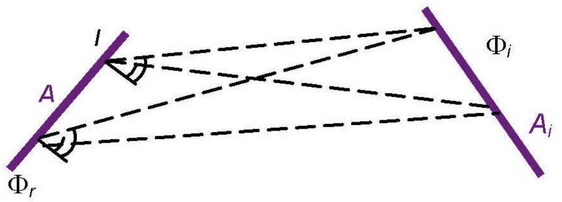
Form Factor and Average Projected Solid Angle

Here the approximations of constant cosines cannot be used.



The angles between the normal to the radiating surface and the directions to points on the illuminating surface vary not only with the locations of the points on the illuminated surface, but also with the locations of points on the radiating surface. The concept of projected solid angle takes the former into account, but not the latter. What is needed is an **average projected solid angle**, $\bar{\Omega}_{r\text{ to }i}$, which is the projected solid angle subtended by the illuminated area and averaged over all points on the radiating area. Then the illuminating flux, Φ_i, from a **Lambertian radiator** is

$$\Phi_i = L\,A_r\,\bar{\Omega}_{r\,to\,i} = \frac{\Phi_r}{\pi}\,\bar{\Omega}_{r\,to\,i}.$$

In practice, the average projected solid angle is not used. However, its geometrical equivalent, called the **form factor**, $F_{a\text{ to }b}$, is used. The only difference between the form factor and the average projected solid angle is a multiplier of π:

$$F_{a\,to\,b} = \bar{\Omega}_{a\,to\,b}/\pi.$$

The form factor measures in hemispheres what the average projected solid angle measures in projected steradians. The form factor also can be interpreted as the portion of the flux leaving a Lambertian radiator, *a*, that illuminates a surface, *b*:

$$\Phi_i = \Phi_r\,F_{r\,to\,i}.$$

Note that the form factor is directional, as are the solid and the projected solid angles. $F_{a\text{ to }b}$ is not in general equal to $F_{b\text{ to }a}$. However, the product of the area and the form factor is constant:

$$A_a\,F_{a\,to\,b} = A_b\,F_{b\,to\,a}.$$

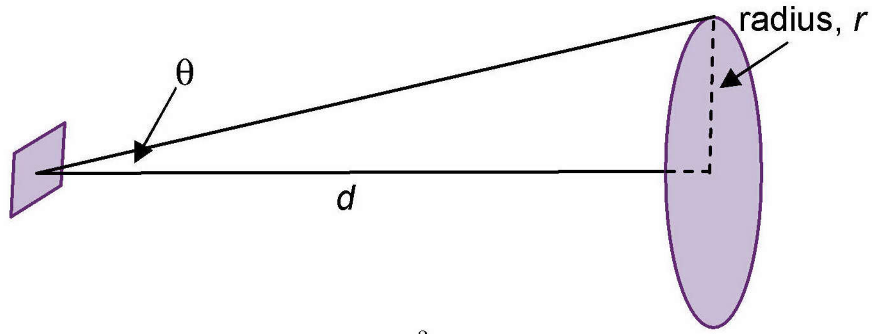
Configuration Factor

The **form factor** and the **average projected solid angle** both link two extended areas. The form factor measures in hemispheres what the average projected solid angle measures in projected steradians. Another term, the **configuration factor**, C, is similarly related to the **projected solid angle**, linking a small area with an extended area. Like the form factor, the configuration factor measures in hemispheres what the projected solid angle measures in projected steradians:

$$C = \Omega/\pi.$$

Tables of configuration factors and form factors for myriad geometries can be found in handbooks on illumination, in books on radiative heat transfer (where the issues are identical to illumination by Lambertian radiators), and on the Internet. Three cases with applicability to many optical situations are listed here:

Case 1: Small area to an extended circular area; both areas parallel and with axial symmetry.



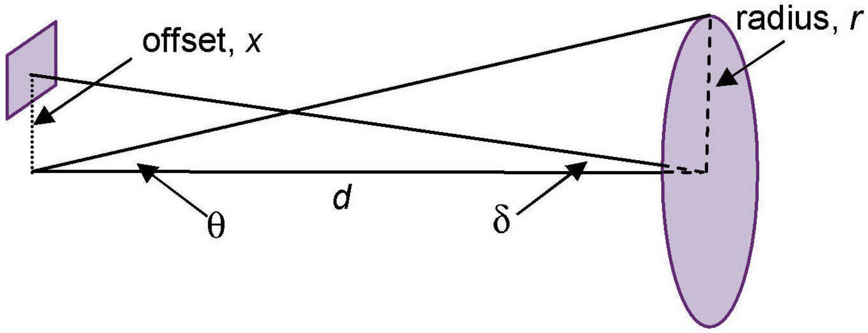
$$C = \frac{r^2}{r^2 + d^2} = \sin^2\theta$$

and

$$\Omega = \frac{\pi r^2}{r^2 + d^2} = \pi \sin^2\theta.$$

Useful Configuration Factor

Case 2: Small area to an extended circular area; both areas parallel, but without axial symmetry.



$$C = \frac{1}{2}\left(1 - \frac{1 + \left(d/x\right)^2 - \left(r/x\right)^2}{\left[\left\{1 + \left(d/x\right)^2 + \left(r/x\right)^2\right\}^2 - 4\left(r/x\right)^2\right]^{1/2}}\right)$$

or, equivalently:

$$C = \frac{1}{2}\left(1 - \frac{1 + \tan^2\delta - \tan^2\theta}{\left[\tan^4\delta + \left(2\tan^2\delta\right)\left(1 - \tan^2\theta\right) + \sec^4\theta\right]^{1/2}}\right)$$

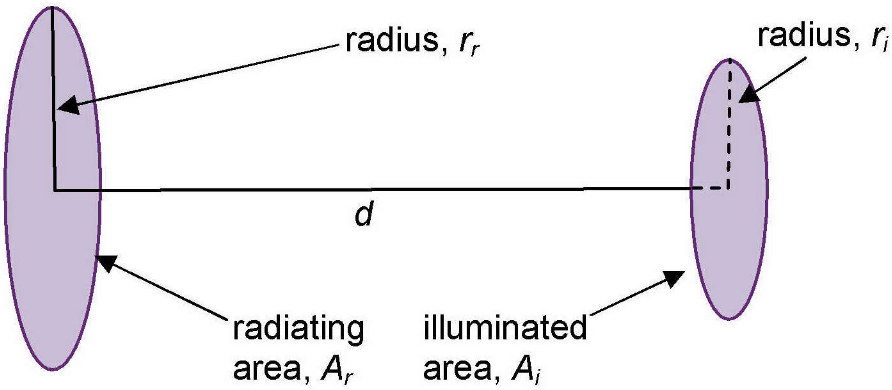
and

$$\Omega = \pi \cdot C.$$

These expressions degenerate to the expressions for case 1 above when x, or equivalently, δ, is equal to zero.

Useful Form Factor

Case 3: An extended circular area illuminating another extended circular area; both areas parallel and centered on the same axis.



$$F_{r\,to\,i} = \frac{1}{2}\left\{1 + \frac{1+\left(\frac{r_i}{d}\right)^2}{\left(\frac{r_r}{d}\right)^2} - \left[\left(1 + \frac{1+\left(\frac{r_i}{d}\right)^2}{\left(\frac{r_r}{d}\right)^2}\right)^2 - 4\left(\frac{r_i}{r_r}\right)^2\right]^{\frac{1}{2}}\right\}.$$

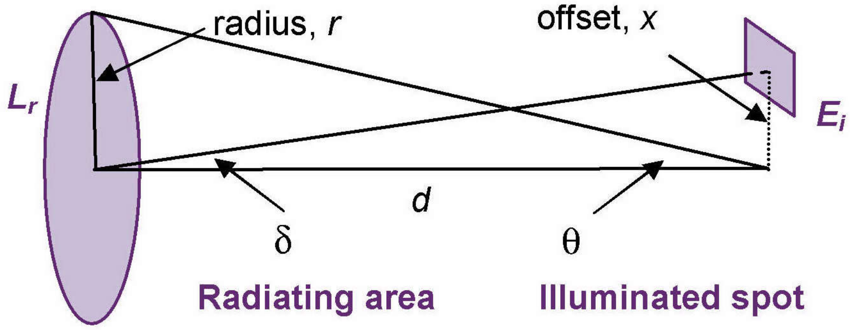
Some numerical values of $F_{r\,to\,i}$ for this case are shown in the table below for several sizes of radiating and illuminated disks (each expressed as a multiple of the distance between the two parallel circular areas that are centered on the same axis).

Form Factor, $F_{r\,to\,i}$

		r_i/d					
		0.03	**0.10**	**0.30**	**1.00**	**3.00**	**10.0**
r_r/d	**0.03**	.001	.010	.083	.500	.900	.990
	0.10	.001	.010	.082	.499	.900	.990
	0.30	.001	.009	.077	.489	.899	.990
	1.00	.000	.005	.044	.382	.890	.990
	3.00	.000	.001	.009	.099	.718	.989
	10.0	.000	.000	.001	.010	.089	.905

Irradiance from a Uniform Lambertian Disk

Many illumination situations can be modeled as illumination by a **uniform circular Lambertian disk**, with the illuminated area parallel to the disk and at some distance from it.



The irradiance at the illuminated spot is equal to the radiance of the radiating area times the projected solid angle of the radiating area when viewed from the illuminated spot:

$$E_i = L_r \ \Omega_i .$$

If the illuminated spot is on axis ($x = 0$, $\delta = 0$), then

$$E_i = \pi L_r \sin^2\theta = \pi L_r \frac{r^2}{r^2 + d^2} .$$

If the spot is offset from the axis, it is necessary to use the projected solid angle or the configuration factor discussed previously for case 2:

$$\Omega_i = \frac{\pi}{2}\left\{1 - \frac{1+\tan^2\delta - \tan^2\theta}{\left[\tan^4\delta + \left(2\tan^2\delta\right)\left(1-\tan^2\theta\right) + \sec^4\theta\right]^{1/2}}\right\}.$$

Note: The configuration factor, form factor, and projected solid angle are useful mainly when the radiation pattern is Lambertian or nearly Lambertian.

Cosine Fourth and Increase Factor

Consider the previous case of illumination by a uniform circular Lambertian disk, with the illuminated area parallel to the disk and at some distance from it. For many values of aperture size (θ) and field angle (δ), the irradiance falls off very nearly at $\cos^4\delta$, a phenomenon often referred to as the **cosine-fourth law**.

> Two of the cosine terms in the $\cos^4$ law are due to the fact that, off axis, the distance increases with the cosine of δ and the inverse square law applies. The third cosine factor comes from the Lambertian source, and the fourth from the fact that the illuminated surface is inclined to the direction of propagation.

In reality, the $\cos^4$ "law" is not exactly true, and is far from true for large values of θ and δ. The table below displays values of the **increase factor**, *F*′, which is the multiplier that must be applied to the irradiance calculated by using the axial irradiance and $\cos^4$ falloff. *F*′ compensates for the inaccuracy in the "cosine-fourth" assumption:

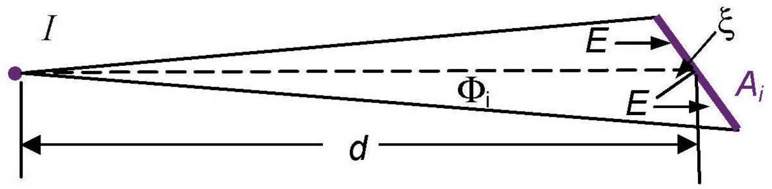
$$E_i = \pi L_r \sin^2\theta \cdot \cos^4\delta \cdot F'$$

Increase Factor, *F*′

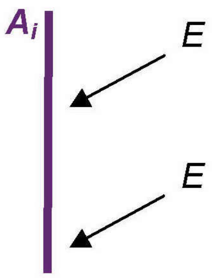
		θ					
	(deg)	**1.8**	**3.6**	**7.2**	**14.5**	**30.0**	**45.0**
	NA	**0.03**	**0.06**	**0.13**	**0.25**	**0.50**	**0.71**
	***f*/#**	**16**	**8**	**4**	**2**	**1**	**0.71**
δ(deg)	**0**	1.00	1.00	1.00	1.00	1.00	1.00
	10	1.00	1.00	1.00	1.01	1.03	1.05
	20	1.00	1.00	1.01	1.03	1.11	1.20
	30	1.00	1.00	1.01	1.05	1.23	1.49
	40	1.00	1.00	1.02	1.08	1.37	1.94
	60	1.00	1.01	1.02	1.09	1.48	2.69

The $\cos^4$ approximation is valid within a few percent up to very large apertures and field angles.

Known Irradiance



If a surface is illuminated by a source of uniform intensity at a distance *d* and the irradiance on the surface is known, then the intensity of the source is



$$I = \frac{E \cdot d^2}{\cos \xi}.$$

For any surface that is illuminated by uniform irradiance, the total flux illuminating the surface is

$$\Phi = E \cdot A_i.$$

The radiance of the surface, caused by the light reflecting from the surface, depends on the reflecting properties of the surface.

If the surface is Lambertian over all angles of reflection (for this incident geometry), then

$$L = \frac{\rho \cdot E}{\pi},$$

where ρ is the reflectance of the surface for the relevant incident geometry.

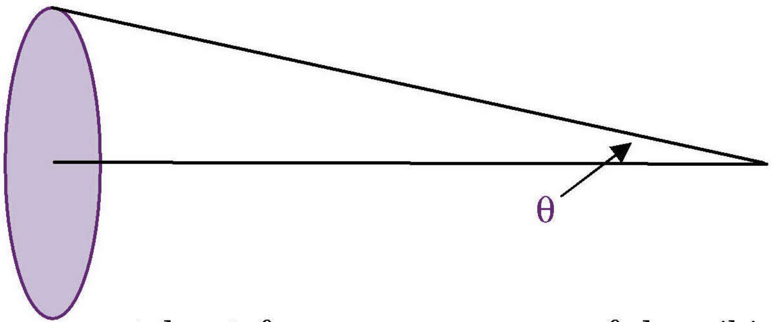
If the surface is not Lambertian over all angles but is Lambertian over the direction of concern, then

$$L = \frac{R \cdot E}{\pi},$$

where *R* is the reflectance factor of the surface for the relevant incident geometry and for the direction of concern.

ω, Ω, NA, and *f*/# for a Circular Cone

The case of a circular disk subtending a known half-angle, θ, shows up often in illumination situations.



There are at least four common ways of describing the cone: **solid angle** (ω), **projected solid angle** (Ω), **numerical aperture (NA)**, and ***f*-number** (*f*/#):

$$\omega = 2\pi(1-\cos\theta) \qquad \Omega = \pi\sin^2\theta$$
$$NA = n\cdot\sin\theta \qquad f/\# = 1/2\sin\theta,$$

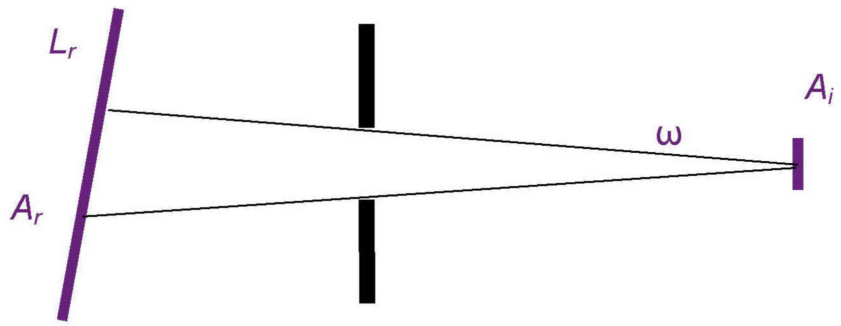
where n is the index of refraction.

Cone subtended by a circular disk				
θ(deg)	ω	Ω	NA/n	f/#
1.8	0.003	0.003	0.03	16.00
3.6	0.012	0.012	0.06	8.00
7.2	0.049	0.049	0.13	4.00
12.7	0.154	0.152	0.22	2.27
14.5	0.200	0.196	0.25	2.00
20.0	0.379	0.367	0.34	1.46
25.0	0.589	0.561	0.42	1.18
30.0	0.842	0.785	0.50	1.00
35.0	1.14	1.03	0.57	0.87
40.0	1.47	1.30	0.64	0.78
45.0	1.84	1.57	0.71	0.71
50.0	2.24	1.84	0.77	0.65
60.0	3.14	2.36	0.87	0.58
70.0	4.13	2.77	0.94	0.53
80.0	5.19	3.05	0.98	0.51
90.0	6.28	3.14	1.00	0.50

Invariance of Radiance

Unlike intensity, which is associated with a specific point, and irradiance, which is associated with a specific surface, radiance is associated with the propagating light rays themselves. This distinction is not trivial and implies that the radiance of a surface can be considered separate from the actual physical emitter or reflector that produces the radiance.

Consider a uniform Lambertian radiating source, A_r, with radiance, L_r, illuminating an area, A_i, through a limiting aperture that limits the solid angle of the source to ω:



The physical location of the radiating source is irrelevant. Only the solid angle matters. In fact, the physical location (and shape) can be assumed to be anywhere (and any shape) as long as the solid angle is the same. All of the following descriptions of the radiating area, A_1, A_2, and A_3, are equivalent to A_r from an illumination point of view:

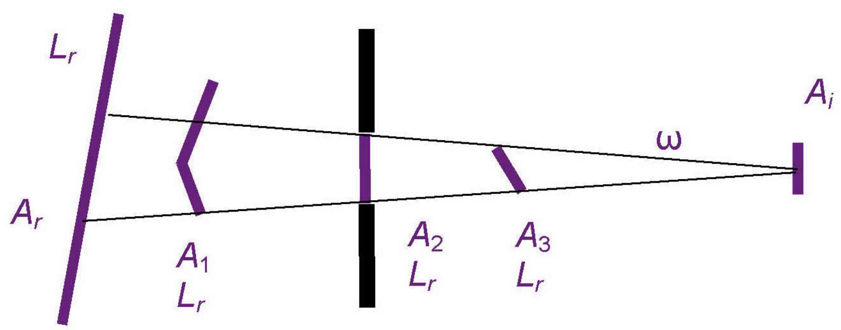
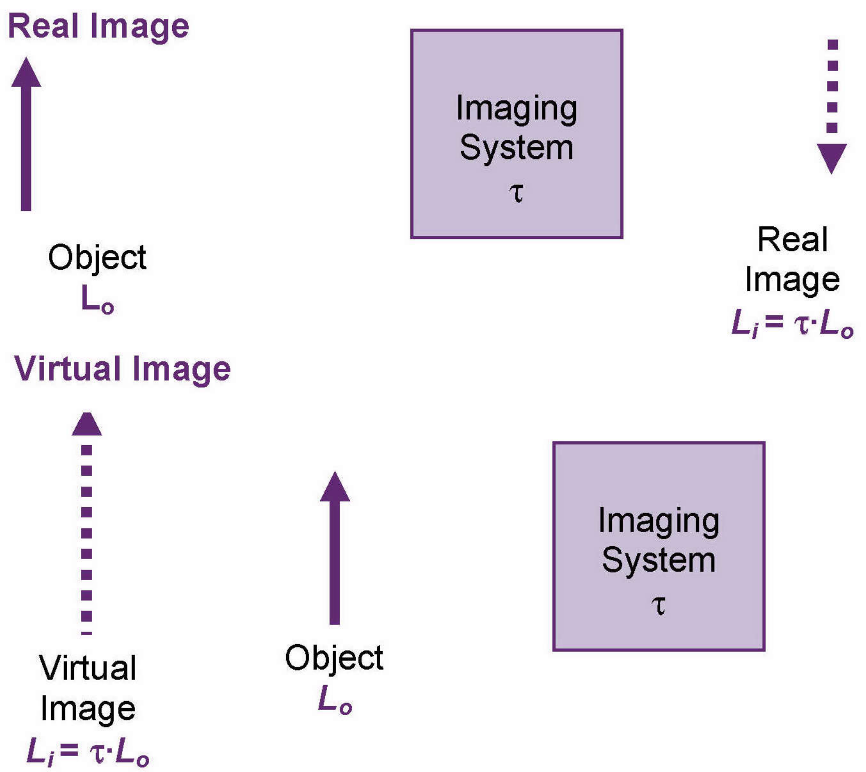
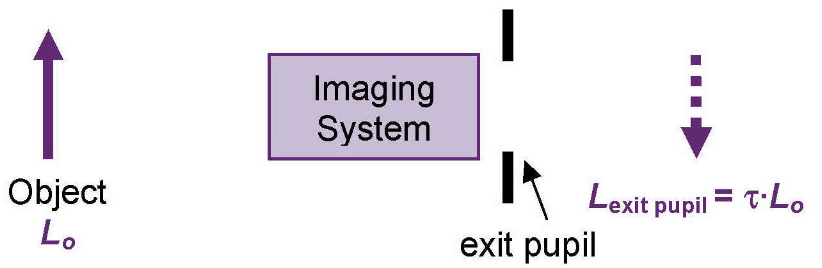


Image Radiance

In an imaging system with no vignetting or significant aberrations, for Lambertian objects, point-by-point, the radiance of an image is equal to the radiance of the object except for losses due to reflection, absorption, and scattering. These losses are usually combined into a single value of **transmittance**, τ. This equivalence of radiance is true for virtual as well as real images, and for reflective or refractive imaging systems.

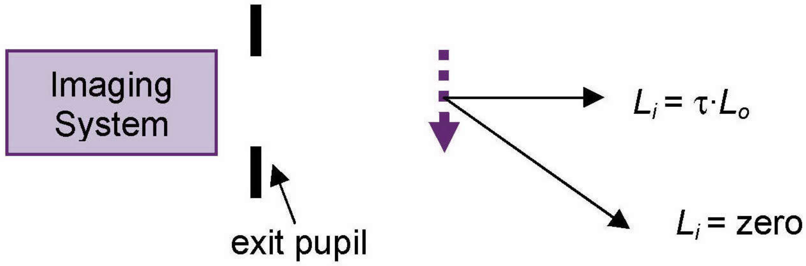


Viewed from any point on a real image, the entire exit pupil of the optical system is also the radiance of the corresponding object point but reduced by τ.



Limitations on Equivalent Radiance

In all cases, the image radiance only exists when the image is viewed through the exit pupil of the imaging system. When viewed in a direction that doesn't include the pupil, the radiance is zero.



If the object is not Lambertian, then the angular distribution of radiance of the image is also not Lambertian. The relationship between the angular distributions of object and image radiances is not straightforward and must be determined by ray tracing on the specific system. However, in many practical cases, the entrance pupil of the imaging system subtends a small angle from the object, and the source is essentially Lambertian over this small angle.

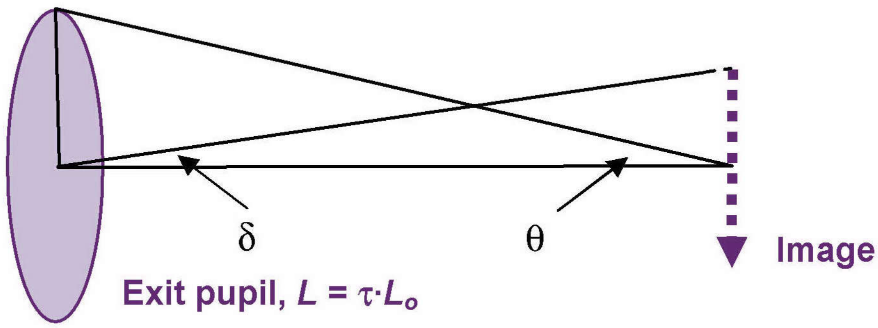
If the object and the image are in media of different refractive indices, n_o for the object and n_i for the image, then the expression for equivalent radiance is

$$\frac{L_i}{n_i^2} = \tau \cdot \frac{L_o}{n_o^2}.$$

The point-by-point equivalence of radiance from object to image is only valid for well-corrected optical systems. For systems that suffer from aberrations or are not in focus, each small point in the object is mapped to a "blur spot" in the image. Thus, the radiance of any small spot in the image is related to the average of the radiances of the corresponding spot in the object and its surrounding area.

Image Irradiance

Since the exit pupil, when viewed from the image, has the radiance of the object, then the irradiance at the image is the same as the irradiance from a source of the same size as the exit pupil and the same radiance as the object (reduced by τ). In most imaging systems, the exit pupil is round and the irradiance is the same as the irradiance from a uniform **Lambertian disk**:



$$E_i = \pi \tau L_o \sin^2\theta \cdot \cos^4\delta \cdot F'.$$

A table of values for the **increase factor**, F, is presented in the section on illumination transfer. F is very close to 1.0 except for a combination of large field angle (δ) and large aperture (θ), which is not a common combination in imaging systems.

The $\cos^4\delta$ term contributes to substantial field darkening in wide-angle imaging systems—for example, $\cos^4 45$ deg = 0.25.

If the physical aperture stop is not the limiting aperture for all the rays converging to an off-axis image point, the light is vignetted. The irradiance at image points where there is **vignetting** will be lower than predicted.

On axis, $\cos^4\delta = 1.0$ and $F = 1.0$. The image irradiance on axis, E_{i0}, is

$$E_{i0} = \pi \tau L_o \sin^2\theta.$$

f/#, Working *f*/#, *T*/#, NA, Ω

For a camera working at infinite conjugates (distant object, magnification, $|m|<<1$), the image irradiance can be expressed in terms of the lens' ***f*-number**, *f*/#:

$$E_{i0} = \frac{\pi \, \tau \, L_o}{4\,(f/\#)^2}.$$

This *f*/#, usually associated with a lens, is an "infinite conjugates" quantity. When a lens is used at finite conjugates, the **working *f*-number**, ***f*/#$_w$**, describes the cone angle illuminating the image:

$$f/\#_w = (f/\#)\cdot(1-m),$$

where m is the lateral magnification of the image (negative for real images), and the axial image irradiance is:

$$E_{i0} = \frac{\pi \, \tau \, L_o}{4\,(f/\#_w)^2}.$$

Note that *f*/#$_w$ degenerates to the conventional "infinite conjugates" *f*/# when the lens is used at infinite conjugates.

Occasionally, a lens will be designated with a ***T*-number**, T/#, which combines the *f*/# and the transmittance into a single quantity,

$$T/\# = \frac{f/\#}{\sqrt{\tau}} \text{ with axial image irradiance: } E_{i0} = \frac{\pi \, L_o}{4\,(T/\#)^2}.$$

Another descriptor of the image illumination cone angle is the **numerical aperture**, NA,

$$NA = \sin\theta \text{ with axial image irradiance: } E_{i0} = \pi \, \tau L_o \, NA^2.$$

In all cases, even without circular symmetry, on or off axis, the cone illuminating the image can be described by its **projected solid angle**, Ω, with image irradiance:

$$E_i = \tau L_o \, \Omega.$$

Flux and Étendue

The total **flux** reaching the image is the product of the image irradiance and the area of the image. The **image irradiance** is proportional to the projected **solid angle** of the exit pupil when viewed from the image:

$$\Phi_i = \tau L_o \, a_i \Omega_i ,$$

where Ω_i is the projected solid angle of the exit pupil viewed from the image, a_i is the area of the image, L_o is the [assumed uniform] radiance of the object, and τL_o is the radiance of the exit pupil.

The flux reaching the image also can be expressed in terms of the radiance of the exit pupil, τL_o, the area of the exit pupil, a_p, and the projected solid angle of the image when viewed from the exit pupil, Ω_p:

$$\Phi_i = \tau L_o \, a_p \Omega_p .$$

The quantity $a\Omega$, representing the area of a plane in the optical system times the projected solid angle of another plane when viewed from it, appears equivalently in both expressions. This **area-solid-angle-product** is a fundamental property of the optical system that determines the amount of light that can get through the system. It is called the **throughput** or **étendue**.

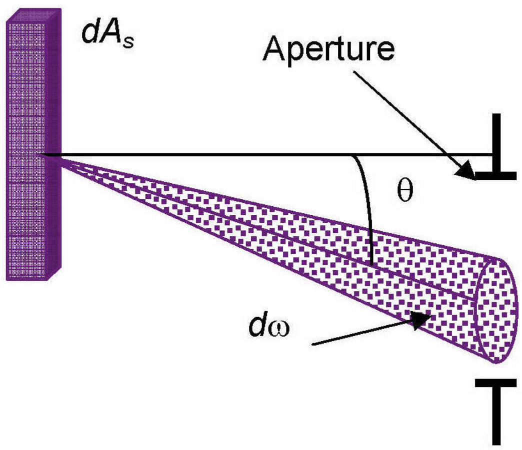
> The radiance of an object is invariant and cannot be increased by an optical system, and the étendue is a fundamental property of an optical system. These two concepts mean that, for a source of given radiance and a given optical system, the maximum flux that can be transmitted through the system is predetermined.

And, without "throwing away" light, the étendue cannot be decreased, but area and solid angle can be traded off.

Generalized Étendue

The terminology for illumination in nonimaging systems is the same as that for imaging systems; however, the range of validity is extended to include all angular space, while that of imaging systems is limited to paraxial systems. With this taken into account, étendue is often called **generalized étendue**. In this domain the étendue cannot be regarded as the simple product of the area and solid angle; it must be integrated per the following equation and figure:

$$\mathcal{E} = n^2 \iint_{\text{aperture}} \cos\theta \, dA_s \, d\omega,$$



where n is the refractive index, θ is the angle from the normal, dA_s is the differential source area, and $d\omega$ is the differential solid angle.

The total flux through the aperture is found by integrating the radiance over the aperture:

$$\Phi = \iint_{\text{aperture}} L(\mathbf{r},\hat{\mathbf{a}}) \cos\theta \, dA_s \, d\omega,$$

where **r** and **â** denote the positional and directional aspects of source emission. Assuming that the source is Lambertian so radiance is independent of angle, then

$$\Phi = L_s \iint_{\text{aperture}} \cos\theta \, dA_s \, d\omega = \frac{L_s \mathcal{E}}{n^2}.$$

Note that total flux is the product of the radiance and the geometrical étendue factor. This also shows the **conservation of étendue** that follows from the conservations of radiance and energy.

Concentration

Concentration (*C*) is a term associated with the generalized étendue. It represents the ability to transfer more light into a desired area by using the conservation of étendue to alter the angle at the output of an optical system. It is defined as the ratio of the input area (A) to the output aperture area (A') that transmits the prescribed flux from area A. For this reason it is called the **concentration ratio**:

$$C = A/A'\,.$$

This expression, a limit factor of the laws of thermodynamics, is a forbearer of the invariance of radiance and étendue.

In a 2D system, which is analogous to an extruded trough, and a 3D system, which is analogous to a well, we find that the respective concentrations are given by

$$C_{\text{2D}} = \frac{a}{a'} = \frac{n'\sin\theta'}{n\sin\theta} \qquad \text{and} \qquad C_{\text{3D}} = \frac{A}{A'} = \left(\frac{n'\sin\theta'}{n\sin\theta}\right)^2,$$

where a and a' are the aperture widths, A and A' are the aperture areas, n is the input index, n' is the output index, θ' is the output angle, and θ is the input angle. **Optimal concentration** is realized when the output angle is $\pi/2$, giving

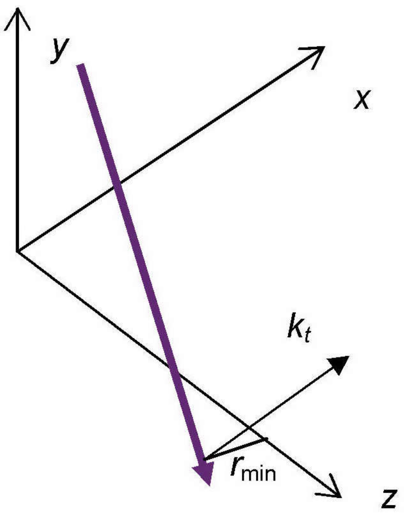
$$C_{\text{2D,opt}} = \frac{n'}{n\sin\theta_a} \qquad \text{and} \qquad C_{\text{3D,opt}} = \left(\frac{n'}{n\sin\theta_a}\right)^2,$$

where θ_a, the **acceptance angle**, is the prescribed upper input angle over which conservation of étendue is maintained.

Skew Invariant

The **skew invariant** is another limiting factor in nonimaging system design. Its definition is rather esoteric:

$$f_{\text{skew}}(s) = \frac{d\mathcal{E}(s)}{ds},$$



where $s = r_{\min}k_t$, and $r_{\min}$ is the ray's closest approach to the optical axis (z, as shown), and k_t is the tangential component of the ray's propagation direction.

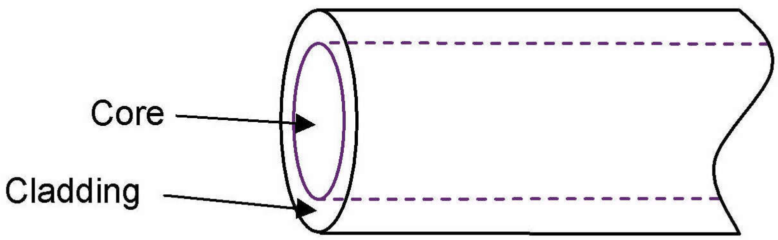
A simpler way to think about the skew invariant is to recognize that in a rotationally symmetric system (e.g., a lens), loss is introduced from the input to the output if the two spatial distributions are not the same shape. For example, if the object shape is a uniform square but a uniform round output is desired, then transfer losses will be produced.

To maximize transfer efficiency with different distributions, the symmetry of the optical system must be broken; or, in other words, there must be a "twist" in the optical components to force rays out of their respective sagittal planes. Many nonimaging optical systems take advantage of this property by including faceted reflectors (e.g., segmented headlights), segmented lenses (e.g., pillow optics for projection displays), or 3D edge-ray concentrators that employ V-wedges near the source (i.e., solar concentrators).

For a rotationally symmetric system, the rotational skewness of each ray is conserved or invariant. This skew invariant is given by the first derivative of the étendue.

Fibers—Basic Description

Optical fibers, **lightpipes**, and **lightguides** are all variations on the same theme. They each contain a central transparent core, usually circular in cross-section, surrounded by an annular cladding. The cladding has a lower index of refraction than the core.



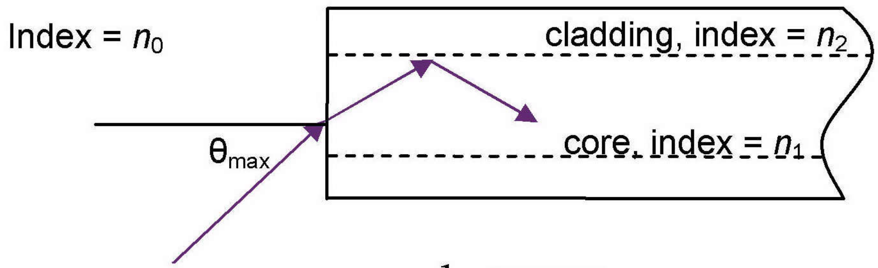
The core can transmit light for long distances with low loss because of total internal reflection at the interface between the core and the cladding. The primary purpose of the cladding is to maintain the integrity of this interface. Without it, total internal reflection would occur at a core-air interface, but dust, nicks, abrasions, oils, and other contamination on the interface would reduce the transmission to unacceptably low levels.

Sometimes layers of buffering and/or jacketing are placed outside the cladding for additional protection.

The core diameter can range from very small, on the order of the wavelength of light, to a centimeter or more. The very thin cores are essentially waveguides and not used for illumination. Flexible glass and quartz fibers have core diameters ranging from approximately 50 microns to about 1 millimeter. If they are thicker than that, they are rigid and called rods or light pipes. Plastic fibers are flexible at thicker core diameters. Sometimes liquid cores and plastic cladding are used to make flexible, high-transmittance lightguides that are over a centimeter in core diameter.

Numerical Aperture and Étendue

The maximum angle that a fiber can accept and transmit depends on the indices of refraction of the core and cladding (as well as the index of the surrounding medium, usually air, $n_0 = 1$).



$$\sin\theta_{max} = \frac{1}{n_0}\sqrt{n_1^2 - n_2^2}$$

and the NA is

$$NA = n_0 \sin\theta_{max} = \sqrt{n_1^2 - n_2^2}.$$

The fiber has a maximum acceptance projected **solid angle**, $\Omega = \pi \sin^2\theta_{max}$, and an acceptance area, the cross-sectional area of the core. Together, they define a throughput or étendue for the fiber in air:

$$\text{Étendue} = \frac{\pi^2}{4} d^2 NA^2,$$

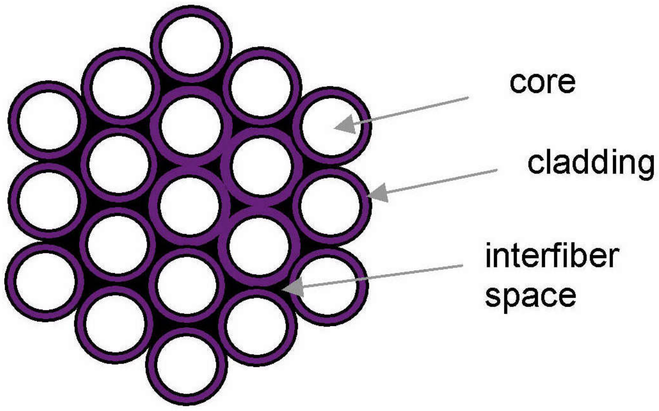
where d is the core diameter.

This étendue defines the maximum flux-carrying capability of the fiber when presented with a source of radiance.

> Note: A fiber illuminated at less than its maximum acceptance angle will, theoretically, preserve the maximum illumination angle at its output. However, bending and scattering at the core-cladding interface broadens this angle toward the maximum allowable. This effect is not important in illumination systems in which it is desirable to utilize the maximum étendue of low-throughput components such as fibers and fill the full input NA.

Fiber Bundles

To achieve high throughput with flexible glass or quartz fibers, multiple fibers are often arranged in a bundle, such as the 19-fiber tightly packed bundle shown below:



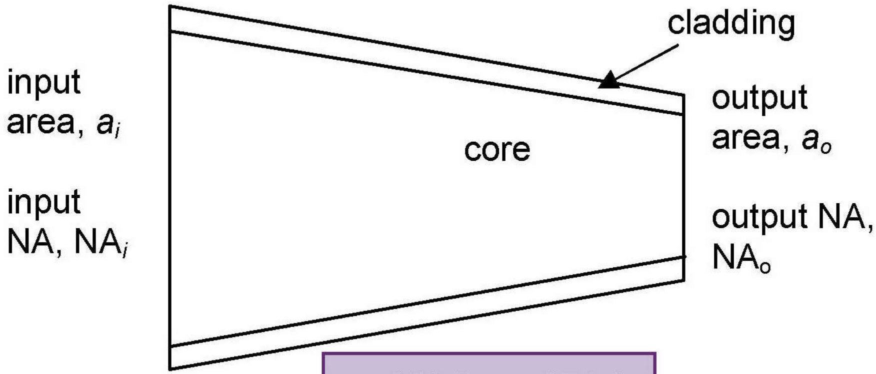
The ratio of the light-carrying core area to the area of the entire bundle is called the **packing fraction (pf)**, and can be as high as 85%. This packing fraction reduces the effective area of the bundle and, correspondingly, its étendue.

In addition to flexibility, fiber bundles have other possible advantages in illumination systems:

- **Shape Conversion:** In some situations, such as when illuminating a spectrometer, it can be useful to convert a circular cross-section of fibers to a line cross-section to align with, or actually become, the entrance slit to the spectrometer.
- **Splitting the Bundle:** By feeding a large fiber bundle with a single light source and splitting the bundle into two or more branches, it is possible to illuminate multiple locations, from multiple angles, with one source.
- **Mixed Bundle:** When illuminating with light over a wide spectral band, such as the full solar spectrum (~250 to 2500 nm), a mixed bundle of high OH silica fibers for good UV transmission and low OH silica fibers for good IR transmission can compensate for the lack of an adequate single-fiber type.

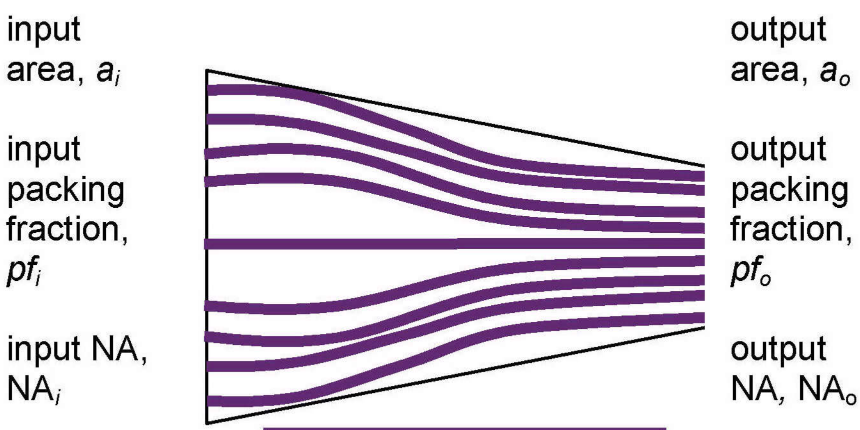
Tapered Fibers and Bundles

By tapering a single fiber, it is possible to trade off between area and solid angle while keeping the product (étendue) approximately constant.



$$a_i \cdot NA_i^2 \approx a_o \cdot NA_o^2$$

On the other hand, when a bundle of straight fibers is tapered, the tradeoff is between the area and packing fraction.

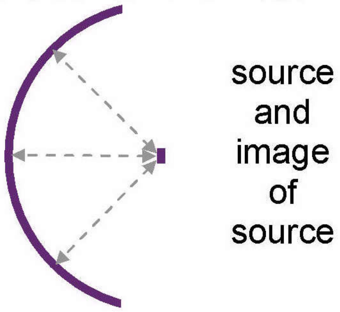


$$NA_o = NA_i = NA_{fiber}$$

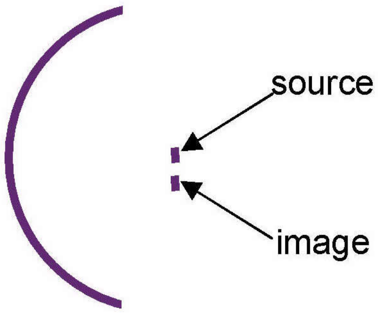
$$a_o \cdot pf_o = a_i \cdot pf_i$$

Spherical Reflector

The light emitted from a source in the direction away from the optical system can be redirected toward the optical system by using a **spherical mirror** with the source located at the center of curvature.



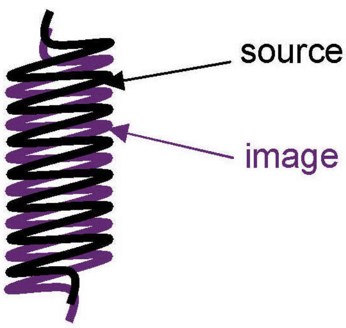
If the source is solid, it is necessary to place the source slightly away from the center of curvature and the image just above, below, or alongside the physical source.



Ignoring losses on reflection, the image has the same radiance as the source, but the effective source area (source plus image) is doubled.

Sometimes this technique is used to place the image of a source in a location where the physical source itself could not fit because of an obstruction such as a lamp envelope or socket.

If the source is not solid, such as a coiled wire tungsten filament, imaging the source almost directly onto itself can help fill in the area between the coils.

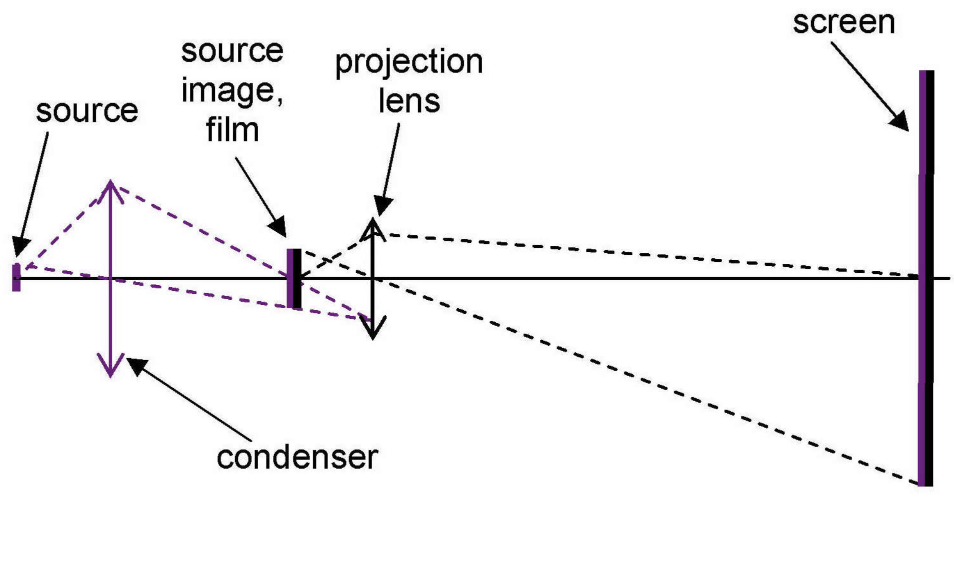


In this case, the effective area of the source is not appreciably increased, but the apparent radiance is nearly doubled.

Abbe Illumination

Abbe illumination is characterized by imaging the source (or imaging an image of the source) directly onto the illuminated area. Since the uniformity of illumination is directly related to the uniformity of source radiance, Abbe illumination requires an extended source of uniform radiance such as a well-controlled arc, a ribbon filament lamp, the output of a clad rod, a frosted bulb, an illuminated diffuser, or the output of an integrating sphere.

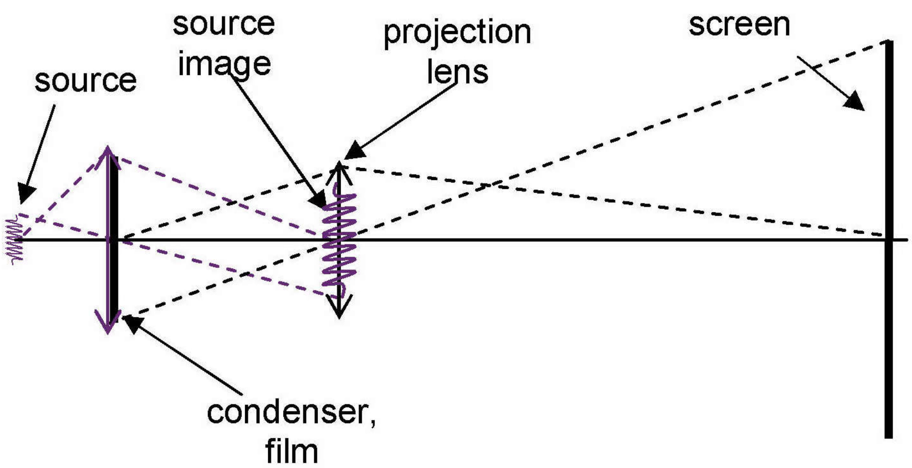
The paraxial layout below shows Abbe illumination used in a projection system. The source is imaged by a condenser onto the film. The projection objective images the film and the image of the source onto the screen. The purple dotted lines show the marginal and chief rays from the source. The black dotted lines show the marginal and chief rays from the film (and the image of the source). The marginal rays go through the on-axis points on the object and image and on the edges of the pupils (which are the lenses in this case). The chief rays go through the edges of the object and image and the on-axis points of the pupils.



Köhler Illumination

Köhler illumination is used when the source is not uniform, such as a coiled tungsten filament. Köhler illumination is characterized by imaging the source through the film onto the projection lens. The film is placed adjacent to the condenser, where the illumination is quite uniform, provided the source has a relative uniform angular distribution of intensity.

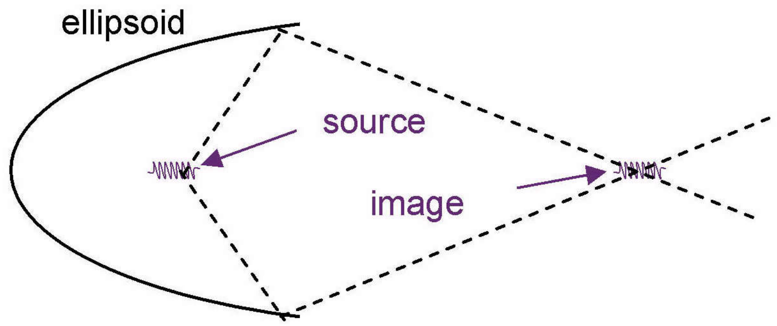
The paraxial layout below shows Köhler illumination used in a projection system. The source is imaged by a condenser onto the projection lens. The projection objective images the film onto the screen. The purple dotted lines show the marginal and chief rays from the source. The black dotted lines show the marginal and chief rays from the film.



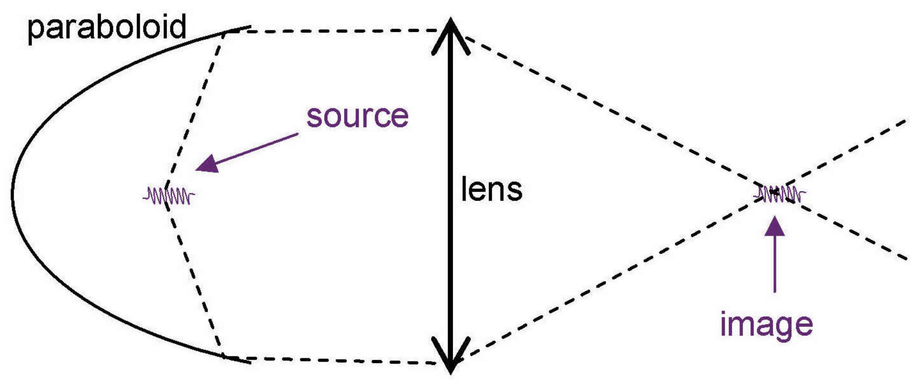
With similar sources, similar condenser NAs, source/condenser étendue as limiting étendue, and similar screen sizes, the average screen irradiance levels are the same for both Abbe and Köhler illumination systems. The choice between the two generally depends upon the type of source available.

Ellipsoidal and Paraboloidal Mirrors

Very efficient collection of light from a source can be achieved using an **ellipsoidal mirror**, placing the source at one of the foci. The source is imaged at the other focus, with light collected over more than a hemisphere.



An alternative is to use a **paraboloidal mirror** to collimate the light from a source and a lens to reimage it. Again, the light from the source is collected over more than a hemisphere.



The forward light is usually ignored in both of these types of designs.

In both cases, the image of the source may not be good quality, but image quality may not be important in illumination systems. Also, obstructions like lamp bases, sockets, and mounting hardware can produce directional anomalies in the radiance of the image.

If the quality of illumination is important, devices such as lenslet arrays or faceted reflectors may be used.

Spectral Control and Heat Management

Specifications for illumination systems often contain spectral requirements. Some of these requirements can be partially met by the selection of lamp type, but usually some sort of filtering is needed. Also, for visual systems, especially those using tungsten lamps, unwanted heat from infrared light may need to be removed. Again, filtering is needed.

The simplest type of filter is the **absorbing filter** placed in front of the light source. Filter glasses with a wide range of spectral characteristics are available from glass manufacturers. The primary concern with absorbing glass filters is cracking from excessive absorbed heat.

> Often a cracked filter will continue to work just fine.

Interference filters use multilayer thin-film coatings that either transmit or reflect light at specific wavelengths. Cracking is generally not a concern unless the filter is made of an absorbing substrate. These filters are available with a much wider variety of spectral properties than absorbing filters, including narrow bandwidth and sharp cut-off, and can be designed and manufactured to achieve specific custom properties. They are also available for different angles of illumination, typically 0 deg and 45 deg.

> Interference filters shift their spectral properties with incident angle and therefore may not be suitable for uncollimated light with a divergence of more than about 10 deg from the axis.

Hot mirrors and **cold mirrors** are excellent ways to manage heat that must be removed from a light source. A hot mirror reflects infrared light and transmits visible light. A cold mirror reflects visible light and transmits infrared light. The reflector behind the light on a dentist's chair is a cold mirror.

Illumination in Visual Afocal Systems

Afocal visual systems, such as binoculars, take the collimated light from an extended distant object and present collimated light to the eye, but with angular magnification. Therefore, the object appears larger. However, the apparent radiance (and therefore perceived brightness) of the object is the same as that of the naked eye, provided the size of the aperture stop is the same with and without the binoculars. Without the binoculars, the aperture stop is merely the eye pupil. With the binoculars, the aperture stop is the smaller of:

- the pupil of the eye magnified by the angular magnification, or
- the aperture of the objective lens.

In other words, if the collimated ray bundle entering the eye from the binoculars is smaller than the eye pupil, the apparent radiance of the object will be less with the binoculars than with the naked eye. If the pupil of the eye is the limiting aperture both with and without the binoculars, the apparent radiance will be the same.

Binoculars are traditionally designated by two numbers, the first being the angular magnification, the second the diameter of the objective lens in mm. A light-adapted eye pupil with a 2-mm diameter would remain the aperture stop for all of the following common sizes of bird-watching binoculars: 8 × 42, 8 × 32, 10 × 42, 6 × 25, and 10 × 25. These binoculars are generally used during the day. However, marine binoculars, which are used under all lighting conditions, are typically 7 × 50 to accommodate a 7-mm-diameter dark-adapted eye pupil.

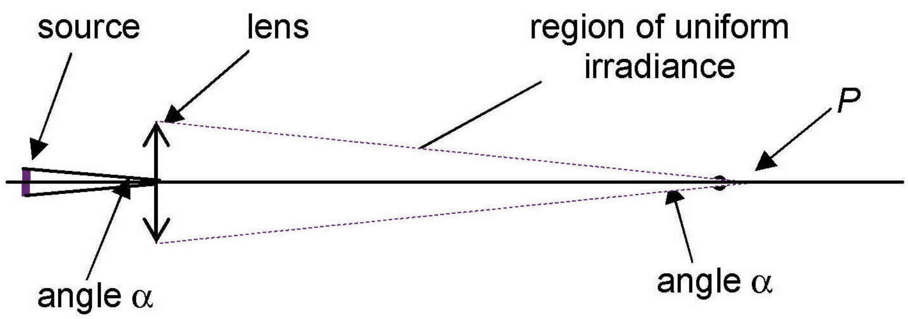
Note that a true point source, such as a star, will have higher apparent intensity (and therefore appear brighter) with binoculars than with the naked eye because more light is collected with the binoculars, but there is no angular magnification.

Searchlight

A **searchlight** can provide uniform irradiance in three dimensions that is extremely insensitive to the position of the irradiated object.

A searchlight consists of a small circular Lambertian source at the focal point of a collimating lens. Anywhere inside the shaded area in the figure below, the source appears as a circular disk at infinity, subtending a full angle α. The entire extent of the source is visible, because it does not completely fill the collimating lens. Since the view of the source is the same anywhere inside this region, the irradiance is the same.

Outside this region and beyond point *P*, the lens restricts the area of the source that is visible. The lens itself appears as a disk of the same radiance as the source. In this region, the irradiance falls off as the square of the distance from the lens.

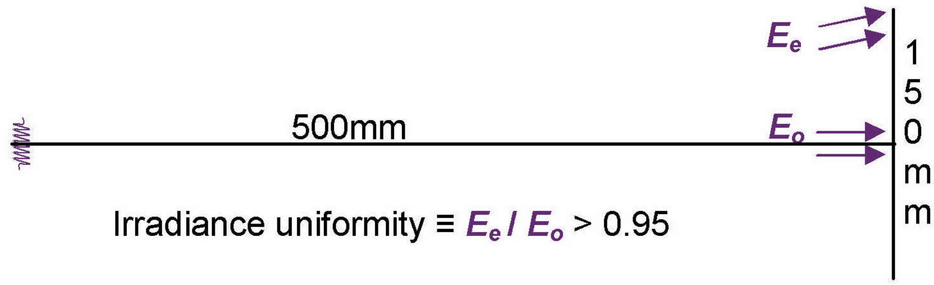


For real searchlights with small sources and large-diameter lenses, the paraxial description above is not exactly valid over the entire shaded region. However, over a relatively small portion of this region, the irradiance is extremely uniform. This region of irradiance uniformity extends not only laterally, but longitudinally as well.

A searchlight provides a volume of uniform irradiance.

Source at a Distance

A small source at a distance from an object can provide reasonably uniform irradiance across the object. It is somewhat counterintuitive that a bare lamp filament, with its obviously terrible radiance uniformity, can produce excellent irradiance uniformity. For example, a small (assumed Lambertian) lamp filament at 500 mm from a flat object whose largest dimension is 150 mm will provide irradiance uniformity across the object of better than 95% (considering only $\cos^4$ falloff). The same lamp and object at 1.0-meter distance produces nearly 99% irradiance uniformity.

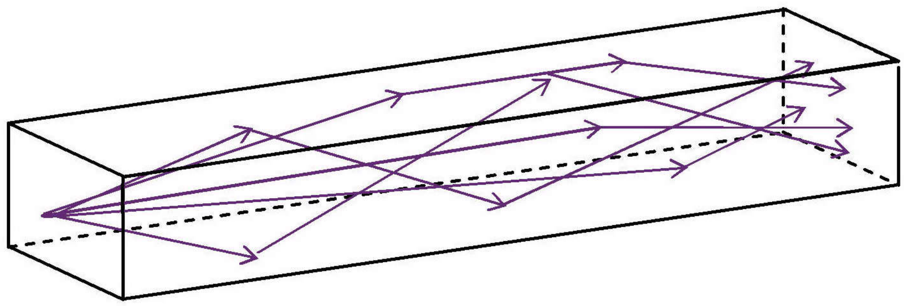


> A source of uniform radiance can be created by illuminating a transmission or reflection diffuser with uniform irradiance.

A common calibration laboratory method used to realize a standard of known radiance is to illuminate a reflection diffuser, typically 50 mm in diameter, with a standard of known irradiance, typically a calibrated 1000-W tungsten halogen lamp (ANSI type FEL), at a 500-mm distance. The irradiance uniformity across the diffuser is better than 99.5%. If the reflectance factor of the diffuser is uniform, the radiance uniformity of the standard is also better than 99.5%.

Mixing Rod

A **mixing rod** is a long piece of clear quartz, glass, or plastic. Light entering one face of the rod undergoes multiple total internal reflections emerging from the other parallel face.



Due to the multiple reflections, the **irradiance** at the exit face can be extremely uniform. In a well-designed and illuminated rod, the **radiance** can be quite directionally uniform as well. The directional uniformity of radiance can be enhanced by placing a diffuser at the exit face of the rod or simply frosting the rod-end itself.

Mixing rods can have any shape desired. The rods with plane sides do a better mixing job in most cases.

Typically the rods have an aspect ratio (length to largest transverse dimension) of about 10:1, and are usually about 75- to 150-mm long. They can be clad like an optical fiber, but generally are not. Unlike a fiber, the number of reflections in a mixing rod is quite small, and losses are not a serious problem.

> Rather than using a rod with polished faces, it is possible to achieve a similar effect using a mirrored tube with a hollow center.

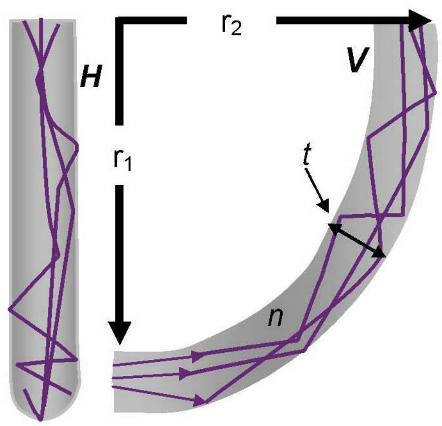
The combination of a rod and its illuminator are sometimes designed by computer simulation. But the degree of uniformity required doesn't always demand this level of complexity, so simple trial-and-error is often sufficient.

Bent Lightpipes

Complex lightpipes made from straight sections, bends, and tapers are common in many industries. **Bent lightpipes** are components used to mix or collect light from different paths that bend around objects or provide light output over an extended region. An example is automotive dashboard illuminators that employ lightpipes coupled to a small incandescent source or an LED. The lightpipe allows the source light to be directed around dials and knobs. The bends allow simple packaging and lower costs at the expense of design complexity.

Any cross-sectional shape, bend angle, bend shape, and so forth is possible, but the simplest is a single, right-angle bend using common-center, circular bends and an arbitrary cross section. A circular cross-section is shown here. Two important slices are called **principal sections**: the vertical (*V*), which shows the bend of the lightpipe, and the horizontal (*H*), which shows the bend going into the page. The vertical slice defines the transmission properties of the lightpipe. For normally incident input light coupled to the lightpipe, there are no propagation losses except Fresnel losses if the **bend ratio**, *R*, is

$$R = r_2/r_1 = 1 + t/r_1 \le n,$$



where r_1 and r_2 are the two bend radii, t is the lightpipe thickness in the vertical section, and n is the lightpipe index in air. As the input angle increases, there are losses at the limit of this equation, but the equation is transcendental. By decreasing the thickness of the lightpipe, one can increase the acceptance angle such that there is no loss.

More complex parameterization of lightpipes, including uncommon bend centers, noncircular bends, and arbitrary cross sections, can be found in the literature.

Integrating Sphere

Integrating spheres produce illumination that has extremely uniform radiance and irradiance. An integrating sphere is a hollow spherical shell coated on the inside with a highly reflecting diffuse coating. The projected solid angle from any point on a sphere to any element of area on the sphere is the same, regardless of location. This fact combined with the diffuse coating and the multiple reflections cause any light introduced into the sphere to produce uniform irradiance on and radiance of the wall of the sphere. A hole or "port" in the sphere allows this uniform illumination to be used in an optical system.

> The radiance at the exit of an integrating sphere extends to a full hemisphere (π projected steradians). The irradiance at the wall of an integrating sphere is incident from a full hemisphere.

The radiance, L, of the wall of an integrating sphere generated by flux, Φ, introduced into the sphere is

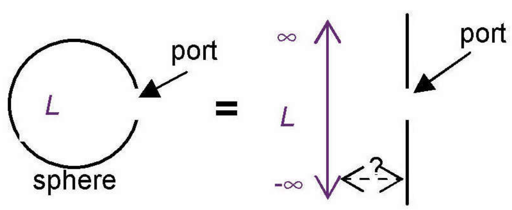
$$L = \frac{\Phi}{\pi \cdot A_s} \cdot M,$$

where A_s is the area of the complete sphere wall, and M is the "sphere multiplier," which is equal to the average numbers of reflections in the sphere. The multiplier, M, is

$$M = \frac{1}{1-\bar{\rho}},$$

where $\bar{\rho}$ is the **average reflectance** of the wall of the sphere, counting the holes as areas of zero reflectance.

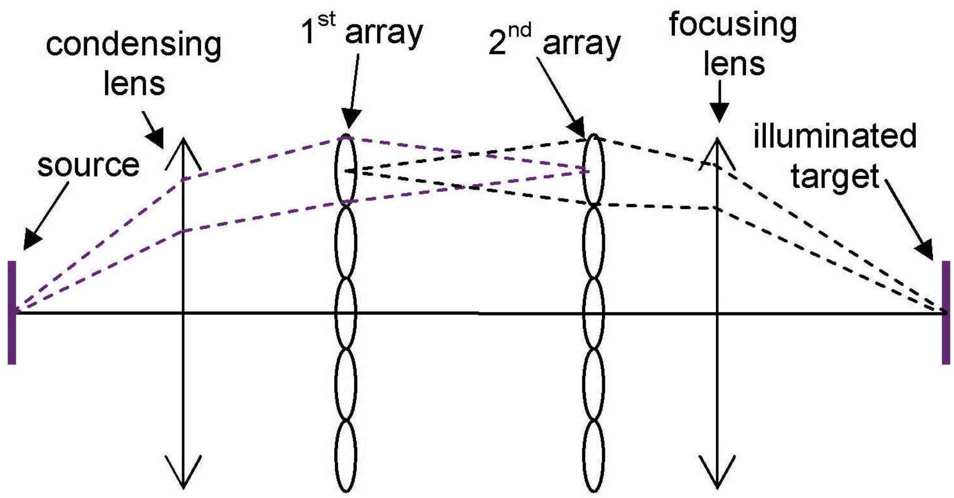
A good working model of an integrating sphere is to consider the port to be a hole in a wall, and, at a totally arbitrary distance behind it, another wall of infinite extent and radiance, L.



Lenslet Arrays

Imaging illumination systems, whether single- or double-lens systems, paraboloidal reflector and lens systems, or single ellipsoidal reflector systems, all suffer from possible nonuniformities in intensity (and consequently also in irradiance). These are due, among other causes, to possible nonuniformities in the source as well as obstructions such as filament support wires, gas discharge electrodes, and LED heat-sink structures.

These nonuniformities can be smoothed out by using a **lenslet array**, an array (usually 2D) of small lenses. Typically, the arrays are used in pairs. In the diagram below, the dotted purple lines show the marginal rays for one of the lenslets in the first array; the black dotted lines show the marginal rays for the corresponding lenslet in the second array.



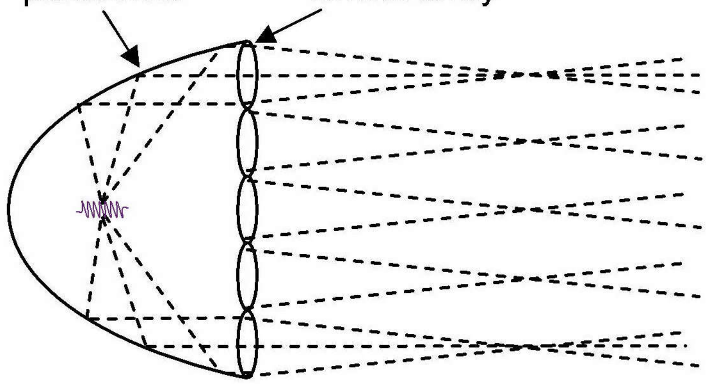
In this configuration, the source is imaged by each lenslet of the first array into the corresponding lenslet of the second array. Each lenslet of the first array is imaged onto the entire target. This overlaying creates uniform illumination of the target. In effect, the lenslet arrays create multiple **Köhler illumination** systems, all superimposed on the target.

Lenslet arrays are generally designed using illumination design software.

Small Reflectors, Lenslet Arrays, and Facets

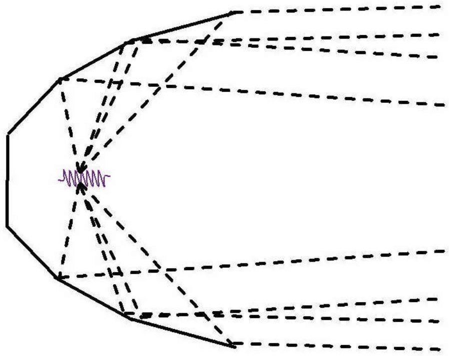
Ellipsoidal and paraboloidal reflectors are often "small" with respect to the lamp dimensions and the distances between the lamp and the reflecting surfaces. In these cases, in addition to the effects of lamp support structures, the size and structure of the lamp itself can produce nonuniformities in illumination.

One method of minimizing these nonuniformities is to include a **lenslet array** in front of the detector. This broadens the beam a little, depending on the *f*# of the lenslets, but it can produce much more uniform illumination than the reflector alone.



> Tandem lenslet arrays also can be used to minimize the effects of small reflectors.

Another approach is to break the reflector into small flat **facets**, either radially, circumferentially, or both.



> Lenslet arrays and faceted reflectors are usually designed with illumination design software.

Source Modeling Overview

A system software model, whether a simple paraxial design or a detailed design of an illumination system, may fail to agree with experimental results due to the lack of a comprehensive **source model**. For the simplest case, where the optics are far away from the source and collect light over a small solid angle, a point source model or a simple geometrical model of the source may be sufficient. The directional distribution of light from these simple models is usually assumed to be isotropic or Lambertian.

For more efficient designs with optics that are close to the source and collect light over a large solid angle, a more complete model of the source is required to obtain meaningful results. These models must reflect the physical size and shape of the source and should contain directional distributions that account for factors such as filament support wires and lamp envelopes.

Source models are made for all types of sources, including LEDs, incandescent, fluorescent, metal vapor, and high-pressure gas discharge sources. The modeling includes spectral, radiance or luminance distributions, and lifetime aspects. For example, accurate source models for the following have been developed:

- The temperature distribution along an incandescent filament varies from its ends to the center. Additionally, the interior of the filament glows "hotter" due to the re-incident radiation.
- Arc emission sources such as metal halide and HID lamps change their radiance distribution and power output over time due to ablation of the electrodes. These lamps have a deposited material to capture this ablation, called the "salt lake" in continuous sources and the "getter" in a pulsed one.

There are essentially four ways of creating complex source models. Three are described on the next page, while the fourth, not presented here, is based on the physics of emission. This method is outside the confines of this text.

Source Modeling Methods

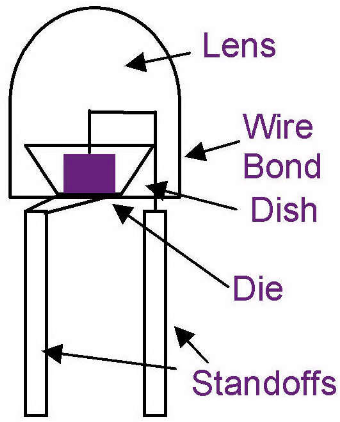
There are three **source modeling methods**, where the accuracy of the model typically increases with number:

1. "Bottom-up" (**geometrical model**): the source geometry starting with the electrodes, supports, and envelope; finishes with the packaging. Emission is assigned to the radiative components.
   - Benefits: No complex measurements; handles re-incident light; provides tolerancing capabilities.
   - Limitations: Emission characteristics assumed; approximate surface and material properties; can include tedious CAD development.
2. "Top-down" (**radiance model**): the optical output of a representative sample of the lamp. These measurements are made with a **goniometer**, which moves a detector around a lamp on two axes. A camera measures the 2D radiance distribution of the lamp from each of many goniometer positions. The resulting 4D model represents a complete description of the lamp that can be used in a computer optical design program.
   - Benefits: Emission is based on physical measurements.
   - Limitations: Does not handle reincident light; is limited by the variance of the number of source samples measured and aligned; and their complex measurement.
3. "Bottom to top" (**system model**): Integrates the bottom-up and top-down approaches to develop a more thorough source model.
   - Benefits: Complete geometrical and radiative models.
   - Limitations: Integration of two submethods.

There are many hybrid methods and methods based on applying the physics of the emission process of a prescribed source. Loosely, the first two methods show agreement to within 25% of experimental results, while the bottom-to-top method shows agreement within 10%. In all cases, rays are assigned, typically in a Monte Carlo approach, to the emission areas.

LED Modeling

The components of an **LED** include the emitting die(s), the lens, the reflecting dish, wire bond and pad, and standoffs. Other components can include phosphors and included detectors.

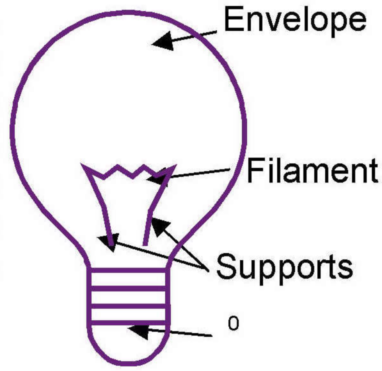


Geometrical modeling is useful to develop LED sources; however, it is difficult to obtain or measure the shapes and sizes of the components within the lens. Radiance modeling suffers because of the large amount of variance between LED samples of one model. The primary issues are the die position within the reflecting dish, the axial position of the die and dish with respect to the lens vertex, and the size and shape of the reflecting dish. Four distinct methods are available for LED modeling:

1. Develop a flat object and assign rays to the surface based upon the intensity distribution provided by the manufacturer. This method ignores spatial variation of the emission.
2. Develop a geometrical model of the LED and assign rays to the emitting surfaces of the die. Optimize the dish shape (typically a cone), size, and the axial offset of the die-dish to the lens vertex. The lens shape must be measured and the die and dish placed at the transverse center of the lens. The model is complete when the intensity pattern from the manufacturer agrees with the ray-trace model.
3. Same as method #2, except develop the layer structure within the die to generate Monte Carlo rays within the active layer(s). This method is tedious for ray tracing due to the index of refraction discontinuity between the die (n = 2.5+) and the epoxy lens (n = 1.45+).
4. Radiance by itself or a system: integrated into #2 or #3.

Incandescent Lamp Modeling

The components of an **incandescent lamp** include the base, filament(s), supports, and the envelope. Other components can include coatings and envelope faceting. Note that the shapes and sizes of components depend on the application. Sources developed for the automotive headlight industry provide the highest level of tolerance from one sample to another.

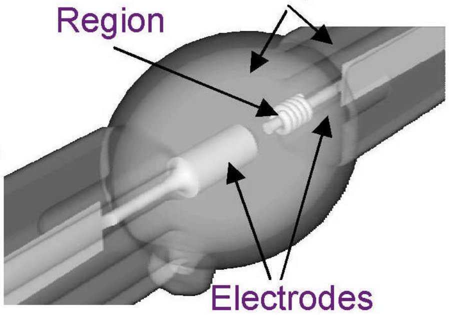


Both geometrical and radiance methods are useful for incandescent source modeling. Radiance modeling is better suited to this source since a goniometer can be focused on the filament source, while the glass envelope supplies little effect on overall optical ray paths. Only light rays that are re-incident (approaching grazing incidence) on the envelope show adverse effects. Geometric modeling involves breaking the glass envelope to gain access to the internal components. This process requires the use of calipers to measure the coil spacing, the thicknesses and lengths of the components, and the number of coils. Provided parameters can help with this process:

- **Maximum overall length (MOL):** Overall distance that includes the base and pins.
- **Light center length (LCL):** Distance between the center of the emitter and a defined reference plane.
- **Filament type:** Designated by @-#, where @ is a series of letters (e.g., C = coiled, CC = coiled coil, and SR = straight ribbon), and # is a number providing an arbitrary pattern for the filament supports.
- **Bulb type:** Designated by @-#, where @ is the bulb shape (e.g., T = tubular), and # is the diameter in eighths of an inch.
- **Base type:** Innumerable types that have no shorthand notation to describe them. Examples include screw, mogul, bipin, and prong.

Arc and Fluorescent Lamp Modeling

The components of an **arc lamp** include the base(s), electrodes, and envelope(s). Other components can include coatings, salt lake (continuous) or getter (pulsed), and ignition wire (flashlamp). The optical radiation is represented by a virtual object called the arc. Note that the aspects of components depend on the application. Automotive headlight arcs provide the highest level of accuracy from one sample to another.

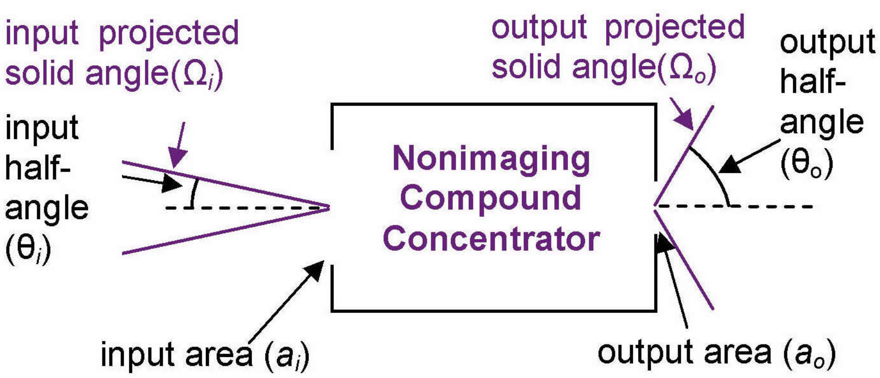


Radiance modeling is especially suited to this source since geometrical modeling cannot effectively represent the arc. The arc must be approximated with a cylinder, tube, or some other geometric shape. Radiance modeling is also suited to this source because a goniometer can be focused on the arc. Due to the typical smaller sizes of these sources compared to incandescent sources, the effect of re-incident rays is more pronounced. Thus, methods to integrate a simplified measurement of the radiance distribution into the geometrical model have been employed. One such method uses the **Abel transform** based on a single image capture of the arc. The Abel transform assumes symmetry of the arc shape and revolves it around a localized centroid of the arc source. Such system models are the most effective way to model such sources.

Fluorescent lamps include the tube and base(s). These are the simplest sources to model other than the complex geometry of compact fluorescent lamps now available. After the geometry is entered, the inner surface of the tube acts as the emitter. Internally, mercury vapor is excited, releasing UV radiation, which is then converted into visible light upon being incident on the phosphor. Geometrical modeling is better suited to this source due to the large size and simplicity of the configurations.

Nonimaging Compound Concentrators

Nonimaging compound concentrators were first developed for solar energy collection to concentrate the **irradiance** from the sun. In solar collection, depending on the degree of sophistication of the sun tracking system, the range of sun input angles can be fairly small. The collectors (solar cells, water pipes, etc.) respond essentially to irradiance and can be illuminated at any angle. The compound concentrator trades off between area and solid angle, presenting a large collection area to the sun (collecting over a narrow solid angle) and delivering the energy to a smaller area (and over a wider solid angle). These devices come close to achieving the theoretical maximum concentration (in three dimensions).



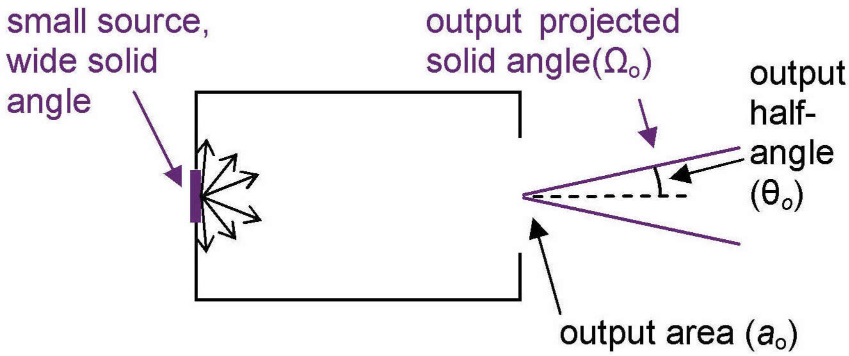
If the output projected **solid angle** is the maximum, π, (θ_o = 90 deg), the concentration is maximum:

$$\frac{a_i}{a_o}(\max) = \frac{1}{\sin^2 \theta_i}.$$

Nonimaging compound concentrators are designed using the **edge-ray principle**, which directs all rays that are at the maximum input angle (θ_i for θ_o = 90 deg in the drawing above) to the edge of the output aperture. All rays at input angles less than this maximum are directed inside the output aperture with no concern for image quality. Often this angle, θ_i, is called the **acceptance angle**, θ_a.

Concentrators as Luminaires

Nonimaging compound concentrators are used in illumination as **luminaires**—devices used to direct the light from a source for illumination. For illumination, they are used in the reverse direction from their configuration in solar collection; they collect light from as large an angle as possible from a small source and direct it over a smaller angle through a larger aperture. For solar collection, they collect energy over a large area and a small angle, delivering it to a small area.



Nonimaging compound concentrators are efficient because:

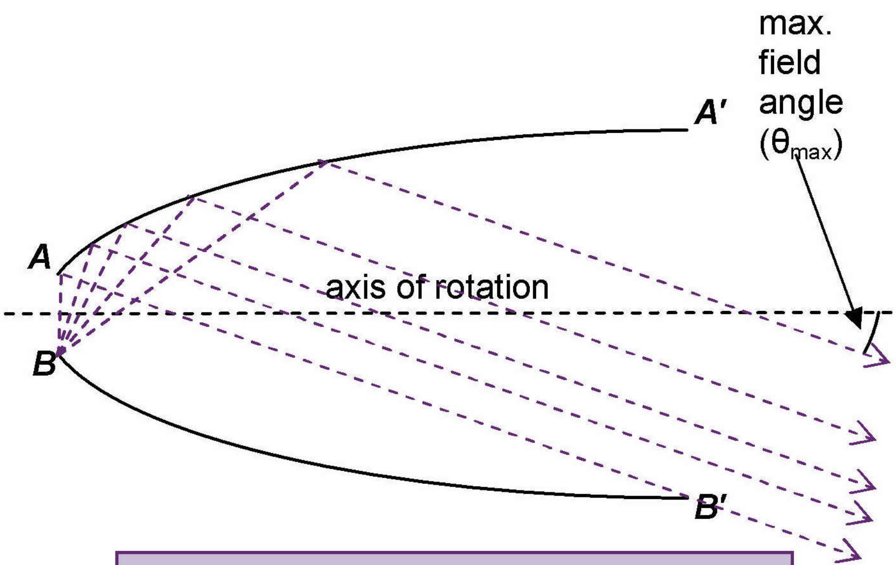
- They collect light from the source over a very large solid angle.
- They are designed using the **edge-ray principle**, keeping all the energy within the intended field.

> Imaging systems are designed to be best on axis, with the edges of the field "spilling over." Nonimaging compound concentrators are designed to be best at the edges of the field, keeping all the energy inside the design boundaries.

Nonimaging concentrators used as luminaires are usually composed of an internal mirror surface with the figure of a **compound parabolic concentrator (CPC)**, **compound elliptical concentrator (CEC)**, or **compound hyperbolic concentrator (CHC)**. Dielectric filled concentrators that employ total internal reflection are also used.

Compound Parabolic Concentrators

The **compound parabolic concentrator (CPC)** is a common shape of nonimaging concentrator used for illumination. A CPC is formed by a parabola with its focus at one edge of the entrance (small) aperture, rotated around an axis that is perpendicular to and through the center of both apertures. CPCs can be quite long.



B is the focus of parabola *AA′*.
A is the focus of parabola *BB′*.

The complete equation for the surface of a CPC can be found in the equation summary in the Appendix.

The ratio of the diameter of the small and large apertures is determined by maximum field angle

$$\frac{d_o}{d_i} = \frac{1}{\sin\theta_{max}},$$

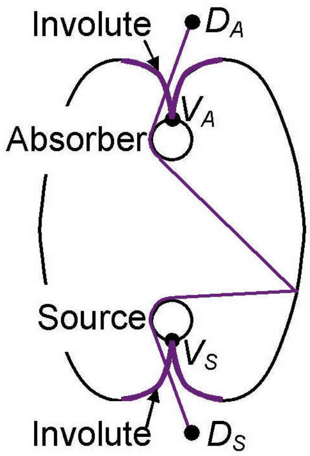
where d_o and d_i are the diameters of the output (large) and input (small) apertures, respectively.

The length of the concentrator is

$$\text{Length} = \frac{d_o + d_i}{2\tan\theta_{max}}.$$

Compound Elliptical and Hyperbolic Concentrators

The **compound elliptical concentrator (CEC)** and **compound hyperbolic concentrator (CHC)** work in the domains of finite and diverging conjugates, while the CPC worked at infinite conjugates. For CECs, the CPC-development methods can be used. To visualize their shapes, consider what is called the **string method**, where the string acts as the edge ray:



- Choose two points, D_A and D_S.
- Select a length of string that allows the pen, P, to read points V_A and V_S.
- While pulling the string taut with P, sweep out the shape on one side of the reflector.
- Flip the string to the other side and repeat.

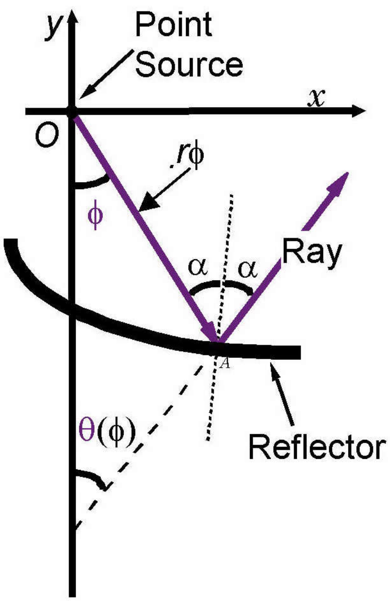
For a CPC, the absorber points D_A and V_A are located at infinity such that a constant angle, θ_a, is obtained. This method is adaptable to handle nonplanar sources. If the source, as shown in the figure, impedes on the string path, then a secondary region called the **involute** is formed. The involute ensures that rays with output angles less than θ_a are transferred by the reflector. Such reflectors with nonplanar sources or absorbers are better denoted as edge-ray reflectors. Note that the terms "absorber" and "source" are swapped for collector design.

The CHC is designed with the **flow-line method**, which treats light rays as fluid flow. Due to the conservation of étendue, we find vectors that define the geometrical flux through the amplitude and their directions define the flow line. The flow lines are hyperbolae. A reflector can be placed along one of the rotationally symmetric flow lines, and due to invariance there is no adverse effect on light emission from a Lambertian source located at the flow line origin.

Tailored-Edge-Ray Design

The involute sections of edge-ray luminaires for nonplanar sources or absorbers indicate that the acceptance angle does not need to be constant over the extent of the reflector. Designs with a functional acceptance angle are called **tailored-edge-ray reflectors**. Using the figure, the equation that governs their shape for a point source design is

$$r(\phi) = r_1 \exp\left[\int_{\phi_1}^{\phi} \tan\left(\frac{s-\theta(s)}{2}\right) ds\right],$$



where r is the distance from the point source to the reflector, r_1 is the distance for the polar angle ϕ_1, and θ is the desired output intensity from the reflector. θ is the variable acceptance angle (note that "acceptance" is a holdover from solar concentrator design). For uniformity at the target,

$$\theta(\phi) = \arctan\left[\tan\theta_1 + \int_{\phi_1}^{\phi} I_{src}(v)dv\right],$$

where I_{src} is the intensity distribution emitted by the point source.

To allow for finite-extent sources, the first equation can be modified, and the reader is encouraged to consult the literature.

Though the formalism presented here might appear daunting, tailored-edge-ray design is a powerful tool to design optimal optics around both the emission aspects of the source and the desired irradiance distribution at the target. The one caveat is that tolerances are quite demanding unless one places sufficient leeway into the source intensity distribution (I_{src}).