\![\mathrm{A}\mathbf{b}]\!] = \mathrm{AB} - \mathrm{BA} \quad (2.14)$$

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{a}^T\mathrm{B}\mathbf{c} \quad (2.15)$$

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathrm{AB}\mathbf{c} \quad (2.16)$$

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \mathrm{AB}\mathbf{c} - \mathrm{BA}\mathbf{c} \quad (2.17)$$

where $[\![\mathrm{A}\mathbf{b}]\!]$ in Eq. (2.14) depicts the skew-symmetric matrix of vector $\mathrm{A}\mathbf{b}$.

2.1.6 Angular Velocity Coordinate Transformations

Just like any other vector, the coordinates of an angular velocity vector can be transformed from one frame to another. Hence the transformation of an angular velocity vector $\boldsymbol{\omega}_{mk}$ from the k-frame to the p-frame can be expressed as

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