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FUNDAMENTALS OF HEAT AND MASS TRANSFER

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FUNDAMENTALS OF HEAT AND MASS TRANSFER

THIRD EDITION

FUNDAMENTALS OF HEAT AND MASS TRANSFER

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Dedicated to those wonderful women in our lives,

Amy, Andrea, Debbie, Donna, Jody,
Karen, Shaunna, and Terri

who, through the years, have blessed us with
their love, patience, and understanding.

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PREFACE

With the passage of approximately nine years since publication of the first edition, this text has been transformed from the status of a newcomer to a mature representative of heat transfer pedagogy. Despite this maturation, however, we like to think that, while remaining true to certain basic tenets, our treatment of the subject is constantly evolving.

Preparation of the first edition was strongly motivated by the belief that, above all, a first course in heat transfer should do two things. First, it should instill within the student a genuine appreciation for the physical origins of the subject. It should then establish the relationship of these origins to the behavior of thermal systems. In so doing, it should develop methodologies which facilitate application of the subject to a broad range of practical problems, and it should cultivate the facility to perform the kind of engineering analysis which, if not exact, still provides useful information concerning the design and/or performance of a particular system or process. Requirements of such an analysis include the ability to discern relevant transport processes and simplifying assumptions, identify important dependent and independent variables, develop appropriate expressions from first principles, and introduce requisite material from the heat transfer knowledge base. In the first edition, achievement of this objective was fostered by couching many of the examples and end-of-chapter problems in terms of actual engineering systems.

The second edition was also driven by the foregoing objectives, as well as by input derived from a questionnaire sent to over 100 colleagues who used, or were otherwise familiar with, the first edition. A major consequence of this input was publication of two versions of the book, *Fundamentals of Heat and Mass Transfer* and *Introduction to Heat Transfer*. As in the first edition, the *Fundamentals* version included mass transfer, providing an integrated treatment of heat, mass and momentum transfer by convection and separate treatments of heat and mass transfer by diffusion. The *Introduction* version of the book was intended for users who embraced the treatment of heat transfer but did not wish to cover mass transfer effects. In both versions, significant improvements were made in the treatments of numerical methods and heat transfer with phase change.

In this latest edition, changes have been motivated by the desire to expand the scope of applications and to enhance the exposition of physical principles. Consideration of a broader range of technically important problems is facilitated by increased coverage of existing material on thermal contact resistance, fin performance, convective heat transfer enhancement, and compact heat exchangers, as well as by the addition of new material on submerged jets (Chapter 7) and free convection in open, parallel plate channels (Chapter 9). Submerged jets are widely used for industrial cooling and drying operations, while free convection in parallel plate channels is pertinent to passive cooling and heating systems. Expanded discussions of physical principles are concentrated in the chapters on single-phase convection (Chapters 7 to 9) and relate, for example, to forced convection in tube banks and to free convection on plates and in cavities. Other improvements relate to the methodology of performing a first law analysis, a more generalized lumped capacitance analysis, transient conduction in semi-infinite media, and finite-difference solutions.

In this edition, the old Chapter 14, which dealt with multimode heat transfer problems, has been deleted and many of the problems have been transferred to earlier chapters. This change was motivated by recognition of the importance of multimode effects and the desirability of impacting student consciousness with this importance at the earliest possible time. Hence, problems involving more than just a superficial consideration of multimode effects begin in Chapter 7 and increase in number through Chapter 13.

The last, but certainly not the least important, improvement in this edition is the inclusion of nearly 300 new problems. In the spirit of our past efforts, we have attempted to address contemporary issues in many of the problems. Hence, as well as relating to engineering applications such as energy conversion and conservation, space heating and cooling, and thermal protection, the problems deal with recent interests in electronic cooling, manufacturing, and material processing. Many of the problems are drawn from our accumulated research and consulting experiences; the solutions, which frequently are not obvious, require thoughtful implementation of the *tools* of heat transfer. It is our hope that in addition to reinforcing the student's understanding of principles and applications, the problems serve a motivational role by relating the subject to real engineering needs.

Over the past nine years, we have been fortunate to have received constructive suggestions from many colleagues throughout the United States and Canada. It is with pleasure that we express our gratitude for this input.

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CONTENTS

SYMBOLS

A	area, m^2
A_c	cross-sectional area, m^2
A_{ff}	free-flow area in compact heat exchanger core (minimum cross-sectional area available for flow through the core), m^2
A_{fr}	heat exchanger frontal area, m^2
A_p	area of prime (unfinned) surface, m^2
A_r	nozzle area ratio
A_s	surface area, m^2
a	acceleration, m/s^2
Bi	Biot number
Bo	Bond number
C	molar concentration, $kmol/m^3$; heat capacity rate, W/K
C_D	drag coefficient
C_f	friction coefficient
C_t	thermal capacitance, J/K
c	specific heat, $J/kg \cdot K$; speed of light, m/s
c_p	specific heat at constant pressure, $J/kg \cdot K$
c_v	specific heat at constant volume, $J/kg \cdot K$
D	diameter, m
D_{AB}	binary mass diffusion coefficient, m^2/s
D_h	hydraulic diameter, m
E	thermal (sensible) internal energy, J; electric potential, V; emissive power, W/m^2
Ec	Eckert number
$\dot{E}_g$	rate of energy generation, W
$\dot{E}_{in}$	rate of energy transfer into a control volume, W
$\dot{E}_{out}$	rate of energy transfer out of control volume, W
$\dot{E}_{st}$	rate of increase of energy stored within a control volume, W
e	thermal internal energy per unit mass, J/kg; surface roughness, m
F	force, N; heat exchanger correction factor; fraction of blackbody radiation in a wavelength band; view factor
Fo	Fourier number
f	friction factor; similarity variable
G	irradiation, W/m^2; mass velocity, $kg/s \cdot m^2$
Gr	Grashof number
Gz	Graetz number
g	gravitational acceleration, m/s^2
g_c	gravitational constant, $1\ kg \cdot m/N \cdot s^2$ or $32.17\ ft \cdot lb_m/lb_f \cdot s^2$
H	nozzle height, m
h	convection heat transfer coefficient, $W/m^2 \cdot K$; Planck's constant
h_{fg}	latent heat of vaporization, J/kg
h_m	convection mass transfer coefficient, m/s
h_{rad}	radiation heat transfer coefficient, $W/m^2 \cdot K$
I	electric current, A; radiation intensity, $W/m^2 \cdot sr$
i	electric current density, A/m^2; enthalpy per unit mass, J/kg
J	radiosity, W/m^2
Ja	Jakob number
J_i^*	diffusive molar flux of species i relative to the mixture molar average velocity, $kmol/s \cdot m^2$
j_i	diffusive mass flux of species i relative to the mixture mass average velocity, $kg/s \cdot m^2$
j_H	Colburn j factor for heat transfer
j_m	Colburn j factor for mass transfer
k	thermal conductivity, $W/m \cdot K$; Boltzmann's constant
k_0	zero-order, homogeneous reaction rate constant, $kmol/s \cdot m^3$
k_1	first-order, homogeneous reaction rate constant, s^{-1}
k_1''	first-order, homogeneous reaction rate constant, m/s
L	characteristic length, m
Le	Lewis number
M	mass, kg; number of heat transfer lanes in a flux plot; reciprocal of the Fourier number for finite-difference solutions
$\dot{M}_i$	rate of transfer of mass for species i, kg/s
$\dot{M}_{i,g}$	rate of increase of mass of species i due to chemical reactions, kg/s
$\dot{M}_{in}$	rate at which mass enters a control volume, kg/s

$\dot{M}_{\text{out}}$	rate at which mass leaves a control volume, kg/s
$\dot{M}_{\text{st}}$	rate of increase of mass stored within a control volume, kg/s
$\mathcal{M}_i$	molecular weight of species i, kg/kmol
m	mass, kg
$\dot{m}$	mass flow rate, kg/s
m_i	mass fraction of species i, ρ_i/ρ
N	number of temperature increments in a flux plot; total number of tubes in a tube bank; number of surfaces in an enclosure
Nu	Nusselt number
NTU	number of transfer units
N_i	molar transfer rate of species i relative to fixed coordinates, kmol/s
N_i''	molar flux of species i relative to fixed coordinates, $\text{kmol/s} \cdot \text{m}^2$
$\dot{N}_i$	molar rate of increase of species i per unit volume due to chemical reactions, $\text{kmol/s} \cdot \text{m}^3$
$\dot{N}_i''$	surface reaction rate of species i, $\text{kmol/s} \cdot \text{m}^2$
n_i''	mass flux of species i relative to fixed coordinates, $\text{kg/s} \cdot \text{m}^2$
$\dot{n}_i$	mass rate of increase of species i per unit volume due to chemical reactions, $\text{kg/s} \cdot \text{m}^3$
N_L, N_T	number of tubes in longitudinal and transverse directions
P_L, P_T	dimensionless longitudinal and transverse pitch of a tube bank
P	perimeter, m; general fluid property designation
Pe	Peclet number ($RePr$)
Pr	Prandtl number
p	pressure, N/m^2
Q	energy transfer, J
q	heat transfer rate, W
$\dot{q}$	rate of energy generation per unit volume, W/m^3
q'	heat transfer rate per unit length, W/m
q''	heat flux, W/m^2
R	cylinder radius, m
$\mathcal{R}$	universal gas constant
Ra	Rayleigh number
Re	Reynolds number
R_e	electric resistance, Ω
R_f	fouling factor, $\text{m}^2 \cdot \text{K/W}$
R_m	mass transfer resistance, s/m^3
$R_{m,n}$	residual for the m, n nodal point
R_t	thermal resistance, K/W
$R_{t,c}$	thermal contact resistance, K/W
r_o	cylinder or sphere radius, m

r, ϕ, z	cylindrical coordinates
r, θ, ϕ	spherical coordinates
S	solubility, $\text{kmol/m}^3 \cdot \text{atm}$; shape factor for two-dimensional conduction, m; nozzle pitch, m; plate spacing, m
S_c	solar constant
Sc	Schmidt number
Sh	Sherwood number
St	Stanton number
S_D, S_L, S_T	diagonal, longitudinal, and transverse pitch of a tube bank, m
T	temperature, K
t	time, s
U	overall heat transfer coefficient, $\text{W/m}^2 \cdot \text{K}$
u, v, w	mass average fluid velocity components, m/s
u^*, v^*, w^*	molar average velocity components, m/s
V	volume, m^3; fluid velocity, m/s
v	specific volume, m^3/kg
W	width of a slot nozzle, m
$\dot{W}$	rate at which work is performed, W
We	Weber number
X, Y, Z	components of the body force per unit volume, N/m^3
x, y, z	rectangular coordinates, m
x_c	critical location for transition to turbulence, m
$x_{\text{fd},c}$	concentration entry length, m
$x_{\text{fd},h}$	hydrodynamic entry length, m
$x_{\text{fd},t}$	thermal entry length, m
x_i	mole fraction of species i, C_i/C

Greek Letters

α	thermal diffusivity, m^2/s; heat exchanger surface area per unit volume, m^2/m^3; absorptivity
β	volumetric thermal expansion coefficient, K^{-1}
Γ	mass flow rate per unit width in film condensation, $\text{kg/s} \cdot \text{m}$
δ	hydrodynamic boundary layer thickness, m
δ_c	concentration boundary layer thickness, m
δ_t	thermal boundary layer thickness, m
ε	emissivity; porosity of a packed bed; heat exchanger effectiveness
ε_f	fin effectiveness
ε_H	turbulent diffusivity for heat transfer, m^2/s
ε_M	turbulent diffusivity for momentum transfer, m^2/s

ε_m	turbulent diffusivity for mass transfer, m^2/s
η	similarity variable
η_f	fin efficiency
η_o	fin temperature effectiveness
θ	zenith angle, rad; temperature difference, K
κ	absorption coefficient, m^{-1}
λ	wavelength, μm
μ	viscosity, $kg/s \cdot m$
ν	kinematic viscosity, m^2/s; frequency of radiation, s^{-1}
ρ	mass density, kg/m^3; reflectivity
σ	Stefan-Boltzmann constant; electrical conductivity, $1/\Omega \cdot m$; normal viscous stress, N/m^2; surface tension, N/m; ratio of heat exchanger minimum cross-sectional area to frontal area
Φ	viscous dissipation function, s^{-2}
ϕ	azimuthal angle, rad
ψ	stream function, m^2/s
τ	shear stress, N/m^2; transmissivity
ω	solid angle, sr

Subscripts

A, B	species in a binary mixture
abs	absorbed
am	arithmetic mean
b	base of an extended surface; blackbody
c	cross-sectional; concentration; cold fluid
cr	critical insulation thickness
cond	conduction
conv	convection
CF	counterflow
D	diameter; drag
dif	diffusion
e	excess; emission
evap	evaporation
f	fluid properties; fin conditions; saturated liquid conditions
fd	fully developed conditions
g	saturated vapor conditions
H	heat transfer conditions
h	hydrodynamic; hot fluid
i	general species designation; inner surface of an annulus; initial condition; tube inlet condition; incident radiation
L	based on characteristic length
l	saturated liquid conditions
lm	log mean condition
M	momentum transfer condition
m	mass transfer condition; mean value over a tube cross section
max	maximum fluid velocity
o	center or midplane condition; tube outlet condition; outer
R	reradiating surface
r, ref	reflected radiation
rad	radiation
S	solar conditions
s	surface conditions; solid properties
sat	saturated conditions
sky	sky conditions
sur	surroundings
t	thermal
tr	transmitted
v	saturated vapor conditions
x	local conditions on a surface
λ	spectral
∞	free stream conditions

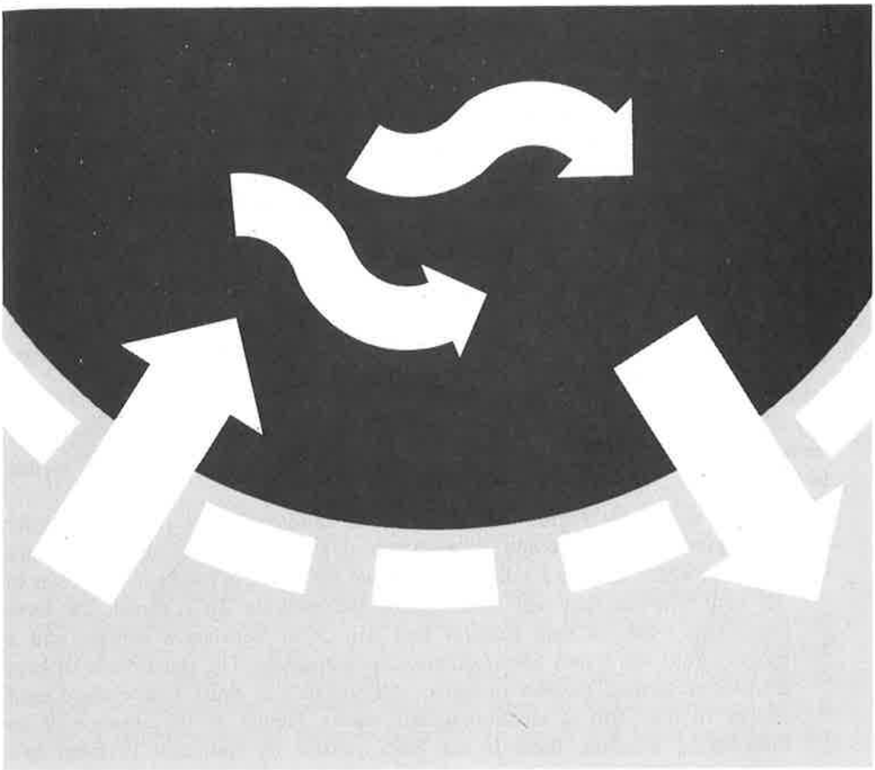
Superscripts

′	fluctuating quantity
*	molar average; dimensionless quantity

Overbar

$\overline{\ }$	surface average conditions; time mean

CHAPTER 1



INTRODUCTION

From the study of thermodynamics, you have learned that energy can be transferred by interactions of a system with its surroundings. These interactions are called work and heat. However, thermodynamics deals with the end states of the process during which an interaction occurs and provides no information concerning the nature of the interaction or the time rate at which it occurs. The objective of this text is to extend thermodynamic analysis through study of the *modes* of heat transfer and through development of relations to calculate heat transfer *rates*. In this chapter we lay the foundation for much of the material treated in the text. We do so by raising several questions. *What is heat transfer*? *How is heat transferred*? *Why is it important to study it*? In answering these questions, we will begin to appreciate the physical mechanisms that underlie heat transfer processes and the relevance of these processes to our industrial and environmental problems.

1.1 WHAT AND HOW?

A simple, yet general, definition provides sufficient response to the question: What is heat transfer?

Heat transfer (or heat) is energy in transit due to a temperature difference.

Whenever there exists a temperature difference in a medium or between media, heat transfer must occur.

As shown in Figure 1.1, we refer to different types of heat transfer processes as *modes*. When a temperature gradient exists in a stationary medium, which may be a solid or a fluid, we use the term *conduction* to refer to the heat transfer that will occur across the medium. In contrast, the term *convection* refers to heat transfer that will occur between a surface and a moving fluid when they are at different temperatures. The third mode of heat transfer is termed *thermal radiation*. All surfaces of finite temperature emit energy in the form of electromagnetic waves. Hence, in the absence of an intervening medium, there is net heat transfer by radiation between two surfaces at different temperatures.

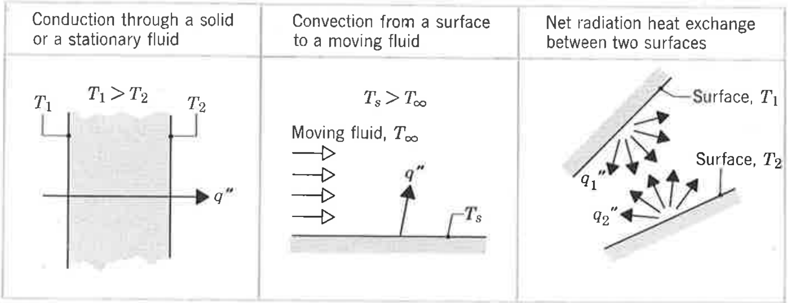


Figure 1.1 Conduction, convection, and radiation heat transfer modes.

1.2 PHYSICAL ORIGINS AND RATE EQUATIONS

As engineers it is important that we understand the *physical mechanisms* that underlie the heat transfer modes and that we be able to use the rate equations that quantify the amount of energy being transferred per unit time.

1.2.1 Conduction

At mention of the word "conduction," we should immediately conjure up concepts of *atomic* and *molecular activity*, for it is processes at these levels that sustain this mode of heat transfer. Conduction may be viewed as the transfer of energy from the more energetic to the less energetic particles of a substance due to interactions between the particles.

The physical mechanism of conduction is most easily explained by considering a gas and using ideas familiar from your thermodynamics background. Consider a gas in which there exists a temperature gradient and assume that there is *no* bulk motion. The gas may occupy the space between two surfaces that are maintained at different temperatures, as shown in Figure 1.2. We associate the temperature at any point with the energy of the gas molecules in the vicinity of the point. This energy is related to the random translational motion, as well as to the internal rotational and vibrational motions, of the molecules. Moreover, higher temperatures are associated with higher molecular energies, and when neighboring molecules collide, as they are constantly doing, a transfer of energy from the more energetic to the less energetic molecules must occur. In the presence of a temperature gradient, energy transfer by conduction must then occur in the direction of decreasing temperature. This transfer is evident from Figure 1.2. The hypothetical plane at x_o is constantly being crossed by molecules from above and below due to their *random* motion. However, molecules from above are associated with a larger temperature than those from below, in which case there must be a *net*

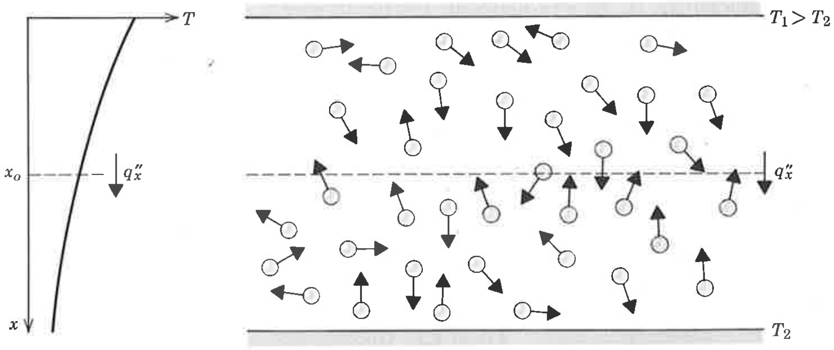


Figure 1.2 Association of conduction heat transfer with diffusion of energy due to molecular activity.

transfer of energy in the positive x direction. We may speak of the net transfer of energy by random molecular motion as a *diffusion* of energy.

The situation is much the same in liquids, although the molecules are more closely spaced and the molecular interactions are stronger and more frequent. Similarly, in a solid, conduction may be attributed to atomic activity in the form of lattice vibrations. The modern view is to ascribe the energy transfer to *lattice waves* induced by atomic motion. In a nonconductor, the energy transfer is exclusively via these lattice waves; in a conductor it is also due to the translational motion of the free electrons. We treat the important properties associated with conduction phenomena in Chapter 2 and in Appendix A.

Examples of conduction heat transfer are legion. The exposed end of a metal spoon suddenly immersed in a cup of hot coffee will eventually be warmed due to the conduction of energy through the spoon. On a winter day there is significant energy loss from a heated room to the outside air. This loss is principally due to conduction heat transfer through the wall that separates the room air from the outside air.

It is possible to quantify heat transfer processes in terms of appropriate *rate equations*. These equations may be used to compute the amount of energy being transferred per unit time. For heat conduction, the rate equation is known as *Fourier's law*. For the one-dimensional plane wall shown in Figure 1.3, having a temperature distribution $T(x)$, the rate equation is expressed as

$$q_x'' = -k\frac{dT}{dx} \tag{1.1}$$

The *heat flux* q_x'' (W/m^2) is the heat transfer rate in the x direction *per* unit area *perpendicular* to the direction of transfer, and it is proportional to the temperature gradient, dT/dx, in this direction. The proportionality constant k is a *transport* property known as the *thermal conductivity* (W/m · K) and is a characteristic of the wall material. The minus sign is a consequence of the fact that heat is transferred in the direction of decreasing temperature. Under the steady-state conditions shown in Figure 1.3, where the temperature distribu-

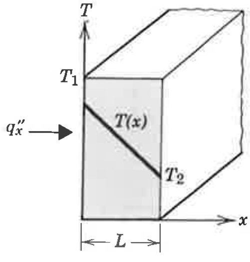


Figure 1.3 One-dimensional heat transfer by conduction (diffusion of energy).

tion is linear, the temperature gradient may be expressed as

$$\frac{dT}{dx} = \frac{T_2 - T_1}{L}$$

and the heat flux is then

$$q''_x = -k\frac{T_2 - T_1}{L}$$

or

$$q''_x = k\frac{T_1 - T_2}{L} = k\frac{\Delta T}{L} \tag{1.2}$$

Note that this equation provides a *heat flux*, that is, the rate of heat transfer per *unit area*. The *heat rate* by conduction, q_x (W), through a plane wall of area A is then the product of the flux and the area, $q_x = q''_x \cdot A$.

EXAMPLE 1.1

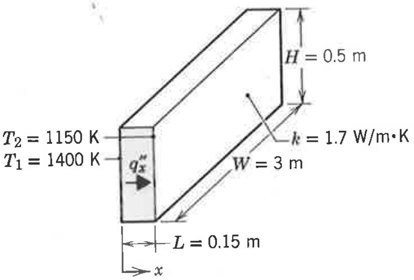
The wall of an industrial furnace is constructed from 0.15 m thick fireclay brick having a thermal conductivity of 1.7 W/m · K. Measurements made during steady-state operation reveal temperatures of 1400 and 1150 K at the inner and outer surfaces, respectively. What is the rate of heat loss through a wall which is 0.5 m by 3 m on a side?

SOLUTION

Known: Steady-state conditions with prescribed wall thickness, area, thermal conductivity, and surface temperatures.

Find: Wall heat loss.

Schematic:



Assumptions:

1. Steady-state conditions.
2. One-dimensional conduction through the wall.
3. Constant properties.

Analysis: Since heat transfer through the wall is by conduction, the heat flux may be determined from Fourier's law. Using Equation 1.2

$$q''_x = k\frac{\Delta T}{L} = 1.7\ \text{W/m}\cdot\text{K} \times \frac{250\ \text{K}}{0.15\ \text{m}} = 2833\ \text{W/m}^2$$

The heat flux represents the rate of heat transfer through a section of unit area. The wall heat loss is then

$$q_x = (HW)q''_x = (0.5\ \text{m} \times 3.0\ \text{m})\ 2833\ \text{W/m}^2 = 4250\ \text{W} \qquad \triangleleft$$

Comments:

1. Note direction of heat flow.
2. Note distinction between heat flux and heat rate.

1.2.2 Convection

The convection heat transfer *mode* is comprised of *two mechanisms*. In addition to energy transfer due to *random molecular motion* (diffusion), there is also energy being transferred by the *bulk*, or *macroscopic*, *motion* of the fluid. This fluid motion is associated with the fact that, at any instant, large numbers of molecules are moving collectively or as aggregates. Such motion, in the presence of a temperature gradient, will give rise to heat transfer. Because the molecules in the aggregate retain their random motion, the total heat transfer is then due to a superposition of energy transport by the random motion of the molecules and by the bulk motion of the fluid. It is customary to use the term *convection* when referring to this cumulative transport and the term *advection* when referring to transport due to bulk fluid motion.

We are especially interested in convection heat transfer, which occurs between a fluid in motion and a bounding surface when the two are at different temperatures. Consider fluid flow over the heated surface of Figure 1.4. A consequence of the fluid–surface interaction is the development of a region in the fluid through which the velocity varies from zero at the surface to a finite value u_∞ associated with the flow. This region of the fluid is known as the *hydrodynamic*, or *velocity*, *boundary layer*. Moreover, if the surface and flow temperatures differ, there will be a region of the fluid through which the

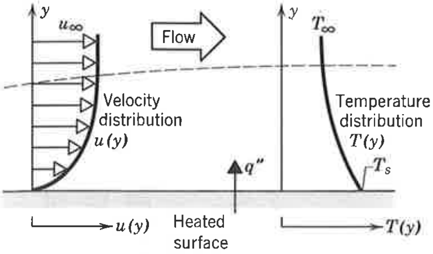


Figure 1.4 Boundary layer development in convection heat transfer.

temperature varies from T_s at $y = 0$ to T_∞ in the outer flow. This region, called the *thermal boundary layer*, may be smaller, larger, or the same size as that through which the velocity varies. In any case, if $T_s > T_\infty$, convection heat transfer will occur between the surface and the outer flow.

The convection heat transfer mode is sustained both by random molecular motion and by the bulk motion of the fluid within the boundary layer. The contribution due to random molecular motion (diffusion) dominates near the surface where the fluid velocity is low. In fact, at the interface between the surface and the fluid ($y = 0$), the fluid velocity is zero and heat is transferred by this mechanism only. The contribution due to bulk fluid motion originates from the fact that the boundary layer *grows* as the flow progresses in the x direction. In effect, the heat that is conducted into this layer is swept downstream and is eventually transferred to the fluid outside the boundary layer. Appreciation of boundary layer phenomena is essential to understanding convection heat transfer. It is for this reason that the discipline of fluid mechanics plays a vital role in our later analysis of convection.

Convection heat transfer may be classified according to the nature of the flow. We speak of *forced convection* when the flow is caused by external means, such as by a fan, a pump, or atmospheric winds. As an example, consider the use of a fan to provide forced convection air cooling of hot electrical components on a stack of printed circuit boards (Figure 1.5*a*). In contrast, for *free* (or *natural*) *convection* the flow is induced by buoyancy forces which arise from density differences caused by temperature variations in the fluid. An example is the free convection heat transfer that occurs from hot components on a vertical array of circuit boards in still air (Figure 1.5*b*). Air that makes contact with the components experiences an increase in temperature and hence a reduction in density. Since it is now lighter than the surrounding air, buoyancy forces induce a vertical motion for which warm air ascending from the boards is replaced by an inflow of cooler ambient air. It is useful to note that, while we have presumed *pure* forced convection in Figure 1.5*a* and *pure* natural convection in Figure 1.5*b*, conditions corresponding to *mixed* (*combined*) *forced* and *natural convection* may exist. For example, if

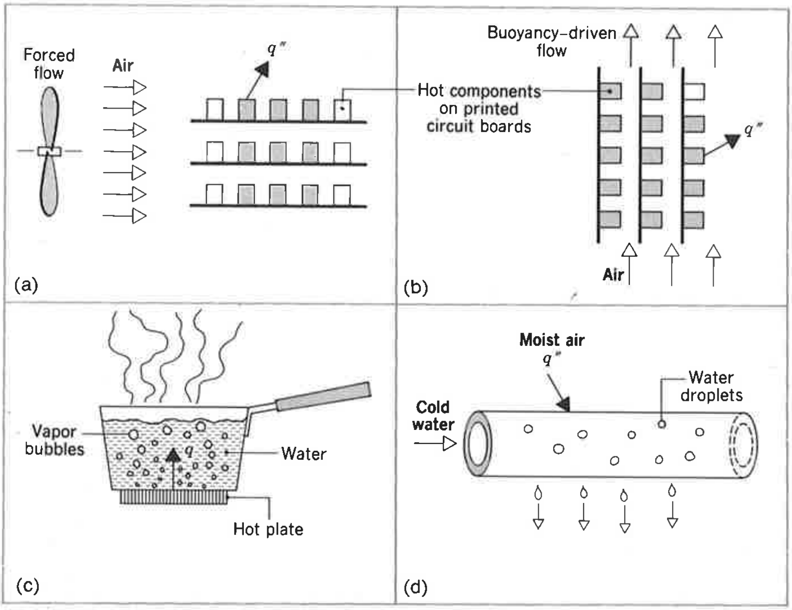


Figure 1.5 Convection heat transfer processes. (*a*) Forced convection. (*b*) Natural convection. (*c*) Boiling. (*d*) Condensation.

velocities associated with the flow of Figure 1.5*a* are small and/or buoyancy forces are large, a secondary flow that is comparable to the imposed forced flow could be induced. The buoyancy induced flow would be normal to the forced flow and would have a significant effect on convection heat transfer from the components. In Figure 1.5*b* mixed convection would result if a fan were used to force air upward through the circuit boards, thereby assisting the buoyancy flow, or downward, thereby opposing the buoyancy flow.

We have described the convection heat transfer mode as energy transfer occurring within a fluid due to the combined effects of conduction and bulk fluid motion. Typically, the energy that is being transferred is the *sensible*, or internal thermal, energy of the fluid. However, there are convection processes for which there is, in addition, *latent* heat exchange. This latent heat exchange is generally associated with a phase change between the liquid and vapor states of the fluid. Two special cases of interest in this text are *boiling* and *condensation*. For example, convection heat transfer results from fluid motion induced by vapor bubbles generated at the bottom of a pan of boiling water (Figure 1.5*c*) or by the condensation of water vapor on the outer surface of a cold water pipe (Figure 1.5*d*).

Regardless of the particular nature of the convection heat transfer process, the appropriate rate equation is of the form

$$q'' = h(T_s - T_\infty) \tag{1.3a}$$

Table 1.1 Typical values of the convection heat transfer coefficient

PROCESS	h $(\mathrm{W/m^2 \cdot K})$
Free convection	
Gases	2–25
Liquids	50–1000
Forced convection	
Gases	25–250
Liquids	50–20,000
Convection with phase change	
Boiling or condensation	2500–100,000

where q'', the convective *heat flux* $(\mathrm{W/m^2})$, is proportional to the difference between the surface and fluid temperatures, T_s and T_∞, respectively. This expression is known as *Newton's law of cooling*, and the proportionality constant h $(\mathrm{W/m^2 \cdot K})$ is referred to as the *convection heat transfer coefficient*, the *film conductance*, or the *film coefficient*. It encompasses all the parameters that influence convection heat transfer. In particular, it depends on conditions in the boundary layer, which are influenced by surface geometry, the nature of the fluid motion, and an assortment of fluid thermodynamic and transport properties. Moreover, any study of convection ultimately reduces to a study of the means by which h may be determined. Although consideration of these means is deferred to Chapter 6, convection heat transfer will frequently appear as a boundary condition in the solution of conduction problems (Chapters 2 to 5). In the solution of such problems we presume h to be known, using typical values given in Table 1.1.

When Equation 1.3a is used, the convection heat flux is presumed to be *positive* if heat is transferred *from* the surface $(T_s > T_\infty)$ and *negative* if heat is transferred *to* the surface $(T_\infty > T_s)$. However, if $T_\infty > T_s$, there is nothing which precludes us from expressing Newton's law of cooling as

$$q'' = h(T_\infty - T_s) \tag{1.3b}$$

in which case heat transfer is positive if it is to the surface.

1.2.3 Radiation

Thermal radiation is energy *emitted* by matter that is at a finite temperature. Although we focus primarily on radiation from solid surfaces, emission may also occur from liquids and gases. Regardless of the form of matter, the emission may be attributed to changes in the electron configurations of the constituent atoms or molecules. The energy of the radiation field is transported by electromagnetic waves (or alternatively, photons). While the transfer of energy by conduction or convection requires the presence of a material medium, radiation does not. In fact, radiation transfer occurs most efficiently in a vacuum.

The *maximum* flux (W/m^2) at which radiation may be emitted from a surface is given by the *Stefan–Boltzmann law*

$$q'' = \sigma T_s^4 \tag{1.4}$$

where T_s is the *absolute* temperature (K) of the surface and σ is the *Stefan–Boltzmann constant* ($\sigma = 5.67 \times 10^{-8}$ W/m$^2 \cdot$ K^4). Such a surface is called an ideal radiator or *blackbody*. The heat flux emitted by a real surface is less than that of the ideal radiator and is given by

$$q'' = \varepsilon \sigma T_s^4 \tag{1.5}$$

where ε is a radiative property of the surface called the *emissivity*. This property, whose value is in the range $0 \leq \varepsilon \leq 1$, indicates how efficiently the surface emits compared to an ideal radiator. Conversely, if radiation is incident upon a surface, a portion will be absorbed, and the rate at which energy is absorbed per unit surface area may be evaluated from knowledge of a surface radiative property termed the *absorptivity* α. That is,

$$q''_{\text{abs}} = \alpha q''_{\text{inc}} \tag{1.6}$$

where $0 \leq \alpha \leq 1$. Whereas radiation emission reduces the thermal energy of matter, absorption increases this energy.

Equations 1.5 and 1.6 determine the rate at which radiant energy is emitted and absorbed, respectively, at a surface. Determination of the *net* rate at which *radiation* is *exchanged between surfaces* is generally a good deal more complicated. However, a special case that occurs frequently in practice involves the net exchange between a small surface and a much larger surface that completely surrounds the smaller one (Figure 1.6). The surface and the surroundings are separated by a gas that has no effect on the radiation transfer. Assuming the surface to be one for which $\alpha = \varepsilon$ (a *gray* surface), the

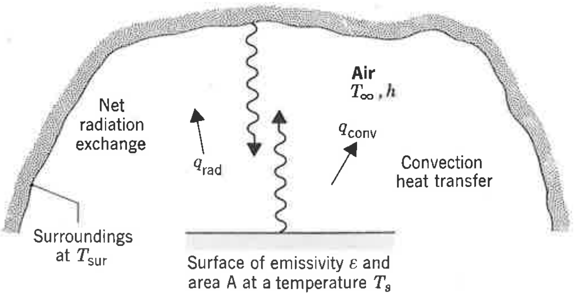


Figure 1.6 Radiation exchange between a surface and its surroundings.

net rate of radiation heat exchange between the surface and its *surroundings*, expressed per unit area of the surface, is

$$q'' = \frac{q}{A} = \varepsilon\sigma\left(T_s^4 - T_{\text{sur}}^4\right) \tag{1.7}$$

In this expression, A is the surface area and ε is its emissivity, while T_{sur} is the temperature of the surroundings. For this special case, the area and emissivity of the surroundings do not influence the net heat exchange rate.

There are many applications for which it is convenient to express the net radiation heat exchange in the form

$$q_{\text{rad}} = h_r A(T_s - T_{\text{sur}}) \tag{1.8}$$

where, from Equation 1.7, the *radiation heat transfer coefficient* h_r is

$$h_r \equiv \varepsilon\sigma(T_s + T_{\text{sur}})\left(T_s^2 + T_{\text{sur}}^2\right) \tag{1.9}$$

Here we have modeled the radiation mode in a manner similar to convection. In this sense we have *linearized* the radiation rate equation, making the heat rate proportional to a temperature difference rather than to the difference between two temperatures to the fourth power. Note, however, that h_r depends strongly on temperature, while the temperature dependence of the convection heat transfer coefficient h is generally weak.

The surface within the surroundings may also simultaneously transfer heat by convection to the adjoining gas (Figure 1.6). The total rate of heat transfer *from* the surface is then the sum of the heat rates due to the two modes. That is,

$$q = q_{\text{conv}} + q_{\text{rad}}$$

or,

$$q = hA(T_s - T_\infty) + \varepsilon A\sigma\left(T_s^4 - T_{\text{sur}}^4\right) \tag{1.10}$$

Note that the convection heat transfer rate q_{conv} is simply the product of the flux given by Equation 1.3a and the surface area.

EXAMPLE 1.2

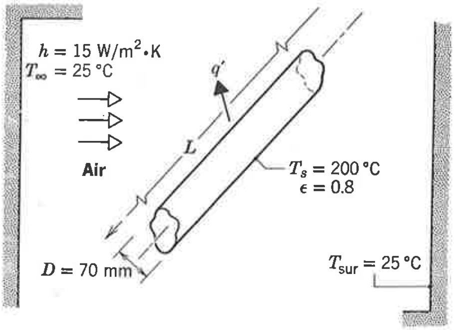
An uninsulated steam pipe passes through a room in which the air and walls are at 25°C. The outside diameter of the pipe is 70 mm, and its surface temperature and emissivity are 200°C and 0.8, respectively. If the coefficient associated with free convection heat transfer from the surface to the air is 15 $\text{W/m}^2 \cdot \text{K}$, what is the rate of heat loss from the surface per unit length of pipe?

SOLUTION

Known: Uninsulated pipe of prescribed diameter, emissivity, and surface temperature in a room with fixed wall and air temperatures.

Find: Pipe heat loss per unit length, q' (W/m).

Schematic:



Assumptions:

1. Steady-state conditions exist.
2. Radiation exchange between the pipe and the room is between a small surface enclosed within a much larger surface.

Analysis: Heat loss from the pipe is by convection to the room air and by radiation exchange with the walls. Hence, from Equation 1.10, with $A = \pi DL$,

$$q = h(\pi DL)(T_s - T_\infty) + \varepsilon(\pi DL)\sigma(T_s^4 - T_{\text{sur}}^4)$$

The heat loss per unit length of pipe is then

$$q' = \frac{q}{L} = 15\ \text{W/m}^2 \cdot \text{K}\ (\pi \times 0.07\ \text{m})(200 - 25)^\circ\text{C}$$

$$+0.8(\pi \times 0.07\ \text{m})\ 5.67 \times 10^{-8}\ \text{W/m}^2 \cdot \text{K}^4\ (473^4 - 298^4)\ \text{K}^4$$

$$q' = 577\ \text{W/m} + 421\ \text{W/m} = 998\ \text{W/m} \qquad \triangleleft$$

Comments:

1. Note that temperature may be expressed in two ways (i.e., in °C or K units) when evaluating the temperature difference for a convection (or conduction) heat transfer rate. However, temperature must be expressed in kelvins (K) when evaluating a radiation transfer rate.
2. In this situation the radiation and convection heat transfer rates are comparable because T_s is large compared to T_{sur} and the coefficient associated with free convection is small. For more moderate values of T_s and the larger values of h associated with forced convection, the effect of radiation may often be neglected. The radiation heat transfer coefficient may be computed from Equation 1.9, and for the conditions of this problem its value is $h_r = 11\ \mathrm{W/m^2 \cdot K}$.

1.2.4 Relationship to Thermodynamics

At this point it is appropriate to take note of the fundamental differences that exist between heat transfer and thermodynamics. Although thermodynamics is concerned with the heat interaction and the vital role it plays in the first and second laws, it considers neither the mechanisms that provide for heat exchange nor the methods that exist for computing the *rate* of heat exchange. Thermodynamics is concerned with *equilibrium* states of matter, where an equilibrium state necessarily precludes the existence of a temperature gradient. Although thermodynamics may be used to determine the amount of energy required in the form of heat for a system to pass from one equilibrium state to another, it does not acknowledge that *heat transfer is inherently a nonequilibrium process*. For heat transfer to occur, there must be a temperature gradient, hence thermodynamic nonequilibrium. The discipline of heat transfer therefore seeks to do what thermodynamics is inherently unable to do. It seeks to quantify the *rate* at which heat transfer occurs in terms of the degree of thermal nonequilibrium. This is done through the rate equations for the three modes, expressed by Equations 1.1, 1.3, and 1.7.

1.3 THE CONSERVATION OF ENERGY REQUIREMENT

The subjects of thermodynamics and heat transfer are highly complementary. For example, heat transfer is an extension of thermodynamics in that it considers the *rate* at which energy is transported. Moreover, in many heat transfer analyses the *first law* of thermodynamics (the *law of conservation of energy*) plays an important role. It is therefore useful to consider general statements for this law.

1.3.1 Conservation of Energy for a Control Volume

In our application of the conservation law, we first need to identify the *control volume*, a region of space bounded by a *control surface* through which energy and matter may pass. In the applications of this text, the region will be *fixed*, implying the existence of a stationary and constant volume. Once the control volume is identified, an appropriate *time basis* must be specified. There are two options. Since the first law must be satisfied at each and every *instant* of time t, one option involves formulating the law on a *rate basis*. That is, at any instant, there must be a balance between all *energy rates*, as measured in joules per second (W). Alternatively, the first law must also be satisfied over any *time interval* Δt. For such an interval, there must be a balance between the *amounts* of all *energy changes*, as measured in joules.

According to the time basis, first law formulations, which are well suited for heat transfer analysis, may be stated as follows.

At an Instant (t)

> *The rate at which thermal and mechanical energy enters a control volume, plus the rate at which thermal energy is generated within the control volume, minus the rate at which thermal and mechanical energy leaves the control volume must equal the rate of increase of energy stored within the control volume.*

Over a Time Interval (Δt)

> *The amount of thermal and mechanical energy which enters a control volume, plus the amount of thermal energy which is generated within the control volume, minus the amount of thermal and mechanical energy which leaves the control volume must equal the increase in the amount of energy stored in the control volume.*

If the inflow and generation of energy exceed the outflow, there will be an increase in the amount of energy stored (accumulated) in the control volume; if the converse is true, there will be a decrease in energy storage. If the inflow and generation of energy equal the outflow, a *steady-state* condition must prevail in which there will be no change in the amount of energy stored in the control volume.

Consider applying energy conservation to the control volume shown in Figure 1.7. The first step is to identify the control surface by drawing a dashed line. The next step is to identify the energy terms. At an instant, these include the rate at which thermal and mechanical energy enter and leave *through* the control surface, $\dot{E}_{\text{in}}$ and $\dot{E}_{\text{out}}$. Also, thermal energy may enter the control volume due to conversion from other energy forms. We refer to this process as *energy generation*, and the rate at which it occurs is designated as $\dot{E}_g$. The rate of change of energy stored within the control volume, dE_{st}/dt, is designated as $\dot{E}_{\text{st}}$. A general form of the energy conservation requirement may then be

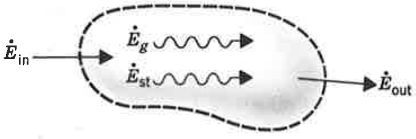


Figure 1.7 Conservation of energy for a control volume. Application at an instant.

expressed on a *rate* basis as

$$\dot{E}_{in} + \dot{E}_g - \dot{E}_{out} = \frac{dE_{st}}{dt} \equiv \dot{E}_{st} \qquad (1.11a)$$

Equation 1.11a may be applied at any *instant of time*. The alternative form that applies for a *time interval* Δt is obtained by integrating Equation 1.11a over time:

$$E_{in} + E_g - E_{out} = \Delta E_{st} \qquad (1.11b)$$

In words this relation says that the amounts of energy inflow and generation act to increase the amount of energy stored within the control volume, whereas outflow acts to decrease the stored energy.

The inflow and outflow rate terms $\dot{E}_{in}$ and $\dot{E}_{out}$ are *surface phenomena*. That is, they are associated exclusively with processes occurring at the control surface, and the rate at which they occur is proportional to surface area. The most common situation will involve energy inflow and outflow due to heat transfer by the conduction, convection, and/or radiation modes. In a situation involving fluid flow across the control surface, $\dot{E}_{in}$ and $\dot{E}_{out}$ also include energy transported by the fluid into and out of the control volume. This energy may be composed of potential, kinetic, and thermal forms. However, for the heat transfer problems encountered in this text, the potential and kinetic energy forms are negligible. The inflow and outflow terms may also include work interactions.

The *thermal energy generation rate term* $\dot{E}_g$ is associated with the rate of conversion from some other energy form (chemical, electrical, electromagnetic, or nuclear) to thermal energy. It is a *volumetric phenomenon*. That is, it occurs within the control volume, and its rate is proportional to the volume. For example, an exothermic chemical reaction may be occurring, converting chemical to thermal energy. The net effect is an increase in the thermal energy of the control volume. Another source of thermal energy is the conversion from electrical energy that occurs due to resistance heating when an electric current is passed through a conductor. That is, if an electric current I passes through a resistance R in the control volume, electrical energy is dissipated at a rate I^2R, which corresponds to the rate at which thermal energy is generated (released) within the volume.

It is important not to confuse the physical process of *energy storage* with that of *energy generation*. Although energy generation may certainly contribute to energy storage, the two processes are fundamentally different. Energy storage is also a *volumetric phenomenon*, but it is simply associated with an increase ($\dot{E}_{st} > 0$) or decrease ($\dot{E}_{st} < 0$) in the energy of the matter occupying the control volume. Under steady-state conditions there is, of course, no change in energy storage ($\dot{E}_{st} = 0$).

EXAMPLE 1.3

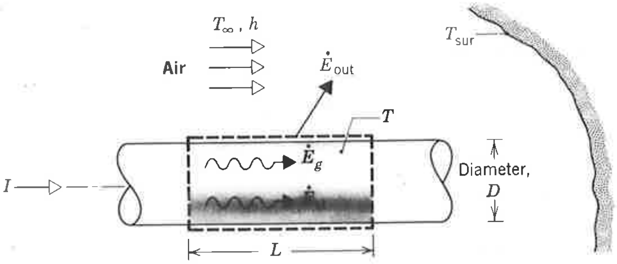
A long conducting rod of diameter D and electrical resistance per unit length R'_e is initially in thermal equilibrium with the ambient air and its surroundings. This equilibrium is disturbed when an electrical current I is passed through the rod. Develop an equation that could be used to compute the variation of the rod temperature with time during passage of the current.

SOLUTION

Known: Temperature of a rod of prescribed diameter and electrical resistance changes with time due to passage of an electrical current.

Find: Equation that governs temperature change with time for the rod.

Schematic:



Assumptions:

1. At any time t the temperature of the rod is uniform.
2. Constant properties exist.
3. Radiation exchange between the outer surface of the rod and the surroundings is between a small surface and a large enclosure.

Analysis: Since we are interested in determining a rate of change, the first law should be applied at an instant of time. Hence, applying Equation 1.11a to a control volume of length L about the rod, it follows that

$$\dot{E}_g - \dot{E}_{\text{out}} = \dot{E}_{\text{st}}$$

where energy generation is due to the electric resistance heating,

$$\dot{E}_g = I^2 R'_e L$$

Energy outflow is due to convection and net radiation from the surface, Equations 1.3a and 1.7, respectively,

$$\dot{E}_{\text{out}} = h(\pi DL)(T - T_\infty) + \varepsilon\sigma(\pi DL)\left(T^4 - T_{\text{sur}}^4\right)$$

and the change in energy storage is due to the temperature change,

$$\dot{E}_{\text{st}} = \frac{d}{dt}(\rho VcT)$$

The term $\dot{E}_{\text{st}}$ is associated with the rate of change in the internal thermal energy of the rod, where ρ and c are the mass density and the specific heat, respectively, of the rod material, and V is the volume of the rod, $V = (\pi D^2/4)L$. Substituting the rate equations into the energy balance, it follows that

$$I^2 R'_e L - h(\pi DL)(T - T_\infty) - \varepsilon\sigma(\pi DL)\left(T^4 - T_{\text{sur}}^4\right) = \rho c\left(\frac{\pi D^2}{4}\right)L\frac{dT}{dt}$$

Hence

$$\frac{dT}{dt} = \frac{I^2 R'_e - \pi Dh(T - T_\infty) - \pi D\varepsilon\sigma\left(T^4 - T_{\text{sur}}^4\right)}{\rho c\left(\pi D^2/4\right)} \qquad \triangleleft$$

Comments: The above equation could be solved for the time dependence of the rod temperature by integrating numerically. A steady-state condition would eventually be reached for which $dT/dt = 0$. The rod temperature is then determined by an algebraic equation of the form

$$\pi Dh(T - T_\infty) + \pi D\varepsilon\sigma\left(T^4 - T_{\text{sur}}^4\right) = I^2 R'_e$$

EXAMPLE 1.4

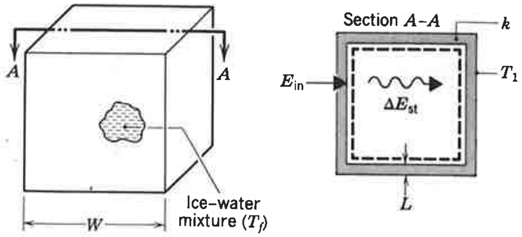
Ice of mass M at the fusion temperature ($T_f = 0°\text{C}$) is enclosed in a cubical cavity of width W on a side. The cavity wall is of thickness L and thermal conductivity k. If at time $t = 0$, the outer surface of the wall is brought to a temperature $T_1 > T_f$, obtain an expression for the time required to completely melt the ice.

SOLUTION

Known: Mass and initial temperature of ice. Dimensions, thermal conductivity, and outer surface temperature of containing wall.

Find: Expression for time needed to melt the ice.

Schematic:



Assumptions:

1. Inner surface of wall is at T_f throughout the process.
2. Constant properties.
3. One-dimensional conduction through each wall.
4. Conduction area of one wall may be approximated as $W^2 (L \ll W)$.

Analysis: Since we must determine the melting time t_m, the first law should be applied over the time interval $\Delta t = t_m$. Hence, applying Equation 1.11b to a control volume about the ice-water mixture, it follows that

$$E_{in} = \Delta E_{st}$$

Since the temperature difference across the wall remains at $(T_1 - T_f)$ throughout the melting process, the wall conduction rate is a constant

$$q_{cond} = k(6W^2)\frac{T_1 - T_f}{L}$$

in which case the amount of energy inflow is

$$E_{in} = \left[k(6W^2)\frac{T_1 - T_f}{L}\right]t_m$$

The increase in energy stored within the control volume is due exclusively to the change in latent energy associated with conversion from the solid to liquid states. The amount of energy required to effect such change per unit mass of solid is termed the *latent heat of fusion* h_{sf}. Hence the increase in energy storage is

$$\Delta E_{\text{st}} = Mh_{sf}$$

Substituting into the first law expression, it follows that

$$t_m = \frac{Mh_{sf}L}{6W^2k(T_1 - T_f)} \qquad \triangleleft$$

Comments: Several complications would arise if the ice were initially subcooled. The storage term would have to include the change in sensible (internal thermal) energy required to take the ice from the subcooled to the fusion temperature. During this process, temperature gradients would develop in the ice.

1.3.2 The Surface Energy Balance

We will frequently have occasion to apply the conservation of energy requirement at the surface of a medium. In this special case the control surface includes no mass or volume and appears as shown in Figure 1.8. Accordingly, the generation and storage terms of the conservation expression, Equation 1.11a, are no longer relevant and it is only necessary to deal with surface phenomena. For this case the conservation requirement then becomes

$$\dot{E}_{\text{in}} - \dot{E}_{\text{out}} = 0 \tag{1.12}$$

Even though thermal energy generation may be occurring in the medium, the process would not affect the energy balance at the control surface. Moreover, this conservation requirement holds for both *steady-state* and *transient* conditions.

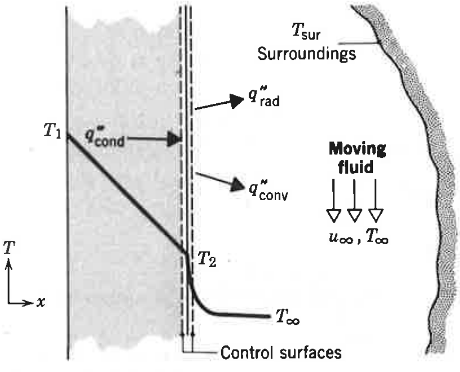


Figure 1.8 The energy balance for conservation of energy at the surface of a medium.

In Figure 1.8 three heat transfer terms are shown for the control surface. On a unit area basis they are conduction from the medium *to* the control surface (q''_{cond}), convection *from* the surface to a fluid (q''_{conv}), and net radiation exchange from the surface to the surroundings (q''_{rad}). The energy balance then takes the form

$$q''_{\text{cond}} - q''_{\text{conv}} - q''_{\text{rad}} = 0 \tag{1.13}$$

and we can express each of the terms according to the appropriate rate equations, Equations 1.2, 1.3a, and 1.7.

EXAMPLE 1.5

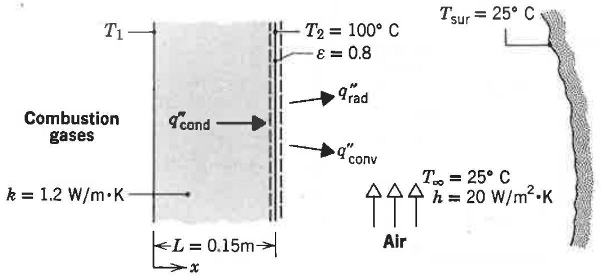
The hot combustion gases of a furnace are separated from the ambient air and its surroundings, which are at 25°C, by a brick wall 0.15 m thick. The brick has a thermal conductivity of 1.2 W/m · K and a surface emissivity of 0.8. Under steady-state conditions an outer surface temperature of 100°C is measured. Free convection heat transfer to the air adjoining this surface is characterized by a convection coefficient $h = 20$ W/m^2 · K. What is the brick inner surface temperature?

SOLUTION

Known: Outer surface temperature of a furnace wall of prescribed thickness, thermal conductivity, and emissivity.

Find: Wall inner surface temperature.

Schematic:



Assumptions:

1. Steady-state conditions.
2. One-dimensional heat transfer by conduction across the wall.
3. Radiation exchange between the outer surface of the wall and the surroundings is between a small surface and a large enclosure.

Analysis: The inside surface temperature may be obtained by performing an energy balance at the outer surface. From Equation 1.12

$$\dot{E}_{\text{in}} - \dot{E}_{\text{out}} = 0$$

it follows that, on a unit area basis,

$$q''_{\text{cond}} - q''_{\text{conv}} - q''_{\text{rad}} = 0$$

or, rearranging and substituting from Equations 1.2, 1.3a, and 1.7,

$$k\frac{T_1 - T_2}{L} = h(T_2 - T_\infty) + \varepsilon\sigma\left(T_2^4 - T_{\text{sur}}^4\right)$$

Therefore, substituting the appropriate numerical values

$$1.2\ \text{W/m}\cdot\text{K}\frac{(T_1 - 373)\ \text{K}}{0.15\ \text{m}} = 20\ \text{W/m}^2\cdot\text{K}\,(373 - 298)\ \text{K}$$

$$+ 0.8\left(5.67\times10^{-8}\ \text{W/m}^2\cdot\text{K}^4\right)\left(373^4 - 298^4\right)\ \text{K}^4$$

$$= 1500\ \text{W/m}^2 + 520\ \text{W/m}^2 = 2020\ \text{W/m}^2$$

Solving for T_1,

$$T_1 = 373\ \text{K} + \frac{0.15\ \text{m}}{1.2\ \text{W/m}\cdot\text{K}}\left(2020\ \text{W/m}^2\right) = 625\ \text{K} = 352^\circ\text{C} \quad \triangleleft$$

Comments:

1. Note that the contribution of radiation to heat transfer from the outer surface is significant. The relative contribution would diminish, however, with increasing h and/or decreasing T_2.
2. When using energy balances involving radiation exchange and other modes, it is good practice to express all temperatures in kelvin units. This practice is *necessary* when the unknown temperature appears in the radiation term and in one or more of the other terms.

1.3.3 Application of the Conservation Laws: Methodology

In addition to being familiar with the transport rate equations described in Section 1.2, the heat transfer analyst must be able to work with the energy conservation requirements of Equations 1.11 and 1.12. The application of these balances is simplified if a few basic rules are followed.

1. The appropriate control volume must be defined, with the control surface represented by a dashed line.
2. The appropriate time basis must be identified.
3. The relevant energy processes should be identified. Each process should be shown on the control volume by an appropriately labeled arrow.
4. The conservation equation should then be written, and appropriate rate expressions should be substituted for the terms in the equation.

In connection with applying the energy conservation equation (1.11) note that this application may be in terms of a *finite* control volume or a *differential* (infinitesimal) control volume. In the first case an equation would be obtained that governs the gross behavior of the system. In contrast, for the second case a differential equation would be obtained that could be solved for conditions at each point in the system. Both types of control volume are used extensively in this text.

1.4 ANALYSIS OF HEAT TRANSFER PROBLEMS: METHODOLOGY

A major objective of this text is to prepare you to solve engineering problems that involve heat transfer processes. To this end numerous problems are provided at the end of each chapter. In working these problems you will gain a deeper appreciation for the fundamentals of the subject, and you will gain confidence in your ability to apply these fundamentals to the solution of engineering problems.

It is our conviction that there exists a preferred approach to problem solving. In contrast to a somewhat random, "plug-and-chug" approach, we advocate the use of a systematic procedure characterized by a prescribed format. We consistently employ the procedure in all the examples of this text, and we require our students to use it in their solution of problems. It consists of the following steps.

1. *Known:* After carefully reading the problem, state briefly and concisely what is known about the problem. Do not repeat the problem statement.
2. *Find:* State briefly and concisely what must be found.
3. *Schematic:* Draw a schematic of the physical system. If application of the conservation laws is anticipated, represent the required control surface by dashed lines on the schematic. Identify relevant heat transfer processes by appropriately labeled arrows on the schematic.
4. *Assumptions:* List all pertinent simplifying assumptions.

5. *Properties:* Compile property values needed for subsequent calculations and identify the source from which they are obtained.
6. *Analysis:* Begin your analysis by applying appropriate conservation laws, and introduce rate equations as needed. Develop the analysis as completely as possible before substituting numerical values. Perform the calculations needed to obtain the desired results.
7. *Comments:* Discuss your results. Such a discussion may include a summary of key conclusions, an inference of trends, and a critique of the original assumptions.

The importance of following steps 1 through 4 should not be underestimated. These steps provide the opportunity to systematically think about a problem before effecting its solution.

1.5 RELEVANCE OF HEAT TRANSFER

In the vernacular of the time, heat transfer is indeed a relevant subject, not to mention an inherently fascinating part of the engineering sciences. We will devote much time to acquiring an understanding of heat transfer effects and to developing the skills needed to predict heat transfer rates. What is the value of this knowledge, and to what kinds of problems may it be applied?

Heat transfer phenomena play an important role in many industrial and environmental problems. As an example, consider the vital area of energy production and conversion. There is not a single application in this area that does not involve heat transfer effects in some way. In the generation of electrical power, whether it be through nuclear fission or fusion, the combustion of fossil fuels, magnetohydrodynamic processes, or the use of geothermal energy sources, there are numerous heat transfer problems that must be solved. These problems involve conduction, convection, and radiation processes and relate to the design of systems such as boilers, condensers, and turbines. One is often confronted with the need to maximize heat transfer rates and to maintain the integrity of materials in high-temperature environments. One is also confronted with the thermal pollution problem associated with the discharge of large amounts of waste heat from a power plant to the environment. Numerous heat transfer considerations relate to the design of cooling towers to alleviate the environmental problems associated with this discharge.

On a smaller scale there are many heat transfer problems related to the development of solar energy conversion systems for space heating and air conditioning, as well as for electric power production. Heat transfer processes also affect the performance of propulsion systems, such as the internal combustion and rocket engines. Heat transfer problems arise in the design of conventional space and water heating systems, in the design of incinerators and cryogenic storage equipment, in the cooling of electronic equipment, in the design of refrigeration and air conditioning systems, and in many other industrial problems. Heat transfer processes are also relevant to air and water pollution and strongly influence local and global climate.

1.6 UNITS AND DIMENSIONS

The physical quantities of heat transfer are specified in terms of *dimensions*, which are measured in terms of units. Four *basic* dimensions are required for the development of heat transfer; they are length (L), mass (M), time (t), and temperature (T). All other physical quantities of interest may be related to these four basic dimensions.

In the United States it has been customary to measure dimensions in terms of an *English system of units*, for which the *base units* are

DIMENSION		UNIT
Length (L)	→	foot (ft)
Mass (M)	→	pound mass (lb_m)
Time (t)	→	second (s)
Temperature (T)	→	degree Fahrenheit (°F)

The units required to specify other physical quantities may then be inferred from this group. For example, the dimension of force is related to mass through Newton's second law of motion

$$F = \frac{1}{g_c} Ma \tag{1.14}$$

where the acceleration a has units of feet per square second and g_c is a proportionality constant. If this constant is arbitrarily set equal to unity and made *dimensionless*, the dimensions of force are $(F) = (M) \cdot (L)/(t)^2$ and the unit of force is

$$1 \text{ poundal} = 1 \text{ lb}_\text{m} \cdot \text{ft/s}^2$$

Alternatively, one could work with a system of basic dimensions that includes both mass and force. However, in this case the proportionality constant must have the dimensions $(M) \cdot (L)/(F) \cdot (t)^2$. Moreover, if one defines the pound force (lb_f) as a unit of force that will accelerate one pound mass by 32.17 ft/s^2, the proportionality constant must be of the form

$$g_c = 32.17 \text{ lb}_\text{m} \cdot \text{ft/lb}_\text{f} \cdot \text{s}^2$$

The units of work may be inferred from its definition as the product of a force times a distance, in which case the units are ft · lb_f. The units of work

Table 1.2 SI base and supplementary units

QUANTITY AND SYMBOL	UNIT AND SYMBOL
Length (L)	meter (m)
Mass (M)	kilogram (kg)
Concentration (C)	mole (mol)
Time (t)	second (s)
Electric current (I)	ampere (A)
Thermodynamic temperature (T)	kelvin (K)
Plane angle[a] (θ)	radian (rad)
Solid angle[a] (ω)	steradian (sr)

[a] Supplementary unit.

and energy are, of course, equivalent, although it is customary to use the British thermal unit (Btu) as the unit of thermal energy. One British thermal unit will raise the temperature of 1 lb_m of water at 68°F by 1°F. It is equivalent to 778.16 ft · lb_f, which is termed the *mechanical equivalent of heat*.

In recent years there has been a strong trend toward worldwide usage of a standard set of units. In 1960 the SI (Système International d'Unités) system of units was defined by the Eleventh General Conference on Weights and Measures and recommended as a worldwide standard. In response to this trend, the American Society of Mechanical Engineers (ASME) has required the use of SI units in all of its publications since July 1, 1974. For this reason and because it is operationally more convenient than the English system, the SI system is used for the calculations of this text. However, because for some time to come, engineers will also have to work with results expressed in the English system, you should be able to convert from one system to the other. For your convenience conversion factors are provided on the inside back cover of the text.

The SI *base* units required for this text are summarized in Table 1.2. With regard to these units note that 1 mol is the amount of substance that has as many atoms or molecules as there are atoms in 12 g of carbon-12 (^{12}C); this is the gram-mole (mol). Although the mole has been recommended as the unit quantity of matter for the SI system, it is more consistent to work with the kilogram-mole (kmol, kg-mole). One kmol is simply the amount of substance that has as many atoms or molecules as there are atoms in 12 kg of ^{12}C. So long as the use is consistent within a given problem, no difficulties arise in using either mol or kmol. The molecular weight of a substance is the mass associated with a mole or kilogram-mole. For oxygen, as an example, the molecular weight $\mathcal{M}$ is 16 g/mol or 16 kg/kmol.

Although the SI unit of temperature is the kelvin, use of the Celsius temperature scale remains widespread. Zero on the Celsius scale (0°C) is

Table 1.3 SI derived units for selected quantities

QUANTITY	NAME AND SYMBOL	FORMULA	EXPRESSION IN SI BASE UNITS
Force	newton (N)	$m \cdot kg/s^2$	$m \cdot kg/s^2$
Pressure and stress	pascal (Pa)	N/m^2	$kg/m \cdot s^2$
Energy	joule (J)	$N \cdot m$	$m^2 \cdot kg/s^2$
Power	watt (W)	J/s	$m^2 \cdot kg/s^3$

Table 1.4 Multiplying prefixes

PREFIX	ABBREVIATION	MULTIPLIER
pico	p	10^{-12}
nano	n	10^{-9}
micro	μ	10^{-6}
milli	m	10^{-3}
centi	c	10^{-2}
hecto	h	10^{2}
kilo	k	10^{3}
mega	M	10^{6}
giga	G	10^{9}
tera	T	10^{12}

equivalent to 273.15 K on the thermodynamic scale,[1] in which case

$$T\ (\mathrm{K}) = T\ (^\circ\mathrm{C}) + 273.15$$

However, temperature *differences* are equivalent for the two scales and may be denoted as °C or K. Also, although the SI unit of time is the second, other units of time (minute, hour, and day) are so common that their use with the SI system is generally accepted.

The SI units comprise a coherent form of the metric system. That is, all remaining units may be derived from the base units using formulas that do not involve any numerical factors. *Derived* units for selected quantities are listed in Table 1.3. Note that force is measured in newtons, where a 1-N force will accelerate a 1-kg mass at 1 m/s^2. Hence 1 N = 1 $kg \cdot m/s^2$. The unit of pressure (N/m^2) is often referred to as the pascal. In the SI system there is one unit of energy (thermal, mechanical, or electrical), called the joule (J), and

---

[1] The degree symbol is retained for designation of the Celsius temperature (°C) to avoid confusion with use of C for the unit of electrical charge (coulomb).

$1 \text{ J} = 1 \text{ N} \cdot \text{m}$. The unit for energy rate, or power, is then J/s, where one joule per second is equivalent to one watt ($1 \text{ J/s} = 1 \text{ W}$). Since it is frequently necessary to work with extremely large or small numbers, a set of standard prefixes has been introduced to simplify matters (Table 1.4). For example, 1 megawatt = 10^6 W, and 1 micrometer (μm) = 10^{-6} m.

1.7 SUMMARY

Although much of the material of this chapter will be discussed in greater detail, you should now have a reasonable overview of heat transfer. You should be aware of the several modes of transfer and their physical origins. Moreover, given a physical situation, you should be able to perceive the relevant transport phenomena. The importance of developing this facility must not be underestimated. You will be devoting much time to acquiring the tools needed to calculate heat transfer phenomena. However, before you can begin to use these tools to solve practical problems, you must have the intuition to determine what is happening physically. In short, you must be able to look at a problem and identify the pertinent transport phenomena. The example and problems at the end of this chapter should help you to begin developing this intuition.

You should also appreciate the significance of the rate equations and feel comfortable in using them to compute transport rates. These equations, summarized in Table 1.5, *should be committed to memory*. You must also recognize the importance of the conservation laws and the need to carefully identify control volumes. With the rate equations, the conservation laws may be used to solve numerous problems involving heat transfer.

Table 1.5 Summary of heat transfer rate processes

MODE	MECHANISM	RATE EQUATION	EQUATION NUMBER	TRANSPORT PROPERTY OR COEFFICIENT
Conduction	Diffusion of energy due to random molecular motion	$q''_x\ (\text{W/m}^2) = -k\dfrac{dT}{dx}$	(1.1)	k (W/m · K)
Convection	Diffusion of energy due to random molecular motion plus energy transfer due to bulk motion (advection)	$q''\ (\text{W/m}^2) = h(T_s - T_\infty)$	(1.3a)	h (W/m² · K)
Radiation	Energy transfer by electromagnetic waves	$q''\ (\text{W/m}^2) = \varepsilon\sigma(T_s^4 - T_{\text{sur}}^4)$ or $q\ (\text{W}) = h_r A(T_s - T_{\text{sur}})$	(1.7) (1.8)	ε h_r (W/m² · K)

EXAMPLE 1.6

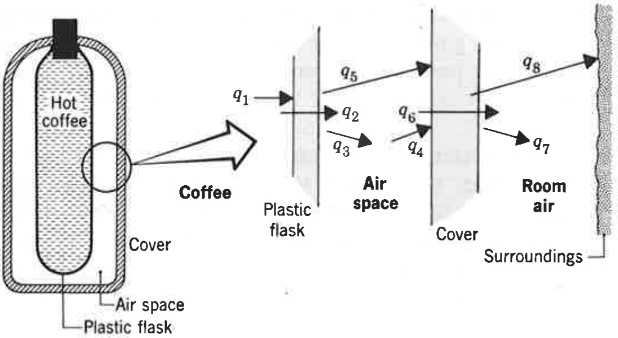
A closed container filled with hot coffee is in a room whose air and walls are at a fixed temperature. Identify all heat transfer processes that contribute to cooling of the coffee. Comment on features that would contribute to a superior container design.

SOLUTION

Known: Hot coffee is separated from its cooler surroundings by a plastic flask, an air space, and a plastic cover.

Find: Relevant heat transfer processes.

Schematic:



Pathways for energy transfer from the coffee are as follows.

q_1: free convection from the coffee to the flask

q_2: conduction through the flask

q_3: free convection from the flask to the air

q_4: free convection from the air to the cover

q_5: net radiation exchange between the outer surface of the flask and the inner surface of the cover

q_6: conduction through the cover

q_7: free convection from the cover to the room air

q_8: net radiation exchange between the outer surface of the cover and the surroundings

Comments: Design improvements are associated with (1) use of aluminized (low emissivity) surfaces for the flask and cover to reduce net radiation, and (2) evacuating the air space or using a filler material to retard free convection.

PROBLEMS

Conduction

1.1 A heat rate of 3 kW is conducted through a section of an insulating material of cross-sectional area 10 m^2 and thickness 2.5 cm. If the inner (hot) surface temperature is 415°C and the thermal conductivity of the material is 0.2 W/m · K, what is the outer surface temperature?

1.2 A concrete wall, which has a surface area of 30 m^2 and is 0.30 m thick, separates warm room air from cold ambient air. The temperature of the inner surface of the wall is known to be 25°C, while the outer surface is at −15°C. The thermal conductivity of the concrete is 1 W/m · K. What is the heat loss through the wall?

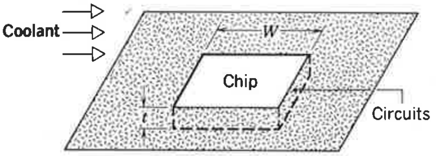
1.3 The heat flux through a wood slab 50 mm thick, whose inner and outer surface temperatures are 40 and 20°C, respectively, has been determined to be 40 W/m^2. What is the thermal conductivity of the wood?

1.4 The inner and outer surface temperatures of a glass window 5 mm thick are 15 and 5°C. What is the heat loss through a window that is 1 m by 3 m on a side? The thermal conductivity of glass is 1.4 W/m · K.

1.5 A freezer compartment consists of a cubical cavity that is 2 m on a side. Assume the bottom to be perfectly insulated. What is the minimum thickness of styrofoam insulation ($k = 0.030$ W/m · K) which must be applied to the top and side walls to ensure a heat load of less than 500 W, when the inner and outer surfaces are −10 and 35°C?

1.6 What is the thickness required of a masonry wall having thermal conductivity 0.75 W/m · K if the heat rate is to be 80% of the heat rate through a composite structural wall having a thermal conductivity 0.25 W/m · K and a thickness of 100 mm? Both walls are subjected to the same surface temperature difference.

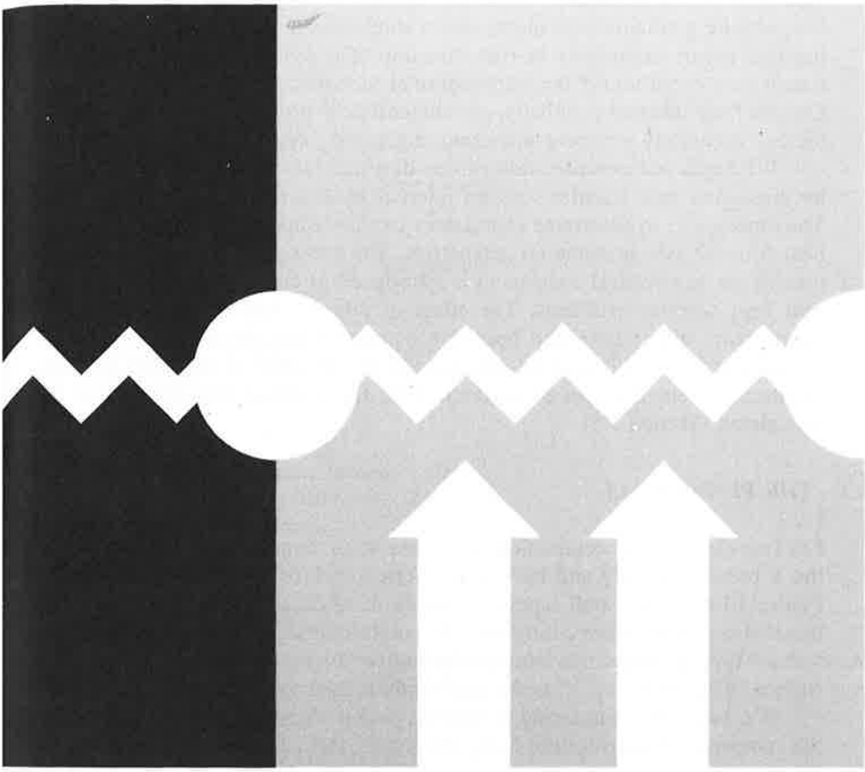
1.7 A square silicon chip ($k = 150$ W/m · K) is of width $W = 5$ mm on a side and of thickness $t = 1$ mm. The chip is mounted in a substrate such that its side and back surfaces are insulated, while the front surface is exposed to a coolant.



If 4 W are being dissipated in circuits mounted to the back surface of the chip, what is the steady-state temperature difference between back and front surfaces?

1.8 A gage for measuring heat flux to a surface or through a laminated material employs thin-film, chromel—alumel (type K) thermocouples deposited on the upper and lower surfaces of a wafer with a thermal conductivity of 1.4 W/m · K and a thickness of 0.25 mm.

CHAPTER 3



ONE-DIMENSIONAL, STEADY-STATE CONDUCTION

In this chapter we treat situations for which heat is transferred by diffusion under *one-dimensional, steady-state* conditions. The term "one-dimensional" refers to the fact that only one coordinate is needed to describe the spatial variation of the dependent variables. Hence, in a *one-dimensional* system, temperature gradients exist along only a single coordinate direction, and heat transfer occurs exclusively in that direction. The system is characterized by *steady-state* conditions if the temperature at each point is independent of time. Despite their inherent simplicity, one-dimensional, steady-state models may be used to accurately represent numerous engineering systems.

We begin our consideration of one-dimensional, steady-state conduction by discussing heat transfer with no internal generation (Sections 3.1 to 3.3). The objective is to determine expressions for the temperature distribution and heat transfer rate in common geometries. The concept of thermal resistance (analogous to electrical resistance) is introduced as an aid to solving conduction heat transfer problems. The effect of internal heat generation on the temperature distribution and heat rate is then treated (Section 3.4). Finally, conduction analysis is used to describe the performance of extended surfaces or fins, wherein the role of convection at the external boundary must be considered (Section 3.5).

3.1 THE PLANE WALL

For one-dimensional conduction in a plane wall, temperature is a function of the x coordinate only and heat is transferred exclusively in this direction. In Figure 3.1*a*, a plane wall separates two fluids of different temperatures. Heat transfer occurs by convection from the hot fluid at $T_{\infty,1}$ to one surface of the wall at $T_{s,1}$, by conduction through the wall, and by convection from the other surface of the wall at $T_{s,2}$ to the cold fluid at $T_{\infty,2}$.

We begin by considering conditions *within* the wall. We first determine the temperature distribution, from which we can then obtain the conduction heat transfer rate.

3.1.1 Temperature Distribution

The temperature distribution in the wall can be determined by solving the heat equation with the proper boundary conditions. For steady-state conditions with no distributed source or sink of energy within the wall, the appropriate form of the heat equation, Equation 2.17, is

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right) = 0 \tag{3.1}$$

Hence, from Equation 2.2, it follows that, for *one-dimensional, steady-state conduction in a plane wall with no heat generation, the heat flux is a constant, independent of x*. If the thermal conductivity of the wall material is assumed to be constant, the equation may be integrated twice to obtain the *general*

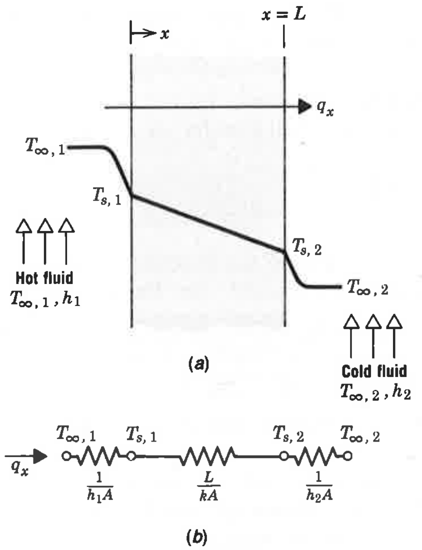


Figure 3.1 Heat transfer through a plane wall. (*a*) Temperature distribution. (*b*) Equivalent thermal circuit.

solution

$$T(x) = C_1x + C_2 \tag{3.2}$$

To obtain the constants of integration, C_1 and C_2, boundary conditions must be introduced. We choose to apply conditions of the first kind at $x = 0$ and $x = L$, in which case

$$T(0) = T_{s,1} \qquad \text{and} \qquad T(L) = T_{s,2}$$

Applying the condition at $x = 0$ to the general solution, it follows that

$$T_{s,1} = C_2$$

Similarly, at $x = L$

$$T_{s,2} = C_1L + C_2 = C_1L + T_{s,1}$$

in which case

$$\frac{T_{s,2} - T_{s,1}}{L} = C_1$$

Substituting into the general solution, the temperature distribution is then

$$T(x) = (T_{s,2} - T_{s,1})\frac{x}{L} + T_{s,1} \tag{3.3}$$

From this result it is evident that, *for one-dimensional, steady-state conduction in a plane wall with no heat generation and constant thermal conductivity, the temperature varies linearly with x.*

Now that we have the temperature distribution, we may use Fourier's law, Equation 2.1, to determine the conduction heat transfer rate. That is,

$$q_x = -kA\frac{dT}{dx} = \frac{kA}{L}(T_{s,1} - T_{s,2}) \tag{3.4}$$

Note that A is the area of the wall *normal* to the direction of heat transfer and, for the plane wall, it is a constant independent of x. The heat flux is then

$$q_x'' = \frac{q_x}{A} = \frac{k}{L}(T_{s,1} - T_{s,2}) \tag{3.5}$$

Equations 3.4 and 3.5 indicate that both the heat rate q_x and heat flux q_x'' are constants, independent of x.

In the foregoing paragraphs we have used the *standard approach* to solving conduction problems. That is, the general solution for the temperature distribution is first obtained by solving the appropriate form of the heat equation. The boundary conditions are then applied to obtain the particular solution, which is used with Fourier's law to determine the heat transfer rate. Note that we have opted to prescribe surface temperatures at $x = 0$ and $x = L$ as boundary conditions, even though it is the fluid temperatures, and not the surface temperatures, that are typically known. However, since adjoining fluid and surface temperatures are easily related through a surface energy balance, it is a simple matter to express Equations 3.3 to 3.5 in terms of fluid, rather than surface, temperatures. Alternatively, equivalent results could be obtained directly by using the surface energy balances as boundary conditions of the third kind in evaluating the constants of Equation 3.2 (see Problem 3.1).

3.1.2 Thermal Resistance

At this point we note that a very important concept is suggested by Equation 3.4. In particular, there exists an analogy between the diffusion of heat and electrical charge. Just as an electrical resistance is associated with the conduction of electricity, a thermal resistance may be associated with the conduction of heat. Defining resistance as the ratio of a driving potential to the corresponding transfer rate, it follows from Equation 3.4 that the *thermal resistance for conduction* is

$$R_{t,\mathrm{cond}} \equiv \frac{T_{s,1} - T_{s,2}}{q_x} = \frac{L}{kA} \tag{3.6}$$

Similarly, for electrical conduction in the same system, Ohm's law provides an electrical resistance of the form

$$R_e = \frac{E_{s,1} - E_{s,2}}{I} = \frac{L}{\sigma A} \tag{3.7}$$

The analogy between Equations 3.6 and 3.7 is obvious. A thermal resistance may also be associated with heat transfer by convection at a surface. From Newton's law of cooling,

$$q = hA(T_s - T_\infty) \tag{3.8}$$

the *thermal resistance for convection* is then

$$R_{t,\text{conv}} \equiv \frac{T_s - T_\infty}{q} = \frac{1}{hA} \tag{3.9}$$

Circuit representations provide a useful tool for both conceptualizing and quantifying heat transfer problems. The *equivalent thermal circuit* for the plane wall with convection surface conditions is shown in Figure 3.1*b*. The heat transfer rate may be determined from separate consideration of each element in the network. That is,

$$q_x = \frac{T_{\infty,1} - T_{s,1}}{1/h_1 A} = \frac{T_{s,1} - T_{s,2}}{L/kA} = \frac{T_{s,2} - T_{\infty,2}}{1/h_2 A} \tag{3.10}$$

In terms of the *overall temperature difference*, $T_{\infty,1} - T_{\infty,2}$, and the *total thermal resistance*, R_{tot}, the heat transfer rate may also be expressed as

$$q_x = \frac{T_{\infty,1} - T_{\infty,2}}{R_{\text{tot}}} \tag{3.11}$$

Because the conduction and convection resistances are in series and may be summed, it follows that

$$R_{\text{tot}} = \frac{1}{h_1 A} + \frac{L}{kA} + \frac{1}{h_2 A} \tag{3.12}$$

Yet another resistance may be pertinent if a surface is separated from *large surroundings* by a gas (Section 1.2.3). In particular, radiation exchange between the surface and its surroundings may be important, and the rate may be determined from Equation 1.8. It follows that a *thermal resistance for radiation* may be defined as

$$R_{t,\text{rad}} = \frac{T_s - T_{\text{sur}}}{q_{\text{rad}}} = \frac{1}{h_r A} \tag{3.13}$$

where h_r is determined from Equation 1.9. Surface radiation and convection resistances act in parallel, and if $T_\infty = T_{\text{sur}}$, they may be combined to obtain a single, effective surface resistance.

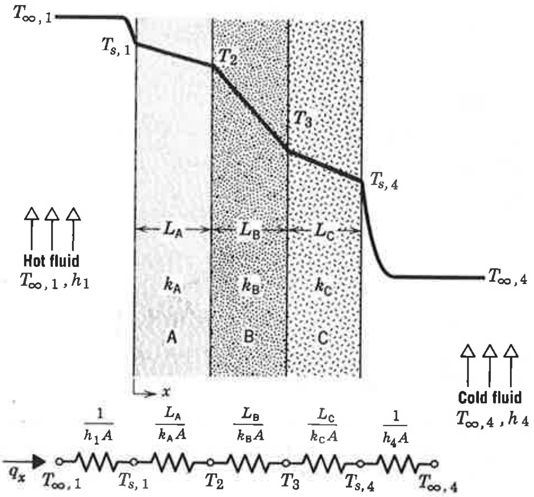


Figure 3.2 Equivalent thermal circuit for a series composite wall.

3.1.3 The Composite Wall

Equivalent thermal circuits may also be used for more complex systems, such as *composite walls*. Such walls may involve any number of series and parallel thermal resistances due to layers of different materials. Consider the series composite wall of Figure 3.2. The one-dimensional heat transfer rate for this system may be expressed as

$$q_x = \frac{T_{\infty,1} - T_{\infty,4}}{\Sigma R_t} \tag{3.14}$$

where $T_{\infty,1} - T_{\infty,4}$ is the *overall* temperature difference and the summation includes all thermal resistances. Hence,

$$q_x = \frac{T_{\infty,1} - T_{\infty,4}}{[(1/h_1\mathrm{A}) + (L_\mathrm{A}/k_\mathrm{A}A) + (L_\mathrm{B}/k_\mathrm{B}A) + (L_\mathrm{C}/k_\mathrm{C}A) + (1/h_4A)]} \tag{3.15}$$

Alternatively, the heat transfer rate can be related to the temperature difference and resistance associated with each element. For example,

$$q_x = \frac{T_{\infty,1} - T_{s,1}}{(1/h_1A)} = \frac{T_{s,1} - T_2}{(L_\mathrm{A}/k_\mathrm{A}A)} = \frac{T_2 - T_3}{(L_\mathrm{B}/k_\mathrm{B}A)} = \cdots \tag{3.16}$$

With composite systems it is often convenient to work with an *overall heat transfer coefficient*, *U*, which is defined by an expression analogous to Newton's law of cooling. Accordingly,

$$q_x \equiv UA\,\Delta T \tag{3.17}$$

where ΔT is the overall temperature difference. The overall heat transfer coefficient is related to the total thermal resistance, and from Equations 3.14 and 3.17 we see that $UA = 1/R_{\text{tot}}$. Hence, for the composite wall of Figure 3.2,

$$U = \frac{1}{R_{\text{tot}}A} = \frac{1}{[(1/h_1) + (L_A/k_A) + (L_B/k_B) + (L_C/k_C) + (1/h_4)]} \tag{3.18}$$

In general, we may write

$$R_{\text{tot}} = \Sigma R_t = \frac{\Delta T}{q} = \frac{1}{UA} \tag{3.19}$$

Composite walls may also be characterized by series–parallel configurations, such as that shown in Figure 3.3. Although the heat flow is now

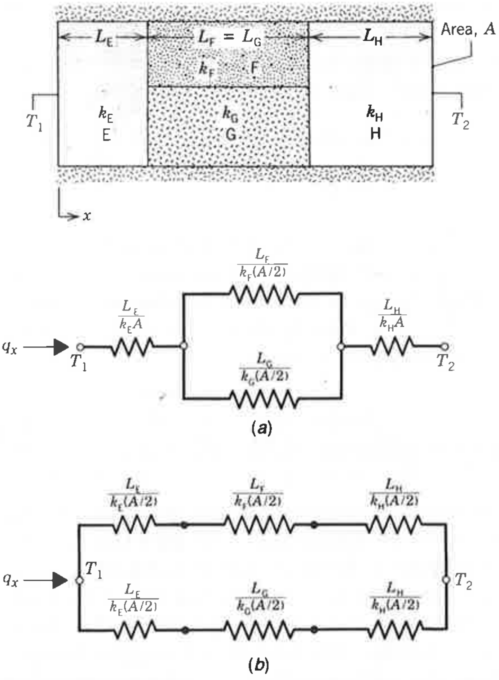


Figure 3.3 Equivalent thermal circuit for a series–parallel composite wall.

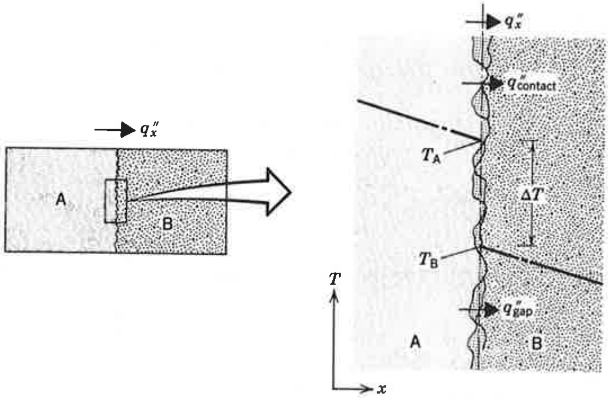


Figure 3.4 Temperature drop due to thermal contact resistance.

two-dimensional, it is often reasonable to assume one-dimensional conditions. Subject to this assumption, two different thermal circuits may be used. For case (a) it is presumed that surfaces normal to the x direction are isothermal, while for case (b) it is assumed that surfaces parallel to the x direction are adiabatic. Different results are obtained for R_{tot}, and the corresponding values of q bracket the actual heat transfer rate. These differences increase with increasing $|k_F - k_G|$, as two-dimensional effects become more significant.

3.1.4 Contact Resistance

Although neglected until now, it is important to recognize that, in composite systems, the temperature drop across the interface between materials may be appreciable. This temperature change is attributed to what is known as the *thermal contact resistance*, $R_{t,c}$. The effect is shown in Figure 3.4, and for a unit area of the interface, the resistance is defined as

$$R''_{t,c} = \frac{T_A - T_B}{q''_x} \tag{3.20}$$

The existence of a finite contact resistance is due principally to surface roughness effects. Contact spots are interspersed with gaps that are, in most instances, air filled. Heat transfer is therefore due to conduction across the actual contact area and to conduction and/or radiation across the gaps. The contact resistance may be viewed as two parallel resistances: that due to the contact spots and that due to the gaps. The contact area is typically small, and especially for rough surfaces, the major contribution to the resistance is made by the gaps.

Table 3.1 Thermal contact resistance for (*a*) metallic interfaces under vacuum conditions and (*b*) aluminum interface (10-μm surface roughness, 10^5 N/m^2) with different interfacial fluids [1]

THERMAL RESISTANCE, $R''_{t,c} \times 10^4$ ($\mathrm{m^2 \cdot K/W}$)				
(a) Vacuum Interface			**(b) Interfacial Fluid**	
Contact pressure	100 kN/m^2	10,000 kN/m^2	Air	2.75
Stainless steel	6–25	0.7–4.0	Helium	1.05
Copper	1–10	0.1–0.5	Hydrogen	0.720
Magnesium	1.5–3.5	0.2–0.4	Silicone oil	0.525
Aluminum	1.5–5.0	0.2–0.4	Glycerine	0.265

For solids whose thermal conductivities exceed that of the interfacial fluid, the contact resistance may be reduced by increasing the area of the contact spots. Such an increase may be effected by increasing the joint pressure and/or by reducing the roughness of the mating surfaces. The contact resistance may also be reduced by selecting an interfacial fluid of large thermal conductivity. In this respect, no fluid (an evacuated interface) eliminates conduction across the gap, thereby increasing the contact resistance.

Although theories have been developed for the prediction of $R''_{t,c}$, the most reliable results are those which have been obtained experimentally. The effect of loading on metallic interfaces can be seen in Table 3.1*a*, which presents the approximate range of thermal resistance values under vacuum conditions. The effect of interfacial fluid on the thermal resistance of an aluminum interface is shown in Table 3.1*b*.

Contrary to the results of Table 3.1, many applications involve contact between dissimilar solids and/or a wide range of possible interstitial (filler) materials (Table 3.2). Any interstitial substance that fills the gap between contacting surfaces and whose thermal conductivity exceeds that of air will decrease the contact resistance. Two classes of materials that are well suited for this purpose are soft metals and thermal greases. The metals, which include indium, lead, tin, and silver, may be inserted as a thin foil or applied as a thin coating to one of the parent materials. Silicon-based thermal greases are attractive on the basis of their ability to completely fill the interstices with a material whose thermal conductivity is as much as 50 times that of air.

Unlike the foregoing interfaces, which are not permanent, many interfaces involve permanently bonded joints. The joint could be formed from an epoxy, a soft solder rich in lead, or a hard solder such as a gold/tin alloy. Due to interface resistances between the parent and bonding materials, the actual thermal resistance of the joint exceeds the theoretical value (L/k) computed from the thickness L and thermal conductivity k of the joint material. The thermal resistance of epoxied and soldered joints is also adversely affected by voids and cracks, which may form during manufacture or as a result of thermal cycling during normal operation.

Table 3.2 Thermal resistance of representative solid/solid interfaces

INTERFACE	$R''_{t,c} \times 10^4$ ($m^2 \cdot K/W$)	SOURCE
Silicon chip/lapped aluminum in air (27–500 kN/m^2)	0.3–0.6	[2]
Aluminum/aluminum with indium foil filler (~ 100 kN/m^2)	~ 0.07	[1, 3]
Stainless/stainless with indium foil filler (~ 3500 kN/m^2)	~ 0.04	[1, 3]
Aluminum/aluminum with metallic (Pb) coating	0.01–0.1	[4]
Aluminum/aluminum with Dow Corning 340 grease (~ 100 kN/m^2)	~ 0.07	[1, 3]
Stainless/stainless with Dow Corning 340 grease (~ 3500 kN/m^2)	~ 0.04	[1, 3]
Silicon chip/aluminum with 0.02-mm epoxy	0.2–0.9	[5]
Brass/brass with 15 μm tin solder	0.025–0.14	[6]

Comprehensive reviews of thermal contact resistance results and models are provided by Snaith et al. [3], Madhusudana and Fletcher [7], and Yovanovich [8].

EXAMPLE 3.1

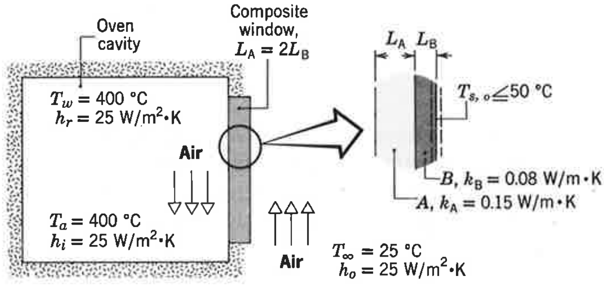
A leading manufacturer of household appliances is proposing a self-cleaning oven design that involves use of a composite window separating the oven cavity from the room air. The composite is to consist of two high temperature plastics (A and B) of thicknesses $L_A = 2L_B$ and thermal conductivities $k_A = 0.15$ W/m · K and $k_B = 0.08$ W/m · K. During the self-cleaning process, the oven wall and air temperatures, T_w and T_a, are 400°C, while the room air temperature T_∞ is 25°C. The inside convection and radiation heat transfer coefficients h_i and h_r, as well as the outside convection coefficient h_o, are each approximately 25 $W/m^2 \cdot K$. What is the minimum window thickness, $L = L_A + L_B$, needed to ensure a temperature that is 50°C or less at the outer surface of the window? This temperature must not be exceeded for safety reasons.

SOLUTION

Known: The properties and relative dimensions of plastic materials used for a composite oven window, and conditions associated with self-cleaning operation.

Find: Composite thickness $L_A + L_B$ needed to ensure safe operation.

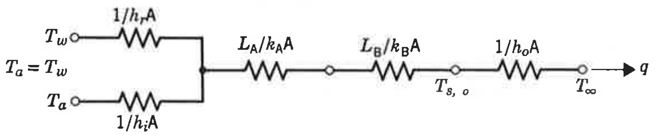
Schematic:



Assumptions:

1. Steady-state conditions exist.
2. Conduction through the window is one-dimensional.
3. Contact resistance is negligible.
4. Radiation absorption *within* the window is negligible; hence no internal heat generation (radiation exchange between window and oven walls occurs at the window inner surface).
5. Radiation exchange between window outer surface and surroundings is negligible.
6. Each plastic is homogeneous with constant properties.

Analysis: The thermal circuit can be constructed by recognizing that resistance to heat flow is associated with convection at the outer surface, conduction in the plastics, and convection and radiation at the inner surface. Accordingly, the circuit and the resistances are of the following form.



Since the outer surface temperature of the window, $T_{s,o}$, is prescribed, the required window thickness may be obtained by applying an energy balance at this surface. That is, from Equation 1.12

$$\dot{E}_{\text{in}} = \dot{E}_{\text{out}}$$

where, from Equation 3.19, with $T_w = T_a$,

$$\dot{E}_{\text{in}} = q = \frac{T_a - T_{s,o}}{\Sigma R_t}$$

and from Equation 3.8

$$\dot{E}_{\text{out}} = q = h_o A (T_{s,o} - T_\infty)$$

The total thermal resistance between the oven cavity and the outer surface of the window includes an effective resistance associated with convection and radiation, which act in parallel at the inner surface of the window, and the conduction resistances of the window materials. Hence

$$\Sigma R_t = \left(\frac{1}{1/h_i A} + \frac{1}{1/h_r A} \right)^{-1} + \frac{L_A}{k_A A} + \frac{L_B}{k_B A}$$

or

$$\Sigma R_t = \frac{1}{A} \left(\frac{1}{h_i + h_r} + \frac{L_A}{k_A} + \frac{L_A}{2k_B} \right)$$

Substituting into the energy balance, it follows that

$$\frac{T_a - T_{s,o}}{(h_i + h_r)^{-1} + (L_A/k_A) + (L_A/2k_B)} = h_o (T_{s,o} - T_\infty)$$

Hence, solving for L_A,

$$L_A = \frac{(1/h_o)(T_a - T_{s,o})/(T_{s,o} - T_\infty) - (h_i + h_r)^{-1}}{(1/k_A + 1/2k_B)}$$

$$L_A = \frac{0.04 \text{ m}^2 \cdot \text{K/W} \left(\dfrac{400 - 50}{50 - 25} \right) - 0.02 \text{ m}^2 \cdot \text{K/W}}{(1/0.15 + 1/0.16) \text{ m} \cdot \text{K/W}} = 0.0418 \text{ m}$$

Since $L_B = L_A/2 = 0.0209$ m,

$$L = L_A + L_B = 0.0627 \text{ m} = 62.7 \text{ mm} \qquad \triangleleft$$

Comments: The self-cleaning operation is a transient process, as far as the thermal response of the window is concerned, and steady-state conditions may not be reached in the time required for cleaning. However, the steady-state condition provides the maximum possible value of $T_{s,o}$ and hence is well suited for the design calculation.

EXAMPLE 3.2

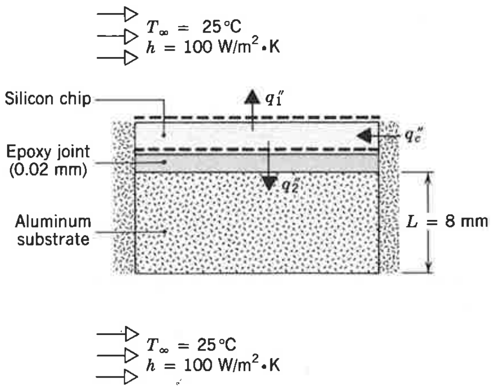
A thin silicon chip and a 8-mm-thick aluminum substrate are separated by a 0.02-mm-thick epoxy joint. The chip and substrate are each 10 mm on a side, and their exposed surfaces are cooled by air which is at a temperature of 25°C and provides a convection coefficient of 100 $W/m^2 \cdot K$. If the chip dissipates 10^4 W/m^2 under normal conditions, will it operate below a maximum allowable temperature of 85°C?

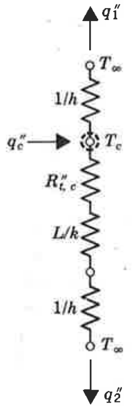
SOLUTION

Known: Dimensions, heat dissipation, and maximum allowable temperature of a silicon chip. Thickness of aluminum substrate and epoxy joint. Convection conditions at exposed chip and substrate surfaces.

Find: Whether maximum allowable temperature is exceeded.

Schematic:





Assumptions:

1. Steady-state conditions.
2. One-dimensional conduction (negligible heat transfer from sides of composite).
3. Negligible chip thermal resistance (an isothermal chip).
4. Constant properties.
5. Negligible radiation exchange with surroundings.

Properties: Table A.1, pure aluminum ($T \sim 350$ K): $k = 238$ W/m · K.

Analysis: Heat dissipated in the chip is transferred to the air directly from the exposed surface and indirectly through the joint and substrate. Performing an energy balance on a control surface about the chip, it follows that, on the basis of a unit surface area,

$$q_c'' = q_1'' + q_2''$$

or

$$q_c'' = \frac{T_c - T_\infty}{(1/h)} + \frac{T_c - T_\infty}{R''_{t,c} + (L/k) + (1/h)}$$

To conservatively estimate T_c, the maximum possible value of $R''_{t,c} = 0.9 \times 10^{-4}$ m² · K/W is obtained from Table 3.2. Hence

$$T_c = T_\infty + q_c''\left[h + \frac{1}{R''_{t,c} + (L/k) + (1/h)}\right]^{-1}$$

or

$$T_c = 25°\text{C} + 10^4 \text{ W/m}^2$$

$$\times\left[100 + \frac{1}{(0.9 + 0.34 + 100) \times 10^{-4}}\right]^{-1} \text{m}^2 \cdot \text{K/W}$$

$$T_c = 25°\text{C} + 50.3°\text{C} = 75.3°\text{C} \qquad \triangleleft$$

Hence, the chip will operate below its maximum allowable temperature.

Comments: The joint and substrate thermal resistances are much less than the convection resistance. The joint resistance would have to increase to the unrealistically large value of 50×10^{-4} m² · K/W, before the maximum allowable chip temperature would be exceeded.

3.2 AN ALTERNATIVE CONDUCTION ANALYSIS

The conduction analysis of Section 3.1 was performed using the *standard approach*. That is, the heat equation was solved to obtain the temperature distribution, Equation 3.3, and Fourier's law was then applied to obtain the heat transfer rate, Equation 3.4. However, an alternative approach may be used for the conditions presently of interest. Considering conduction in the system of Figure 3.5, we recognize that, for *steady-state conditions* with *no heat generation* and *no heat loss from the sides*, the heat transfer rate q_x must be a constant independent of x. That is, for any differential element dx, $q_x = q_{x+dx}$.

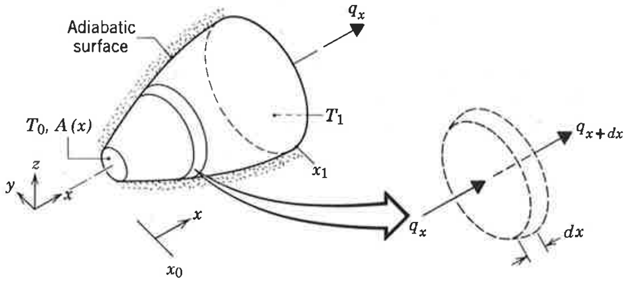


Figure 3.5 System with a constant conduction heat transfer rate.

This condition is, of course, a consequence of the energy conservation requirement, and it must apply even if the area varies with position $A(x)$ and the thermal conductivity varies with temperature $k(T)$. Moreover, even though the temperature distribution may be two dimensional, varying with x and y, it is often reasonable to neglect the y variation and to assume a *one-dimensional* distribution in x.

For the above conditions it is possible to work exclusively with Fourier's law when performing a conduction analysis. In particular, since the conduction rate is a *constant*, the rate equation may be *integrated*, even though neither the rate nor the temperature distribution is known. Consider Fourier's law, Equation 2.1, which may be applied to the system of Figure 3.5. Although we may have no knowledge of the value of q_x or the form of $T(x)$, we do know that q_x is a constant. Hence we may express Fourier's law in the integral form

$$q_x \int_{x_0}^{x} \frac{dx}{A(x)} = -\int_{T_0}^{T} k(T)\, dT \tag{3.21}$$

The cross-sectional area may be a known function of x, and the material thermal conductivity may vary with temperature in a known manner. If the integration is performed from a point x_0 at which the temperature T_0 is known, the resulting equation provides the functional form of $T(x)$. Moreover, if the temperature $T = T_1$ at some $x = x_1$ is also known, integration between x_0 and x_1 provides an expression from which q_x may be computed. Note that, if the area A is uniform and k is independent of temperature, Equation 3.21 reduces to

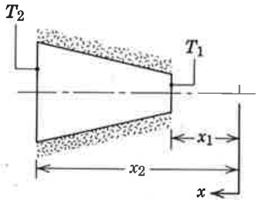
$$\frac{q_x\, \Delta x}{A} = -k\, \Delta T \tag{3.22}$$

where $\Delta x = x_1 - x_0$ and $\Delta T = T_1 - T_0$.

We frequently elect to solve diffusion problems by working with integrated forms of the diffusion rate equations. However, the limiting conditions for which this may be done should be firmly fixed in our minds: *steady-state* and *one-dimensional* transfer with *no heat generation*.

EXAMPLE 3.3

The diagram shows a conical section fabricated from pyroceram. It is of circular cross section with the diameter $D = ax$, where $a = 0.25$. The small end is at $x_1 = 50$ mm and the large end at $x_2 = 250$ mm. The end temperatures are $T_1 = 400$ K and $T_2 = 600$ K, while the lateral surface is well insulated.



1. Derive an expression for the temperature distribution $T(x)$ in symbolic form, assuming one-dimensional conditions. Sketch the temperature distribution.
2. Calculate the heat rate q_x through the cone.

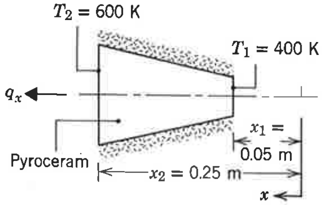
SOLUTION

Known: Conduction in a circular conical section having a diameter $D = ax$, where $a = 0.25$.

Find:

1. Temperature distribution $T(x)$.
2. Heat transfer rate q_x.

Schematic:



Assumptions:

1. Steady-state conditions.
2. One-dimensional conduction in the x direction.
3. No internal heat generation.
4. Constant properties.

Properties: Table A.2, pyroceram (500 K): $k = 3.46\ \mathrm{W/m \cdot K}$.

Analysis:

1. Since heat conduction occurs under steady-state, one-dimensional conditions with no internal heat generation, the heat transfer rate q_x is a constant independent of x. Accordingly, Fourier's law, Equation 2.1, may be used to determine the temperature distribution

$$q_x = -kA\frac{dT}{dx}$$

With $A = \pi D^2/4 = \pi a^2 x^2/4$ and separating variables

$$\frac{4q_x\,dx}{\pi a^2 x^2} = -k\,dT$$

Integrating from x_1 to any x within the cone, and recalling that q_x and k are constants, it follows that

$$\frac{4q_x}{\pi a^2}\int_{x_1}^{x}\frac{dx}{x^2} = -k\int_{T_1}^{T} dT$$

Hence

$$\frac{4q_x}{\pi a^2}\left(-\frac{1}{x} + \frac{1}{x_1}\right) = -k(T - T_1)$$

or solving for T

$$T(x) = T_1 - \frac{4q_x}{\pi a^2 k}\left(\frac{1}{x_1} - \frac{1}{x}\right)$$

Although q_x is a constant, it is as yet an unknown. However, it may be determined by evaluating the above expression at $x = x_2$ where $T(x_2) = T_2$. Hence

$$T_2 = T_1 - \frac{4q_x}{\pi a^2 k}\left(\frac{1}{x_1} - \frac{1}{x_2}\right)$$

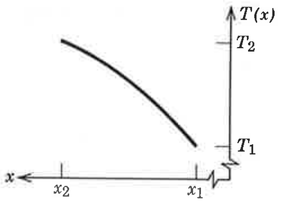
and solving for q_x

$$q_x = \frac{\pi a^2 k(T_1 - T_2)}{4[(1/x_1) - (1/x_2)]}$$

Substituting for q_x into the expression for $T(x)$, the temperature distribution becomes

$$T(x) = T_1 + (T_1 - T_2)\left[\frac{(1/x) - (1/x_1)}{(1/x_1) - (1/x_2)}\right] \qquad \triangleleft$$

From this result, temperature may be calculated as a function of x and the distribution is as shown.



Note that, since $dT/dx = -4q_x/k\pi a^2 x^2$ from Fourier's law, it follows that the temperature gradient and heat flux decrease with increasing x.

2. Substituting numerical values into the foregoing result for the heat transfer rate, it follows that

$$q_x = \frac{\pi(0.25)^2 \times 3.46\ \mathrm{W/m \cdot K}\ (400 - 600)\ \mathrm{K}}{4\left(\dfrac{1}{0.05\ \mathrm{m}} - \dfrac{1}{0.25\ \mathrm{m}}\right)} = -2.12\ \mathrm{W} \qquad \triangleleft$$

Comments: When the parameter a increases, the one-dimensional assumption becomes less appropriate. That is, the assumption worsens when the cross-sectional area change with distance is more pronounced.

3.3 RADIAL SYSTEMS

Cylindrical and spherical systems often experience temperature gradients in the radial direction only and may therefore be treated as one dimensional. Moreover, under steady-state conditions with no heat generation, such systems may be analyzed by using the *standard* method, which begins with the appropriate form of the heat equation, or the *alternative* method, which begins with the appropriate form of Fourier's law. In this section, the cylindrical

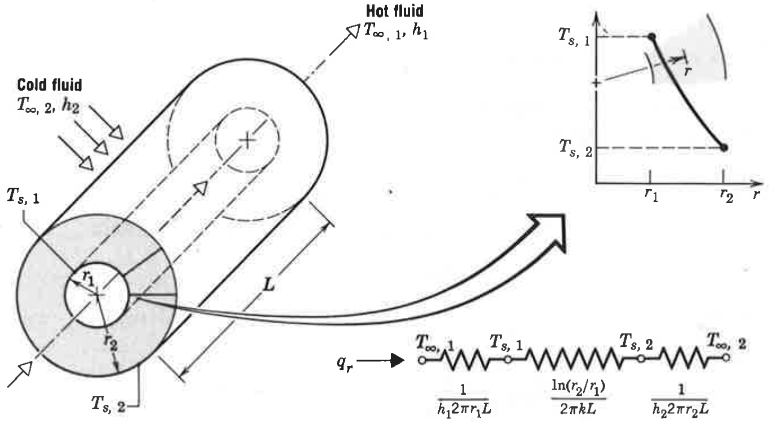


Figure 3.6 Hollow cylinder with convective surface conditions.

system is analyzed by means of the standard method and the spherical system by means of the alternative method.

3.3.1 The Cylinder

A common example is the hollow cylinder, whose inner and outer surfaces are exposed to fluids at different temperatures (Figure 3.6). For steady-state conditions with no heat generation, the appropriate form of the heat equation, Equation 2.20, is

$$\frac{1}{r}\frac{d}{dr}\left(kr\frac{dT}{dr}\right) = 0 \tag{3.23}$$

where, for the moment, k is treated as a variable. The physical significance of this result becomes evident if we also consider the appropriate form of Fourier's law. The rate at which energy is conducted across any cylindrical surface in the solid may be expressed as

$$q_r = -kA\frac{dT}{dr} = -k(2\pi rL)\frac{dT}{dr} \tag{3.24}$$

where $A = 2\pi rL$ is the area normal to the direction of heat transfer. From Equations 3.23 and 3.24 it follows that the conduction *heat transfer rate* q_r (*not* the heat flux q_r'') is a *constant in the radial direction*.

We may determine the temperature distribution in the cylinder by solving Equation 3.23 and applying appropriate boundary conditions. Assuming the value of k to be constant, Equation 3.23 may be integrated twice to obtain the general solution

$$T(r) = C_1 \ln r + C_2 \tag{3.25}$$

To obtain the constants of integration C_1 and C_2, we introduce the following boundary conditions.

$$T(r_1) = T_{s,1} \qquad \text{and} \qquad T(r_2) = T_{s,2}$$

Applying these conditions to the general solution, we then obtain

$$T_{s,1} = C_1 \ln r_1 + C_2 \qquad \text{and} \qquad T_{s,2} = C_1 \ln r_2 + C_2$$

Solving for C_1 and C_2 and substituting into the general solution, we then obtain

$$T(r) = \frac{T_{s,1} - T_{s,2}}{\ln(r_1/r_2)} \ln\left(\frac{r}{r_2}\right) + T_{s,2} \tag{3.26}$$

Note that the temperature distribution associated with radial conduction through a cylinder is logarithmic, not linear, as it is for the plane wall under the same conditions. The logarithmic distribution is sketched in the insert of Figure 3.6.

If the temperature distribution, Equation 3.26, is now used with Fourier's law, Equation 3.24, we obtain the following expression for the heat transfer rate

$$q_r = \frac{2\pi Lk(T_{s,1} - T_{s,2})}{\ln(r_2/r_1)} \tag{3.27}$$

From this result it is evident that, for radial conduction in hollow cylinders, the thermal resistance is of the form

$$R_{t,\text{cond}} = \frac{\ln(r_2/r_1)}{2\pi Lk} \tag{3.28}$$

This resistance is shown in the series circuit of Figure 3.6. Note that since the value of q_r is independent of r, the foregoing result could have been obtained by using the alternative method, that is, by integrating Equation 3.24.

Consider now the composite system of Figure 3.7. Recalling how we treated the composite plane wall and neglecting the interfacial contact resistances, the heat transfer rate may be expressed as

$$q_r = \frac{T_{\infty,1} - T_{\infty,4}}{\dfrac{1}{2\pi r_1 L h_1} + \dfrac{\ln(r_2/r_1)}{2\pi k_A L} + \dfrac{\ln(r_3/r_2)}{2\pi k_B L} + \dfrac{\ln(r_4/r_3)}{2\pi k_C L} + \dfrac{1}{2\pi r_4 L h_4}} \tag{3.29}$$

The foregoing result may also be expressed in terms of an overall heat transfer

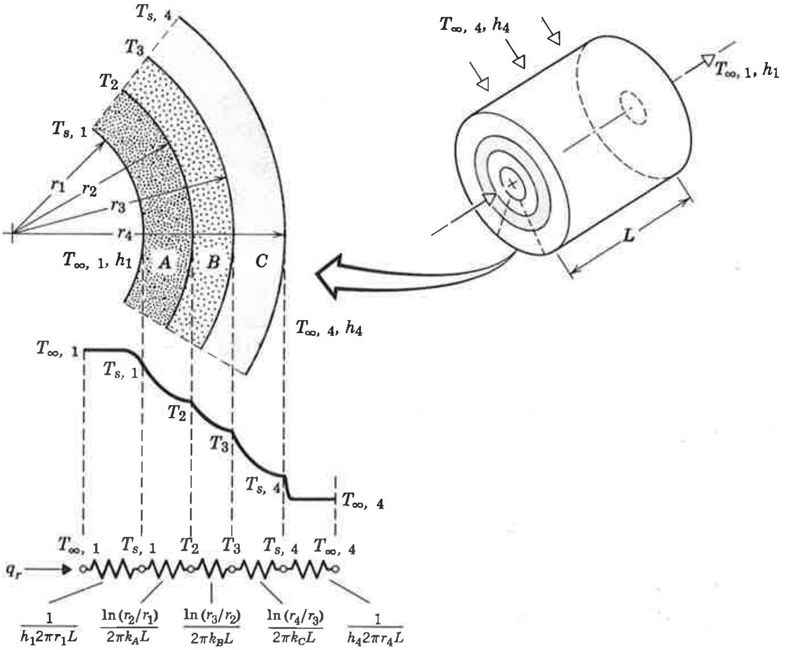


Figure 3.7 Temperature distribution for a composite cylindrical wall.

coefficient. That is,

$$q_r = \frac{T_{\infty,1} - T_{\infty,4}}{R_{\text{tot}}} = U_1 A_1 (T_{\infty,1} - T_{\infty,4}) \tag{3.30}$$

where $A_1 = 2\pi r_1 L$ and

$$U_1 = \frac{1}{\dfrac{1}{h_1} + \dfrac{r_1}{k_A} \ln \dfrac{r_2}{r_1} + \dfrac{r_1}{k_B} \ln \dfrac{r_3}{r_2} + \dfrac{r_1}{k_C} \ln \dfrac{r_4}{r_3} + \dfrac{r_1}{r_4} \dfrac{1}{h_4}} \tag{3.31}$$

Equation 3.30 defines U in terms of the inside surface area A_1 of the composite cylinder. This definition is *arbitrary*, and the overall coefficient may also be defined in terms of A_4 or any of the intermediate areas. Note that

$$U_1 A_1 = U_2 A_2 = U_3 A_3 = U_4 A_4 = (\Sigma R_t)^{-1} \tag{3.32}$$

and the specific forms of U_2, U_3, and U_4 may be inferred from Equation 3.29.

EXAMPLE 3.4

The possible existence of an optimum insulation thickness for radial systems is suggested by the presence of competing effects associated with an increase in this thickness. In particular, although the conduction resistance increases with the addition of insulation, the convection resistance decreases due to increasing outer surface area. Hence there may exist an insulation thickness that minimizes heat loss by maximizing the total resistance to heat transfer. Resolve this question by considering the following system.

1. A thin-walled copper tube of radius r_i is used to transport a low temperature refrigerant and is at a temperature T_i that is less than that of the ambient air at T_∞ around the tube. Is there an optimum thickness associated with application of insulation to the tube?
2. Confirm the above result by computing the total thermal resistance per unit length of tube for a 10-mm-diameter tube having the following insulation thicknesses: 0, 2, 5, 10, 20, and 40 mm. The insulation is composed of cellular glass, and the outer surface convection coefficient is $5 \text{ W/m}^2 \cdot \text{K}$.

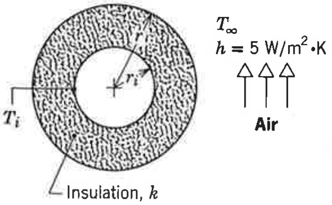
SOLUTION

Known: Radius r_i and temperature T_i of a thin-walled copper tube to be insulated from the ambient air.

Find:

1. Whether there exists an optimum insulation thickness that minimizes the heat transfer rate.
2. Thermal resistance associated with using cellular glass insulation of varying thickness.

Schematic:



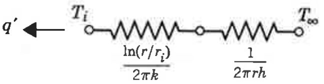
Assumptions:

1. Steady-state conditions.
2. One-dimensional heat transfer in the radial (cylindrical) direction.
3. Negligible tube wall thermal resistance.
4. Constant properties for insulation.
5. Negligible radiation exchange between insulation outer surface and surroundings.

Properties: Table A.3, cellular glass (285 K, assumed): $k = 0.055$ W/m · K.

Analysis:

1. The resistance to heat transfer between the refrigerant and the air is dominated by conduction in the insulation and convection in the air. The thermal circuit is therefore



where the conduction and convection resistances per unit length follow from Equations 3.28 and 3.9, respectively. The total thermal resistance per unit length of tube is then

$$R'_{\text{tot}} = \frac{\ln(r/r_i)}{2\pi k} + \frac{1}{2\pi rh}$$

where the rate of heat transfer per unit length of tube is

$$q' = \frac{T_\infty - T_i}{R'_{\text{tot}}}$$

An optimum insulation thickness would be associated with the value of r that minimized q' or maximized R'_{tot}. Such a value could be obtained from the requirement that

$$\frac{dR'_{\text{tot}}}{dr} = 0$$

Hence

$$\frac{1}{2\pi kr} - \frac{1}{2\pi r^2 h} = 0$$

or

$$r = \frac{k}{h}$$

To determine whether the foregoing result maximizes or minimizes the total resistance, the second derivative must be evaluated. Hence

$$\frac{d^2R'_{\text{tot}}}{dr^2} = -\frac{1}{2\pi k r^2} + \frac{1}{\pi r^3 h}$$

or, at $r = k/h$

$$\frac{d^2R'_{\text{tot}}}{dr^2} = \frac{1}{\pi(k/h)^2}\left(\frac{1}{k} - \frac{1}{2k}\right) = \frac{1}{2\pi k^3/h^2} > 0$$

Since this result is always positive, it follows that $r = k/h$ is the insulation radius for which the total resistance is a minimum, not a maximum. Hence an *optimum* insulation thickness *does not exist.*

From the above result it makes more sense to think in terms of a *critical insulation radius*

$$r_{\text{cr}} \equiv \frac{k}{h}$$

below which q' increases with increasing r and above which q' decreases with increasing r.

2. With $h = 5\ \text{W/m}^2 \cdot \text{K}$ and $k = 0.055\ \text{W/m} \cdot \text{K}$, the critical radius is

$$r_{\text{cr}} = \frac{0.055\ \text{W/m} \cdot \text{K}}{5\ \text{W/m}^2 \cdot \text{K}} = 0.011\ \text{m}$$

Hence $r_{\text{cr}} > r_i$ and heat transfer will increase with the addition of insulation up to a thickness of

$$r_{cr} - r_i = (0.011 - 0.005)\ \text{m} = 0.006\ \text{m}$$

The thermal resistances corresponding to the prescribed insulation thicknesses may be calculated and are summarized as follows.

INSULATION THICKNESS $(r - r_i)$ (mm)	INSULATION RADIUS r (m)	THERMAL RESISTANCES (m · K/W) R'_{cond}	R'_{conv}	R'_{tot}
0	0.005	0	6.37	6.37
2	0.007	0.97	4.55	5.52
5	0.010	2.00	3.18	5.18
6	r_{cr} = 0.011	2.28	2.89	5.17
10	0.015	3.18	2.12	5.30
20	0.025	4.66	1.27	5.93
40	0.045	6.35	0.71	7.06

Comments:

1. The effect of the critical radius is revealed by the fact that, even for 20 mm of insulation, the total resistance is not as large as the value for no insulation.
2. If $r_i < r_{cr}$, as it is in this case, the total resistance decreases and the heat rate therefore increases with the addition of insulation. This trend continues until the outer radius of the insulation corresponds to the critical radius. The trend is desirable for electrical current flow through a wire, since the addition of electrical insulation would aid in transferring heat dissipated in the wire to the surroundings. Conversely, if $r_i > r_{cr}$, any addition of insulation would increase the total resistance and therefore decrease the heat loss. This behavior would be desirable for steam flow through a pipe, where insulation is added to reduce heat loss to the surroundings.
3. For radial systems, the problem of reducing the total resistance through the application of insulation exists only for small diameter wires or tubes and for small convection coefficients, such that $r_{cr} > r_i$. For a typical insulation ($k \approx 0.03\ \mathrm{W/m \cdot K}$) and free convection in air ($h \approx 10\ \mathrm{W/m^2 \cdot K}$), $r_{cr} = (k/h) \approx 0.003$ m. Such a small value tells us that, normally, $r_i > r_{cr}$ and we need not be concerned with the effects of a critical radius.
4. The existence of a critical radius requires that the heat transfer area change in the direction of transfer, as for radial conduction in a cylinder (or a sphere). In a plane wall, for example, the area perpendicular to the direction of heat flow is constant and there is no critical insulation thickness (the total resistance always increases with increasing insulation thickness).

3.3.2 The Sphere

Now consider applying the alternative method to analyzing conduction in the hollow sphere of Figure 3.8. For the differential control volume of the figure, energy conservation requires that $q_r = q_{r+dr}$ for steady-state, one-dimensional conditions with no heat generation. The appropriate form of Fourier's law is

$$q_r = -kA\frac{dT}{dr} = -k(4\pi r^2)\frac{dT}{dr} \tag{3.33}$$

where $A = 4\pi r^2$ is the area normal to the direction of heat transfer.

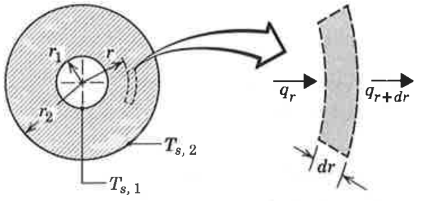


Figure 3.8 Conduction in a spherical shell.

Acknowledging that q_r is a constant, independent of r, Equation 3.33 may be expressed in the integral form

$$\frac{q_r}{4\pi}\int_{r_1}^{r_2}\frac{dr}{r^2} = -\int_{T_{s,1}}^{T_{s,2}} k(T)\,dT \tag{3.34}$$

Assuming constant k, we then obtain

$$q_r = \frac{4\pi k(T_{s,1} - T_{s,2})}{(1/r_1) - (1/r_2)} \tag{3.35}$$

Remembering that the thermal resistance is defined as the temperature difference divided by the heat transfer rate, we obtain

$$R_{t,\text{cond}} = \frac{1}{4\pi k}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) \tag{3.36}$$

Note that the temperature distribution and Equations 3.35 and 3.36 could have been obtained by using the standard approach, which begins with the appropriate form of the heat equation.

Spherical composites may be treated in much the same way as composite walls and cylinders, where appropriate forms of the total resistance and overall heat transfer coefficient may be determined.

EXAMPLE 3.5

A spherical, thin-walled metallic container is used to store liquid nitrogen at 77 K. The container has a diameter of 0.5 m and is covered with an evacuated, reflective insulation composed of silica powder. The insulation is 25 mm thick, and its outer surface is exposed to ambient air at 300 K. The convection coefficient is known to be 20 W/m^2 · K. The latent heat of vaporization and the density of liquid nitrogen are 2×10^5 J/kg and 804 kg/m^3, respectively.

1. What is the rate of heat transfer to the liquid nitrogen?
2. What is the rate of liquid boil-off?

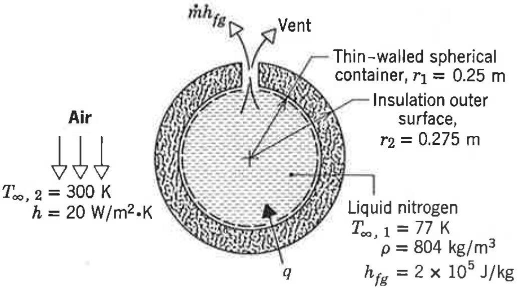
SOLUTION

Known: Liquid nitrogen is stored in a spherical container that is insulated and exposed to ambient air.

Find:

1. The rate of heat transfer to the nitrogen.
2. The mass rate of nitrogen boil-off.

Schematic:



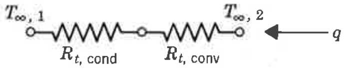
Assumptions:

1. Steady-state conditions.
2. One-dimensional transfer in the radial direction.
3. Negligible resistance to heat transfer through the container wall and from the container to the nitrogen.
4. Constant properties.
5. Negligible radiation exchange between outer surface of insulation and surroundings.

Properties: Table A.3, evacuated silica powder (300 K): $k = 0.0017$ W/m · K.

Analysis:

1. The thermal circuit involves a conduction and convection resistance in series and is of the form



where, from Equation 3.36,

$$R_{t,\text{cond}} = \frac{1}{4\pi k}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

and from Equation 3.9

$$R_{t,\text{conv}} = \frac{1}{h4\pi r_2^2}$$

The rate of heat transfer to the liquid nitrogen is then

$$q = \frac{T_{\infty,2} - T_{\infty,1}}{(1/4\pi k)[(1/r_1) - (1/r_2)] + (1/h4\pi r_2^2)}$$

Hence

$$q = [(300 - 77)\text{ K}] \div \left[\frac{1}{4\pi(0.0017\text{ W/m}\cdot\text{K})}\left(\frac{1}{0.25\text{ m}} - \frac{1}{0.275\text{ m}}\right) + \frac{1}{(20\text{ W/m}^2\cdot\text{K})4\pi(0.275\text{ m})^2}\right]$$

$$q = \frac{223}{17.02 + 0.05}\text{ W} = 13.06\text{ W} \qquad \triangleleft$$

2. Performing an energy balance for a control surface about the nitrogen, it follows from Equation 1.12 that

$$\dot{E}_{\text{in}} - \dot{E}_{\text{out}} = 0$$

where $\dot{E}_{\text{in}} = q$ and $\dot{E}_{\text{out}} = \dot{m}h_{fg}$ is associated with the loss of latent energy due to boiling. Hence

$$q - \dot{m}h_{fg} = 0$$

and the boil-off $\dot{m}$ is

$$\dot{m} = \frac{q}{h_{fg}}$$

$$\dot{m} = \frac{13.06\text{ J/s}}{2 \times 10^5\text{ J/kg}} = 6.53 \times 10^{-5}\text{ kg/s}$$

The loss per day is

$$\dot{m} = 6.53 \times 10^{-5}\ \text{kg/s} \times 3600\ \text{s/h} \times 24\ \text{h/day}$$

$$\dot{m} = 5.64\ \text{kg/day} \qquad \triangleleft$$

or on a volumetric basis

$$\dot{V} = \frac{\dot{m}}{\rho} = \frac{5.64\ \text{kg/day}}{804\ \text{kg/m}^3} = 0.007\ \text{m}^3/\text{day} = 7\ \text{liters/day}$$

Comments:

1. $R_{t,\text{conv}} \ll R_{t,\text{cond}}$.
2. With a container volume of $(4/3)(\pi r_1^3) = 0.065\ \text{m}^3 = 65$ liters, the daily loss amounts to (7 liters/65 liters) 100% = 10.8% of capacity.

3.4 SUMMARY OF ONE-DIMENSIONAL CONDUCTION RESULTS

Many important problems are characterized by one-dimensional, steady-state conduction in plane, cylindrical, or spherical walls without thermal energy generation. Key results for these three geometries are summarized in Table 3.3, where ΔT refers to the temperature difference, $T_{s,1} - T_{s,2}$, between the inner and outer surfaces identified in Figures 3.1, 3.6, and 3.8. In each case, beginning with the heat equation, you should be able to derive the corresponding expressions for the temperature distribution, heat flux, heat rate, and thermal resistance.

Table 3.3 One-dimensional, steady-state solutions to the heat equation with no generation

	PLANE WALL	CYLINDRICAL WALL[a]	SPHERICAL WALL[a]
Heat equation	$\frac{d^2T}{dx^2} = 0$	$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) = 0$	$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dT}{dr}\right) = 0$
Temperature distribution	$T_{s,1} - \Delta T\frac{x}{L}$	$T_{s,2} + \Delta T\frac{\ln(r/r_2)}{\ln(r_1/r_2)}$	$T_{s,1} - \Delta T\left[\frac{1-(r_1/r)}{1-(r_1/r_2)}\right]$
Heat flux (q'')	$k\frac{\Delta T}{L}$	$\frac{k\,\Delta T}{r\ln(r_2/r_1)}$	$\frac{k\,\Delta T}{r^2[(1/r_1)-(1/r_2)]}$
Heat rate (q)	$kA\frac{\Delta T}{L}$	$\frac{2\pi Lk\,\Delta T}{\ln(r_2/r_1)}$	$\frac{4\pi k\,\Delta T}{(1/r_1)-(1/r_2)}$
Thermal resistance ($R_{t,\text{cond}}$)	$\frac{L}{kA}$	$\frac{\ln(r_2/r_1)}{2\pi Lk}$	$\frac{(1/r_1)-(1/r_2)}{4\pi k}$

[a] The critical radius of insulation is $r_{cr} = k/h$ for the cylinder and $r_{cr} = 2k/h$ for the sphere.

3.5 CONDUCTION WITH THERMAL ENERGY GENERATION

In the preceding section we considered conduction problems for which the temperature distribution in a medium was determined solely by conditions at the boundaries of the medium. We now want to consider the additional effect on the temperature distribution of processes that may be occurring *within* the medium. In particular, we wish to consider situations for which thermal energy is being *generated* due to *conversion* from some other energy form.

A common thermal energy generation process involves the conversion from *electrical to thermal energy* in a current-carrying medium (*ohmic* or *resistance heating*). The rate at which energy is generated by passing a current I through a medium of electrical resistance R_e is

$$\dot{E}_g = I^2 R_e \tag{3.37}$$

If this power generation (W) occurs uniformly throughout the medium of volume V, the volumetric generation rate (W/m^3) is then

$$\dot{q} \equiv \frac{\dot{E}_g}{V} = \frac{I^2 R_e}{V} \tag{3.38}$$

Energy generation may also occur as a result of the deceleration and absorption of neutrons in the fuel element of a nuclear reactor or exothermic chemical reactions occurring within a medium. Endothermic reactions would, of course, have the inverse effect (a thermal energy sink) of converting thermal energy to chemical bonding energy. Finally, a conversion from electromagnetic to thermal energy may occur due to the absorption of radiation within the medium. The process may occur, for example, because gamma rays are absorbed in external nuclear reactor components (cladding, thermal shields, pressure vessels, etc.) or because visible radiation is absorbed in a semitransparent medium. Remember not to confuse energy generation with energy storage (Section 1.3.1).

3.5.1 The Plane Wall

Consider the plane wall of Figure 3.9*a*, in which there is *uniform* energy generation per unit volume ($\dot{q}$ is constant) and the surfaces are maintained at $T_{s,1}$ and $T_{s,2}$. For constant thermal conductivity k, the appropriate form of the heat equation, Equation 2.16, is

$$\frac{d^2T}{dx^2} + \frac{\dot{q}}{k} = 0 \tag{3.39}$$

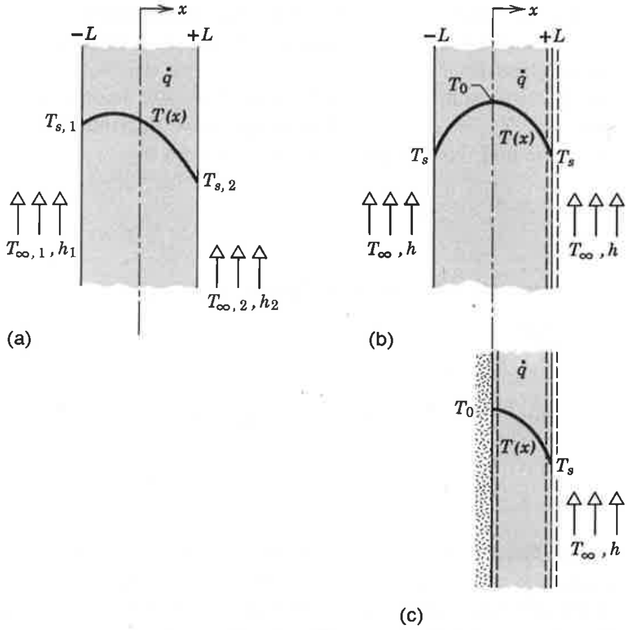


Figure 3.9 Conduction in a plane wall with uniform heat generation. (*a*) Asymmetrical boundary conditions. (*b*) Symmetrical boundary conditions. (*c*) Adiabatic surface at midplane.

The general solution is

$$T = -\frac{\dot{q}}{2k}x^2 + C_1x + C_2 \tag{3.40}$$

where C_1 and C_2 are the constants of integration. For the prescribed boundary conditions,

$$T(-L) = T_{s,1} \qquad \text{and} \qquad T(L) = T_{s,2}$$

The constants may be evaluated and are of the form

$$C_1 = \frac{T_{s,2} - T_{s,1}}{2L} \qquad \text{and} \qquad C_2 = \frac{\dot{q}}{2k}L^2 + \frac{T_{s,1} + T_{s,2}}{2}$$

in which case the temperature distribution is

$$T(x) = \frac{\dot{q}L^2}{2k}\left(1 - \frac{x^2}{L^2}\right) + \frac{T_{s,2} - T_{s,1}}{2}\frac{x}{L} + \frac{T_{s,1} + T_{s,2}}{2} \tag{3.41}$$

The heat flux at any point in the wall may, of course, be determined by using Equation 3.41 with Fourier's law. Note, however, that *with generation the heat flux is no longer independent of* x.

The preceding result simplifies when both surfaces are maintained at a common temperature, $T_{s,1} = T_{s,2} \equiv T_s$. The temperature distribution is then *symmetrical* about the midplane, Figure 3.9*b*, and is given by

$$T(x) = \frac{\dot{q}L^2}{2k}\left(1 - \frac{x^2}{L^2}\right) + T_s \tag{3.42}$$

The maximum temperature exists at the midplane

$$T(0) \equiv T_0 = \frac{\dot{q}L^2}{2k} + T_s \tag{3.43}$$

in which case the temperature distribution, Equation 3.42, may be expressed as

$$\frac{T(x) - T_0}{T_s - T_0} = \left(\frac{x}{L}\right)^2 \tag{3.44}$$

It is important to note that at the plane of symmetry in Figure 3.9*b*, the temperature gradient is zero, $(dT/dx)_{x=0} = 0$. Accordingly, this plane may be represented by the *adiabatic* surface shown in Figure 3.9*c*. One implication of this result is that Equation 3.42 also applies to plane walls that are perfectly insulated on one side ($x = 0$) and maintained at a fixed temperature T_s on the other side ($x = L$).

To use the foregoing results the surface temperature(s) T_s must be known. However, a common situation is one for which it is the temperature of an adjoining fluid, T_∞, and not T_s, which is known. It then becomes necessary to relate T_s to T_∞. This relation may be developed by applying a surface energy balance. Consider the surface at $x = L$ for the symmetrical plane wall (Figure 3.9*b*) or the insulated plane wall (Figure 3.9*c*). Neglecting radiation and substituting the appropriate rate equations, the energy balance given by Equation 1.12 reduces to

$$-k\left.\frac{dT}{dx}\right|_{x=L} = h(T_s - T_\infty) \tag{3.45}$$

Substituting from Equation 3.42 to obtain the temperature gradient at $x = L$, it follows that

$$T_s = T_\infty + \frac{\dot{q}L}{h} \tag{3.46}$$

Hence T_s may be computed from knowledge of T_∞, $\dot{q}$, L, and h.

Equation 3.46 may also be obtained by applying an *overall* energy balance to the plane wall of Figure 3.9*b* or 3.9*c*. For example, relative to a control surface about the wall of Figure 3.9*c*, the rate at which energy is generated within the wall must be balanced by the rate at which energy leaves via convection at the boundary. Equation 1.11a reduces to

$$\dot{E}_g = \dot{E}_{\text{out}} \tag{3.47}$$

or, for a unit surface area,

$$\dot{q}L = h(T_s - T_\infty) \tag{3.48}$$

Solving for T_s, Equation 3.46 is obtained.

Equation 3.46 may be combined with Equation 3.42 to eliminate T_s from the temperature distribution, which is then expressed in terms of the known quantities $\dot{q}$, L, k, h, and T_∞. The same result may be obtained directly by using Equation 3.45 as a boundary condition to evaluate the constants of integration appearing in Equation 3.40 (see Problem 3.58).

EXAMPLE 3.6

A plane wall is a composite of two materials, A and B. The wall of material A has uniform heat generation $\dot{q} = 1.5 \times 10^6$ W/m^3, $k_A = 75$ W/m · K, and thickness $L_A = 50$ mm. The wall material B has no generation with $k_B = 150$ W/m · K and thickness $L_B = 20$ mm. The inner surface of material A is well insulated, while the outer surface of material B is cooled by a water stream with $T_\infty = 30$°C and $h = 1000$ W/m^2 · K.

1. Sketch the temperature distribution that exists in the composite under steady-state conditions.
2. Determine the temperature T_0 of the insulated surface and the temperature T_2 of the cooled surface.

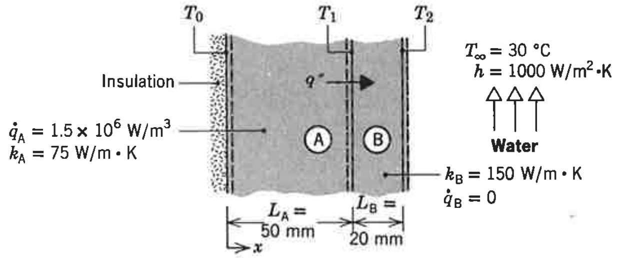
SOLUTION

Known: Plane wall of material A with internal heat generation is insulated on one side and bounded by a second wall of material B, which is without heat generation and is subjected to convection cooling.

Find:

1. Sketch of steady-state temperature distribution in the composite.
2. Inner and outer surface temperatures of the composite.

Schematic:

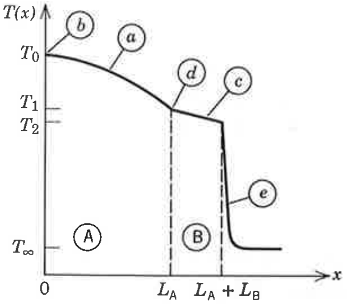


Assumptions:

1. Steady-state conditions.
2. One-dimensional conduction in x direction.
3. Negligible contact resistance between walls.
4. Inner surface of A adiabatic.
5. Constant properties for materials A and B.

Analysis:

1. From the prescribed physical conditions, the temperature distribution in the composite is known to have the following features, as shown.
   (a) Parabolic in material A.
   (b) Zero slope at insulated boundary.
   (c) Linear in material B.
   (d) Slope change $= k_B/k_A = 2$ at interface.
   The temperature distribution in the water is characterized by
   (e) Large gradients near the surface.



2. The outer surface temperature T_2 may be obtained by performing an energy balance on a control volume about material B. Since there is no generation in this material, it follows that, for steady-state conditions and a unit surface area, the heat flux into the material at $x = L_A$ must equal the heat flux from the material due to convection at $x = L_A + L_B$. Hence

$$q'' = h(T_2 - T_\infty) \tag{1}$$

The heat flux q'' may be determined by performing a second energy balance on a control volume about material A. In particular, since the surface at $x = 0$ is adiabatic, there is no inflow and the rate at which energy is generated must equal the outflow. Accordingly, for a unit surface area,

$$\dot{q}L_A = q'' \tag{2}$$

Combining Equations 1 and 2, the outer surface temperature is

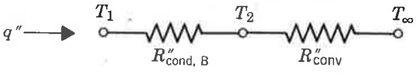
$$T_2 = T_\infty + \frac{\dot{q}L_A}{h}$$

$$T_2 = 30°\text{C} + \frac{1.5 \times 10^6\ \text{W/m}^3 \times 0.05\ \text{m}}{1000\ \text{W/m}^2 \cdot \text{K}} = 105°\text{C} \qquad \triangleleft$$

From Equation 3.43 the temperature at the insulated surface is

$$T_0 = \frac{\dot{q}L_A^2}{2k_A} + T_1 \tag{3}$$

where T_1 may be obtained from the following thermal circuit.



That is,

$$T_1 = T_\infty + (R''_{\text{cond,B}} + R''_{\text{conv}})q''$$

where the resistances for a unit surface area are

$$R''_{\text{cond,B}} = \frac{L_B}{k_B} \qquad R''_{\text{conv}} = \frac{1}{h}$$

Hence

$$T_1 = 30°\text{C} + \left(\frac{0.02\ \text{m}}{150\ \text{W/m}\cdot\text{K}} + \frac{1}{1000\ \text{W/m}^2\cdot\text{K}}\right)$$

$$\times 1.5 \times 10^6\ \text{W/m}^3 \times 0.05\ \text{m}$$

$$T_1 = 30°\text{C} + 85°\text{C} = 115°\text{C}$$

Substituting into Equation 3,

$$T_0 = \frac{1.5 \times 10^6\ \text{W/m}^3 (0.05\ \text{m})^2}{2 \times 75\ \text{W/m}\cdot\text{K}} + 115°\text{C}$$

$$T_0 = 25°\text{C} + 115°\text{C} = 140°\text{C} \qquad \triangleleft$$

Comments:

1. Material A, having heat generation, cannot be represented by a thermal circuit element.
2. $(T_2 - T_\infty)/(T_1 - T_2) = R''_{\text{conv}}/R''_{\text{cond, B}} = 7.5$.

3.5.2 Radial Systems

Heat generation may occur in a variety of radial geometries. Consider the long, solid cylinder of Figure 3.10, which could represent a current-carrying wire or a fuel element in a nuclear reactor. For steady-state conditions the rate at which heat is generated within the cylinder must equal the rate at which heat is convected from the surface of the cylinder to a moving fluid. This condition allows the surface temperature to be maintained at a fixed value of T_s.

To determine the temperature distribution in the cylinder, we begin with the appropriate form of the heat equation. For constant thermal conductivity

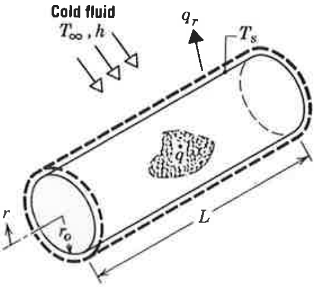


Figure 3.10 Conduction in a solid cylinder with uniform heat generation.

k, Equation 2.20 reduces to

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) + \frac{\dot{q}}{k} = 0 \tag{3.49}$$

Separating variables and assuming uniform generation, this expression may be integrated to obtain

$$r\frac{dT}{dr} = -\frac{\dot{q}}{2k}r^2 + C_1 \tag{3.50}$$

Repeating the procedure, the general solution for the temperature distribution becomes

$$T(r) = -\frac{\dot{q}}{4k}r^2 + C_1 \ln r + C_2 \tag{3.51}$$

To obtain the constants of integration C_1 and C_2, we apply the boundary conditions

$$\left.\frac{dT}{dr}\right|_{r=0} = 0 \qquad \text{and} \qquad T(r_o) = T_s$$

The first condition results from the symmetry of the situation. That is, for the solid cylinder the centerline is a line of symmetry for the temperature distribution and the temperature gradient must be zero. Recall that similar conditions existed at the midplane of a wall having symmetrical boundary conditions (Figure 3.9*b*). From the symmetry condition at $r = 0$ and Equation 3.50, it is evident that $C_1 = 0$. Using the surface boundary condition at $r = r_o$ with Equation 3.51, we then obtain

$$C_2 = T_s + \frac{\dot{q}}{4k}r_o^2 \tag{3.52}$$

The temperature distribution is therefore

$$T(r) = \frac{\dot{q}r_o^2}{4k}\left(1 - \frac{r^2}{r_o^2}\right) + T_s \tag{3.53}$$

Evaluating Equation 3.53 at the centerline and dividing the result into Equation 3.53, we obtain the temperature distribution in nondimensional form

$$\frac{T(r) - T_s}{T_o - T_s} = 1 - \left(\frac{r}{r_o}\right)^2 \tag{3.54}$$

where T_o is the centerline temperature. The heat rate at any radius in the cylinder may, of course, be evaluated by using Equation 3.53 with Fourier's law.

To relate the surface temperature, T_s, to the temperature, T_∞, of the cold fluid, either a surface energy balance or an overall energy balance may be used. Choosing the second approach, we obtain

$$\dot{q}(\pi r_o^2 L) = h(2\pi r_o L)(T_s - T_\infty)$$

or

$$T_s = T_\infty + \frac{\dot{q}r}{2h} \qquad (3.55)$$

The foregoing approach may also be used to obtain the temperature distribution in solid spheres and in cylindrical and spherical shells for a variety of boundary conditions.

EXAMPLE 3.7

Consider a long solid tube, insulated at the outer radius r_o and cooled at the inner radius r_i, with uniform heat generation $\dot{q}$ (W/m^3) within the solid.

1. Obtain the general solution for the temperature distribution in the tube.
2. In a practical application a limit would be placed on the maximum temperature that is permissible at the insulated surface ($r = r_o$). Specifying this limit as T_o, identify appropriate boundary conditions that could be used to determine the arbitrary constants appearing in the general solution. Determine these constants and the corresponding form of the temperature distribution.
3. Determine the heat removal rate per unit length of tube.
4. If the coolant is available at a temperature T_∞, obtain an expression for the convection coefficient that would have to be maintained at the inner surface to allow for operation at prescribed values of T_o and $\dot{q}$.

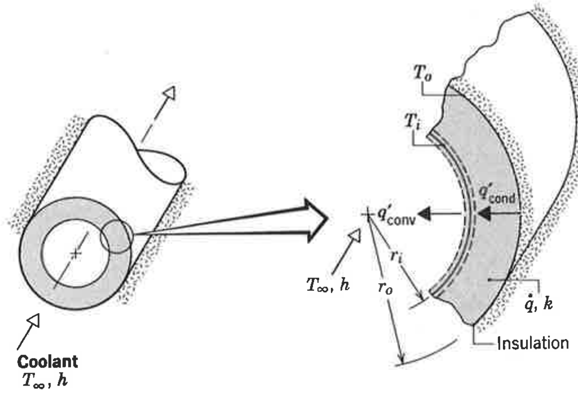
SOLUTION

Known: Solid tube with uniform heat generation is insulated at the outer surface and cooled at the inner surface.

Find:

1. General solution for the temperature distribution $T(r)$.
2. Appropriate boundary conditions and the corresponding form of the temperature distribution.
3. Heat removal rate.
4. Convection coefficient at the inner surface.

Schematic:



Assumptions:

1. Steady-state conditions.
2. One-dimensional radial conduction.
3. Constant properties.
4. Uniform volumetric heat generation.
5. Outer surface adiabatic.

Analysis:

1. To determine $T(r)$, the appropriate form of the heat equation, Equation 2.20, must be solved. For the prescribed conditions, this expression reduces to Equation 3.49, and the general solution is given by Equation 3.51. Hence this solution applies in a cylindrical shell, as well as in a solid cylinder (Figure 3.10).
2. Two boundary conditions are needed to evaluate C_1 and C_2, and in this problem it is appropriate to specify both conditions at r_o. Invoking the prescribed temperature limit,

$$T(r_o) = T_o \tag{1}$$

and applying Fourier's law, Equation 3.24, at the adiabatic outer surface

$$\left.\frac{dT}{dr}\right|_{r_o} = 0 \tag{2}$$

Using Equations 3.51 and 1, it follows that

$$T_o = -\frac{\dot{q}}{4k}r_o^2 + C_1 \ln r_o + C_2 \tag{3}$$

Similarly, from Equations 3.50 and 2

$$0 = -\frac{\dot{q}}{2k}r_o^2 + C_1 \tag{4}$$

Hence, from Equation 4,

$$C_1 = \frac{\dot{q}}{2k}r_o^2 \tag{5}$$

and from Equation 3

$$C_2 = T_o + \frac{\dot{q}}{4k}r_o^2 - \frac{\dot{q}}{2k}r_o^2 \ln r_o \tag{6}$$

Substituting Equations 5 and 6 into the general solution, Equation 3.51, it follows that

$$T(r) = T_o + \frac{\dot{q}}{4k}\left(r_o^2 - r^2\right) - \frac{\dot{q}}{2k}r_o^2 \ln \frac{r_o}{r} \tag{7}$$

3. The heat removal rate may be determined by obtaining the conduction rate at r_i or by evaluating the total generation rate for the tube. From Fourier's law

$$q_r' = -k2\pi r\frac{dT}{dr}$$

Hence substituting from Equation 7 and evaluating the result at r_i,

$$q_r'(r_i) = -k2\pi r_i\left(-\frac{\dot{q}}{2k}r_i + \frac{\dot{q}}{2k}\frac{r_o^2}{r_i}\right) = -\pi\dot{q}\left(r_o^2 - r_i^2\right) \tag{8}$$

Alternatively, because the tube is insulated at r_o, the rate at which heat is generated in the tube must equal the rate of removal at r_i. That is, for a control volume about the tube, the energy conservation requirement, Equation 1.11a, reduces to $\dot{E}_g - \dot{E}_{\text{out}} = 0$, where $\dot{E}_g = \dot{q}\pi(r_o^2 - r_i^2)L$ and $\dot{E}_{\text{out}} = q'_{\text{cond}}L = -q_r'(r_i)L$. Hence

$$q_r'(r_i) = -\pi\dot{q}\left(r_o^2 - r_i^2\right) \tag{9}$$

4. Applying the energy conservation requirement, Equation 1.12, to the inner surface, it follows that

$$q'_{\text{cond}} = q'_{\text{conv}}$$

or

$$\pi\dot{q}\left(r_o^2 - r_i^2\right) = h2\pi r_i(T_i - T_\infty)$$

Hence

$$h = \frac{\dot{q}\left(r_o^2 - r_i^2\right)}{2r_i(T_i - T_\infty)} \tag{10}$$

where T_i may be obtained by evaluating Equation 7 at $r = r_i$.

Comments: Note that, through application of Fourier's law in part 3, the sign on $q_r'(r_i)$ was found to be negative, Equation 8, implying that heat flow is in the negative r direction. However, in applying the energy balance, we acknowledged that heat flow was *out of* the wall. Hence we expressed q_{cond}' as $-q_r'(r_i)$ and we expressed q_{conv}' in terms of $(T_i - T_\infty)$, rather than $(T_\infty - T_i)$.

3.5.3 Application of Resistance Concepts

We conclude our discussion of heat generation effects with a word of caution. In particular, when such effects are present, the heat transfer rate is not a constant, independent of the spatial coordinate. Accordingly, it would be *incorrect* to use the thermal resistance concepts and the related heat rate equations developed in Sections 3.1 and 3.3.

3.6 HEAT TRANSFER FROM EXTENDED SURFACES

The term *extended surface* is commonly used in reference to a solid that experiences energy transfer by conduction within its boundaries, as well as energy transfer by convection (and/or radiation) between its boundaries and the surroundings. Such a system is shown schematically in Figure 3.11. A strut

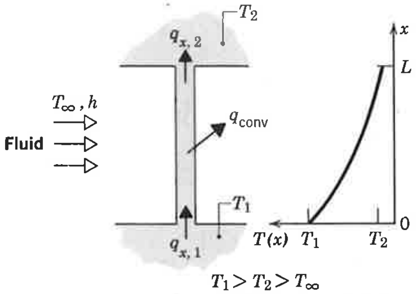


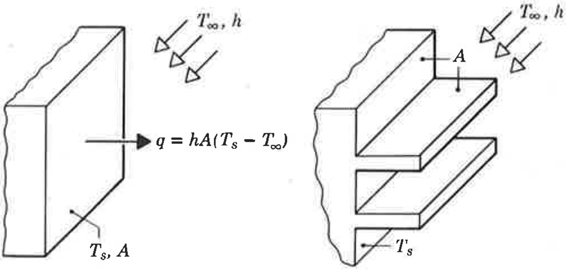
Figure 3.11 Combined conduction and convection in a structural element.

is used to provide mechanical support to two walls that are at different temperatures. A temperature gradient in the x direction sustains heat transfer by conduction internally, at the same time that there is energy transfer by convection from the surface.

Although there are many different situations that involve combined conduction–convection effects, the most frequent application is one in which an extended surface is used specifically to *enhance* the heat transfer rate between a solid and an adjoining fluid. Such an extended surface is termed a *fin*.

Consider the plane wall of Figure 3.12*a*. If T_s is fixed, there are two ways in which the heat transfer rate may be increased. The convection coefficient h could be increased by increasing the fluid velocity, and/or the fluid temperature T_∞ could be reduced. However, many situations would be encountered in which increasing h to the maximum possible value is either insufficient to obtain the desired heat transfer rate or the associated costs are prohibitive. Such costs are related to the blower or pump power requirements needed to increase h through increased fluid motion. Moreover, the second option of reducing T_∞ is often impractical. Examining Figure 3.12*b*, however, we see that there exists a third option. That is, the heat transfer rate may be increased by increasing the surface area across which the convection occurs. This may be done by employing *fins* that *extend* from the wall into the surrounding fluid. The thermal conductivity of the fin material has a strong effect on the temperature distribution along the fin and therefore influences the degree to which the heat transfer rate is enhanced. Ideally, the fin material should have a large thermal conductivity to minimize temperature variations from its base to its tip. In the limit of infinite thermal conductivity, the entire fin would be at the temperature of the base surface, thereby providing the maximum possible heat transfer enhancement.

You are already familiar with several fin applications. Consider the arrangement for cooling engine heads on motorcycles and lawnmowers or for cooling electric power transformers. Consider also the tubes with attached fins



(*a*) (*b*)

Figure 3.12 Use of fins to enhance heat transfer from a plane wall. (*a*) Bare surface. (*b*) Finned surface.

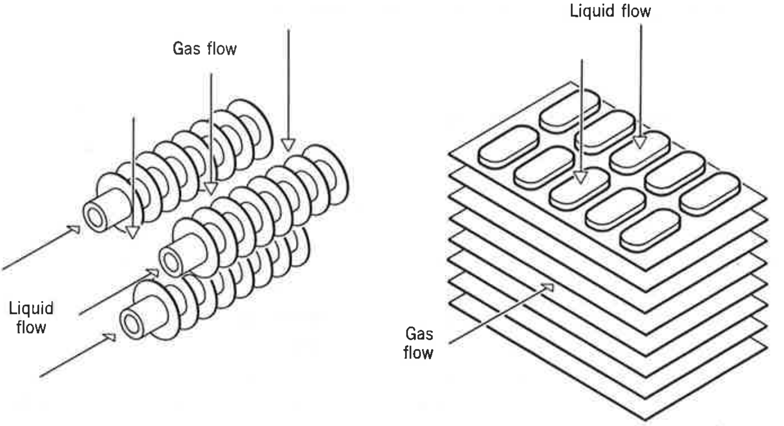


Figure 3.13 Schematic of typical finned-tube heat exchangers.

used to promote heat exchange between air and the working fluid of an air conditioner. Two common finned-tube arrangements are shown in Figure 3.13.

Different fin configurations are illustrated in Figure 3.14. A *straight fin* is any extended surface that is attached to a *plane wall*. It may be of uniform cross-sectional area, or its cross-sectional area may vary with the distance x from the wall. An *annular fin* is one that is circumferentially attached to a cylinder, and its cross section varies with radius from the centerline of the cylinder. The foregoing fin types have rectangular cross sections, whose area may be expressed as a product of the fin thickness t and the width w for straight fins or the circumference $2\pi r$ for annular fins. In contrast a *pin fin*, or *spine*, is an extended surface of circular cross section. Pin fins may also be of uniform or nonuniform cross section. In any application, selection of a particular fin configuration may depend on space, weight, manufacturing, and cost considerations, as well as on the extent to which the fins reduce the

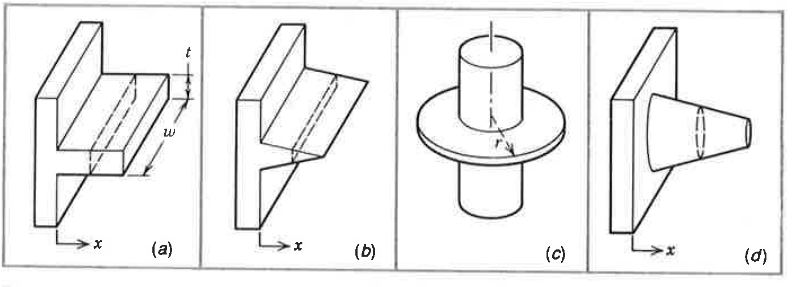


Figure 3.14 Fin configurations. (a) Straight fin of uniform cross section. (b) Straight fin of nonuniform cross section. (c) Annular fin. (d) Pin fin.

surface convection coefficient and increase the pressure drop associated with flow over the fins.

3.6.1 A General Conduction Analysis

As engineers we are primarily interested in knowing the extent to which particular extended surfaces or fin arrangements could improve heat *dissipation* from a surface to the surrounding fluid. To determine the heat transfer rate associated with a fin, we must first obtain the temperature distribution along the fin. As we have done for previous systems, we begin by performing an energy balance on an appropriate differential element. Consider the extended surface of Figure 3.15. The analysis is simplified if certain assumptions are made. We choose to assume one-dimensional conditions in the longitudinal (x) direction, even though conduction within the fin is actually two dimensional. The rate at which energy is convected to the fluid from any point on the fin surface must be balanced by the rate at which energy reaches that point due to conduction in the transverse (y, z) direction. However, in practice the fin is thin and temperature changes in the longitudinal direction are much larger than those in the transverse direction. Hence we may assume one-dimensional conduction in the x direction. We will consider steady-state conditions and also assume that the thermal conductivity is constant, that radiation from the surface is negligible, that heat generation effects are absent, and that the convection heat transfer coefficient h is uniform over the surface.

Applying the conservation of energy requirement, Equation 1.11a, to the differential element of Figure 3.15, we obtain

$$q_x = q_{x+dx} + dq_{\text{conv}} \tag{3.56}$$

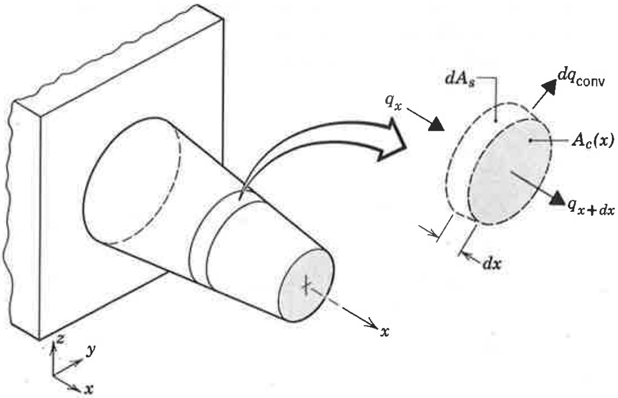


Figure 3.15 Energy balance for an extended surface.

From Fourier's law we know that

$$q_x = -kA_c\frac{dT}{dx} \tag{3.57}$$

where A_c is the *cross-sectional* area, which may vary with x. Since the conduction heat rate at $x + dx$ may be expressed as

$$q_{x+dx} = q_x + \frac{dq_x}{dx}\,dx \tag{3.58}$$

it follows that

$$q_{x+dx} = -kA_c\frac{dT}{dx} - k\frac{d}{dx}\left(A_c\frac{dT}{dx}\right)dx \tag{3.59}$$

The convection heat transfer rate may be expressed as

$$dq_{\text{conv}} = h\,dA_s(T - T_\infty) \tag{3.60}$$

where dA_s is the *surface* area of the differential element. Substituting the foregoing rate equations into the energy balance, Equation 3.56, we obtain

$$\frac{d}{dx}\left(A_c\frac{dT}{dx}\right) - \frac{h}{k}\frac{dA_s}{dx}(T - T_\infty) = 0$$

or

$$\frac{d^2T}{dx^2} + \left(\frac{1}{A_c}\frac{dA_c}{dx}\right)\frac{dT}{dx} - \left(\frac{1}{A_c}\frac{h}{k}\frac{dA_s}{dx}\right)(T - T_\infty) = 0 \tag{3.61}$$

This result provides a general form of the energy equation for one-dimensional conditions in an extended surface. Its solution for appropriate boundary conditions would provide the temperature distribution, which could then be used with Equation 3.57 to calculate the conduction rate at any x.

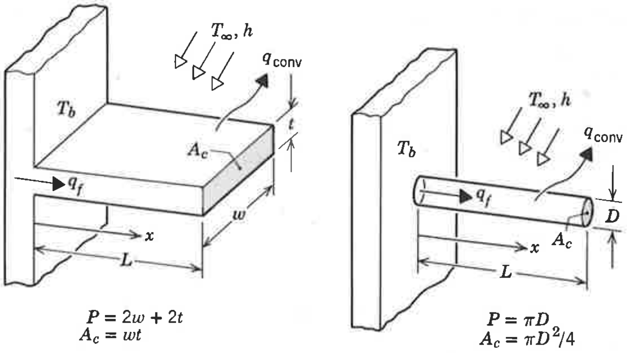
3.6.2 Fins of Uniform Cross-Sectional Area

To solve Equation 3.61 it is necessary to be more specific about the geometry. We begin with the simplest case of straight rectangular and pin fins of uniform cross section (Figure 3.16). Each fin is attached to a base surface of temperature $T(0) = T_b$ and extends into a fluid of temperature T_∞.

For the prescribed fins, A_c is a constant and $A_s = Px$, where A_s is the surface area measured from the base to x and P is the fin perimeter. Accordingly, with $dA_c/dx = 0$ and $dA_s/dx = P$, Equation 3.61 reduces to

$$\frac{d^2T}{dx^2} - \frac{hP}{kA_c}(T - T_\infty) = 0 \tag{3.62}$$

To simplify the form of this equation, we transform the dependent variable by



(a) (b)

Figure 3.16 Straight fins of uniform cross section. (*a*) Rectangular fin. (*b*) Pin fin.

defining an *excess temperature* θ as

$$\theta(x) \equiv T(x) - T_\infty \tag{3.63}$$

where, since T_∞ is a constant, $d\theta/dx = dT/dx$. Substituting Equation 3.63 into Equation 3.62, we then obtain

$$\frac{d^2\theta}{dx^2} - m^2\theta = 0 \tag{3.64}$$

where

$$m^2 \equiv \frac{hP}{kA_c} \tag{3.65}$$

Equation 3.64 is a linear, homogeneous, second-order differential equation with constant coefficients. Its general solution is of the form

$$\theta(x) = C_1 e^{mx} + C_2 e^{-mx} \tag{3.66}$$

By substitution it may be readily verified that Equation 3.66 is indeed a solution to Equation 3.64.

To evaluate the constants C_1 and C_2 of Equation 3.66, it is necessary to specify appropriate boundary conditions. One such condition may be specified in terms of the temperature at the *base* of the fin ($x = 0$)

$$\theta(0) = T_b - T_\infty \equiv \theta_b \tag{3.67}$$

The second condition, specified at the fin tip ($x = L$), may correspond to any one of four different physical situations.

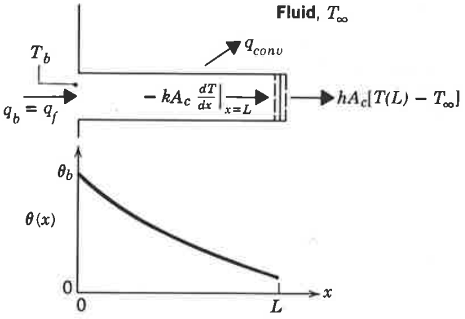


Figure 3.17 Conduction and convection in a fin of uniform cross section.

The first condition, case A, considers convection heat transfer from the fin tip. Applying an energy balance to a control surface about this tip (Figure 3.17), we obtain

$$hA_c[T(L) - T_\infty] = -kA_c \frac{dT}{dx}\bigg|_{x=L}$$

or

$$h\theta(L) = -k\frac{d\theta}{dx}\bigg|_{x=L} \tag{3.68}$$

That is, the rate at which energy is transferred to the fluid by convection from the tip must equal the rate at which energy reaches the tip by conduction through the fin. Substituting Equation 3.66 into Equations 3.67 and 3.68, we obtain, respectively,

$$\theta_b = C_1 + C_2 \tag{3.69}$$

and

$$h\left(C_1 e^{mL} + C_2 e^{-mL}\right) = km\left(C_2 e^{-mL} - C_1 e^{mL}\right)$$

Solving for C_1 and C_2, it may be shown, after some manipulation, that

$$\frac{\theta}{\theta_b} = \frac{\cosh m(L-x) + (h/mk)\sinh m(L-x)}{\cosh mL + (h/mk)\sinh mL} \tag{3.70}$$

The form of this temperature distribution is shown schematically in Figure 3.17. Note that the magnitude of the temperature gradient decreases with increasing x. This trend is a consequence of the reduction in the conduction heat transfer $q_x(x)$ with increasing x due to continuous convection losses from the fin surface.

We are also interested in the total heat transferred by the fin. From Figure 3.17 it is evident that the fin heat transfer rate q_f may be evaluated in two alternative ways, both of which involve use of the temperature distribution. The simpler procedure, and the one which we will use, involves applying Fourier's law at the fin base. That is,

$$q_f = q_b = -kA_c \frac{dT}{dx}\bigg|_{x=0} = -kA_c \frac{d\theta}{dx}\bigg|_{x=0} \tag{3.71}$$

Hence, knowing the temperature distribution, $\theta(x)$, q_f may be evaluated, giving

$$q_f = \sqrt{hPkA_c}\,\theta_b \frac{\sinh mL + (h/mk)\cosh mL}{\cosh mL + (h/mk)\sinh mL} \tag{3.72}$$

However, conservation of energy dictates that the rate at which heat is transferred by convection from the fin must equal the rate at which it is conducted through the base of the fin. Accordingly, the alternative formulation for q_f is

$$q_f = \int_{A_f} h[T(x) - T_\infty]\, dA_s$$

$$q_f = \int_{A_f} h\theta(x)\, dA_s \tag{3.73}$$

where A_f is the *total*, including the tip, *fin surface area*. Substitution of Equation 3.70 into Equation 3.73 would yield Equation 3.72.

The second tip condition, case B, corresponds to the assumption that the convective heat loss from the fin tip is negligible, in which case the tip may be treated as adiabatic and

$$\frac{d\theta}{dx}\bigg|_{x=L} = 0 \tag{3.74}$$

Substituting from Equation 3.66 and dividing by m, we then obtain

$$C_1 e^{mL} - C_2 e^{-mL} = 0$$

Using this expression with Equation 3.69 to solve for C_1 and C_2 and substituting the results into Equation 3.66, we obtain

$$\frac{\theta}{\theta_b} = \frac{\cosh m(L-x)}{\cosh mL} \tag{3.75}$$

Using this temperature distribution with Equation 3.71, the fin heat transfer rate is then

$$q_f = \sqrt{hPkA_c}\,\theta_b \tanh mL \tag{3.76}$$

In the same manner, we can obtain the fin temperature distribution and heat transfer rate for case C, where the temperature is prescribed at the fin tip. That is, the second boundary condition is $\theta(L) = \theta_L$, and the resulting expressions are of the form

$$\frac{\theta}{\theta_b} = \frac{(\theta_L/\theta_b)\sinh mx + \sinh m(L-x)}{\sinh mL} \tag{3.77}$$

$$q_f = \sqrt{hPkA_c}\,\theta_b \frac{\cosh mL - \theta_L/\theta_b}{\sinh mL} \tag{3.78}$$

The *very long fin*, case D, is an interesting extension of these results. In particular, as $L \to \infty$, $\theta_L \to 0$ and it is easily verified that

$$\frac{\theta}{\theta_b} = e^{-mx} \tag{3.79}$$

$$q_f = \sqrt{hPkA_c}\,\theta_b \tag{3.80}$$

The foregoing results are summarized in Table 3.4. A table of hyperbolic functions is provided in Appendix B.1.

Table 3.4 Temperature distribution and heat loss for fins of uniform cross section

CASE	TIP CONDITION ($X = L$)	TEMPERATURE DISTRIBUTION θ/θ_b	FIN HEAT TRANSFER RATE q_f
A	Convection heat transfer: $h\theta(L) = -k\,d\theta/dx\vert_{x=L}$	$\dfrac{\cosh m(L-x) + (h/mk)\sinh m(L-x)}{\cosh mL + (h/mk)\sinh mL}$ (3.70)	$M\dfrac{\sinh mL + (h/mk)\cosh mL}{\cosh mL + (h/mk)\sinh mL}$ (3.72)
B	Adiabatic: $d\theta/dx\vert_{x=L} = 0$	$\dfrac{\cosh m(L-x)}{\cosh mL}$ (3.75)	$M \tanh mL$ (3.76)
C	Prescribed temperature: $\theta(L) = \theta_L$	$\dfrac{(\theta_L/\theta_b)\sinh mx + \sinh m(L-x)}{\sinh mL}$ (3.77)	$M\dfrac{(\cosh mL - \theta_L/\theta_b)}{\sinh mL}$ (3.78)
D	Infinite fin ($L \to \infty$): $\theta(L) = 0$	e^{-mx} (3.79)	M (3.80)

$\theta \equiv T - T_\infty$ $\qquad m^2 \equiv hP/kA_c$

$\theta_b = \theta(0) = T_b - T_\infty$ $\qquad M \equiv \sqrt{hPkA_c}\,\theta_b$

Analysis of fin thermal behavior is, of course, a good deal more complex if the fin is of nonuniform cross section. For such cases the second term of Equation 3.61 must be retained, and the solutions are no longer in the form of simple exponential or hyperbolic functions. The development of such solutions is beyond the scope of this text (although working results are presented in the following section), and the interested student is referred to Schneider [3].

EXAMPLE 3.8

A very long rod 25 mm in diameter has one end maintained at 100°C. The surface of the rod is exposed to ambient air at 25°C with a convection heat transfer coefficient of 10 W/m$^2 \cdot$ K.

1. What are the heat losses from rods constructed of pure copper and type AISI 316 stainless steel?
2. Estimate how long the rods must be to be considered infinite.

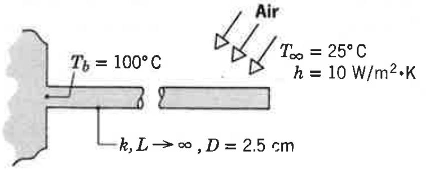
SOLUTION

Known: A long circular fin exposed to ambient air.

Find:

1. Heat loss q when rod is fabricated from copper or stainless steel.
2. How long rods must be to assume infinite length.

Schematic:



Assumptions:

1. Steady-state conditions.
2. One-dimensional conduction along the rod.
3. Constant properties.
4. Negligible radiation exchange with surroundings.

5. Uniform heat transfer coefficient.
6. Infinitely long rod.

Properties: Table A.1, copper [$T = (T_b + T_\infty)/2 = 62.5°\text{C} \approx 335$ K]: $k = 398$ W/m · K. Table A.1, stainless steel, AISI 316 (335 K): $k = 14$ W/m · K.

Analysis:

1. From Equation 3.80 the heat loss is

$$q_f = \sqrt{hPkA_c}\,\theta_b$$

Hence for copper,

$$q_f = \left[10\ \text{W/m}^2\cdot\text{K} \times \pi \times 0.025\ \text{m} \times 398\ \text{W/m}\cdot\text{K} \times \frac{\pi}{4}(0.025\ \text{m})^2\right]^{1/2}(100 - 25)°\text{C}$$

$$= 29.4\ \text{W} \quad \triangleleft$$

and for stainless steel,

$$q_f = \left[10\ \text{W/m}^2\cdot\text{K} \times \pi \times 0.025\ \text{m} \times 14\ \text{W/m}\cdot\text{K} \times \frac{\pi}{4}(0.025\ \text{m})^2\right]^{1/2}(100 - 25)°\text{C}$$

$$= 5.5\ \text{W} \quad \triangleleft$$

2. Since there is no heat loss from the tip of an infinitely long rod, an estimate of the validity of this approximation may be made by comparing Equations 3.76 and 3.80. To a satisfactory approximation, the expressions provide equivalent results if tanh $mL \geq 0.99$ or $mL \geq 2.65$. Hence a rod may be assumed to be infinitely long if

$$L \geq L_\infty \equiv \frac{2.65}{m} = 2.65\left(\frac{kA_c}{hP}\right)^{1/2}$$

For copper,

$$L_\infty = 2.65\left[\frac{398\ \text{W/m}\cdot\text{K} \times (\pi/4)(0.025\ \text{m})^2}{10\ \text{W/m}^2\cdot\text{K} \times \pi(0.025\ \text{m})}\right]^{1/2} = 1.32\ \text{m} \quad \triangleleft$$

For stainless steel,

$$L_\infty = 2.65\left[\frac{14\ \mathrm{W/m\cdot K} \times (\pi/4)(0.025\ \mathrm{m})^2}{10\ \mathrm{W/m^2\cdot K} \times \pi(0.025\ \mathrm{m})}\right]^{1/2} = 0.25\ \mathrm{m} \qquad \triangleleft$$

Comments: The foregoing results suggest that the fin heat transfer rate may be accurately predicted from the infinite fin approximation if $mL \gtrsim 2.65$. However, if the infinite fin approximation is to accurately predict the temperature distribution $T(x)$, a larger value of mL would be required. Note the effect of thermal conductivity on q_f and L_∞.

3.6.3 Fin Performance

Recall that fins are used to increase the heat transfer from a surface by increasing the effective surface area. However, the fin itself represents a conduction resistance to heat transfer from the original surface. For this reason, there is no assurance that the heat transfer rate will be increased through the use of fins. An assessment of this matter may be made by evaluating the *fin effectiveness* ε_f. It is defined as the *ratio of the fin heat transfer rate to the heat transfer rate that would exist without the fin*. Therefore

$$\varepsilon_f = \frac{q_f}{hA_{c,b}\theta_b} \tag{3.81}$$

where $A_{c,b}$ is the fin cross-sectional area at the base. In any rational design the value of ε_f should be as large as possible, and in general, the use of fins may rarely be justified unless $\varepsilon_f \gtrsim 2$.

Subject to any one of the four tip conditions that have been considered, the effectiveness for a fin of uniform cross section may be obtained by dividing the appropriate expression for q_f in Table 3.4 by $hA_{c,b}\theta_b$. For the infinite fin approximation (case D), the result is

$$\varepsilon_f = \left(\frac{kP}{hA_c}\right)^{1/2} \tag{3.82}$$

and several important trends may be inferred. Obviously, fin effectiveness is enhanced by the choice of a material of high thermal conductivity. Aluminum alloys and copper come to mind. However, although copper is superior from the standpoint of thermal conductivity, aluminum alloys are the more common choice because of additional benefits related to lower cost and weight. Fin effectiveness is also enhanced by increasing the ratio of the perimeter to the cross-sectional area. For this reason the use of *thin*, but closely spaced, fins is preferred in most engineering applications.

Equation 3.82 also suggests that the use of fins can better be justified under conditions for which the convection coefficient h is small. Hence from Table 1.1 it is evident that the need for fins is stronger when the fluid is a gas rather than a liquid and particularly when the surface heat transfer is by *free* convection. If fins are to be used on a surface separating a gas and a liquid, they are generally placed on the gas side, which is the side of lower convection coefficient. A common example is the tubing in an automobile radiator. Fins are applied to the outer tube surface, over which there is flow of ambient air (small h), and not to the inner surface, through which there is flow of water (large h). Note that, if $\varepsilon_f > 2$ is used as a criterion to justify the implementation of fins, Equation 3.82 yields the requirement that $(kP/hA_c)^{1/2} > 4$.

Equation 3.81 provides an upper limit to ε_f, which is reached as L approaches infinity. However, it is certainly not necessary to use very long fins to achieve near maximum heat transfer enhancement. When an adiabatic tip condition is considered, Equation 3.76 and Table B.1 tell us that 98% of the maximum possible fin heat transfer rate is achieved for $mL = 2.3$. Hence, it would make little sense to extend the fins beyond $L = 2.3/m$.

Fin performance may also be quantified in terms of a thermal resistance. Treating the difference between the base and fluid temperatures as the driving potential, a *fin resistance* may be defined as

$$R_{t,f} = \frac{\theta_b}{q_f} \tag{3.83}$$

Dividing Equation 3.83 into the expression for the thermal resistance of the exposed base

$$R_{t,b} = \frac{1}{hA_{c,b}} \tag{3.84}$$

and substituting from Equation 3.81, it follows that

$$\varepsilon_f = \frac{R_{t,b}}{R_{t,f}} \tag{3.85}$$

Hence, the fin effectiveness may be interpreted as a ratio of thermal resistances, and to increase ε_f it is necessary to reduce the conduction/convection resistance of the fin. If the fin is to enhance heat transfer, its resistance must not exceed that of the exposed base.

Another measure of fin thermal performance is provided by the *fin efficiency* η_f. The maximum driving potential for convection is the temperature difference between the base ($x = 0$) and the fluid, $\theta_b = T_b - T_\infty$. Hence, the maximum rate at which a fin could dissipate energy is the rate that would exist *if* the entire fin surface were at the base temperature. However, since any fin is characterized by a finite conduction resistance, a temperature gradient must exist along the fin and the above condition is an idealization. A logical definition of fin efficiency is therefore

$$\eta_f \equiv \frac{q_f}{q_{\max}} = \frac{q_f}{hA_f\theta_b} \tag{3.86}$$

where A_f is the surface area of the fin. For a straight fin of uniform cross section and an adiabatic tip, Equations 3.76 and 3.86 yield

$$\eta_f = \frac{M \tanh mL}{hPL\theta_b} = \frac{\tanh mL}{mL} \tag{3.87}$$

Referring to Table B.1, this result tells us that η_f approaches its maximum and minimum values of 1 and 0, respectively, as L approaches 0 and ∞.

In lieu of the somewhat cumbersome expression for heat transfer from a straight rectangular fin with an active tip, Equation 3.72, it has been suggested that approximate, yet accurate, predictions may be obtained by using the adiabatic tip result, Equation 3.76, with a corrected fin length of the form $L_c = L + (t/2)$ [9]. Hence, with tip convection the fin heat rate may be approximated as

$$q_f = M \tanh mL_c \tag{3.88}$$

and the corresponding efficiency as

$$\eta_f = \frac{\tanh mL_c}{mL_c} \tag{3.89}$$

Errors associated with the approximation are negligible if $(ht/k) < 0.0625$ [10]. If the fin width is much larger than its thickness, $w \gg t$, the perimeter may be approximated as $P = 2w$, and

$$mL_c = \left(\frac{hP}{kA_c}\right)^{1/2} L_c = \left(\frac{2h}{kt}\right)^{1/2} L_c$$

Multiplying numerator and denominator by $L_c^{1/2}$ and introducing a corrected fin profile area, $A_p = L_c t$, it follows that

$$mL_c = \left(\frac{2h}{kA_p}\right)^{1/2} L_c^{3/2} \tag{3.90}$$

Hence, as shown in Figure 3.18, the efficiency of a rectangular fin with tip convection may be represented as a function of $L_c^{3/2}(h/kA_p)^{1/2}$.

Although the foregoing discussion has been limited to fins of uniform cross section, other geometries are commonly used. However, with A_c now a function of x and A_s varying nonlinearly with x, exact solutions to Equation 3.61 become much more difficult to obtain. However, the results of such solutions may still be represented graphically, as shown in Figures 3.18 and 3.19. A straight *triangular* $(y \sim x)$ fin is attractive because, for equivalent heat

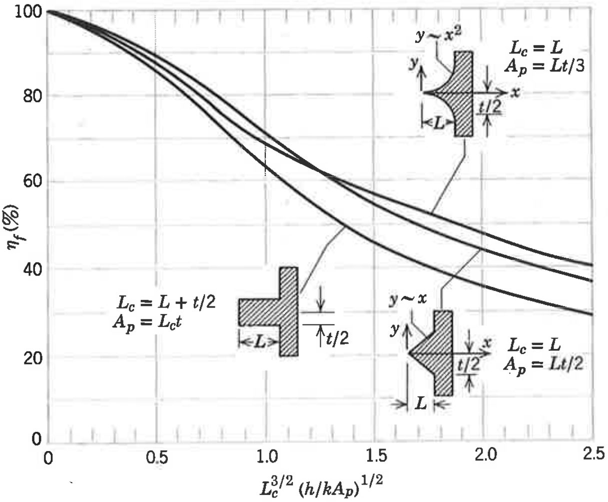


Figure 3.18 Efficiency of straight fins (rectangular, triangular, and parabolic profiles).

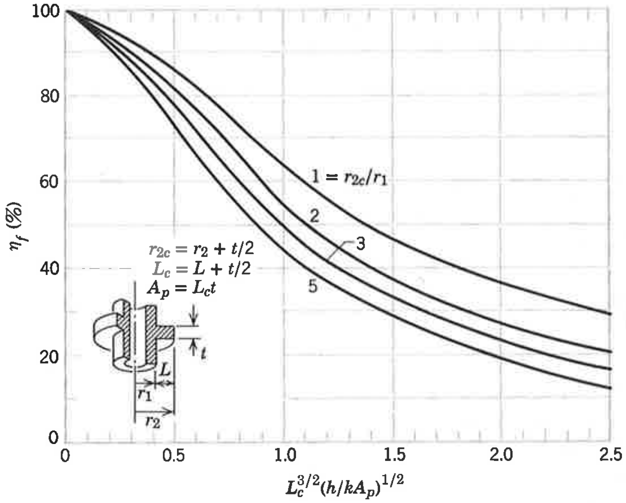


Figure 3.19 Efficiency of annular fins of rectangular profile.

dissipation, it requires much less volume (fin material) than a rectangular profile. In this regard, heat dissipation per unit volume, $(q/V)_f$, is largest for a *parabolic* $(y \sim x^2)$ profile. However, since $(q/V)_f$ for the parabolic profile is only slightly larger than that for a triangular profile, its use can rarely be justified in view of its much larger manufacturing costs. The *annular* fin of rectangular profile is commonly used to enhance heat transfer to or from circular tubes.

In Figures 3.18 and 3.19 fin efficiencies are plotted as a function of the parameter $L_c^{3/2}(h/kA_p)^{1/2}$ inferred for the straight rectangular fin. For the annular fin, Figure 3.19, a corrected fin length is used to represent the effects of convection from the fin tip. Fin efficiencies obtained from the figures may be used to calculate the actual fin heat transfer rate from the expression

$$q_f = \eta_f q_{\max} = \eta_f h A_f \theta_b \tag{3.91}$$

where, respectively, surface areas for the rectangular, triangular, parabolic, and annular fins may be approximated as

$$A_{f(\text{rect})} = 2wL_c \tag{3.92a}$$

$$A_{f(\text{tri})} = 2w\left[L^2 + \left(\frac{t}{2}\right)^2\right]^{1/2} \tag{3.92b}$$

$$A_{f(\text{par})} = 2.05w\left[L^2 + \left(\frac{t}{2}\right)^2\right]^{1/2} \tag{3.93a}$$

$$A_{f(\text{ann})} = 2\pi\left(r_{2c}^2 - r_1^2\right) \tag{3.93b}$$

Extended surfaces find use in numerous engineering applications. For comprehensive discussions of thermal effects, the treatments by Schneider [10] and by Kern and Kraus [11] should be consulted.

3.6.4 Overall Surface Efficiency

In contrast to the fin efficiency η_f, which characterizes the performance of a single fin, the *overall surface efficiency* η_o characterizes an *array* of fins and the base surface to which they are attached. Representative arrays are shown in Figure 3.20, where S designates the fin pitch. In each case the overall efficiency is defined as

$$\eta_o = \frac{q_t}{q_{\max}} = \frac{q_t}{hA_t\theta_b} \tag{3.94}$$

where q_t and A_t are the total heat rate and exposed area of the finned and unfinned surfaces. The maximum possible heat rate would result if the entire fin surface, as well as the exposed base, were maintained at T_b.

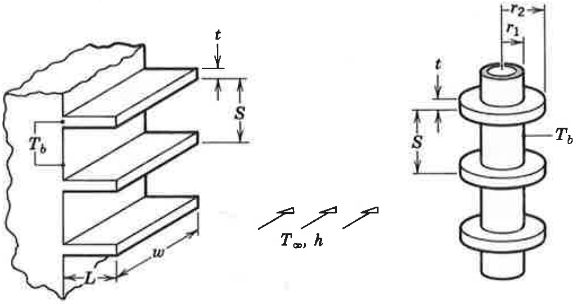


Figure 3.20 Representative fin arrays. (*a*) Rectangular fins. (*b*) Annular fins.

Decomposing the total surface area into contributions due to the fin and base (unfinned) surfaces ($A_t = A_f + A_b$), the total heat rate may be expressed as

$$q_t = hA_b\theta_b + hA_f\eta_f\theta_b \tag{3.95}$$

where η_f is the efficiency of a single fin. Hence,

$$q_t = h\left[\left(A_t - A_f\right) + A_f\eta_f\right]\theta_b = hA_t\left[1 - \frac{A_f}{A_t}\left(1 - \eta_f\right)\right]\theta_b \tag{3.96}$$

Combining Equations 3.96 and 3.94, it follows that

$$\eta_o = 1 - \frac{A_f}{A_t}\left(1 - \eta_f\right) \tag{3.97}$$

From knowledge of η_o, Equation 3.94 may be used to calculate the total heat rate for a fin array.

EXAMPLE 3.9

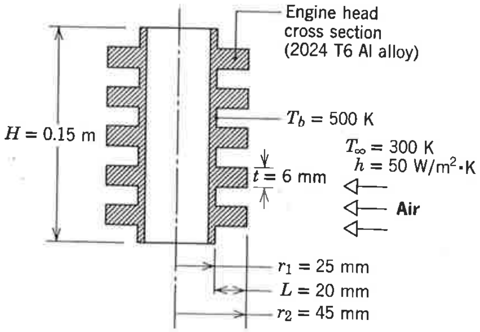
The cylinder barrel of a motorcycle is constructed of 2024-T6 aluminum alloy and is of height $H = 0.15$ m and outside diameter $D = 50$ mm. Under typical operating conditions the outer surface of the cylinder is at a temperature of 500 K and is exposed to ambient air at 300 K, with a convection coefficient of 50 W/m$^2\cdot$K. Annular fins of rectangular profile are typically added to increase heat transfer to the surroundings. Assume that five such fins, which are of thickness $t = 6$ mm, length $L = 20$ mm and equally spaced, are added. What is the increase in heat transfer due to addition of the fins?

SOLUTION

Known: Operating conditions of a finned motorcycle cylinder barrel.

Find: Increase in heat transfer associated with using fins.

Schematic:



Assumptions:

1. Steady-state conditions.
2. One-dimensional radial conduction in fins.
3. Constant properties.
4. No internal heat generation.
5. Negligible radiation exchange with surroundings.
6. Uniform convection coefficient over outer surface (with or without fins).

Properties: Table A.1, 2024-T6 aluminum ($T = 400$ K): $k = 186$ W/m · K.

Analysis: With the fins in place, the heat transfer rate is

$$q = q_f + q_b$$

From Equation 3.91, the fin heat transfer rate is

$$q_f = N\eta_f q_{\max}$$

where N is the number of fins and from Equations 3.91 and 3.93b

$$q_{\max} = 2\pi h\left(r_{2c}^2 - r_1^2\right)\left(T_b - T_\infty\right)$$

Heat transfer from the exposed cylinder surface is

$$q_b = hA_b(T_b - T_\infty)$$

where

$$A_b = (H - Nt)(2\pi r_1)$$

Hence

$$q = N\eta_f h 2\pi\left(r_{2c}^2 - r_1^2\right)(T_b - T_\infty) + h(H - Nt)(2\pi r_1)(T_b - T_\infty)$$

The fin efficiency may be obtained from Figure 3.19 with

$$r_{2c} = r_2 + \frac{t}{2} = 0.048 \text{ m}, \quad L_c = L + \frac{t}{2} = 0.023 \text{ m}$$

$$\frac{r_{2c}}{r_1} = 1.92, \quad A_p = L_c t = 1.38 \times 10^{-4} \text{ m}^2, \quad L_c^{3/2}\left(\frac{h}{kA_p}\right)^{1/2} = 0.15$$

Hence from Figure 3.19, $\eta_f \approx 0.95$. It follows that

$$q = 5\left\{0.95 \times 50 \text{ W/m}^2 \cdot \text{K} \times 2\pi\left[(0.048^2 - 0.025^2) \text{ m}^2\right] \times (500 - 300) \text{ K}\right\}$$
$$+ 50 \text{ W/m}^2 \cdot \text{K } (0.15 - 5 \times 0.006)(2\pi \times 0.025) \text{ m}^2 \times (500 - 300) \text{ K}$$

Hence

$$q = 5\{100.22\} \text{ W} + 188.50 \text{ W} = 690 \text{ W}$$ ◁

Without the fins, the heat transfer rate is

$$q_{wo} = hA_{wo}(T_b - T_\infty)$$

where

$$A_{wo} = H \times 2\pi r_1$$

Hence

$$q_{wo} = 50 \text{ W/m}^2 \cdot \text{K } (0.15 \times 2\pi \times 0.025) \text{ m}^2 \text{ } (200 \text{ K}) = 236 \text{ W}$$ ◁

Comments:

1. Although the fins significantly increase heat dissipation from the cylinder, considerable improvement could still be obtained by increasing the number of fins (by reducing both t and the separation between fins).
2. Alternatively, heat transfer from the fin array could be obtained from Equations 3.94 and 3.97. With $A_f = N2\pi(r_{2c}^2 - r_1^2) = 0.0527$ m² and $A_t = A_f + (H - Nt)(2\pi r_1) = 0.0716$ m², Equation 3.97 yields $\eta_o = 1 - 0.736(1 - 0.95) = 0.963$. Hence, from Equation 3.94, $q_t = \eta_o h A_t \theta_b$ or $q_t = 0.963 \times 50$ W/m² · K × 0.0716 m² × 200 K = 690 W.

3.6.5 Fin Contact Resistance

If fins are machined as an integral part of the wall from which they extend, there is no contact resistance at their base. However, more commonly, fins are manufactured separately and are attached to the wall by a metallurgical or adhesive joint. Alternatively, the attachment may involve a *press fit*, for which the fins are forced into slots machined on the wall material. In such cases, there is a thermal contact resistance, $R_{t,c}$, which may adversely influence overall thermal performance. In manufacturing, care must be taken to render $R_{t,c} \ll R_{t,f}$.

EXAMPLE 3.10

Heat transfer from a transistor may be enhanced by inserting it in an aluminum sleeve ($k = 200$ W/m · K) having 12 integrally machined longitudinal fins on its outer surface. The transistor radius and height are $r_1 = 2$ mm and $H = 6$ mm, respectively, while the fins are of length $L = r_3 - r_2 = 10$ mm and uniform thickness $t = 0.7$ mm. The thickness of the sleeve base is $r_2 - r_1 = 1$ mm, and the contact resistance of the sleeve–transistor interface is $R''_{t,c} = 10^{-3}$ m^2 · K/W. Air at $T_\infty = 20°$C flows over the fin surface, providing a uniform convection coefficient of $h = 25$ W/m^2 · K.

1. Assuming one-dimensional transfer in the radial direction, sketch the equivalent thermal circuit for heat transfer from the transistor case ($r = r_1$) to the air. Clearly label each resistance.
2. Evaluate each of the resistances in the foregoing circuit. If the temperature of the transistor case is $T_1 = 80°$C, what is the rate of heat transfer from the sleeve?

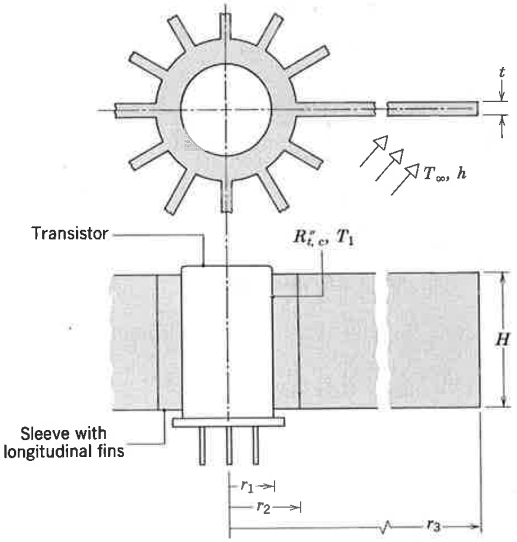
SOLUTION

Known: Dimensions of finned aluminum sleeve inserted over a transistor. Contact resistance between sleeve and transistor. Surface convection conditions and temperature of transistor case.

Find:

1. Equivalent thermal circuit.
2. Rate of heat transfer from sleeve.

Schematic:

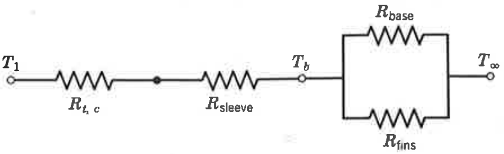


Assumptions:

1. Steady-state conditions.
2. One-dimensional radial conduction.
3. Constant properties.
4. Negligible radiation.

Analysis:

1. The circuit accounts for the contact resistance, conduction in the sleeve, convection from the exposed base, and conduction–convection from the fins.



2. Thermal resistances for the contact joint and sleeve are

$$R_{t,c} = \frac{R''_{t,c}}{2\pi r_1 H} = \frac{10^{-3}\ \text{m}^2 \cdot \text{K/W}}{2\pi\ (0.002\ \text{m})(0.006\ \text{m})} = 13.3\ \text{K/W}$$

$$R_{\text{sleeve}} = \frac{\ln(r_2/r_1)}{2\pi k H} = \frac{\ln(3/2)}{2\pi\ (200\ \text{W/m} \cdot \text{K})(0.006\ \text{m})} = 0.054\ \text{K/W}$$

For a single fin, $R_{\text{fin}} = \theta_b/q_f$, where from Table 3.4,

$$q_f = (hPkA_c)^{1/2}\theta_b \frac{\sinh mL + (h/mk)\cosh mL}{\cosh mL + (h/mk)\sinh mL}$$

With

$$P = 2(H + t) = 13.4\ \text{mm} = 0.0134\ \text{m}$$

$$A_c = t \times H = 4.2 \times 10^{-6}\ \text{m}^2$$

$$m = \frac{hP}{kA_c} = \left(\frac{25\ \text{W/m}^2 \cdot \text{K} \times 0.0134\ \text{m}}{200\ \text{W/m} \cdot \text{K} \times 4.2 \times 10^{-6}\ \text{m}^2}\right)^{1/2} = 20.0\ \text{m}^{-1}$$

$$mL = 20\ \text{m}^{-1} \times 0.01\ \text{m} = 0.20$$

$$\frac{h}{mk} = \frac{25\ \text{W/m}^2 \cdot \text{K}}{20\ \text{m}^{-1} \times 200\ \text{W/m} \cdot \text{K}} = 0.00625$$

and

$$(hPkA_c)^{1/2} = (25\ \text{W/m}^2 \cdot \text{K} \times 0.0134\ \text{m} \times 200\ \text{W/m} \cdot \text{K} \times 4.2 \times 10^{-6}\ \text{m}^2)^{1/2} = 0.0168\ \text{W/K}$$

use of Table B.1 yields, for a *single fin*,

$$R_{\text{fin}} = \frac{1.020 + 0.00625 \times 0.201}{0.0168\ \text{W/K}\ (0.201 + 0.00625 \times 1.020)} = 293\ \text{K/W}$$

Hence, for 12 fins,

$$R_{\text{fins}} = \frac{R_{\text{fin}}}{12} = 24.4\ \text{K/W}$$

For the exposed base,

$$R_{\text{base}} = \frac{1}{h(2\pi r_2 - 12t)H}$$

$$= \frac{1}{25\ \text{W/m}^2 \cdot \text{K}\ (2\pi \times 0.003 - 12 \times 0.0007)\ \text{m} \times 0.006\ \text{m}}$$

$$R_{\text{base}} = 638\ \text{K/W}$$

With

$$R_{\text{equiv}} = \left[(24.4)^{-1} + (638)^{-1}\right]^{-1} = 23.5\ \text{K/W}$$

it follows that

$$R_{\text{tot}} = (13.3 + 0.054 + 23.5)\ \text{K/W} = 36.9\ \text{K/W}$$

and

$$q = \frac{T_1 - T_\infty}{R_{\text{tot}}} = \frac{(80 - 20)^\circ\text{C}}{36.9\ \text{K/W}} = 1.63\ \text{W}$$

Comments:

1. The sleeve resistance is negligible, but the contact resistance is significant relative to that of the fins.
2. Without the finned sleeve, the convection resistance of the transistor case is $R_{\text{tran}} = (2\pi r_1 H h)^{-1} = 531$ K/W. Hence, there is considerable advantage to using the fins.
3. The fin efficiency is $\eta_f = 0.988$.
4. If an adiabatic fin tip is assumed, $\tanh mL = 0.197$ and $R_{\text{fin}} = 302$. Hence, the fin resistance is within 3% of that obtained for the actual convecting tip.

3.7 SUMMARY

Despite the inherent mathematical simplicity, one-dimensional, steady-state heat transfer occurs in numerous engineering applications. Although one-dimensional, steady-state conditions may not apply exactly, the assumptions may often be made to obtain results of reasonable accuracy. You should therefore be thoroughly familiar with the means by which such problems are treated. In particular you should be comfortable with the use of equivalent thermal circuits and with the expressions for the conduction resistances that pertain to each of the three common geometries. You should also be familiar with how the heat equation and Fourier's law may be used to obtain temperature distributions and the corresponding fluxes. The implications of an internally distributed source of energy should also be clearly understood. Finally, you should appreciate the important role that extended surfaces can play in the design of thermal systems and should have the facility to effect design and performance calculations for such surfaces.

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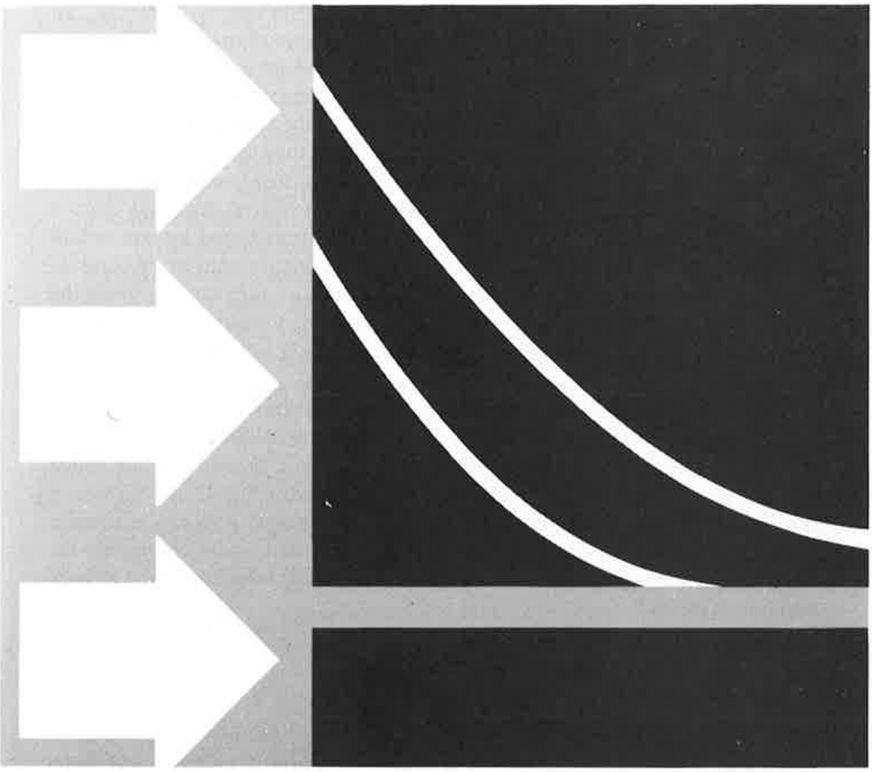
PROBLEMS

Plane Wall

3.1 Consider the plane wall of Figure 3.1 separating hot and cold fluids at temperatures $T_{\infty,1}$ and $T_{\infty,2}$, respectively. Using surface energy balances as boundary conditions at $x = 0$ and $x = L$ (see Equation 2.27), obtain the temperature

CHAPTER 5



TRANSIENT CONDUCTION

In our treatment of conduction we have gradually considered more complicated conditions. We began with the simple case of one-dimensional, steady-state conduction with no internal generation, and we subsequently considered complications due to multidimensional and generation effects. However, we have not yet considered situations for which conditions change with time.

We now recognize that many heat transfer problems are time dependent. Such *unsteady*, or *transient*, problems typically arise when the boundary conditions of a system are changed. For example, if the surface temperature of a system is altered, the temperature at each point in the system will also begin to change. The changes will continue to occur until a *steady-state* temperature distribution is reached. Consider a hot metal billet that is removed from a furnace and exposed to a cool airstream. Energy is transferred by convection and radiation from its surface to the surroundings. Energy transfer by conduction also occurs from the interior of the metal to the surface, and the temperature at each point in the billet decreases until a steady-state condition is reached. Such time-dependent effects occur in many industrial heating and cooling processes.

To determine the time dependence of the temperature distribution within a solid during a transient process, we could begin by solving the appropriate form of the heat equation, for example, Equation 2.13. Some cases for which solutions have been obtained are discussed in Sections 5.4 to 5.8. However, such solutions are often difficult to obtain, and where possible a simpler approach is preferred. One such approach may be used under conditions for which temperature gradients within the solid are small. It is termed the *lumped capacitance method*.

5.1 THE LUMPED CAPACITANCE METHOD

A simple, yet common, transient conduction problem is one in which a solid experiences a sudden change in its thermal environment. Consider a hot metal forging that is initially at a uniform temperature T_i and is quenched by immersing it in a liquid of lower temperature $T_\infty < T_i$ (Figure 5.1). If the quenching is said to begin at time $t = 0$, the temperature of the solid will

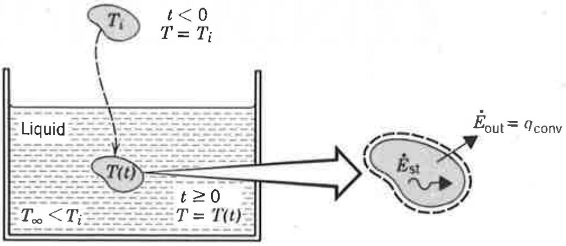


Figure 5.1 Cooling of a hot metal forging.

decrease for time $t > 0$, until it eventually reaches T_∞. This reduction is due to convection heat transfer at the solid–liquid interface. The essence of the lumped capacitance method is the assumption that the temperature of the solid is *spatially uniform* at any instant during the transient process. This assumption implies that temperature gradients within the solid are negligible.

From Fourier's law, heat conduction in the absence of a temperature gradient implies the existence of infinite thermal conductivity. Such a condition is clearly impossible. However, although the condition is never satisfied exactly, it is closely approximated if the resistance to conduction within the solid is small compared with the resistance to heat transfer between the solid and its surroundings. For now we assume that this is, in fact, the case.

In neglecting temperature gradients within the solid, we can no longer consider the problem from within the framework of the heat equation. Instead, the transient temperature response is determined by formulating an overall energy balance on the solid. This balance must relate the rate of heat loss at the surface to the rate of change of the internal energy. Applying Equation 1.11a to the control volume of Figure 5.1, this requirement takes the form

$$-\dot{E}_{\text{out}} = \dot{E}_{\text{st}} \tag{5.1}$$

or

$$-hA_s(T - T_\infty) = \rho Vc\frac{dT}{dt} \tag{5.2}$$

Introducing the temperature difference

$$\theta \equiv T - T_\infty \tag{5.3}$$

and recognizing that $(d\theta/dt) = (dT/dt)$, it follows that

$$\frac{\rho Vc}{hA_s}\frac{d\theta}{dt} = -\theta$$

Separating variables and integrating from the initial condition, for which $t = 0$ and $T(0) = T_i$, we then obtain

$$\frac{\rho Vc}{hA_s}\int_{\theta_i}^{\theta}\frac{d\theta}{\theta} = -\int_0^t dt$$

where

$$\theta_i \equiv T_i - T_\infty \tag{5.4}$$

Evaluating the integrals it follows that

$$\frac{\rho Vc}{hA_s}\ln\frac{\theta_i}{\theta} = t \tag{5.5}$$

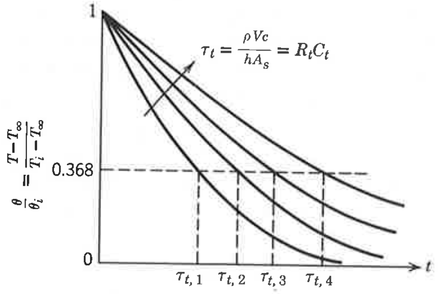


Figure 5.2 Transient temperature response of lumped capacitance solids corresponding to different thermal time constants τ_t.

or

$$\frac{\theta}{\theta_i} = \frac{T - T_\infty}{T_i - T_\infty} = \exp\left[-\left(\frac{hA_s}{\rho Vc}\right)t\right] \tag{5.6}$$

Equation 5.5 may be used to determine the time required for the solid to reach some temperature T, or, conversely, Equation 5.6 may be used to compute the temperature reached by the solid at some time t.

The foregoing results indicate that the difference between the solid and fluid temperatures must decay exponentially to zero as t approaches infinity. This behavior is shown in Figure 5.2. From Equation 5.6 it is also evident that the quantity $(\rho Vc/hA_s)$ may be interpreted as a *thermal time constant*. This time constant may be expressed as

$$\tau_t = \left(\frac{1}{hA_s}\right)(\rho Vc) = R_tC_t \tag{5.7}$$

where R_t is the resistance to convection heat transfer and C_t is the *lumped thermal capacitance* of the solid. Any increase in R_t or C_t will cause a solid to respond more slowly to changes in its thermal environment and will increase the time required to reach thermal equilibrium ($\theta = 0$).

It is useful to note that the foregoing behavior is analogous to the voltage decay that occurs when a capacitor is discharged through a resistor in an electrical RC circuit. Accordingly, the process may be represented by an *equivalent thermal circuit*, which is shown in Figure 5.3. With the switch closed the solid is charged to the temperature θ_i. When the switch is opened, the energy that is stored in the solid is discharged through the thermal resistance and the temperature of the solid decays with time. This analogy suggests that RC electrical circuits may be used to determine the transient behavior of thermal systems. In fact, before the advent of digital computers, RC circuits were widely used to simulate transient thermal behavior.

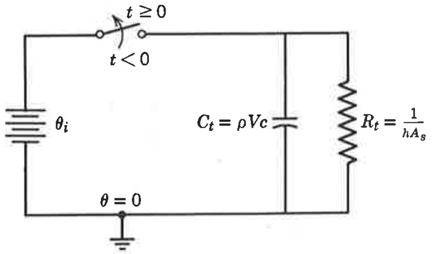


Figure 5.3 Equivalent thermal circuit for a lumped capacitance solid.

To determine the total energy transfer Q occurring up to some time t, we simply write

$$Q = \int_0^t q\,dt = hA_s \int_0^t \theta\,dt$$

Substituting for θ from Equation 5.6 and integrating, we obtain

$$Q = (\rho Vc)\theta_i\left[1 - \exp\left(-\frac{t}{\tau_t}\right)\right] \tag{5.8a}$$

The quantity Q is, of course, related to the change in the internal energy of the solid, and from Equation 1.11b

$$-Q = \Delta E_{\text{st}} \tag{5.8b}$$

For quenching Q is positive and the solid experiences a decrease in energy. Equations 5.5, 5.6, and 5.8a also apply to situations where the solid is heated ($\theta < 0$), in which case Q is negative and the internal energy of the solid increases.

5.2 VALIDITY OF THE LUMPED CAPACITANCE METHOD

From the foregoing results it is easy to see why there is a strong preference for using the lumped capacitance method. It is certainly the simplest and most convenient method that can be used to solve transient conduction problems. Hence it is important to determine under what conditions it may be used with reasonable accuracy.

To develop a suitable criterion consider steady-state conduction through the plane wall of area A (Figure 5.4). Although we are assuming steady-state conditions, this criterion is readily extended to transient processes. One surface is maintained at a temperature $T_{s,1}$ and the other surface is exposed to a fluid of temperature $T_\infty < T_{s,1}$. The temperature of this surface will be some

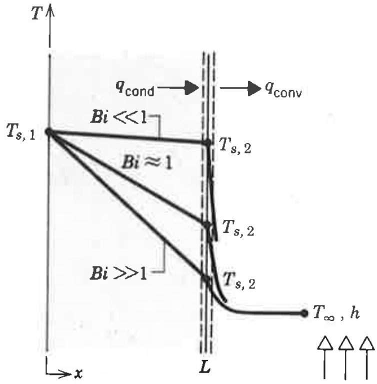


Figure 5.4 Effect of Biot number on steady-state temperature distribution in a plane wall with surface convection.

intermediate value, $T_{s,2}$, for which $T_\infty < T_{s,2} < T_{s,1}$. Hence under steady-state conditions the surface energy balance, Equation 1.12, reduces to

$$\frac{kA}{L}(T_{s,1} - T_{s,2}) = hA(T_{s,2} - T_\infty)$$

where k is the thermal conductivity of the solid. Rearranging, we then obtain

$$\frac{T_{s,1} - T_{s,2}}{T_{s,2} - T_\infty} = \frac{(L/kA)}{(1/hA)} = \frac{R_{\text{cond}}}{R_{\text{conv}}} = \frac{hL}{k} \equiv Bi \qquad (5.9)$$

The quantity (hL/k) appearing in Equation 5.9 is a *dimensionless parameter*. It is termed the *Biot number*, and it plays a fundamental role in conduction problems that involve surface convection effects. According to Equation 5.9 and as illustrated in Figure 5.4, the Biot number provides a measure of the temperature drop in the solid relative to the temperature difference between the surface and the fluid. Note especially the conditions corresponding to $Bi \ll 1$. The results suggest that, for these conditions, it is reasonable to *assume* a uniform temperature distribution across a solid at any time during a transient process. This result may also be associated with interpretation of the Biot number as a ratio of thermal resistances, Equation 5.9. *If $Bi \ll 1$, the resistance to conduction within the solid is much less than the resistance to convection across the fluid boundary layer*. Hence the assumption of a uniform temperature distribution is reasonable.

We have introduced the Biot number because of its significance to transient conduction problems. Consider the plane wall of Figure 5.5, which is initially at a uniform temperature T_i and experiences convection cooling when it is immersed in a fluid of $T_\infty < T_i$. The problem may be treated as one dimensional in x, and we are interested in the temperature variation with position and time, $T(x, t)$. This variation is a strong function of the Biot

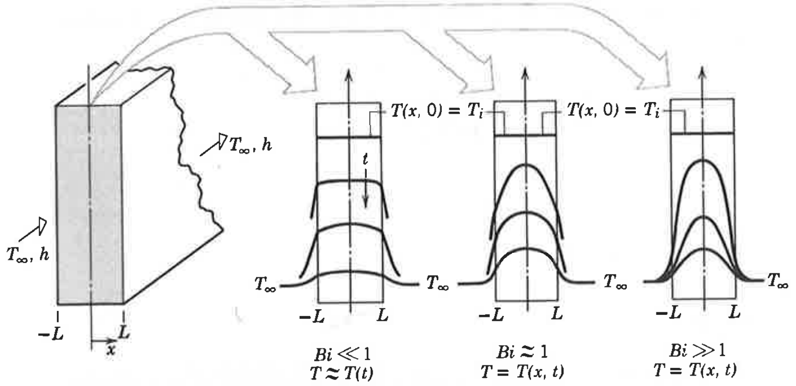


Figure 5.5 Transient temperature distribution for different Biot numbers in a plane wall symmetrically cooled by convection.

number, and three conditions are shown in Figure 5.5. For $Bi \ll 1$ the temperature gradient in the solid is small and $T(x, t) \approx T(t)$. Virtually all the temperature difference is between the solid and the fluid, and the solid temperature remains nearly uniform as it decreases to T_∞. For moderate to large values of the Biot number, however, the temperature gradients within the solid are significant. Hence $T = T(x, t)$. Note that for $Bi \gg 1$, the temperature difference across the solid is now much larger than that between the surface and the fluid.

We conclude this section by emphasizing the importance of the lumped capacitance method. Its inherent simplicity renders it the preferred method for solving transient conduction problems. Hence, when confronted with such a problem, *the very first thing that one should do is calculate the Biot number.* If the following condition is satisfied

$$Bi = \frac{hL_c}{k} < 0.1 \qquad (5.10)$$

the error associated with using the lumped capacitance method is small. For convenience, it is customary to define the *characteristic length* of Equation 5.10 as the ratio of the solid's volume to surface area, $L_c \equiv V/A_s$. Such a definition facilitates calculation of L_c for solids of complicated shape and reduces to the half-thickness L for a plane wall of thickness $2L$ (Figure 5.5), to $r_o/2$ for a long cylinder, and to $r_o/3$ for a sphere. However, if one wishes to implement the criterion in a conservative fashion, L_c should be associated with the length scale corresponding to the maximum spatial temperature difference. Accordingly, for a symmetrically heated (or cooled) plane wall of thickness $2L$, L_c would remain equal to the half-thickness L. However, for a long cylinder or sphere, L_c would equal the actual radius r_o, rather than $r_o/2$ or $r_o/3$.

Finally, we note that, with $L_c \equiv V/A_s$, the exponent of Equation 5.6 may be expressed as

$$\frac{hA_s t}{\rho V c} = \frac{ht}{\rho c L_c} = \frac{hL_c}{k}\frac{k}{\rho c}\frac{t}{L_c^2} = \frac{hL_c}{k}\frac{\alpha t}{L_c^2}$$

or

$$\frac{hA_s t}{\rho V c} = Bi \cdot Fo \tag{5.11}$$

where

$$Fo \equiv \frac{\alpha t}{L_c^2} \tag{5.12}$$

is termed the Fourier number. It is a *dimensionless time*, which, with the Biot number, characterizes transient conduction problems. Substituting Equation 5.11 into 5.6, we obtain

$$\frac{\theta}{\theta_i} = \frac{T - T_\infty}{T_i - T_\infty} = \exp(-Bi \cdot Fo) \tag{5.13}$$

EXAMPLE 5.1

A thermocouple junction, which may be approximated as a sphere, is to be used for temperature measurement in a gas stream. The convection coefficient between the junction surface and the gas is known to be $h = 400\ \text{W/m}^2 \cdot \text{K}$, and the junction thermophysical properties are $k = 20\ \text{W/m} \cdot \text{K}$, $c = 400\ \text{J/kg} \cdot \text{K}$, and $\rho = 8500\ \text{kg/m}^3$. Determine the junction diameter needed for the thermocouple to have a time constant of 1 s. If the junction is at 25°C and is placed in a gas stream that is at 200°C, how long will it take for the junction to reach 199°C?

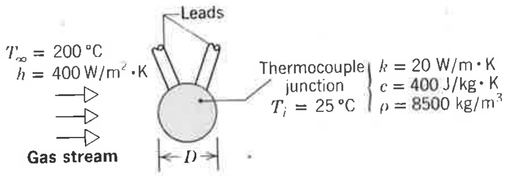
SOLUTION

Known: Thermophysical properties of thermocouple junction used to measure temperature of a gas stream.

Find:

1. Junction diameter needed for a time constant of 1 s.
2. Time required to reach 199°C in gas stream at 200°C.

Schematic:



Assumptions:

1. Temperature of junction is uniform at any instant.
2. Radiation exchange with the surroundings is negligible.
3. Losses by conduction through the leads are negligible.
4. Constant properties.

Analysis:

1. Because the junction diameter is unknown, it is not possible to begin the solution by determining whether the criterion for using the lumped capacitance method, Equation 5.10, is satisfied. However, a reasonable approach is to use the method to find the diameter and to then determine whether the criterion is satisfied. From Equation 5.7 and the fact that $A_s = \pi D^2$ and $V = \pi D^3/6$ for a sphere, it follows that

$$\tau_t = \frac{1}{h\pi D^2} \times \frac{\rho \pi D^3}{6} c$$

Rearranging and substituting numerical values,

$$D = \frac{6h\tau_t}{\rho c} = \frac{6 \times 400\ \mathrm{W/m^2 \cdot K} \times 1\ \mathrm{s}}{8500\ \mathrm{kg/m^3} \times 400\ \mathrm{J/kg \cdot K}} = 7.06 \times 10^{-4}\ \mathrm{m} \qquad \triangleleft$$

With $L_c = r_o/3$ it then follows from Equation 5.10 that

$$Bi = \frac{h(r_o/3)}{k} = \frac{400\ \mathrm{W/m^2 \cdot K} \times 3.53 \times 10^{-4}\ \mathrm{m}}{3 \times 20\ \mathrm{W/m \cdot K}} = 2.35 \times 10^{-4}$$

Accordingly, Equation 5.10 is satisfied (for $L_c = r_o$, as well as for $L_c = r_o/3$) and the lumped capacitance method may be used to an excellent approximation.

2. From Equation 5.5 the time required for the junction to reach $T = 199°\text{C}$ is

$$t = \frac{\rho(\pi D^3/6)c}{h(\pi D^2)} \ln \frac{T_i - T_\infty}{T - T_\infty} = \frac{\rho D c}{6h} \ln \frac{T_i - T_\infty}{T - T_\infty}$$

$$t = \frac{8500\ \text{kg/m}^3 \times 7.06 \times 10^{-4}\ \text{m} \times 400\ \text{J/kg}\cdot\text{K}}{6 \times 400\ \text{W/m}^2\cdot\text{K}} \ln \frac{25 - 200}{199 - 200}$$

$$t = 5.2\ \text{s} \approx 5\tau_t \qquad \triangleleft$$

Comments: Heat losses due to radiation exchange between the junction and the surroundings and conduction through the leads would necessitate using a smaller junction diameter to achieve the desired time response.

5.3 GENERAL LUMPED CAPACITANCE ANALYSIS

Although transient conduction in a solid is commonly initiated by convection heat transfer to or from an adjoining fluid, other processes may induce transient thermal conditions within the solid. For example, a solid may be separated from large surroundings by a gas or vacuum. If the temperatures of the solid and surroundings differ, radiation exchange could cause the internal thermal energy, and hence the temperature, of the solid to change. Temperature changes could also be induced by applying a heat flux at a portion, or all, of the surface and/or by initiating thermal energy generation within the solid. Surface heating could, for example, be applied by attaching a film or sheet electrical heater to the surface, while thermal energy could be generated by passing an electrical current through the solid.

Figure 5.6 depicts a situation for which thermal conditions within a solid may be simultaneously influenced by convection, radiation, an applied surface

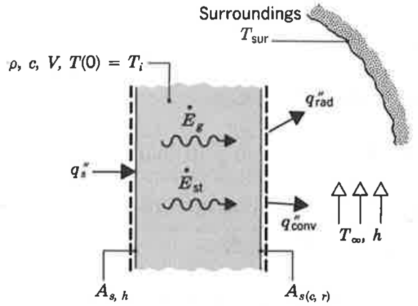


Figure 5.6 Contral surface for general lumped capacitance analysis.

heat flux, and internal energy generation. It is presumed that, initially ($t = 0$), the temperature of the solid (T_i) differs from that of the fluid, T_∞, and the surroundings, T_{sur}, and that both surface and volumetric heating (q_s'' and $\dot{q}$) are initiated. The imposed heat flux q_s'' and the convection–radiation heat transfer occur at mutually exclusive portions of the surface, $A_{s(h)}$ and $A_{s(c,r)}$, respectively, and convection–radiation transfer is presumed to be *from* the surface. Applying conservation of energy at any instant t, it follows from Equation 1.11a that

$$q_s'' A_{s,h} + \dot{E}_g - \left(q''_{\text{conv}} + q''_{\text{rad}}\right) A_{s(c,r)} = \rho V c \frac{dT}{dt} \tag{5.14}$$

or, from Equations 1.3a and 1.7,

$$q_s'' A_{s,h} + \dot{E}_g - \left[h(T - T_\infty) + \varepsilon\sigma\left(T^4 - T_{\text{sur}}^4\right)\right] A_{s(c,r)} = \rho V c \frac{dT}{dt} \tag{5.15}$$

Unfortunately, Equation 5.15 is a nonlinear, first-order, nonhomogeneous, ordinary differential equation which cannot be integrated to obtain an exact solution.[1] However, exact solutions may be obtained for simplified versions of the equation. For example, if there is no imposed heat flux or generation and convection is either nonexistent (a vacuum) or negligible relative to radiation, Equation 5.15 reduces to

$$\rho V c \frac{dT}{dt} = -\varepsilon A_{s,r}\sigma\left(T^4 - T_{\text{sur}}^4\right) \tag{5.16}$$

Separating variables and integrating from the initial condition to any time t, it follows that

$$\frac{\varepsilon A_{s,r}\sigma}{\rho V c}\int_0^t dt = \int_{T_i}^{T} \frac{dT}{T_{\text{sur}}^4 - T^4} \tag{5.17}$$

Evaluating both integrals and rearranging, the time required to reach the temperature T becomes

$$t = \frac{\rho V c}{4\varepsilon A_{s,r}\sigma T_{\text{sur}}^3}\left\{\ln\left|\frac{T_{\text{sur}} + T}{T_{\text{sur}} - T}\right| - \ln\left|\frac{T_{\text{sur}} + T_i}{T_{\text{sur}} - T_i}\right| + 2\left[\tan^{-1}\left(\frac{T}{T_{\text{sur}}}\right) - \tan^{-1}\left(\frac{T_i}{T_{\text{sur}}}\right)\right]\right\} \tag{5.18}$$

This expression cannot be used to evaluate T explicitly in terms of t, T_i, and T_{sur}, nor does it readily reduce to the limiting result for $T_{\text{sur}} = 0$ (radiation to

---

[1] An approximate, finite-difference solution may be obtained by *discretizing* the time derivative (Section 5.9) and *marching* the solution out in time.

deep space). Returning to Equation 5.17, it is readily shown that, for $T_{\text{sur}} = 0$,

$$t = \frac{\rho V c}{3\varepsilon A_{s,r}\sigma}\left(\frac{1}{T^3} - \frac{1}{T_i^3}\right) \tag{5.19}$$

An exact solution to Equation 5.15 may also be obtained if radiation may be neglected and h is independent of time. Introducing a reduced temperature, $\theta \equiv T - T_\infty$, where $d\theta/dt = dT/dt$, Equation 5.15 reduces to a linear, first-order, nonhomogeneous differential equation of the form

$$\frac{d\theta}{dt} + a\theta - b = 0 \tag{5.20}$$

where $a \equiv (hA_{s,c}/\rho V c)$ and $b \equiv [(q_s'' A_{s,h} + \dot{E}_g)/\rho V c]$. Although Equation 5.20 may be solved by summing its homogeneous and particular solutions, an alternative approach is to eliminate the nonhomogeneity by introducing the transformation

$$\theta' \equiv \theta - \frac{b}{a} \tag{5.21}$$

Recognizing that $d\theta'/dt = d\theta/dt$, Equation 5.21 may be substituted into (5.20) to yield

$$\frac{d\theta'}{dt} + a\theta' = 0 \tag{5.22}$$

Separating variables and integrating from 0 to t (θ_i' to θ'), it follows that

$$\frac{\theta'}{\theta_i'} = \exp(-at) \tag{5.23}$$

or substituting for θ' and θ,

$$\frac{T - T_\infty - (b/a)}{T_i - T_\infty - (b/a)} = \exp(-at) \tag{5.24}$$

Hence,

$$\frac{T - T_\infty}{T_i - T_\infty} = \exp(-at) + \frac{b/a}{T_i - T_\infty}[1 - \exp(-at)] \tag{5.25}$$

As it must, Equation 5.25 reduces to (5.6) when $b = 0$ and yields $T = T_i$ at $t = 0$. As $t \to \infty$, Equation 5.25 reduces to $(T - T_\infty) = (b/a)$, which could also be obtained by performing an energy balance on the control surface of Figure 5.6 for steady-state conditions.

5.4 SPATIAL EFFECTS

Situations frequently arise for which the lumped capacitance method is inappropriate, and alternative methods must be used. Regardless of the particular form of the method, we must now cope with the fact that gradients within the medium are no longer negligible.

In their most general form, transient conduction problems are described by the heat equation, Equation 2.13 for rectangular coordinates or Equations 2.20 and 2.23, respectively, for cylindrical and spherical coordinates. The solution to these partial differential equations provides the variation of temperature with both time and the spatial coordinates. However, in many problems, such as the plane wall of Figure 5.5, only one spatial coordinate is needed to describe the internal temperature distribution. With no internal generation and the assumption of constant thermal conductivity, Equation 2.13 then reduces to

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{5.26}$$

To solve Equation 5.26 for the temperature distribution $T(x, t)$, it is necessary to specify an *initial* condition and two *boundary conditions*. For the typical transient conduction problem of Figure 5.5, the initial condition is

$$T(x, 0) = T_i \tag{5.27}$$

and the boundary conditions are

$$\left.\frac{\partial T}{\partial x}\right|_{x=0} = 0 \tag{5.28}$$

and

$$-k\left.\frac{\partial T}{\partial x}\right|_{x=L} = h[T(L, t) - T_\infty] \tag{5.29}$$

Equation 5.27 presumes a uniform temperature distribution at time $t = 0$; Equation 5.28 reflects the *symmetry requirement* for the midplane of the wall; and Equation 5.29 describes the surface condition experienced for time $t > 0$. From Equations 5.26 to 5.29, it is evident that, in addition to depending on x and t, temperatures in the wall also depend on a number of physical parameters. In particular

$$T = T(x, t, T_i, T_\infty, L, k, \alpha, h) \tag{5.30}$$

The foregoing problem may be solved analytically or numerically. These methods will be considered in subsequent sections, but first it is important to note the advantages that may be obtained by *nondimensionalizing* the governing equations. This may be done by arranging the relevant variables into suitable *groups*. Consider the dependent variable T. If the temperature difference $\theta \equiv T - T_\infty$ is divided by the *maximum possible temperature difference* $\theta_i \equiv T_i - T_\infty$, a dimensionless form of the dependent variable may be defined as

$$\theta^* \equiv \frac{\theta}{\theta_i} = \frac{T - T_\infty}{T_i - T_\infty} \tag{5.31}$$

Accordingly, θ^* must lie in the range $0 \le \theta^* \le 1$. A dimensionless spatial coordinate may be defined as

$$x^* \equiv \frac{x}{L} \tag{5.32}$$

where L is the half-thickness of the plane wall, and a dimensionless time may be defined as

$$t^* \equiv \frac{\alpha t}{L^2} \equiv Fo \tag{5.33}$$

where t^* is equivalent to the dimensionless *Fourier number*, Equation 5.12.

Substituting the definitions of Equations 5.31 to 5.33 into Equations 5.26 to 5.29, the heat equation becomes

$$\frac{\partial^2 \theta^*}{\partial x^{*2}} = \frac{\partial \theta^*}{\partial Fo} \tag{5.34}$$

and the initial and boundary conditions become

$$\theta^*(x^*, 0) = 1 \tag{5.35}$$

$$\left.\frac{\partial \theta^*}{\partial x^*}\right|_{x^*=0} = 0 \tag{5.36}$$

and

$$\left.\frac{\partial \theta^*}{\partial x^*}\right|_{x^*=1} = -Bi\,\theta^*(1, t^*) \tag{5.37}$$

where the Biot number is $Bi \equiv hL/k$. In dimensionless form the functional dependence may now be expressed as

$$\theta^* = f(x^*, Fo, Bi) \tag{5.38}$$

Recall that this functional dependence, without the x^* variation, was obtained for the lumped capacitance method, as shown in Equation 5.13.

Comparing Equations 5.30 and 5.38, the considerable advantage associated with casting the problem in dimensionless form becomes apparent.

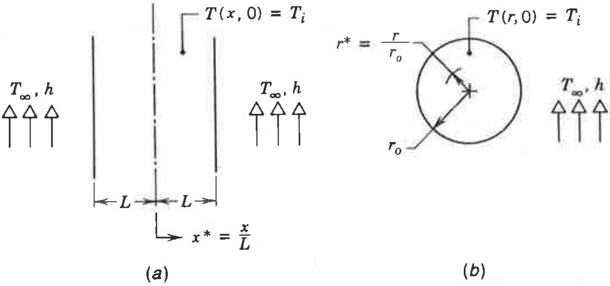
Equation 5.38 implies that *for a prescribed geometry, the transient temperature distribution is a universal function of* x^*, *Fo, and Bi*. That is, the *dimensionless solution* assumes a prescribed form that does not depend on the particular value of T_i, T_∞, L, k, α, or h. Since this generalization greatly simplifies the presentation and utilization of transient solutions, the dimensionless variables are used extensively in subsequent sections.

5.5 THE PLANE WALL WITH CONVECTION

Exact, analytical solutions to transient conduction problems have been obtained for many simplified geometries and boundary conditions and are well documented in the literature [1–4]. Several mathematical techniques, including the method of separation of variables (Section 4.2), may be used for this purpose, and typically the solution for the dimensionless temperature distribution, Equation 5.38, is in the form of an infinite series. However, except for very small values of the Fourier number, this series may be approximated by a single term and the results may be represented in a convenient graphical form.

5.5.1 Exact Solution

Consider the *plane wall* of thickness $2L$ (Figure 5.7*a*). If the thickness is small relative to the width and height of the wall, it is reasonable to assume that conduction occurs exclusively in the x direction. If the wall is initially at a uniform temperature, $T(x, 0) = T_i$, and is suddenly immersed in a fluid of $T_\infty \neq T_i$, the resulting temperatures may be obtained by solving Equation 5.34 subject to the conditions of Equations 5.35 to 5.37. Since the convection conditions for the surfaces at $x^* = \pm 1$ are the same, the temperature distribution at any instant must be symmetrical about the midplane ($x^* = 0$). An



(*a*) (*b*)

Figure 5.7 One-dimensional systems with an initial uniform temperature subjected to sudden convection conditions. (*a*) Plane wall. (*b*) Infinite cylinder or sphere.

exact solution to this problem has been obtained and is of the form [2]

$$\theta^* = \sum_{n=1}^{\infty} C_n \exp\left(-\zeta_n^2 Fo\right) \cos\left(\zeta_n x^*\right) \tag{5.39a}$$

where the coefficient C_n is

$$C_n = \frac{4 \sin \zeta_n}{2\zeta_n + \sin\left(2\zeta_n\right)} \tag{5.39b}$$

and the discrete values (*eigenvalues*) of ζ_n are positive roots of the transcendental equation

$$\zeta_n \tan \zeta_n = Bi \tag{5.39c}$$

The first four roots of this equation are given in Appendix B.3.

5.5.2 Approximate Solution

It can be shown (Problem 5.24) that for values of $Fo \geq 0.2$, the infinite series solution, Equation 5.39a, can be approximated by the first term of the series. Invoking this approximation, the dimensionless form of the temperature distribution becomes

$$\theta^* = C_1 \exp\left(-\zeta_1^2 Fo\right) \cos\left(\zeta_1 x^*\right) \tag{5.40a}$$

or

$$\theta^* = \theta_o^* \cos\left(\zeta_1 x^*\right) \tag{5.40b}$$

where θ_o^* represents the midplane ($x^* = 0$) temperature

$$\theta_o^* = C_1 \exp\left(-\zeta_1^2 Fo\right) \tag{5.41}$$

An important implication of Equation 5.40b is that *the time dependence of the temperature at any location within the wall is the same as that of the midplane temperature*. The coefficients C_1 and ζ_1 are evaluated from Equations 5.39b and 5.39c, respectively, and are given in Table 5.1 for a range of Biot numbers.

5.5.3 Total Energy Transfer

In many situations it is useful to know the total energy that has left the wall up to any time t in the transient process. The conservation of energy requirement, Equation 1.11b, may be applied for the time interval bounded by the initial condition ($t = 0$) and time $t > 0$

$$E_{\text{in}} - E_{\text{out}} = \Delta E_{\text{st}} \tag{5.42}$$

Table 5.1 Coefficients used in the one-term approximation to the series solutions for transient one-dimensional conduction

	PLANE WALL		INFINITE CYLINDER		SPHERE	
Bi^a	ζ_1 (rad)	C_1	ζ_1 (rad)	C_1	ζ_1 (rad)	C_1
0.01	0.0998	1.0017	0.1412	1.0025	0.1730	1.0030
0.02	0.1410	1.0033	0.1995	1.0050	0.2445	1.0060
0.03	0.1732	1.0049	0.2439	1.0075	0.2989	1.0090
0.04	0.1987	1.0066	0.2814	1.0099	0.3450	1.0120
0.05	0.2217	1.0082	0.3142	1.0124	0.3852	1.0149
0.06	0.2425	1.0098	0.3438	1.0148	0.4217	1.0179
0.07	0.2615	1.0114	0.3708	1.0173	0.4550	1.0209
0.08	0.2791	1.0130	0.3960	1.0197	0.4860	1.0239
0.09	0.2956	1.0145	0.4195	1.0222	0.5150	1.0268
0.10	0.3111	1.0160	0.4417	1.0246	0.5423	1.0298
0.15	0.3779	1.0237	0.5376	1.0365	0.6608	1.0445
0.20	0.4328	1.0311	0.6170	1.0483	0.7593	1.0592
0.25	0.4801	1.0382	0.6856	1.0598	0.8448	1.0737
0.30	0.5218	1.0450	0.7465	1.0712	0.9208	1.0880
0.4	0.5932	1.0580	0.8516	1.0932	1.0528	1.1164
0.5	0.6533	1.0701	0.9408	1.1143	1.1656	1.1441
0.6	0.7051	1.0814	1.0185	1.1346	1.2644	1.1713
0.7	0.7506	1.0919	1.0873	1.1539	1.3525	1.1978
0.8	0.7910	1.1016	1.1490	1.1725	1.4320	1.2236
0.9	0.8274	1.1107	1.2048	1.1902	1.5044	1.2488
1.0	0.8603	1.1191	1.2558	1.2071	1.5708	1.2732
2.0	1.0769	1.1795	1.5995	1.3384	2.0288	1.4793
3.0	1.1925	1.2102	1.7887	1.4191	2.2889	1.6227
4.0	1.2646	1.2287	1.9081	1.4698	2.4556	1.7201
5.0	1.3138	1.2402	1.9898	1.5029	2.5704	1.7870
6.0	1.3496	1.2479	2.0490	1.5253	2.6537	1.8338
7.0	1.3766	1.2532	2.0937	1.5411	2.7165	1.8674
8.0	1.3978	1.2570	2.1286	1.5526	2.7654	1.8921
9.0	1.4149	1.2598	2.1566	1.5611	2.8044	1.9106
10.0	1.4289	1.2620	2.1795	1.5677	2.8363	1.9249
20.0	1.4961	1.2699	2.2881	1.5919	2.9857	1.9781
30.0	1.5202	1.2717	2.3261	1.5973	3.0372	1.9898
40.0	1.5325	1.2723	2.3455	1.5993	3.0632	1.9942
50.0	1.5400	1.2727	2.3572	1.6002	3.0788	1.9962
100.0	1.5552	1.2731	2.3809	1.6015	3.1102	1.9990

[a] $Bi = hL/k$ for the plane wall and hr_o/k for the infinite cylinder and sphere. See Figure 5.7.

Equating the energy transferred from the wall Q to E_{out} and setting $E_{\text{in}} = 0$ and $\Delta E_{\text{st}} = E(t) - E(0)$, it follows that

$$Q = -[E(t) - E(0)] \tag{5.43a}$$

or

$$Q = -\int \rho c[T(r,t) - T_i]\, dV \tag{5.43b}$$

where the integration is performed over the volume of the wall. It is convenient to nondimensionalize this result by introducing the quantity

$$Q_o = \rho c V(T_i - T_\infty) \tag{5.44}$$

which may be interpreted as the initial internal energy of the wall relative to the fluid temperature. It is also the *maximum* amount of energy transfer which could occur if the process were continued to time $t = \infty$. Hence, assuming constant properties, the ratio of the total energy transferred from the wall over the time interval t to the maximum possible transfer is

$$\frac{Q}{Q_o} = \int \frac{-[T(r,t) - T_i]}{T_i - T_\infty} \frac{dV}{V} = \frac{1}{V}\int (1 - \theta^*)\, dV \tag{5.45}$$

Employing the approximate form of the temperature distribution for the plane wall, Equation 5.40b, the integration prescribed by Equation 5.45 can be performed to obtain

$$\frac{Q}{Q_o} = 1 - \frac{\sin \zeta_1}{\zeta_1}\theta_o^* \tag{5.46}$$

where θ_o^* can be determined from Equation 5.41, using Table 5.1 for values of the coefficients C_1 and ζ_1.

5.5.4 Graphical Representations

Graphical representations of the *approximate* relations for the transient temperature distribution and energy transfer were first presented by Heisler [5] and Gröber et al. [6]. The graphs have been widely used for nearly four decades; in addition to offering computational convenience, they illustrate the functional dependence of the transient, dimensionless temperature distribution on the Biot and Fourier numbers.

Results for the plane wall are presented in Figures 5.8 to 5.10. Figure 5.8 may be used to obtain the *midplane* temperature of the wall, $T(0, t) = T_o(t)$, at any time during the transient process. If T_o is known for particular values of Fo and Bi, Figure 5.9 may be used to determine the corresponding temperature at any location *off the midplane*. Hence, Figure 5.9 must be used

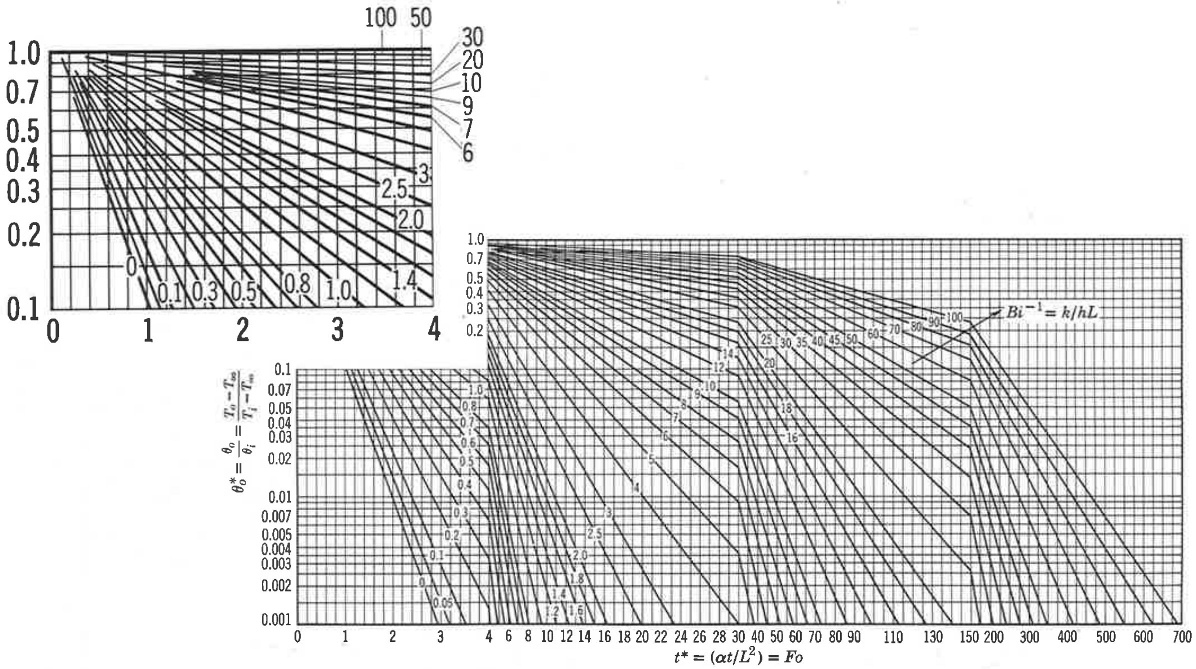


Figure 5.8 Midplane temperature as a function of time for a plane wall of thickness $2L$ [5]. Used with permission.

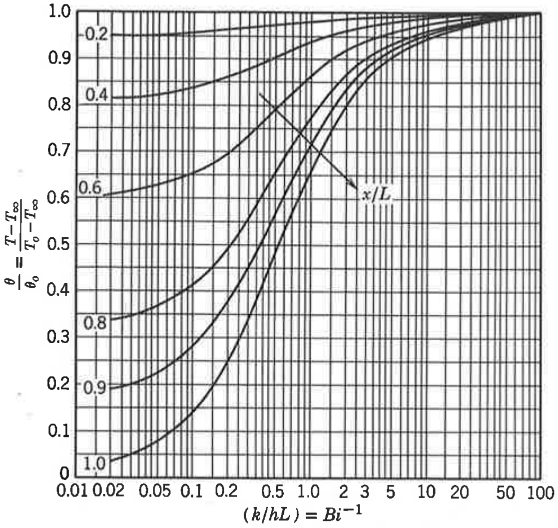


Figure 5.9 Temperature distribution in a plane wall of thickness $2L$ [5]. Used with permission.

in conjunction with Figure 5.8. For example, if one wishes to determine the surface temperature ($x^* = \pm 1$) at some time t, Figure 5.8 would first be used to determine T_o at t. Figure 5.9 would then be used to determine the surface temperature from knowledge of T_o. The procedure would be inverted if the problem were one of determining the time required for the surface to reach a prescribed temperature.

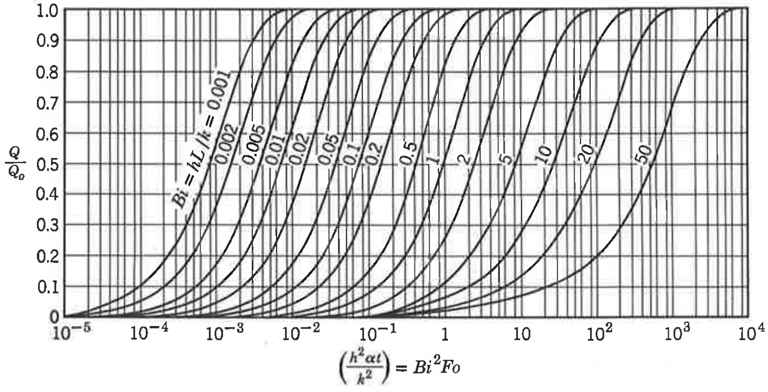


Figure 5.10 Internal energy change as a function of time for a plane wall of thickness $2L$ [6]. Adapted with permission.

Absence of the Fourier number in Figure 5.9 implies that the time dependence of any temperature off the midplane corresponds to the time dependence of the midplane temperature. This result is, of course, a consequence of the approximation that led to Equation 5.40b and is valid for all but the earliest stages of the transient process ($Fo \geq 0.2$).

Graphical results for the energy transferred from a plane wall over the time interval t are presented in Figure 5.10. These results were generated from Equation 5.46. The dimensionless energy transfer Q/Q_o is expressed exclusively in terms of Fo and Bi.

Because the mathematical problem is precisely the same, the foregoing results may also be applied to a plane wall of thickness L, which is insulated on one side ($x^* = 0$) and experiences convective transport on the other side ($x^* = +1$). This equivalence is a consequence of the fact that, regardless of whether a symmetrical or an adiabatic requirement is prescribed at $x^* = 0$, the boundary condition is of the form $\partial\theta^*/\partial x^* = 0$.

5.6 RADIAL SYSTEMS WITH CONVECTION

For an infinite cylinder or sphere of radius r_o (Figure 5.7*b*), which is at an initial uniform temperature and experiences a change in convective conditions, results similar to those of Section 5.5 may be developed. That is, an exact series solution may be obtained for the time dependence of the radial temperature distribution; a one-term approximation may be used for most conditions; and the approximation may be conveniently represented in graphical form. The infinite cylinder is an idealization that permits the assumption of one-dimensional conduction in the radial direction. It is a reasonable approximation for cylinders having $L/r_o \gtrsim 10$.

5.6.1 Exact Solutions

Exact solutions to the transient, one-dimensional form of the heat equation have been developed for the infinite cylinder and for the sphere. For a uniform initial temperature and convective boundary conditions, the solutions [2] are as follows.

Infinite Cylinder In dimensionless form, the temperature is

$$\theta^* = \sum_{n=1}^{\infty} C_n \exp\left(-\zeta_n^2 Fo\right) J_0(\zeta_n r^*) \tag{5.47a}$$

where

$$C_n = \frac{2}{\zeta_n} \frac{J_1(\zeta_n)}{J_0^2(\zeta_n) + J_1^2(\zeta_n)} \tag{5.47b}$$

and the discrete values of ζ_n are positive roots of the transcendental equation

$$\zeta_n \frac{J_1(\zeta_n)}{J_0(\zeta_n)} = Bi \tag{5.47c}$$

The quantities J_1 and J_0 are Bessel functions of the first kind and their values are tabulated in Appendix B.4. Roots of the transcendental equation (5.47c) are tabulated by Schneider [2].

Sphere Similarly, for the sphere

$$\theta^* = \sum_{n=1}^{\infty} C_n \exp\left(-\zeta_n^2 Fo\right) \frac{1}{\zeta_n r^*} \sin\left(\zeta_n r^*\right) \tag{5.48a}$$

where

$$C_n = \frac{4[\sin(\zeta_n) - \zeta_n \cos(\zeta_n)]}{2\zeta_n - \sin(2\zeta_n)} \tag{5.48b}$$

and the discrete values of ζ_n are positive roots of the transcendental equation

$$1 - \zeta_n \cot \zeta_n = Bi \tag{5.48c}$$

Roots of the transcendental equation are tabulated by Schneider [2].

5.6.2 Approximate Solutions

For the infinite cylinder and sphere, Heisler [5] has shown that for $Fo \geq 0.2$, the foregoing series solutions can be approximated by a single term. Hence, as for the case of the plane wall, the time dependence of the temperature at any location within the radial system is the same as that of the centerline or centerpoint.

Infinite Cylinder The one-term approximation to Equation 5.47 is

$$\theta^* = C_1 \exp\left(-\zeta_1^2 Fo\right) J_0(\zeta_1 r^*) \tag{5.49a}$$

or

$$\theta^* = \theta_o^* J_0(\zeta_1 r^*) \tag{5.49b}$$

where θ_o^* represents the centerline temperature and is of the form

$$\theta_o^* = C_1 \exp\left(-\zeta_1^2 Fo\right) \tag{5.49c}$$

Values of the coefficients C_1 and ζ_1 have been determined and are listed in Table 5.1 for a range of Biot numbers.

Sphere From Equation 5.48a, the one-term approximation is

$$\theta^* = C_1 \exp\left(-\zeta_1^2 Fo\right) \frac{1}{\zeta_1 r^*} \sin\left(\zeta_1 r^*\right) \tag{5.50a}$$

or

$$\theta^* = \theta_o^* \frac{1}{\zeta_1 r^*} \sin\left(\zeta_1 r^*\right) \tag{5.50b}$$

where θ_o^* represents the center temperature and is of the form

$$\theta_o^* = C_1 \exp\left(-\zeta_1^2 Fo\right) \tag{5.50c}$$

Values of the coefficients C_1 and ζ_1 have been determined and are listed in Table 5.1 for a range of Biot numbers.

5.6.3 Total Energy Transfer

As in Section 5.5.3, an energy balance may be performed to determine the total energy transfer from the infinite cylinder or sphere over the time interval $\Delta t = t$. Substituting from the approximate solutions, Equations 5.49b and 5.50b, and introducing Q_o from Equation 5.44, the results are as follows.

Infinite Cylinder

$$\frac{Q}{Q_o} = 1 - \frac{2\theta_o^*}{\zeta_1} J_1(\zeta_1) \tag{5.51}$$

Sphere

$$\frac{Q}{Q_o} = 1 - \frac{3\theta_o^*}{\zeta_1^3}\left[\sin(\zeta_1) - \zeta_1 \cos(\zeta_1)\right] \tag{5.52}$$

Values of the center temperature θ_o^* are determined from Equation 5.49c or 5.50c, using the coefficients of Table 5.1 for the appropriate system.

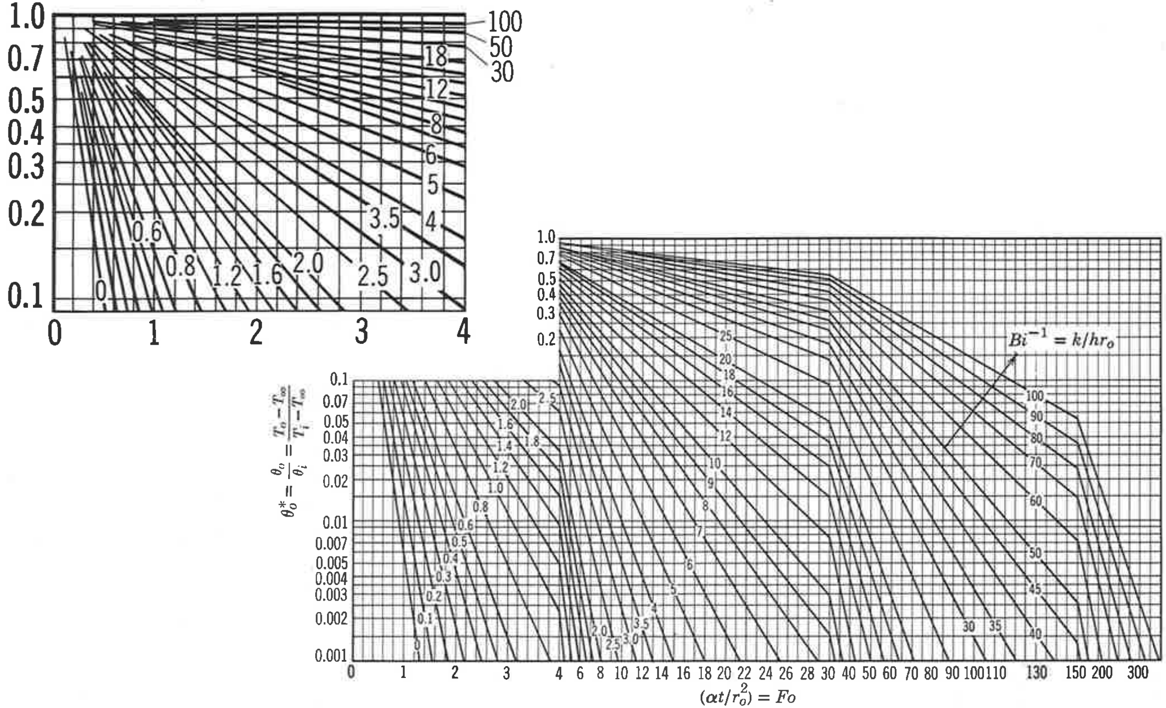


Figure 5.11 Centerline temperature as a function of time for an infinite cylinder of radius r_o [5]. Used with permission.

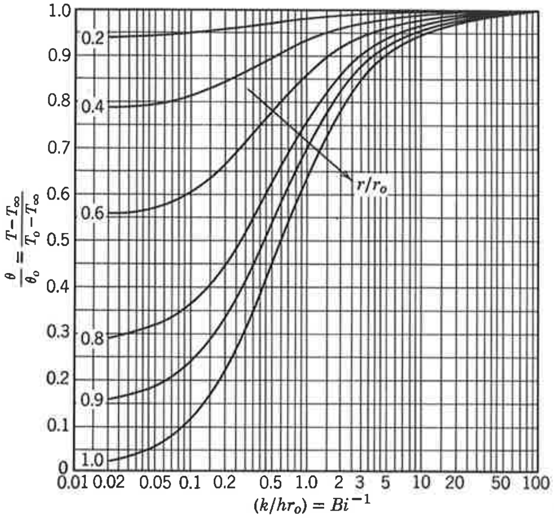


Figure 5.12 Temperature distribution in an infinite cylinder of radius r_o [5]. Used with permission.

5.6.4 Graphical Representation

Graphical representations similar to those for the plane wall (Figures 5.8 to 5.10) have also been generated by Heisler [5] and Gröber et al. [6] for an infinite cylinder and a sphere. Results for the infinite cylinder are presented in Figures 5.11 to 5.13, and those for the sphere are presented in Figures 5.14 to 5.16. *Note that, with respect to the use of these figures, the Biot number is*

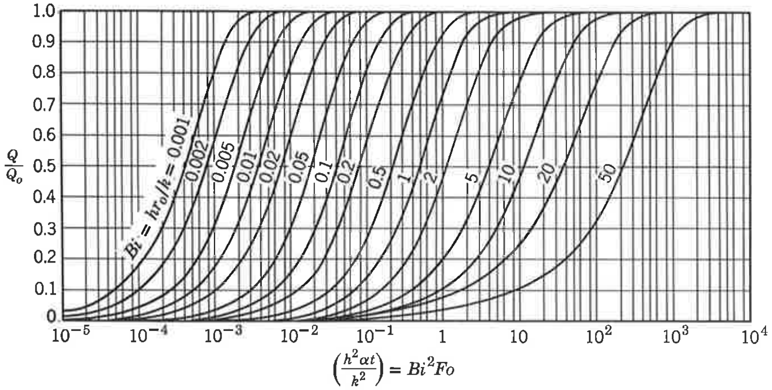


Figure 5.13 Internal energy change as a function of time for an infinite cylinder of radius r_o [6]. Adapted with permission.

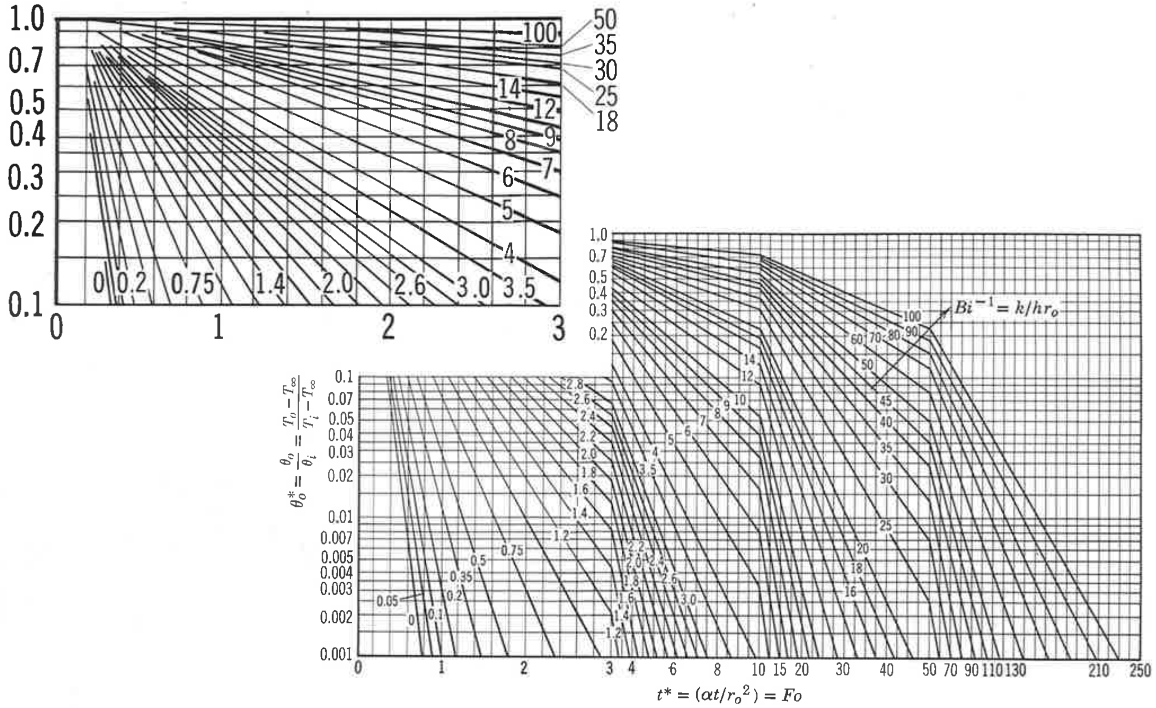


Figure 5.14 Center temperature as a function of time in a sphere of radius r_o [5]. Used with permission.

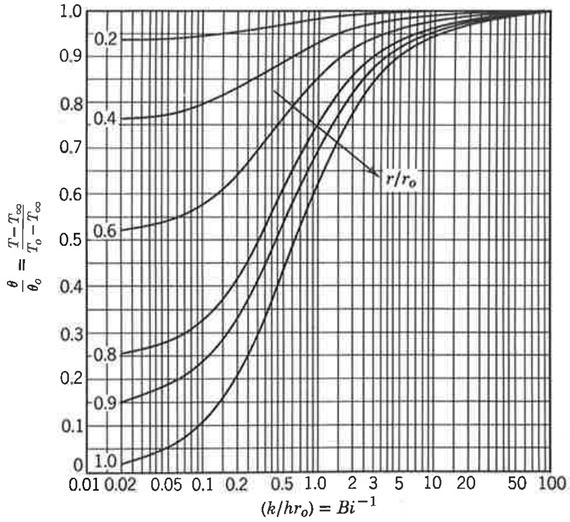


Figure 5.15 Temperature distribution in a sphere of radius r_o [5]. Used with permission.

defined in terms of r_o. In contrast recall that, for the lumped capacitance method, the characteristic length in the Biot number is customarily defined as $r_o/2$ for the cylinder and $r_o/3$ for the sphere.

In closing it should be noted that the *Heisler charts* may also be used to determine the transient response of a plane wall, an infinite cylinder, or a sphere subjected to a *sudden change in surface temperature*. For such a

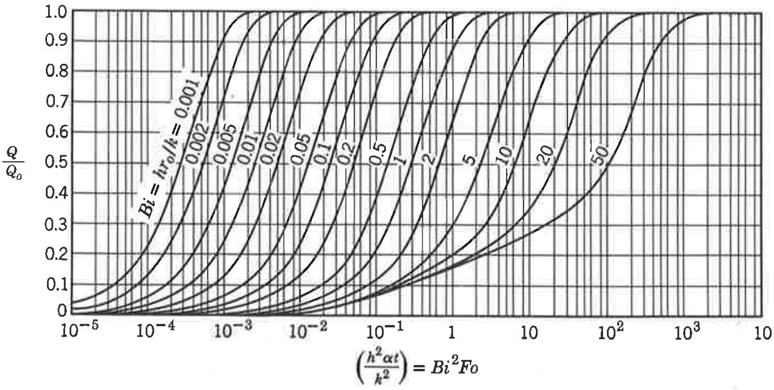


Figure 5.16 Internal energy change as a function of time for a sphere of radius r_o [6]. Adapted with permission.

condition it is only necessary to replace T_∞ by the prescribed surface temperature T_s and to set Bi^{-1} equal to zero. In so doing the convection coefficient is tacitly assumed to be infinite, in which case $T_\infty = T_s$.

EXAMPLE 5.2

Consider a steel pipeline (AISI 1010) that is 1 m in diameter and has a wall thickness of 40 mm. The pipe is heavily insulated on the outside, and before the initiation of flow, the walls of the pipe are at a uniform temperature of $-20°\text{C}$. With the initiation of flow, hot oil at 60°C is pumped through the pipe creating a convective surface condition corresponding to $h = 500$ $\text{W/m}^2 \cdot \text{K}$ at the inner surface of the pipe.

1. What are the appropriate Biot and Fourier numbers 8 min after the initiation of flow?
2. At $t = 8$ min, what is the temperature of the exterior pipe surface covered by the insulation?
3. What is the heat flux q'' (W/m^2) to the pipe from the oil at $t = 8$ min?
4. How much energy per meter of pipe length has been transferred from the oil to the pipe at $t = 8$ min?

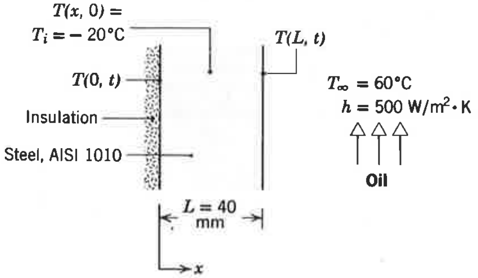
SOLUTION

Known: Wall subjected to sudden change in convective surface condition.

Find:

1. Biot and Fourier numbers after 8 min.
2. Temperature of exterior pipe surface after 8 min.
3. Heat flux to the wall at 8 min.
4. Energy transferred to pipe per unit length after 8 min.

Schematic:



Assumptions:

1. Pipe wall can be approximated as plane wall, since thickness is much less than diameter.
2. Constant properties.
3. Outer surface of pipe is adiabatic.

Properties: Table A.1, steel type AISI 1010 [$T = (-20 + 60)°\text{C}/2 \approx 300$ K]: $\rho = 7823$ kg/m^3, $c = 434$ J/kg · K, $k = 63.9$ W/m · K, $\alpha = 18.8 \times 10^{-6}$ m^2/s.

Analysis:

1. At $t = 8$ min, the Biot and Fourier numbers are computed from Equations 5.10 and 5.12, respectively, with $L_c = L$. Hence

$$Bi = \frac{hL}{k} = \frac{500\ \text{W/m}^2 \cdot \text{K} \times 0.04\ \text{m}}{63.9\ \text{W/m} \cdot \text{K}} = 0.313 \qquad \triangleleft$$

$$Fo = \frac{\alpha t}{L^2} = \frac{18.8 \times 10^{-6}\ \text{m}^2/\text{s} \times 8\ \text{min} \times 60\ \text{s/min}}{(0.04\ \text{m})^2} = 5.64 \qquad \triangleleft$$

2. With $Bi = 0.313$, use of the lumped capacitance method is inappropriate. However, since transient conditions in the insulated pipe wall of thickness L correspond to those in a plane wall of thickness $2L$ experiencing the same surface condition, the desired results may be obtained from the charts for the plane wall. Using Figure 5.8, with $Bi^{-1} = 3.2$, it follows that

$$\frac{\theta_o}{\theta_i} = \frac{T(0, t) - T_\infty}{T_i - T_\infty} \approx 0.22$$

Hence after 8 min, the temperature of the exterior pipe surface, which corresponds to the midplane temperature of a plane wall, is

$$T_o = T(0, 480\ \text{s}) \approx T_\infty + 0.22(T_i - T_\infty)$$

$$T_o = 60°\text{C} + 0.22(-20 - 60)°\text{C} \approx 42°\text{C} \qquad \triangleleft$$

3. Heat transfer to the inner surface at $x = L$ is by convection, and at any time t the heat flux may be obtained from Newton's law of cooling. Hence at $t = 480$ s,

$$q''_x(L, 480\ \text{s}) \equiv q''_L = h[T(L, 480\ \text{s}) - T_\infty]$$

The surface temperature $T(L, 480\ \text{s})$ may be obtained from Figure 5.9.

For the prescribed conditions

$$\frac{x}{L} = 1 \qquad \text{and} \qquad Bi^{-1} = 3.2$$

it follows that

$$\frac{\theta(L, 480\text{ s})}{\theta_o(480\text{ s})} = \frac{T(L, 480\text{ s}) - T_\infty}{T_o(480\text{ s}) - T_\infty} \approx 0.86$$

Hence

$$T(L, 480\text{ s}) \approx T_\infty + 0.86[T_o(480\text{ s}) - T_\infty]$$

$$T(L, 480\text{ s}) \approx 60°\text{C} + 0.86[42 - 60]°\text{C} \approx 45°\text{C}$$

The heat flux at $t = 8$ min is then

$$q''_L = 500\text{ W/m}^2 \cdot \text{K}\ (45 - 60)°\text{C} = -7500\text{ W/m}^2 \qquad \triangleleft$$

4. The energy transfer to the pipewall over the 8-min interval may be obtained from Figure 5.10 and Equation 5.44. With

$$Bi = 0.313 \qquad Bi^2 Fo = 0.55$$

it follows that

$$\frac{Q}{Q_o} \approx 0.78$$

Hence

$$Q \approx 0.78\rho c V(T_i - T_\infty)$$

or with a volume per unit pipe length of $V' = \pi D L$,

$$Q' \approx 0.78\rho c \pi D L(T_i - T_\infty)$$

$$Q' \approx 0.78 \times 7823\text{ kg/m}^3 \times 434\text{ J/kg} \cdot \text{K} \times \pi \times 1\text{ m} \times 0.04\text{ m}\ (-20 - 60)°\text{C}$$

$$Q' \approx -2.7 \times 10^7\text{ J/m} \qquad \triangleleft$$

Comments:

1. The minus sign associated with q'' and Q' simply implies that the direction of heat transfer is from the oil to the pipe (into the pipe wall).
2. Since $Fo > 0.2$, the one-term approximation can be used to calculate wall temperatures and the total energy transfer. The midplane tempera-

ture can be determined from Equation 5.41

$$\theta_o^* = \frac{T_o - T_\infty}{T_i - T_\infty} = C_1 \exp\left(-\zeta_1^2 Fo\right)$$

where, with $Bi = 0.313$, $C_1 = 1.047$ and $\zeta_1 = 0.531$ rad from Table 5.1. With $Fo = 5.64$,

$$\theta_o^* = 1.047 \exp\left[-(0.531 \text{ rad})^2 \times 5.64\right] = 0.214$$

This result is in good agreement with the value of 0.22 obtained from Figure 5.8. Hence,

$$T(0, 8 \text{ min}) = T_\infty + \theta_o^*(T_i - T_\infty) = 60°\text{C} + 0.214(-20 - 60)°\text{C} = 42.9°\text{C}$$

which is within 2% of the value determined from the Heisler chart.

3. Using the one-term approximation for the surface temperature, Equation 5.40b with $x^* = 1$ has the form

$$\theta^* = \theta_o^* \cos(\zeta_1)$$

$$T(L, t) = T_\infty + (T_i - T_\infty)\theta_o^* \cos(\zeta_1)$$

$$T(L, 8 \text{ min}) = 60°\text{C} + (-20 - 60)°\text{C} \times 0.214 \times \cos(0.531 \text{ rad})$$

$$T(L, 8 \text{ min}) = 45.2°\text{C}$$

which is within 1% of the value determined from the Heisler chart.

4. The total energy transferred during the transient process can be determined from the result associated with the one-term approximation, Equation 5.46.

$$\frac{Q}{Q_o} = 1 - \frac{\sin(\zeta_1)}{\zeta_1}\theta_o^*$$

$$\frac{Q}{Q_o} = 1 - \frac{\sin(0.531 \text{ rad})}{0.531 \text{ rad}} \times 0.214 = 0.80$$

which is within 3% of the value determined from the Gröber chart.

EXAMPLE 5.3

A new process for treatment of a special material is to be evaluated. The material, a sphere of radius $r_o = 5$ mm, is initially in equilibrium at 400°C in a furnace. It is suddenly removed from the furnace and subjected to a two-step cooling process.

Step 1 Cooling in air at 20°C for a period of time t_a until the center temperature reaches a critical value, $T_a(0, t_a) = 335°\text{C}$. For this situation, the convective heat transfer coefficient is $h_a = 10\ \text{W/m}^2 \cdot \text{K}$.

After the sphere has reached this critical temperature, the second step is initiated.

Step 2 Cooling in a well-stirred water bath at 20°C, with a convective heat transfer coefficient of $h_w = 6000\ \text{W/m}^2 \cdot \text{K}$.

The thermophysical properties of the material are $\rho = 3000\ \text{kg/m}^3$, $k = 20\ \text{W/m} \cdot \text{K}$, $c = 1000\ \text{J/kg} \cdot \text{K}$, and $\alpha = 6.66 \times 10^{-6}\ \text{m}^2/\text{s}$.

1. Calculate the time t_a required for step 1 of the cooling process to be completed.
2. Calculate the time t_w required during step 2 of the process for the center of the sphere to cool from 335°C (the condition at the completion of step 1) to 50°C.

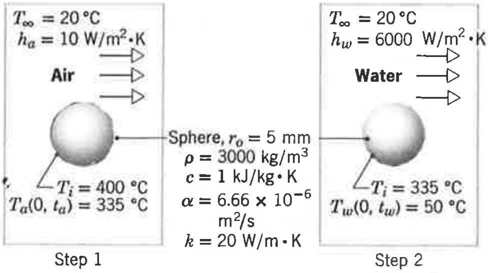
SOLUTION

Known: Temperature requirements for cooling a sphere.

Find:

1. Time t_a required to accomplish desired cooling in air.
2. Time t_w required to complete cooling in water bath.

Schematic:



Assumptions:

1. One-dimensional conduction in r.
2. Constant properties.

Analysis:

1. To determine whether the lumped capacitance method can be used, the Biot number is calculated. From Equation 5.10, with $L_c = r_o/3$,

$$Bi = \frac{h_a r_o}{3k} = \frac{10\ \mathrm{W/m^2 \cdot K} \times 0.005\ \mathrm{m}}{3 \times 20\ \mathrm{W/m \cdot K}} = 8.33 \times 10^{-4}$$

Accordingly, the lumped capacitance method may be used, and the temperature is nearly uniform throughout the sphere. From Equation 5.5 it follows that

$$t_a = \frac{\rho V c}{h_a A_s} \ln \frac{\theta_i}{\theta_a} = \frac{\rho r_o c}{3h_a} \ln \frac{T_i - T_\infty}{T_a - T_\infty}$$

where $V = (4/3)\pi r_o^3$ and $A_s = 4\pi r_o^2$. Hence

$$t_a = \frac{3000\ \mathrm{kg/m^3} \times 0.005\ \mathrm{m} \times 1000\ \mathrm{J/kg \cdot K}}{3 \times 10\ \mathrm{W/m^2 \cdot K}} \ln \frac{400 - 20}{335 - 20} = 94\ \mathrm{s}$$

◁

2. To determine whether the lumped capacitance method may also be used for the second step of the cooling process, the Biot number is again calculated. In this case

$$Bi = \frac{h_w r_o}{3k} = \frac{6000\ \mathrm{W/m^2 \cdot K} \times 0.005\ \mathrm{m}}{3 \times 20\ \mathrm{W/m \cdot K}} = 0.50$$

and the lumped capacitance method is not appropriate. However, to an excellent approximation, the temperature of the sphere is uniform at $t = t_a$ and the Heisler charts may be used for the calculations from $t = t_a$ to $t = t_a + t_w$. Using Figure 5.14 with

$$Bi^{-1} = \frac{k}{h_w r_o} = \frac{20\ \mathrm{W/m \cdot K}}{6000\ \mathrm{W/m^2 \cdot K} \times 0.005\ \mathrm{m}} = 0.67$$

$$\frac{\theta_o}{\theta_i} = \frac{T_o - T_\infty}{T_i - T_\infty} = \frac{50 - 20}{335 - 20} = 0.095$$

it follows that $Fo \approx 0.80$, and

$$t_w = Fo\frac{r_o^2}{\alpha} \approx 0.80\frac{(0.005\text{ m})^2}{6.66 \times 10^{-6}\text{ m}^2/\text{s}} \approx 3.0\text{ s} \qquad \triangleleft$$

Comments:

1. If the temperature distribution in the sphere at the conclusion of step 1 were not uniform, the Heisler chart could not be used for the calculations of step 2.
2. The surface temperature of the sphere at the conclusion of step 2 may be obtained from Figure 5.15. With

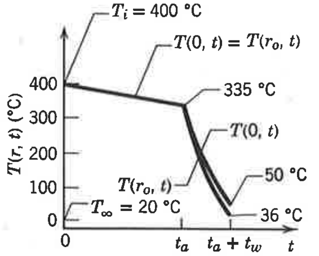
$$Bi^{-1} = 0.67 \qquad \text{and} \qquad \frac{r}{r_o} = 1$$

$$\frac{\theta(r_o)}{\theta_o} = \frac{T(r_o) - T_\infty}{T_o - T_\infty} \approx 0.52$$

Hence

$$T(r_o) \approx 20°\text{C} + 0.52(50 - 20)°\text{C} \approx 36°\text{C}$$

The variation of the center and surface temperature with time is then as follows.



3. For the step 2 transient process in water, the one-term approximation is appropriate for determining the time t_w at which the center temperature reaches 50°C, that is, $T(0, t_w) = 50°\text{C}$. Rearranging Equation 5.50c,

$$Fo = -\frac{1}{\zeta_1^2}\ln\left[\frac{\theta_o^*}{C_1}\right] = -\frac{1}{\zeta_1^2}\ln\left[\frac{1}{C_1} \times \frac{T(0, t_w) - T_\infty}{T_i - T_\infty}\right]$$

Using Table 5.1 to obtain the coefficients for $Bi = 1/0.67 = 1.50$ ($C_1 = 1.376$ and $\zeta_1 = 1.800$ rad) and substituting appropriate temperatures, it follows that

$$Fo = -\frac{1}{(1.800 \text{ rad})^2} \ln\left[\frac{1}{1.376} \times \frac{(50-20)^\circ\text{C}}{(335-20)^\circ\text{C}}\right] = 0.82$$

Substituting for r_o and α, it follows that $t_w = 3.1$ s, which is within 3% of the value of 3.0 s obtained from the Heisler chart.

5.7 THE SEMI-INFINITE SOLID

Another simple geometry for which analytical solutions may be obtained is the *semi-infinite solid*. Since such a solid extends to infinity in all but one direction, it is characterized by a single identifiable surface (Figure 5.17). If a sudden change of conditions is imposed at this surface, transient, one-dimensional conduction will occur within the solid. The semi-infinite solid provides a *useful idealization* for many practical problems. It may be used to determine transient heat transfer near the surface of the earth or to approximate the transient response of a finite solid, such as a thick slab. For this second situation the approximation would be reasonable for the early portion of the transient, during which temperatures in the slab interior (well removed from the surface) are uninfluenced by the change in surface conditions.

The heat equation for transient conduction in a semi-infinite solid is given by Equation 5.26. The initial condition is prescribed by Equation 5.27, and the interior boundary condition is of the form

$$T(\infty, t) = T_i \tag{5.53}$$

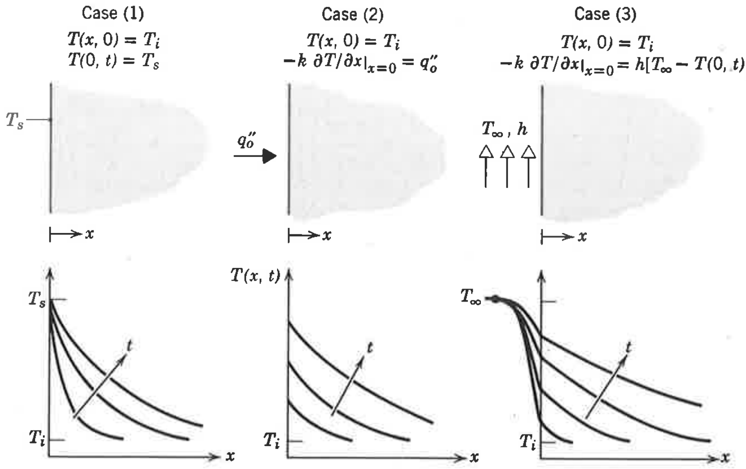


Figure 5.17 Transient temperature distributions in a semi-infinite solid for three surface conditions: constant surface temperature, constant surface heat flux, and surface convection.

Closed-form solutions have been obtained for three important surface conditions, instantaneously applied at $t = 0$ [1, 2]. These conditions are shown in Figure 5.17. They include application of a constant surface temperature $T_s \neq T_i$, application of a constant surface heat flux q''_o, and exposure of the surface to a fluid characterized by $T_\infty \neq T_i$ and the convection coefficient h. The solutions are summarized as follows.

Case 1 Constant Surface Temperature

$$T(0, t) = T_s \tag{5.54}$$

$$\frac{T(x, t) - T_s}{T_i - T_s} = \operatorname{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right) \tag{5.55}$$

$$q''_s(t) = -k\frac{\partial T}{\partial x}\bigg|_{x=0} = \frac{k(T_s - T_i)}{\sqrt{\pi\alpha t}} \tag{5.56}$$

Case 2 Constant Surface Heat Flux

$$q''_s = q''_o \tag{5.57}$$

$$T(x, t) - T_i = \frac{2q''_o(\alpha t/\pi)^{1/2}}{k}\exp\left(\frac{-x^2}{4\alpha t}\right) - \frac{q''_o x}{k}\operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right) \tag{5.58}$$

Case 3 Surface Convection

$$-k\frac{\partial T}{\partial x}\bigg|_{x=0} = h[T_\infty - T(0, t)] \tag{5.59}$$

$$\frac{T(x, t) - T_i}{T_\infty - T_i} = \operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right) - \left[\exp\left(\frac{hx}{k} + \frac{h^2\alpha t}{k^2}\right)\right]\left[\operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}} + \frac{h\sqrt{\alpha t}}{k}\right)\right] \tag{5.60}$$

The quantity erf w appearing in Equation 5.55 is the *Gaussian error function*, which is tabulated in Section B.1 of Appendix B. The *complementary error function*, erfc w, is defined as

$$\operatorname{erfc} w \equiv 1 - \operatorname{erf} w$$

Temperature histories for the three cases are also shown in Figure 5.17. Carefully note their distinguishing features. For case 3 the specific temperature histories computed from Equation 5.60 are plotted in Figure 5.18. Note that the curve corresponding to $h = \infty$ is equivalent to the result that would be obtained for a sudden change in the *surface temperature* to $T_s = T_\infty$. That is, for $h = \infty$ the second term on the right-hand side of Equation 5.60 goes to zero, and the result is equivalent to Equation 5.55.

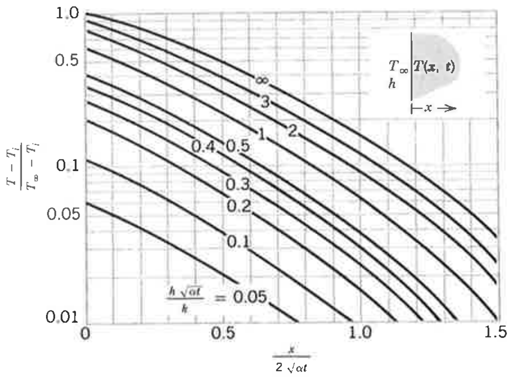


Figure 5.18 Temperature histories in a semi-infinite solid with surface convection [2]. Adapted with permission.

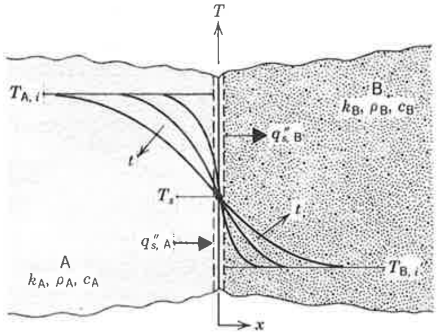


Figure 5.19 Interfacial contact between two semi-infinite solids at different initial temperatures.

An interesting permutation of case 1 results when two semi-infinite solids, initially at uniform temperatures $T_{A,i}$ and $T_{B,i}$, are placed in contact at their free surfaces (Figure 5.19). If the contact resistance is negligible, the requirement of thermal equilibrium dictates that, at the instant of contact ($t = 0$), both surfaces must assume the same temperature T_s, for which $T_{B,i} < T_s < T_{A,i}$. Since T_s does not change with increasing time, it follows that the transient thermal response and the surface heat flux of each of the solids is determined by Equations 5.55 and 5.56, respectively.

The equilibrium surface temperature of Figure 5.19 may be determined from a surface energy balance, which requires that

$$q''_{s,A} = q''_{s,B} \tag{5.61}$$

Substituting from Equation 5.56 for $q''_{s,A}$ and $q''_{s,B}$ and recognizing that the x

coordinate of Figure 5.19 requires a sign change for $q''_{s,\mathrm{A}}$, it follows that

$$\frac{-k_{\mathrm{A}}(T_s - T_{\mathrm{A},i})}{(\pi\alpha_{\mathrm{A}} t)^{1/2}} = \frac{k_{\mathrm{B}}(T_s - T_{\mathrm{B},i})}{(\pi\alpha_{\mathrm{B}} t)^{1/2}} \tag{5.62}$$

or, solving for T_s,

$$T_s = \frac{(k\rho c)_{\mathrm{A}}^{1/2} T_{\mathrm{A},i} + (k\rho c)_{\mathrm{B}}^{1/2} T_{\mathrm{B},i}}{(k\rho c)_{\mathrm{A}}^{1/2} + (k\rho c)_{\mathrm{B}}^{1/2}} \tag{5.63}$$

Hence, the quantity $m \equiv (k\rho c)^{1/2}$ is a weighting factor which determines whether T_s will more closely approach $T_{\mathrm{A},i}(m_{\mathrm{A}} > m_{\mathrm{B}})$ or $T_{\mathrm{B},i}(m_{\mathrm{B}} > m_{\mathrm{A}})$.

EXAMPLE 5.4

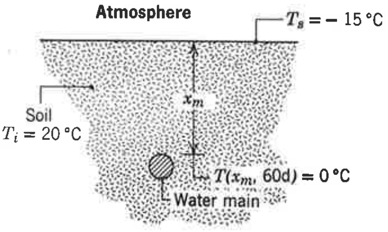
In laying water mains, utilities must be concerned with the possibility of freezing during cold periods. Although the problem of determining the temperature in soil as a function of time is complicated by changing surface conditions, reasonable estimates can be based on the assumption of a constant surface temperature over a prolonged period of cold weather. What minimum burial depth x_m would you recommend to avoid freezing under conditions for which soil, initially at a uniform temperature of 20°C, is subjected to a constant surface temperature of −15°C for 60 days?

SOLUTION

Known: Temperature imposed at the surface of soil that is initially at 20°C.

Find: The depth x_m to which the soil has frozen after 60 days.

Schematic:



Assumptions:

1. One-dimensional conduction in x.
2. Soil is a semi-infinite medium.
3. Constant properties.

Properties: Table A.3, soil (300 K): $\rho = 2050\ \text{kg/m}^3$, $k = 0.52\ \text{W/m}\cdot\text{K}$, $c = 1840\ \text{J/kg}\cdot\text{K}$, $\alpha = (k/\rho c) = 0.138 \times 10^{-6}\ \text{m}^2/\text{s}$.

Analysis: The prescribed conditions correspond to those of case 1 of Figure 5.17, and the transient temperature response of the soil is governed by Equation 5.55. Hence at the time $t = 60$ days after the surface temperature change,

$$\frac{T(x_m, t) - T_s}{T_i - T_s} = \operatorname{erf}\left(\frac{x_m}{2\sqrt{\alpha t}}\right)$$

or

$$\frac{0 - (-15)}{20 - (-15)} = 0.429 = \operatorname{erf}\left(\frac{x_m}{2\sqrt{\alpha t}}\right)$$

Hence from Appendix B.1

$$\frac{x_m}{2\sqrt{\alpha t}} = 0.40$$

and

$$x_m = 0.80\sqrt{\alpha t} = 0.80(0.138 \times 10^{-6}\ \text{m}^2/\text{s} \times 60\ \text{days} \times 24\ \text{h/day} \times 3600\ \text{s/h})^{1/2} = 0.68\ \text{m} \qquad \triangleleft$$

Comments: The properties of soil are highly variable, depending on the nature of the soil and its moisture content.

5.8 MULTIDIMENSIONAL EFFECTS

Transient problems are frequently encountered for which two- and even three-dimensional effects are significant. Solution to a class of such problems can be obtained from the one-dimensional results of Sections 5.6 and 5.7.

Consider immersing the *short* cylinder of Figure 5.20, which is initially at a uniform temperature T_i, in a fluid of temperature $T_\infty \neq T_i$. Because the length and diameter are comparable, the subsequent transfer of energy by conduction will be significant for both the r and x coordinate directions. The temperature within the cylinder will therefore depend on r, x, and t.

Assuming constant properties and no generation, the appropriate form of the heat equation is, from Equation 2.20,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

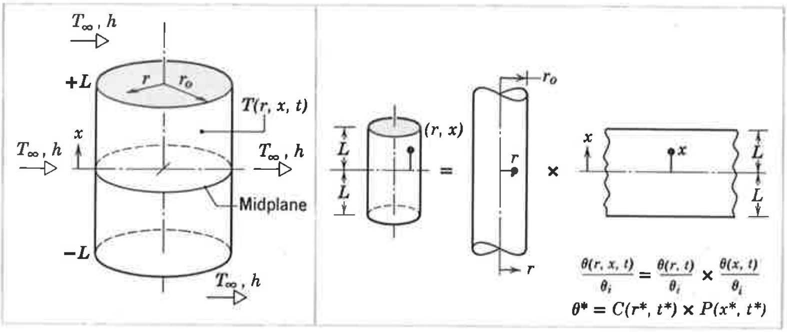


Figure 5.20 Two-dimensional, transient conduction in a short cylinder. (*a*) Geometry. (b) Form of the product solution.

where x has been used in place of z to designate the axial coordinate. A closed-form solution to this equation may be obtained by the separation of variables method. Although we will not consider the details of this solution, it is important to note that the end result may be expressed in the following form.

$$\frac{T(r,x,t)-T_\infty}{T_i-T_\infty} = \left.\frac{T(x,t)-T_\infty}{T_i-T_\infty}\right|_{\text{Plane wall}} \cdot \left.\frac{T(r,t)-T_\infty}{T_i-T_\infty}\right|_{\text{Infinite cylinder}}$$

That is, the two-dimensional solution may be expressed as a *product* of one-dimensional solutions that correspond to those for a plane wall of thickness $2L$ and an infinite cylinder of radius r_o. These solutions are available from Figures 5.8 and 5.9 for the plane wall and Figures 5.11 and 5.12 for the infinite cylinder. They are also available from the one-term approximations given by Equations 5.40 and 5.49.

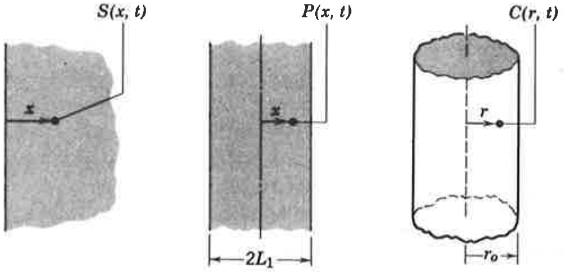
Results for other multidimensional geometries are summarized in Figure 5.21. In each case the multidimensional solution is prescribed in terms of a product involving one or more of the following one-dimensional solutions.

$$S(x,t) \equiv \left.\frac{T(x,t)-T_\infty}{T_i-T_\infty}\right|_{\text{Semi-infinite solid}} \tag{5.64}$$

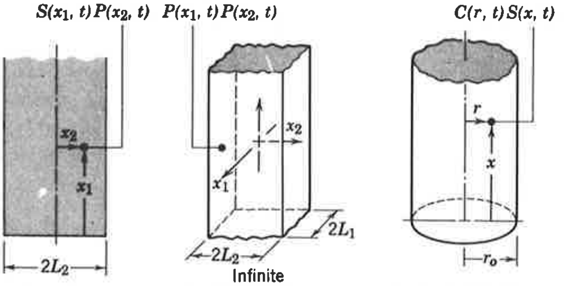
$$P(x,t) \equiv \left.\frac{T(x,t)-T_\infty}{T_i-T_\infty}\right|_{\text{Plane wall}} \tag{5.65}$$

$$C(r,t) \equiv \left.\frac{T(r,t)-T_\infty}{T_i-T_\infty}\right|_{\text{Infinite cylinder}} \tag{5.66}$$

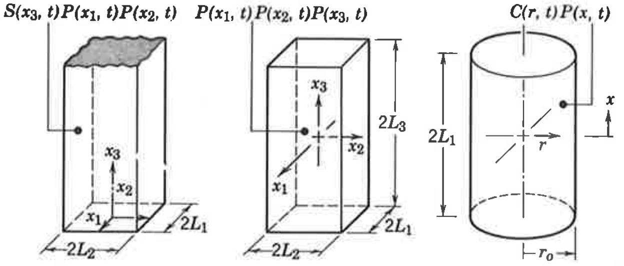
The x coordinate for the semi-infinite solid is measured from the surface,



(a) Semi-infinite solid (b) Plane wall (c) Infinite cylinder



(d) Semi-infinite plate (e) Infinite rectangular bar (f) Semi-infinite cylinder



(g) Semi-infinite rectangular bar (h) Rectangular parallelepiped (i) Short cylinder

Figure 5.21 Solutions for multidimensional systems expressed as products of one-dimensional results.

whereas for the plane wall it is measured from the midplane. In using Figure 5.21 the coordinate origins should be carefully noted. The transient, three-dimensional temperature distribution in a rectangular parallelepiped, Figure 5.21*h*, is then, for example, the product of three one-dimensional solutions for plane walls of thicknesses $2L_1$, $2L_2$, and $2L_3$. That is,

$$\frac{T(x_1, x_2, x_3, t) - T_\infty}{T_i - T_\infty} = P(x_1, t) \cdot P(x_2, t) \cdot P(x_3, t)$$

The distances x_1, x_2, and x_3 are all measured with respect to a rectangular coordinate system whose origin is at the center of the parallelepiped.

The amount of energy Q transferred to or from a solid during a multidimensional transient conduction process may also be determined by combining one-dimensional results, as shown by Langston [7].

EXAMPLE 5.5

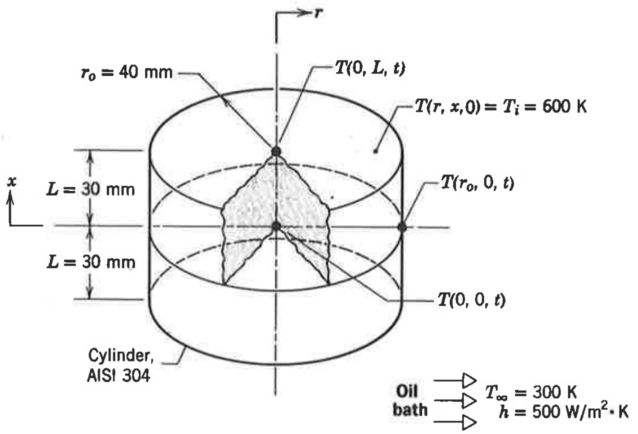
In a manufacturing process stainless steel cylinders (AISI 304) initially at 600 K are quenched by submersion in an oil bath maintained at 300 K with $h = 500 \text{ W/m}^2 \cdot \text{K}$. Each cylinder is of length $2L = 60$ mm and diameter $D = 80$ mm. Consider a time 3 min into the cooling process and determine temperatures at the center of the cylinder, at the center of a circular face, and at the midheight of the side.

SOLUTION

Known: Initial temperature and dimensions of cylinder and temperature and convection conditions of an oil bath.

Find: Temperatures $T(r, x, t)$ after 3 min at the cylinder center, $T(0, 0, 3 \text{ min})$, at the center of a circular face, $T(0, L, 3 \text{ min})$, and at the midheight of the side, $T(r_o, 0, 3 \text{ min})$.

Schematic:



Assumptions:

1. Two-dimensional conduction in r and x.
2. Constant properties.

Properties: Table A.1, stainless steel, AISI 304 [$T = (600 + 300)/2 = 450$ K]: $\rho = 7900\ \mathrm{kg/m^3}$, $c = 526\ \mathrm{J/kg \cdot K}$, $k = 17.4\ \mathrm{W/m \cdot K}$, $\alpha = k/\rho c = 4.19 \times 10^{-6}\ \mathrm{m^2/s}$.

Analysis: The solid steel cylinder corresponds to case i of Figure 5.21, and the temperature at any point in the cylinder may be expressed as the following product of one-dimensional solutions.

$$\frac{T(r, x, t) - T_\infty}{T_i - T_\infty} = P(x, t)C(r, t)$$

where $P(x, t)$ and $C(r, t)$ are defined by Equations 5.65 and 5.66, respectively. Accordingly, for the center of the cylinder,

$$\frac{T(0, 0, 3\ \text{min}) - T_\infty}{T_i - T_\infty} = \left.\frac{T(0, 3\ \text{min}) - T_\infty}{T_i - T_\infty}\right|_{\substack{\text{Plane}\\ \text{wall}}} \cdot \left.\frac{T(0, 3\ \text{min}) - T_\infty}{T_i - T_\infty}\right|_{\substack{\text{Infinite}\\ \text{cylinder}}}$$

Hence, for the plane wall, with

$$Bi^{-1} = \frac{k}{hL} = \frac{17.4\ \mathrm{W/m \cdot K}}{500\ \mathrm{W/m^2 \cdot K} \times 0.03\ \mathrm{m}} = 1.16$$

$$Fo = \frac{\alpha t}{L^2} = \frac{4.19 \times 10^{-6}\ \mathrm{m^2/s} \times 180\ \mathrm{s}}{(0.03\ \mathrm{m})^2} = 0.84$$

it follows from Figure 5.8 that

$$\frac{\theta_o}{\theta_i} = \left.\frac{T(0, 3\ \text{min}) - T_\infty}{T_i - T_\infty}\right|_{\substack{\text{Plane}\\ \text{wall}}} \approx 0.64$$

Similarly, for the infinite cylinder, with

$$Bi^{-1} = \frac{k}{hr_o} = \frac{17.4\ \mathrm{W/m \cdot K}}{500\ \mathrm{W/m^2 \cdot K} \times 0.04\ \mathrm{m}} = 0.87$$

$$Fo = \frac{\alpha t}{r_o^2} = \frac{4.19 \times 10^{-6}\ \mathrm{m^2/s} \times 180\ \mathrm{s}}{(0.04\ \mathrm{m})^2} = 0.47$$

it follows from Figure 5.11 that

$$\frac{\theta_o}{\theta_i} = \left.\frac{T(0, 3\text{ min}) - T_\infty}{T_i - T_\infty}\right|_{\substack{\text{Infinite}\\\text{cylinder}}} \approx 0.55$$

Hence, for the center of the cylinder,

$$\frac{T(0, 0, 3\text{ min}) - T_\infty}{T_i - T_\infty} \approx 0.64 \times 0.55 \approx 0.35$$

$$T(0, 0, 3\text{ min}) \approx 300\text{ K} + 0.35(600 - 300)\text{ K} \approx 405\text{ K} \qquad \triangleleft$$

The temperature at the center of a circular face may be obtained from the requirement that

$$\frac{T(0, L, 3\text{ min}) - T_\infty}{T_i - T_\infty} = \left.\frac{T(L, 3\text{ min}) - T_\infty}{T_i - T_\infty}\right|_{\substack{\text{Plane}\\\text{wall}}} \cdot \left.\frac{T(0, 3\text{ min}) - T_\infty}{T_i - T_\infty}\right|_{\substack{\text{Infinite}\\\text{cylinder}}}$$

where, from Figure 5.9 with $(x/L) = 1$ and $Bi^{-1} = 1.16$,

$$\frac{\theta(L)}{\theta_o} = \left.\frac{T(L, 3\text{ min}) - T_\infty}{T(0, 3\text{ min}) - T_\infty}\right|_{\substack{\text{Plane}\\\text{wall}}} \approx 0.68$$

Hence

$$\left.\frac{T(L, 3\text{ min}) - T_\infty}{T_i - T_\infty}\right|_{\substack{\text{Plane}\\\text{wall}}} = \left.\frac{T(L, 3\text{ min}) - T_\infty}{T(0, 3\text{ min}) - T_\infty}\right|_{\substack{\text{Plane}\\\text{wall}}} \cdot \left.\frac{T(0, 3\text{ min}) - T_\infty}{T_i - T_\infty}\right|_{\substack{\text{Plane}\\\text{wall}}}$$

$$\left.\frac{T(L, 3\text{ min}) - T_\infty}{T_i - T_\infty}\right|_{\substack{\text{Plane}\\\text{wall}}} \approx 0.68 \times 0.64 \approx 0.44$$

Hence

$$\frac{T(0, L, 3\text{ min}) - T_\infty}{T_i - T_\infty} \approx 0.44 \times 0.55 \approx 0.24$$

$$T(0, L, 3\text{ min}) \approx 300\text{ K} + 0.24(600 - 300)\text{ K} \approx 372\text{ K} \qquad \triangleleft$$

The temperature at the midheight of the side may be obtained from the requirement that

$$\frac{T(r_o, 0, 3\text{ min}) - T_\infty}{T_i - T_\infty} = \left.\frac{T(0, 3\text{ min}) - T_\infty}{T_i - T_\infty}\right|_{\substack{\text{Plane}\\\text{wall}}} \cdot \left.\frac{T(r_o, 3\text{ min}) - T_\infty}{T_i - T_\infty}\right|_{\substack{\text{Infinite}\\\text{cylinder}}}$$

where, from Figure 5.12 with $(r/r_o) = 1$ and $Bi^{-1} = 0.87$,

$$\frac{\theta(r_o)}{\theta_o} = \frac{T(r_o, 3\text{ min}) - T_\infty}{T(0, 3\text{ min}) - T_\infty}\bigg|_{\substack{\text{Infinite}\\\text{cylinder}}} \approx 0.61$$

Hence

$$\frac{T(r_o, 3\text{ min}) - T_\infty}{T_i - T_\infty}\bigg|_{\substack{\text{Infinite}\\\text{cylinder}}} = \frac{T(r_o, 3\text{ min}) - T_\infty}{T(0, 3\text{ min}) - T_\infty}\bigg|_{\substack{\text{Infinite}\\\text{cylinder}}} \cdot \frac{T(0, 3\text{ min}) - T_\infty}{T_i - T_\infty}\bigg|_{\substack{\text{Infinite}\\\text{cylinder}}}$$

$$\frac{T(r_o, 3\text{ min}) - T_\infty}{T_i - T_\infty}\bigg|_{\substack{\text{Infinite}\\\text{cylinder}}} \approx 0.61 \times 0.55 \approx 0.34$$

Hence

$$\frac{T(r_o, 0, 3\text{ min}) - T_\infty}{T_i - T_\infty} \approx 0.64 \times 0.34 \approx 0.22$$

$$T(r_o, 0, 3\text{ min}) \approx 300\text{ K} + 0.22(600 - 300)\text{ K} \approx 366\text{ K} \qquad \triangleleft$$

Comments:

1. Verify that the temperature at the edge of the cylinder is $T(r_o, L, 3\text{ min}) \approx 345$ K.
2. The one-term approximations can be used to calculate the dimensionless temperatures read from the Heisler charts. For the *plane wall*, the midplane temperature can be determined from Equation 5.41

$$\theta_o^* = \frac{\theta_o}{\theta_i} = C_1 \exp\left(-\zeta_1^2 Fo\right)$$

where, with $Bi = 0.862$, $C_1 = 1.109$ and $\zeta_1 = 0.814$ rad from Table 5.1. With $Fo = 0.84$,

$$\frac{\theta_o}{\theta_i}\bigg|_{\substack{\text{Plane}\\\text{wall}}} = 1.109 \exp\left[-(0.814\text{ rad})^2 \times 0.84\right] = 0.636$$

The surface temperature can be evaluated using Equation 5.40b

$$\frac{\theta^*}{\theta_o^*} = \frac{\theta}{\theta_o} = \cos\left(\zeta_1 x^*\right)$$

with $x^* = 1$ to give

$$\frac{\theta^*(1, Fo)}{\theta_o^*} = \frac{\theta(L, t)}{\theta_o} = \cos\left(0.814 \text{ rad} \times 1\right) = 0.687$$

For the *infinite cylinder*, the centerline temperature can be determined from Equation 5.49c.

$$\theta_o^* = \frac{\theta_o}{\theta_i} = C_1 \exp\left(-\zeta_1^2 Fo\right)$$

where, with $Bi = 1.15$, $C_1 = 1.227$ and $\zeta_1 = 1.307$ from Table 5.1. With $Fo = 0.47$,

$$\left.\frac{\theta_o}{\theta_i}\right|_{\substack{\text{Infinite}\\ \text{cylinder}}} = 1.109 \exp\left[-(1.307 \text{ rad})^2 \times 0.47\right] = 0.550$$

The surface temperature can be evaluated using Equation 5.49b

$$\frac{\theta^*}{\theta_o^*} = \frac{\theta}{\theta_o} = J_0(\zeta_1 r^*)$$

with $r^* = 1$ and the value of the Bessel function determined from Table B.4,

$$\frac{\theta^*(1, Fo)}{\theta_o^*} = \frac{\theta(L, t)}{\theta_o} = J_0(1.307 \text{ rad} \times 1) = 0.616$$

The one-term approximations are in good agreement with results from the Heisler charts.

5.9 FINITE-DIFFERENCE METHODS

Analytical solutions to transient problems are restricted to simple geometries and boundary conditions, such as those considered in the preceding sections. Extensive coverage of these and other solutions is treated in the literature [1–4]. However, in many cases the geometry and/or boundary conditions preclude the use of analytical techniques, and recourse must be made to *finite-difference* methods. Such methods, introduced in Section 4.4 for steady-state conditions, are readily extended to transient problems. In this section we consider *explicit* and *implicit* forms of finite-difference solutions to transient conduction problems. More detailed treatments, as well as related algorithms, may be found in the literature [8–10].

5.9.1 Discretization of the Heat Equation: The Explicit Method

Once again consider the two-dimensional system of Figure 4.5. Under transient conditions with constant properties and no internal generation, the appropriate form of the heat equation, Equation 2.15, is

$$\frac{1}{\alpha}\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \tag{5.67}$$

To obtain the finite-difference form of this equation, we may use the *central-difference* approximations to the spatial derivatives prescribed by Equations 4.31 and 4.32. Once again the m and n subscripts may be used to designate the x and y locations of *discrete nodal points*. However, in addition to being discretized in space, the problem must be discretized in time. The integer p is introduced for this purpose, where

$$t = p\,\Delta t \tag{5.68}$$

and the finite-difference approximation to the time derivative in Equation 5.67 is expressed as

$$\left.\frac{\partial T}{\partial t}\right|_{m,n} \approx \frac{T^{p+1}_{m,n} - T^{p}_{m,n}}{\Delta t} \tag{5.69}$$

The superscript p is used to denote the time dependence of T, and the time derivative is expressed in terms of the difference in temperatures associated with the *new* $(p + 1)$ and *previous* (p) times. Hence calculations must be performed at successive times separated by the interval Δt, and just as a finite-difference solution restricts temperature determination to discrete points in space, it also restricts it to discrete points in time.

If Equation 5.69 is substituted into Equation 5.67, the nature of the finite-difference solution will depend on the specific time at which temperatures are evaluated in the finite-difference approximations to the spatial derivatives. In the *explicit method* of solution, these temperatures are evaluated at the *previous* (p) time. Hence Equation 5.69 is considered to be a *forward-difference* approximation to the time derivative. Evaluating terms on the right-hand side of Equations 4.31 and 4.32 at p and substituting into Equation 5.67, the explicit form of the finite-difference equation for the interior node m, n is

$$\frac{1}{\alpha}\frac{T_{m,n}^{p+1} - T_{m,n}^{p}}{\Delta t} = \frac{T_{m+1,n}^{p} + T_{m-1,n}^{p} - 2T_{m,n}^{p}}{(\Delta x)^2} + \frac{T_{m,n+1}^{p} + T_{m,n-1}^{p} - 2T_{m,n}^{p}}{(\Delta y)^2} \tag{5.70}$$

Solving for the nodal temperature at the new $(p + 1)$ time and assuming that $\Delta x = \Delta y$, it follows that

$$T_{m,n}^{p+1} = Fo\left(T_{m+1,n}^{p} + T_{m-1,n}^{p} + T_{m,n+1}^{p} + T_{m,n-1}^{p}\right) + (1 - 4Fo)T_{m,n}^{p} \tag{5.71}$$

where Fo is a finite-difference form of the Fourier number

$$Fo = \frac{\alpha\,\Delta t}{(\Delta x)^2} \tag{5.72}$$

If the system is one-dimensional in x, the explicit form of the finite-difference equation for an interior node m reduces to

$$T_{m}^{p+1} = Fo\left(T_{m+1}^{p} + T_{m-1}^{p}\right) + (1 - 2Fo)T_{m}^{p} \tag{5.73}$$

Equations 5.71 and 5.73 are *explicit* because *unknown* nodal temperatures for the new time are determined exclusively by *known* nodal temperatures at the previous time. Hence calculation of the unknown temperatures is straightforward. Since the temperature of each interior node is known at $t = 0$ $(p = 0)$ from prescribed initial conditions, the calculations begin at $t = \Delta t$ $(p = 1)$, where Equation 5.71 or 5.73 is applied to each interior node to determine its temperature. With temperatures known for $t = \Delta t$, the appropriate finite-difference equation is then applied at each node to determine its temperature at $t = 2\,\Delta t$ $(p = 2)$. In this way, the transient temperature distribution is obtained by *marching out in time*, using intervals of Δt.

The accuracy of the finite-difference solution may be improved by decreasing the values of Δx and Δt. Of course, the number of interior nodal points that must be considered increases with decreasing Δx, and the number of time intervals required to carry the solution to a prescribed final time increases with decreasing Δt. Hence, the computation time increases with decreasing Δx and Δt. The choice of Δx is typically based on a compromise between accuracy and computational requirements. Once this selection has been made, however, the value of Δt may not be chosen independently. It is, instead, determined by *stability* requirements.

An undesirable feature of the explicit method is that it is not unconditionally *stable*. In a transient problem, the solution for the nodal temperatures should continuously approach final (steady-state) values with increasing time. However, with the explicit method, this solution may be characterized by numerically induced oscillations, which are physically impossible. The oscillations may become *unstable*, causing the solution to diverge from the actual steady-state conditions. To prevent such erroneous results, the prescribed value of Δt must be maintained below a certain limit, which depends on Δx and other parameters of the system. This dependence is termed a *stability criterion*, which may be obtained mathematically [8] or demonstrated from a thermodynamic argument (see Problem 5.69). For the problems of interest in this text, *the criterion is determined by requiring that the coefficient associated with the node of interest at the previous time is greater than or equal to zero*. In general, this is done by collecting all terms involving $T_{m,n}^{p}$ to obtain the form of the coefficient. This result is then used to obtain a limiting relation involving Fo, from which the maximum allowable value of Δt may be determined. For example, with Equations 5.71 and 5.73 already expressed in the desired form, it follows that the stability criterion for a one-dimensional interior node is $(1 - 2Fo) \geq 0$, or

$$Fo \leq \tfrac{1}{2} \tag{5.74}$$

and for a two-dimensional node, it is $(1 - 4Fo) \geq 0$, or

$$Fo \leq \tfrac{1}{4} \tag{5.75}$$

For prescribed values of Δx and α, these criteria may be used to determine upper limits to the value of Δt.

Equations 5.71 and 5.73 may also be derived by applying the energy balance method of Section 4.4.3 to a control volume about the interior node. Accounting for changes in thermal energy storage, a general form of the energy balance equation may be expressed as

$$\dot{E}_{\text{in}} + \dot{E}_{\text{g}} = \dot{E}_{\text{st}} \tag{5.76}$$

In the interest of adopting a consistent methodology, it is again assumed that all heat flow is *into* the node.

To illustrate application of Equation 5.76, consider the surface node of the one-dimensional system shown in Figure 5.22. To more accurately determine thermal conditions near the surface, this node has been assigned a thickness which is one-half that of the interior nodes. Assuming convection transfer from an adjoining fluid and no generation, it follows from Equation 5.76 that

$$hA(T_\infty - T_0^p) + \frac{kA}{\Delta x}(T_1^p - T_0^p) = \rho c A \frac{\Delta x}{2} \frac{T_0^{p+1} - T_0^p}{\Delta t}$$

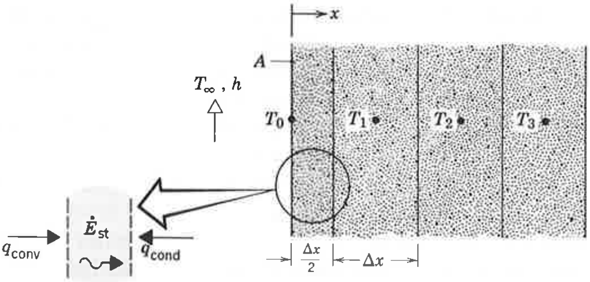


Figure 5.22 Surface node with convection and one-dimensional transient conduction.

or, solving for the surface temperature at $t + \Delta t$,

$$T_0^{p+1} = \frac{2h\,\Delta t}{\rho c\,\Delta x}(T_\infty - T_0^p) + \frac{2\alpha\,\Delta t}{\Delta x^2}(T_1^p - T_0^p) + T_0^p$$

Recognizing that $(2h\,\Delta t/\rho c\,\Delta x) = 2(h\,\Delta x/k)(\alpha\,\Delta t/\Delta x^2) = 2BiFo$ and grouping terms involving T_0^p, it follows that

$$T_0^{p+1} = 2Fo(T_1^p + BiT_\infty) + (1 - 2Fo - 2BiFo)T_0^p \qquad (5.77)$$

The finite-difference form of the Biot number is

$$Bi = \frac{h\,\Delta x}{k} \qquad (5.78)$$

Recalling the procedure for determining the stability criterion, we require that the coefficient for T_0^p be greater than or equal to zero. Hence

$$1 - 2Fo - 2BiFo \geq 0$$

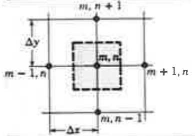
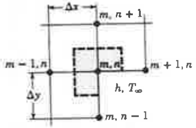
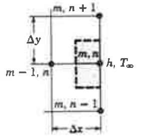
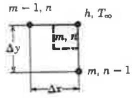
or

$$Fo(1 + Bi) \leq \tfrac{1}{2} \qquad (5.79)$$

Since the complete finite-difference solution requires the use of Equation 5.73 for the interior nodes, as well as Equation 5.77 for the surface node, Equation 5.79 must be contrasted with Equation 5.74 to determine which requirement is the more stringent. Since $Bi \geq 0$, it is apparent that the limiting value of Fo for Equation 5.79 is less than that for Equation 5.74. To ensure stability for all nodes, Equation 5.79 should therefore be used to select the maximum allowable value of Fo, and hence Δt, to be used in the calculations.

Forms of the explicit finite-difference equation for several common geometries are presented in Table 5.2. Each equation may be derived by applying the energy balance method to a control volume about the corresponding node. To develop confidence in your ability to apply this method, you should attempt to verify at least one of these equations.

Table 5.2 Summary of transient, two-dimensional finite-difference equations ($\Delta x = \Delta y$)

	EXPLICIT METHOD		
CONFIGURATION	FINITE-DIFFERENCE EQUATION	STABILITY CRITERION	IMPLICIT METHOD
	$T^{p+1}_{m,n} = Fo\big(T^p_{m+1,n} + T^p_{m-1,n} + T^p_{m,n+1} + T^p_{m,n-1}\big) + (1 - 4Fo)T^p_{m,n}$ (5.71) 1. Interior node	$Fo \leq \frac{1}{4}$ (5.75)	$(1 + 4Fo)T^{p+1}_{m,n} - Fo\big(T^{p+1}_{m+1,n} + T^{p+1}_{m-1,n} + T^{p+1}_{m,n+1} + T^{p+1}_{m,n-1}\big) = T^p_{m,n}$ (5.87)
	$T^{p+1}_{m,n} = \frac{2}{3}Fo\big(T^p_{m+1,n} + 2T^p_{m-1,n} + 2T^p_{m,n+1} + T^p_{m,n-1} + 2BiT_\infty\big) + \big(1 - 4Fo - \frac{4}{3}BiFo\big)T^p_{m,n}$ (5.80) 2. Node at interior corner with convection	$Fo(3 + Bi) \leq \frac{3}{4}$ (5.81)	$\big(1 + 4Fo\big(1 + \frac{1}{3}Bi\big)\big)T^{p+1}_{m,n} - \frac{2}{3}Fo \big(T^{p+1}_{m+1,n} + 2T^{p+1}_{m-1,n} + 2T^{p+1}_{m,n+1} + T^{p+1}_{m,n-1}\big) = T^p_{m,n} + \frac{4}{3}BiFoT_\infty$ (5.90)
	$T^{p+1}_{m,n} = Fo\big(2T^p_{m-1,n} + T^p_{m,n+1} + T^p_{m,n-1} + 2BiT_\infty\big) + (1 - 4Fo - 2BiFo)T^p_{m,n}$ (5.82) 3. Node at plane surface with convection[a]	$Fo(2 + Bi) \leq \frac{1}{2}$ (5.83)	$(1 + 2Fo(2 + Bi))T^{p+1}_{m,n} - Fo\big(2T^{p+1}_{m-1,n} + T^{p+1}_{m,n+1} + T^{p+1}_{m,n-1}\big) = T^p_{m,n} + 2BiFoT_\infty$ (5.91)
	$T^{p+1}_{m,n} = 2Fo\big(T^p_{m-1,n} + T^p_{m,n-1} + 2BiT_\infty\big) + (1 - 4Fo - 4BiFo)T^p_{m,n}$ (5.84) 4. Node at exterior corner with convection	$Fo(1 + Bi) \leq \frac{1}{4}$ (5.85)	$(1 + 4Fo(1 + Bi))T^{p+1}_{m,n} - 2Fo\big(T^{p+1}_{m-1,n} + T^{p+1}_{m,n-1}\big) = T^p_{m,n} + 4BiFoT_\infty$ (5.92)

[a] To obtain the finite-difference equation and/or stability criterion for an adiabatic surface (or surface of symmetry), simply set Bi equal to zero.

EXAMPLE 5.6

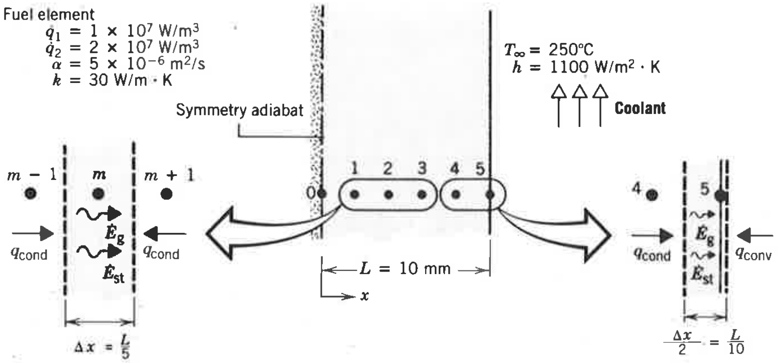
A fuel element of a nuclear reactor is in the shape of a plane wall of thickness $2L = 20$ mm and is convectively cooled at both surfaces, with $h = 1100$ $\mathrm{W/m^2 \cdot K}$ and $T_\infty = 250°\mathrm{C}$. At normal operating power, heat is generated uniformly within the element at a volumetric rate of $\dot{q}_1 = 10^7\ \mathrm{W/m^3}$. A departure from the steady-state conditions associated with normal operation will occur if there is a change in the generation rate. Consider a sudden change to $\dot{q}_2 = 2 \times 10^7\ \mathrm{W/m^3}$, and use the explicit finite-difference method to determine the fuel element temperature distribution after 1.5 s. The fuel element thermal properties are $k = 30\ \mathrm{W/m \cdot K}$ and $\alpha = 5 \times 10^{-6}\ \mathrm{m^2/s}$.

SOLUTION

Known: Conditions associated with heat generation in a rectangular fuel element with surface cooling.

Find: Temperature distribution 1.5 s after a change in operating power.

Schematic:



Assumptions:

1. One-dimensional conduction in x.
2. Uniform generation.
3. Constant properties.

Analysis: A numerical solution will be obtained using a space increment of $\Delta x = 2$ mm. Since there is symmetry about the midplane, the nodal network yields six unknown nodal temperatures. Using the energy balance method, Equation 5.76, an explicit finite-difference equation may be derived for any interior node m.

$$kA\frac{T_{m-1}^p - T_m^p}{\Delta x} + kA\frac{T_{m+1}^p - T_m^p}{\Delta x} + \dot{q}A\,\Delta x = \rho A\,\Delta x c\frac{T_m^{p+1} - T_m^p}{\Delta t}$$

Solving for T_m^{p+1} and rearranging,

$$T_m^{p+1} = Fo\left[T_{m-1}^p + T_{m+1}^p + \frac{\dot{q}(\Delta x)^2}{k}\right] + (1 - 2Fo)T_m^p \qquad (1)$$

This equation may be used for node 0, with $T_{m-1}^p = T_{m+1}^p$, as well as for nodes 1, 2, 3, and 4. Applying energy conservation to a control volume about node 5,

$$hA(T_\infty - T_5^p) + kA\frac{T_4^p - T_5^p}{\Delta x} + \dot{q}A\frac{\Delta x}{2} = \rho A\frac{\Delta x}{2}c\frac{T_5^{p+1} - T_5^p}{\Delta t}$$

or

$$T_5^{p+1} = 2Fo\left[T_4^p + BiT_\infty + \frac{\dot{q}(\Delta x)^2}{2k}\right] + (1 - 2Fo - 2BiFo)T_5^p \qquad (2)$$

Since the most restrictive stability criterion is associated with Equation 2, we select Fo from the requirement that

$$Fo(1 + Bi) \le \tfrac{1}{2}$$

Hence, with

$$Bi = \frac{h\,\Delta x}{k} = \frac{1100\ \text{W/m}^2\cdot\text{K}\ (0.002\ \text{m})}{30\ \text{W/m}\cdot\text{K}} = 0.0733$$

it follows that

$$Fo \le 0.466$$

or

$$\Delta t = \frac{Fo(\Delta x)^2}{\alpha} \le \frac{0.466(2\times10^{-3}\ \text{m})^2}{5\times10^{-6}\ \text{m}^2/\text{s}} \le 0.373\ \text{s}$$

To be well within the stability limit, we select $\Delta t = 0.3$ s, which corresponds to

$$Fo = \frac{5\times10^{-6}\ \text{m}^2/\text{s}(0.3\ \text{s})}{(2\times10^{-3}\ \text{m})^2} = 0.375$$

Substituting numerical values, including $\dot{q} = \dot{q}_2 = 2 \times 10^7\ \mathrm{W/m^3}$, the nodal equations become

$$T_0^{p+1} = 0.375(2T_1^p + 2.67) + 0.250T_0^p$$

$$T_1^{p+1} = 0.375(T_0^p + T_2^p + 2.67) + 0.250T_1^p$$

$$T_2^{p+1} = 0.375(T_1^p + T_3^p + 2.67) + 0.250T_2^p$$

$$T_3^{p+1} = 0.375(T_2^p + T_4^p + 2.67) + 0.250T_3^p$$

$$T_4^{p+1} = 0.375(T_3^p + T_5^p + 2.67) + 0.250T_4^p$$

$$T_5^{p+1} = 0.750(T_4^p + 19.67) + 0.195T_5^p$$

To begin the marching solution, the initial temperature distribution must be known. This distribution is given by Equation 3.42, with $\dot{q} = \dot{q}_1$. Obtaining $T_s = T_5$ from Equation 3.46,

$$T_5 = T_\infty + \frac{\dot{q}L}{h} = 250^\circ\mathrm{C} + \frac{10^7\ \mathrm{W/m^3} \times 0.01\ \mathrm{m}}{1100\ \mathrm{W/m^2 \cdot K}} = 340.91^\circ\mathrm{C}$$

it follows that

$$T(x) = 16.67\left(1 - \frac{x^2}{L^2}\right) + 340.91^\circ\mathrm{C}$$

Computed temperatures for the nodal points of interest are shown in the first row of the accompanying table.

Using the finite-difference equations, the nodal temperatures may be sequentially calculated with a time increment of 0.3 s until the desired final time is reached. The results are illustrated in rows 2 through 6 of the table and may be contrasted with the new steady-state condition (row 7), which was obtained by using Equations 3.42 and 3.46 with $\dot{q} = \dot{q}_2$,

Tabulated nodal temperatures

p	t (s)	T_0	T_1	T_2	T_3	T_4	T_5
0	0	357.58	356.91	354.91	351.58	346.91	340.91
1	0.3	358.08	357.41	355.41	352.08	347.41	341.41
2	0.6	358.58	357.91	355.91	352.58	347.91	341.88
3	0.9	359.08	358.41	356.41	353.08	348.41	342.35
4	1.2	359.58	358.91	356.91	353.58	348.89	342.82
5	1.5	360.08	359.41	357.41	354.07	349.37	343.27
∞	∞	465.15	463.82	459.82	453.15	443.82	431.82

Comments: It is evident that at 1.5 s, the wall is in the early stages of the transient process and that many additional calculations would have to be made to reach steady-state conditions with the finite-difference solution. The computation time could be slightly reduced by using the maximum allowable time increment ($\Delta t = 0.373$ s), but with some loss of accuracy. In the interest of maximizing accuracy, the time interval should be reduced until the computed results become independent of further reductions in Δt.

5.9.2 Discretization of the Heat Equation: The Implicit Method

In the *explicit* finite-difference scheme, the temperature of any node at $t + \Delta t$ may be calculated from knowledge of temperatures at the same and neighboring nodes for the *preceding time t*. Hence, determination of a nodal temperature at some time is *independent* of temperatures at other nodes for the *same time*. Although the method offers computational convenience, it suffers from limitations on the selection of Δt. For a given space increment, the time interval must be compatible with stability requirements. Frequently, this dictates the use of extremely small values of Δt, and a very large number of time intervals may be necessary to obtain a solution.

A reduction in the amount of computation time may often be realized by employing an *implicit*, rather than explicit, finite-difference scheme. The implicit form of a finite-difference equation may be derived by using Equation 5.69 to approximate the time derivative, while evaluating all other temperatures at the *new* $(p + 1)$ time, instead of the previous (p) time. Equation 5.69 is then considered to provide a *backward-difference* approximation to the time derivative. In contrast to Equation 5.70, the implicit form of the finite-difference equation for the interior node of a two-dimensional system is then

$$\frac{1}{\alpha}\frac{T_{m,n}^{p+1} - T_{m,n}^{p}}{\Delta t} = \frac{T_{m+1,n}^{p+1} + T_{m-1,n}^{p+1} - 2T_{m,n}^{p+1}}{(\Delta x)^2} + \frac{T_{m,n+1}^{p+1} + T_{m,n-1}^{p+1} - 2T_{m,n}^{p+1}}{(\Delta y)^2} \tag{5.86}$$

Rearranging and assuming $\Delta x = \Delta y$, it follows that

$$(1 + 4Fo)T_{m,n}^{p+1} - Fo\left(T_{m+1,n}^{p+1} + T_{m-1,n}^{p+1} + T_{m,n+1}^{p+1} + T_{m,n-1}^{p+1}\right) = T_{m,n}^{p} \tag{5.87}$$

From Equation 5.87 it is evident that the *new* temperature of the m, n node depends on the *new* temperatures of its adjoining nodes, which are, in general, unknown. Hence, to determine the unknown nodal temperatures at $t + \Delta t$, the corresponding nodal equations must be *solved simultaneously*. Such a solution may be effected by using Gauss–Seidel iteration or matrix inversion, as discussed in Section 4.5. The *marching solution* would then involve simultaneously solving the nodal equations at each time $t = \Delta t, 2\Delta t, \ldots,$ until the desired final time was reached.

Although computations involving the implicit method are more complicated than those of the explicit method, the implicit formulation has the important advantage of being *unconditionally stable*. That is, the solution remains stable for all space and time intervals, in which case there are no restrictions on Δx and Δt. Since larger values of Δt may therefore be used with an implicit method, computation times may often be reduced, with little loss of accuracy. Nevertheless, to maximize accuracy, Δt should be sufficiently small to ensure that the results are independent of further reductions in its value.

The implicit form of a finite-difference equation may also be derived from the energy balance method. For the surface node of Figure 5.22, it is readily shown that

$$(1 + 2Fo + 2FoBi)T_0^{p+1} - 2FoT_1^{p+1} = 2FoBiT_\infty + T_0^p \tag{5.88}$$

For any interior node of Figure 5.22, it may also be shown that

$$(1 + 2Fo)T_m^{p+1} - Fo\left(T_{m-1}^{p+1} + T_{m+1}^{p+1}\right) = T_m^p \tag{5.89}$$

Forms of the implicit finite-difference equation for other common geometries are presented in Table 5.2. Each equation may be derived by applying the energy balance method.

EXAMPLE 5.7

A thick slab of copper initially at a uniform temperature of 20°C is suddenly exposed to radiation at one surface such that the net heat flux is maintained at a constant value of 3×10^5 W/m². Using the explicit and implicit finite-difference techniques with a space increment of $\Delta x = 75$ mm, determine the temperature at the irradiated surface and at an interior point that is 150 mm from the surface after 2 min have elapsed. Compare the results with those obtained from an appropriate analytical solution.

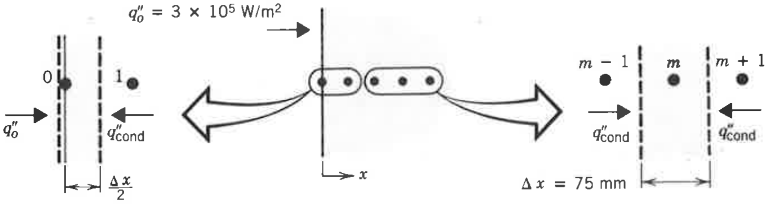
SOLUTION

Known: Thick slab of copper, initially at a uniform temperature, is subjected to a constant net heat flux at one surface.

Find:

1. Using the explicit finite-difference method, determine temperatures at the surface and 150 mm from the surface after an elapsed time of 2 min.
2. Repeat the calculations using the implicit finite-difference method.
3. Determine the same temperatures analytically.

Schematic:



Assumptions:

1. One-dimensional conduction in x.
2. Thick slab may be approximated as a semi-infinite medium with constant surface heat flux.
3. Constant properties.

Properties: Table A.1, copper (300 K): $k = 401$ W/m · K, $\alpha = 117 \times 10^{-6}$ m²/s.

Analysis:

1. An explicit form of the finite-difference equation for the surface node may be obtained by applying an energy balance to a control volume about the node.

$$q''_o A + kA\frac{T_1^p - T_0^p}{\Delta x} = \rho A \frac{\Delta x}{2} c \frac{T_0^{p+1} - T_0^p}{\Delta t}$$

or

$$T_0^{p+1} = 2Fo\left(\frac{q''_o \Delta x}{k} + T_1^p\right) + (1 - 2Fo)T_0^p$$

The finite-difference equation for any interior node is given by Equation 5.73. Both the surface and interior nodes are governed by the

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stability criterion

$$Fo \le \tfrac{1}{2}$$

Noting that the finite-difference equations are simplified by choosing the maximum allowable value of Fo, we select $Fo = \frac{1}{2}$. Hence

$$\Delta t = Fo \frac{(\Delta x)^2}{\alpha} = \frac{1}{2} \frac{(0.075\ \text{m})^2}{117 \times 10^{-6}\ \text{m}^2/\text{s}} = 24\ \text{s}$$

With

$$\frac{q''_o\, \Delta x}{k} = \frac{3 \times 10^5\ \text{W/m}^2\ (0.075\ \text{m})}{401\ \text{W/m} \cdot \text{K}} = 56.1°\text{C}$$

the finite-difference equations become

$$T_0^{p+1} = 56.1°\text{C} + T_1^p \qquad \text{and} \qquad T_m^{p+1} = \frac{T_{m+1}^p + T_{m-1}^p}{2}$$

for the surface and interior nodes, respectively. Performing the calculations, the results are tabulated as follows.

Explicit finite-difference solution for $Fo = \frac{1}{2}$

p	t (s)	T_0	T_1	T_2	T_3	T_4
0	0	20	20	20	20	20
1	24	76.1	20	20	20	20
2	48	76.1	48.1	20	20	20
3	72	104.2	48.1	34.1	20	20
4	96	104.2	69.1	34.1	27.1	20
5	120	125.3	69.1	48.1	27.1	20

After 2 min, the surface temperature and the desired interior temperature are $T_0 = 125.3°\text{C}$ and $T_2 = 48.1°\text{C}$.

Note that calculation of identical temperatures at successive times for the same node is an idiosyncrasy of using the maximum allowable value of Fo with the explicit finite-difference technique. The actual physical condition is, of course, one in which the temperature changes continuously with time. The idiosyncrasy is eliminated and the accuracy of the calculations is improved by reducing the value of Fo.

To determine the extent to which the accuracy may be improved by reducing Fo, let us redo the calculations for $Fo = \frac{1}{4}(\Delta t = 12\ \text{s})$. The finite-difference equations are then of the form

$$T_0^{p+1} = \tfrac{1}{2}(56.1°\mathrm{C} + T_1^p) + \tfrac{1}{2}T_0^p$$

$$T_m^{p+1} = \tfrac{1}{4}(T_{m+1}^p + T_{m-1}^p) + \tfrac{1}{2}T_m^p$$

and the results of the calculations are tabulated as follows.

Explicit finite-difference solution for *Fo* = 1/4

p	t (s)	T_0	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8
0	0	20	20	20	20	20	20	20	20	20
1	12	48.1	20	20	20	20	20	20	20	20
2	24	62.1	27.0	20	20	20	20	20	20	20
3	36	72.6	34.0	21.8	20	20	20	20	20	20
4	48	81.4	40.6	24.4	20.4	20	20	20	20	20
5	60	89.0	46.7	27.5	21.3	20.1	20	20	20	20
6	72	95.9	52.5	30.7	22.6	20.4	20.0	20	20	20
7	84	102.3	57.9	34.1	24.1	20.8	20.1	20.0	20	20
8	96	108.1	63.1	37.6	25.8	21.5	20.3	20.0	20.0	20
9	108	113.7	68.0	41.0	27.6	22.2	20.5	20.1	20.0	20.0
10	120	118.9	72.6	44.4	29.6	23.2	20.8	20.2	20.0	20.0

After 2 min, the desired temperatures are $T_0 = 118.9°\mathrm{C}$ and $T_2 = 44.4°\mathrm{C}$. Comparing the above results with those obtained for $Fo = \frac{1}{2}$, it is clear that by reducing *Fo* we have eliminated the problem of recurring temperatures. We have also predicted greater thermal penetration (to node 6 instead of node 3). An assessment of the improvement in accuracy must await a comparison with results based on an exact solution.

2. Performing an energy balance on a control volume about the surface node, the implicit form of the finite-difference equation is

$$q_o'' + k\frac{T_1^{p+1} - T_0^{p+1}}{\Delta x} = \rho\frac{\Delta x}{2}c\frac{T_0^{p+1} - T_0^p}{\Delta t}$$

or,

$$(1 + 2Fo)T_0^{p+1} - 2Fo\,T_1^{p+1} = \frac{2\alpha q_o''\,\Delta t}{k\,\Delta x} + T_0^p$$

Arbitrarily choosing $Fo = \frac{1}{2}(\Delta t = 24\text{ s})$, it follows that

$$2T_0^{p+1} - T_1^{p+1} = 56.1 + T_0^p$$

From Equation 5.89, the finite-difference equation for any interior node is then of the form

$$-T_{m-1}^{p+1} + 4T_m^{p+1} - T_{m+1}^{p+1} = 2T_m^p$$

Since we are dealing with a semi-infinite solid, the number of nodes is, in principle, infinite. In practice, however, the number may be limited to the nodes that are affected by the change in the boundary condition for the time period of interest. From the results of the explicit method, it is evident that we are safe in choosing nine nodes corresponding to $T_0, T_1, \ldots, T_8$. We are thereby assuming that, at $t = 120$ s, there has been no change in T_8.

We now have a set of nine equations that must be solved simultaneously for each time increment. Using the matrix inversion method, we express the equations in the form $[A][T] = [C]$, where

$$[A] = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 4 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 4 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 4 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 4 \end{bmatrix}$$

$$[C] = \begin{bmatrix} 56.1 + T_0^p \\ 2T_1^p \\ 2T_2^p \\ 2T_3^p \\ 2T_4^p \\ 2T_5^p \\ 2T_6^p \\ 2T_7^p \\ 2T_8^p + T_9^{p+1} \end{bmatrix}$$

Note that numerical values for the components of $[C]$ are determined from previous values of the nodal temperatures. Note also how the finite-difference equation for node 8 appears in matrices $[A]$ and $[C]$.

A table of nodal temperatures may be compiled, beginning with the first row ($p = 0$) corresponding to the prescribed initial condition. To obtain nodal temperatures for subsequent times, the inverse of the coefficient matrix $[A]^{-1}$ must first be found. At each time $p + 1$, it is then multiplied by the column vector $[C]$, which is evaluated at p, to obtain the temperatures $T_0^{p+1}, T_1^{p+1}, \dots, T_8^{p+1}$. For example, multiplying $[A]^{-1}$ by the column vector corresponding to $p = 0$,

$$[C]_{p=0} = \begin{bmatrix} 76.1 \\ 40 \\ 40 \\ 40 \\ 40 \\ 40 \\ 40 \\ 40 \\ 60 \end{bmatrix}$$

the second row of the table is obtained. Updating $[C]$, the process is repeated four more times to determine the nodal temperatures at 120 s. The desired temperatures are $T_0 = 114.7°\text{C}$ and $T_2 = 44.2°\text{C}$.

Implicit finite-difference solution for $Fo = \frac{1}{2}$

p	t (s)	T_0	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8
0	0	20.0	20.0	20.0	20.0	20.0	20.0	20.0	20.0	20.0
1	24	52.4	28.7	22.3	20.6	20.2	20.0	20.0	20.0	20.0
2	48	74.0	39.5	26.6	22.1	20.7	20.2	20.1	20.0	20.0
3	72	90.2	50.3	32.0	24.4	21.6	20.6	20.2	20.1	20.0
4	96	103.4	60.5	38.0	27.4	22.9	21.1	20.4	20.2	20.1
5	120	114.7	70.0	44.2	30.9	24.7	21.9	20.8	20.3	20.1

3. Approximating the slab as a semi-infinite medium, the appropriate analytical expression is given by Equation 5.58, which may be applied to any point in the slab.

$$T(x,t) - T_i = \frac{2q_o''(\alpha t/\pi)^{1/2}}{k} \exp\left(-\frac{x^2}{4\alpha t}\right) - \frac{q_o'' x}{k} \operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right)$$

At the surface, this expression yields

$$T(0, 120\ \text{s}) - 20°\text{C} = \frac{2 \times 3 \times 10^5\ \text{W/m}^2}{401\ \text{W/m}\cdot\text{K}} \left(117 \times 10^{-6}\ \text{m}^2/\text{s} \times 120\ \text{s}/\pi\right)^{1/2}$$

or

$$T(0, 120\ \text{s}) = 120.0°\text{C}$$

◁

At the interior point ($x = 0.15$ m)

$$T(0.15\text{ m}, 120\text{ s}) - 20°\text{C} = \frac{2 \times 3 \times 10^5\text{ W/m}^2}{401\text{ W/m} \cdot \text{K}}$$

$$\times \left(117 \times 10^{-6}\text{ m}^2\text{/s} \times 120\text{ s}/\pi\right)^{1/2}$$

$$\times \exp\left[-\frac{(0.15\text{ m})^2}{4 \times 117 \times 10^{-6}\text{ m}^2\text{/s} \times 120\text{ s}}\right] - \frac{3 \times 10^5\text{ W/m}^2 \times 0.15\text{ m}}{401\text{ W/m} \cdot \text{K}}$$

$$\times \left[1 - \text{erf}\left(\frac{0.15\text{ m}}{2\sqrt{117 \times 10^{-6}\text{ m}^2\text{/s} \times 120\text{ s}}}\right)\right] = 45.4°\text{C} \qquad \triangleleft$$

Comments:

1. Comparing the exact results with those obtained from the three approximate solutions, it is clear that the explicit method with $Fo = 1/4$ provides the most accurate predictions.

METHOD	$T_0 = T(0, 120\text{ s})$	$T_2 = T(0.15\text{ m}, 120\text{ s})$
Explicit ($Fo = \frac{1}{2}$)	125.3	48.1
Explicit ($Fo = \frac{1}{4}$)	118.9	44.4
Implicit ($Fo = \frac{1}{2}$)	114.7	44.2
Exact	120.0	45.4

   This is not unexpected, since the corresponding value of Δt is 50% smaller than that used in the other two methods.

2. Although computations are simplified by using the maximum allowable value of *Fo* in the explicit method, the accuracy of the results is seldom satisfactory.
3. Note that the coefficient matrix $[A]$ is *tridiagonal.* That is, all elements are zero except those which are on, or to either side of, the main diagonal. Tridiagonal matrices are associated with one-dimensional conduction problems. In such cases the problem of solving for the unknown temperatures is greatly simplified, and stock computer programs may readily be obtained for this purpose.
4. A more general radiative heating condition would be one in which the surface is suddenly exposed to large surroundings at an elevated temperature T_{sur} (Problem 5.84). The net rate at which radiation is transferred to the surface may then be calculated from Equation 1.7. Allowing for convection heat transfer to the surface, application of conservation of energy to the surface node yields an explicit finite-

difference equation of the form

$$\varepsilon\sigma\left[T_{\text{sur}}^4 - \left(T_0^p\right)^4\right] + h\left(T_\infty - T_0^p\right) + k\frac{T_1^p - T_0^p}{\Delta x} = \rho\frac{\Delta x}{2}c\frac{T_0^{p+1} - T_0^p}{\Delta t}$$

Use of this finite-difference equation in a numerical solution is complicated by the fact that it is *nonlinear*. However, the equation may be *linearized* by introducing the radiation heat transfer coefficient h_r defined by Equation 1.9, and the finite-difference equation is

$$h_r^p\left(T_{\text{sur}} - T_0^p\right) + h\left(T_\infty - T_0^p\right) + k\frac{T_1^p - T_0^p}{\Delta x} = \rho\frac{\Delta x}{2}c\frac{T_0^{p+1} - T_0^p}{\Delta t}$$

The solution may proceed in the usual manner, although the effect of a radiative Biot number ($Bi_r \equiv h_r\,\Delta x/k$) must be included in the stability criterion and the value of h_r must be updated at each step in the calculations. If the implicit method is used, h_r is calculated at $p + 1$, in which case an iterative calculation must be made at each time step.

5.10 SUMMARY

Transient conduction occurs in numerous engineering applications, and it is important to appreciate the different methods for dealing with it. There is certainly much to be said for simplicity, in which case, when confronted with a transient problem, the first thing you should do is calculate the Biot number. If this number is much less than unity, you may use the lumped capacitance method to obtain accurate results with minimal computational requirements. However, if the Biot number is not much less than unity, spatial effects must be considered, and some other method must be used. Analytical results are available in convenient graphical and equation form for the plane wall, the infinite cylinder, the sphere, and the semi-infinite solid. You should know when and how to use these results. If geometrical complexities and/or the form of the boundary conditions preclude their use, recourse must be made to finite-difference methods. With the digital computer, such methods may be used to solve any conduction problem, regardless of complexity.

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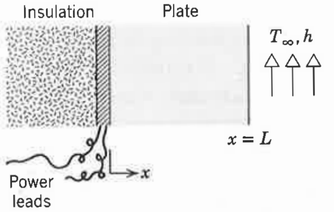
PROBLEMS

Qualitative Considerations

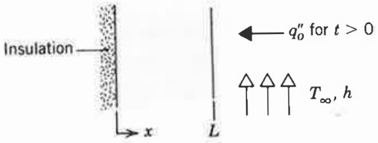
5.1 Consider a thin electrical heater attached to a plate and backed by insulation. Initially, the heater and plate are at the temperature of the ambient air, T_∞. Suddenly, the power to the heater is switched on giving rise to a constant heat flux q_o'' (W/m^2) at the inner surface of the plate.



(a) Sketch and label, on T–x coordinates, the temperature distributions: initial, steady-state, and at two intermediate times.

(b) Sketch the heat flux at the outer surface $q_x''(L, t)$ as a function of time.

5.2 The inner surface of a plane wall is insulated while the outer surface is exposed to an airstream at T_∞. The wall is at a uniform temperature corresponding to that of the airstream. Suddenly, a radiation heat source is switched on applying a uniform flux q_o'' to the outer surface.



(a) Sketch and label, on T–x coordinates, the temperature distributions: initial, steady-state, and at two intermediate times.

(b) Sketch the heat flux at the outer surface $q_x''(L, t)$ as a function of time.