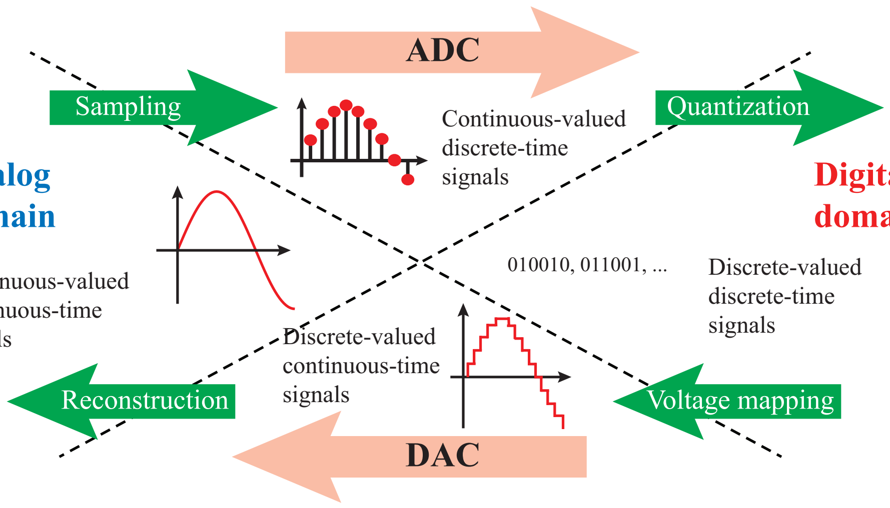

Figure 2-15: Data conversion between the analog and digital domains. Analog signals are converted into digital by sampling and quantization. Digital signals are converted into analog by mapping numbers into voltage levels, followed by reconstruction filtering.

In other words,

$$A\cos(2\pi f_0 t) \longleftrightarrow \frac{A}{2}[\delta(f-f_0)+\delta(f+f_0)].$$

The PSD is

$$S_X(f) = \frac{A^2}{4}\left[\delta(f-f_0)+\delta(f+f_0)\right],$$

and the average power is

$$P_x = \int_{-\infty}^{\infty} S_X(f)\,df = \frac{A^2}{2}\,.$$

The autocorrelation function is the inverse Fourier transform of the PSD, which is straightforward to determine:

$$R_x(\tau) = \frac{A^2}{2}\cos(2\pi f_c \tau).$$

2-7 Fundamentals of Analog-to-Digital Conversion and Vice-Versa

This section covers the conversion of analog to digital signals, and vice-versa. There are two parts of this conversion:

(1) ***sampling*** a continous-time signal to a discrete-time signal, which takes on continuous values at times that are rational numbers, and

(2) ***quantization*** of the continuous values that the signal takes on into values that are rational numbers. The quantized and sampled signal is called a ***digital signal***. This is illustrated in **Fig. 2-15**.

Some sources produce analog signals, such as speech and audio. Conversion between analog and digital representations utilize the fundamental operations illustrated in **Fig. 2-15**. An ***analog-to-digital converters*** (*ADC*) converts analog signals, which are continuous in both time and amplitude, to digital signals, which are discrete in both time and amplitude. A ***digital-to-analog converter*** (***DAC***) converts a digital representation into an analog signal and can be thought of as the inverse of the ADC. The analog signal can then be used to drive a speaker or other analog output device.

An ADC typically requires three subsystems—an ***anti-aliasing filter*** (***AAF***), a sampler, and a quantizer (**Fig. 2-16**). Each one of these three subsystems is examined next.

Figure 2-16: High-level block diagram of an ADC.

2-7.1 Sampling

Sampling a signal $x(t)$ at a ***sampling period*** T_s is described by

$$x[nT_s] = x(t)\big|_{t=nT_s}. \qquad (2.118)$$

Sampling can also be represented as an inner product with a Dirac delta function at the sampling instant $t = nT_s$:

$$\begin{aligned} x[nT_s] &= \langle x(t), \delta(t-nT_s)\rangle \\ &= \int_{-\infty}^{\infty} x(t)\, \delta(t-nT_s)\, dt. \end{aligned} \qquad (2.119)$$

The main parameter of a sampler is the ***sampling frequency*** $f_s = 1/T_s$. The time between two samples is the sampling period T_s. Samplers take regularly separated analog measurements of the signal's amplitude. Most samplers operate on voltage, but some can operate on current. While conceptually, sampling is performed instantaneously, in practice it can take a small but nonzero amount of time to "sample." Most samplers return the average amplitude during the time they actively "sample."

Sampling is typically performed by electronic circuits called ***sample-and-hold*** (***S/H***) circuits. The

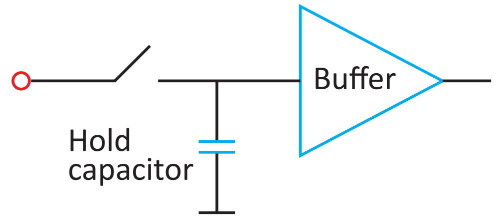

Figure 2-17: Block diagram of a sample-and-hold circuit.

purpose of the circuit is to keep the input signal constant during the ADC conversion. Their input is the analog signal to be sampled and the output is a piecewise constant signal. While there are many different implementations, all S/H circuits include a switching circuit, a holding capacitor, and an output buffer which is an amplifier with a gain equal to one.

A block diagram of a sample-and-hold circuit is shown in **Fig. 2-17**. There could also be an input buffer (not shown) to supply the necessary current to charge the hold capacitor. When the switch closes, the circuit samples the input signal. Ideally the time to sample is infinitely small. When the switch opens, the S/H circuit operates in hold mode. The output buffer has a very high input impedance and the capacitor does not discharge; i.e., the capacitor holds the voltage. The circuit remains in hold mode for the duration of the sampling period T_s. Therefore, the impulse response of an ideal S/H circuit is

$$h(t) = u(t) - u(t-T_s) = \Pi\left(\frac{t}{T_s} - \frac{1}{2}\right), \qquad (2.120)$$

and the frequency response is

$$H(f) = T_s \frac{\sin \pi f T_s}{\pi f T_s}\, e^{-j\pi f T_s}. \qquad (2.121)$$

Suppose that the input to the sampler is the analog signal $x(t)$ with a corresponding Fourier transform $X(f)$, and the output of the sampler is the sampled signal $x_s(t)$. The sampled signal can be viewed in the time domain as a product between $x(t)$ and a sequence of Dirac delta functions:

$$x_s(t) = x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_s). \tag{2.122}$$

The function $\sum_{n=-\infty}^{n=\infty} \delta(t - nT)$ is called an ***impulse train*** or a ***sampling function***.

Because the product $x(t)\,\delta(t - nT_s)$ is zero everywhere except at the sampling instances nT_s, $x(t)$ can be replaced with the discrete time signal $x[nT_s]$ without changing the result:

$$x_s(t) = \sum_{n=-\infty}^{\infty} x[nT_s]\,\delta(t - nT_s). \tag{2.123}$$

Example 2-11: Fourier transform of a pulse train

Determine the Fourier transform of the impulse train $\sum_{n=-\infty}^{n=\infty} \delta(t - nT)$. This will be used in the derivation of the sampling theorem below.

Solution: From Appendix A, the Fourier transform relationship of the Dirac delta function is

$$\delta(t - nT) \longleftrightarrow e^{-j2\pi f nT},$$

since

$$\int_{-\infty}^{\infty} \delta(t - nT)\, e^{-j2\pi ft}\, dt = e^{-j2\pi ft}\big|_{t=nT} = e^{-j2\pi f nT}. \tag{2.124}$$

Because the Fourier transform is linear,

$$\sum_{n=-\infty}^{\infty} \delta(t - nT) \longleftrightarrow \sum_{n=-\infty}^{\infty} e^{-j2\pi f nT}. \tag{2.125}$$

The above result leads to another useful relationship. The signal $\sum_{n=-\infty}^{\infty} \delta(t - nT)$ is periodic and can be represented using a Fourier series with $e^{-j2\pi nt/T}$ as basis functions:

$$\sum_{n=-\infty}^{\infty} \delta(t - nT) = \sum_{n=-\infty}^{\infty} x_n e^{-j2\pi nt/T}, \tag{2.126}$$

where the coefficients x_n are equal to the inner products

$$\begin{aligned} x_n &= \left\langle \sum_{n=-\infty}^{\infty} \delta(t - nT),\, e^{-j2\pi nt/T} \right\rangle \\ &= \frac{1}{T} \int_T \delta(t)\, e^{-j2\pi nt/T}\, dt = \frac{1}{T}. \end{aligned} \tag{2.127}$$

Therefore, the Fourier series of the impulse train is

$$\sum_{n=-\infty}^{\infty} \delta(t - nT) = \frac{1}{T} \sum_{n=-\infty}^{\infty} e^{-j2\pi nt/T}. \tag{2.128}$$

From the property given by Eq. (2.43) with $f_0 = n/T$, it follows that $e^{j2\pi tn/T} \longleftrightarrow \delta(f - n/T)$. In conclusion, there are two Fourier transform representations:

$$\sum_{n=-\infty}^{\infty} \delta(t - nT) \longleftrightarrow \begin{cases} \displaystyle\sum_{n=-\infty}^{\infty} e^{-j2\pi f nT}, \\ \displaystyle\frac{1}{T} \sum_{n} \delta(f - n/T). \end{cases} \tag{2.129}$$

Therefore the Fourier transform of an impulse train with period T is another impulse train with period T^{-1} Hz.

Using the Fourier transform of a product is the convolution of the Fourier transforms (see Appendix A). Because in the time domain the sampled signal given by Eq. (2.122) is a product, its spectrum is given by the convolution

$$\begin{aligned} X_s(f) &= X(f) * \frac{1}{T_s} \sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n}{T_s}\right) \\ &= \frac{1}{T_s} \sum_{n=-\infty}^{\infty} X\left(f - \frac{n}{T_s}\right) \\ &= \frac{1}{T_s} [\cdots + X(f + f_s) + X(f) + X(f - f_s) + \cdots] \\ &= f_s\, X(f) + f_s \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} X(f - nf_s). \end{aligned} \tag{2.130}$$

From this equation we can conclude that the spectrum of the sampled signal is periodic with a period in frequency equal to the sampling frequency $f_s = 1/T_s$ (**Fig. 2-18**). In the frequency domain, there are infinitely many replicas of the spectrum of the continuous-time signal. These replicas are frequency-translated versions of the spectrum of the original signal. When these replicas overlap, the effect is known as ***aliasing***. Sampling maps analog frequencies in the range $[0, \infty)$ to discrete-time frequencies in the range $[0, f_s/2)$. The frequency $f_s/2$ is called the ***Nyquist frequency***.

Let us consider in more detail the sampling of a real-valued signal. Recall that a real-valued signal is absolutely bandlimited if there exists an f_{max} such that $X(f) = 0$ for frequencies $|f| > |f_{max}|$. **Figure 2-18** illustrates the spectrum of a real-valued signal, where the signal is absolutely bandlimited to f_{max}. It also shows the spectrum of the signal sampled at $f_{max} > f_s/2$ (so that there is aliasing), but the sampling frequency is greater than the bandwidth of the signal:

$$\frac{f_s}{2} < f_{max} < f_s. \tag{2.131}$$

If there is aliasing, some higher-frequency components of the analog signal appear "on top" of some of the lower-frequency components. Only one component is formed as a result; i.e., the higher-frequency components become indistinguishable from the lower-frequency components and cannot be separated by filtering later. In the aliasing example in **Fig. 2-18(b)**, the components from $f_s/2$ to f_{max} overlap with the components from $f_s/2$ down to $f_s - f_{max}$.

The presence of aliasing over part of the spectrum does not mean that the entire signal is useless. If the signal of interest is in the bandwidth $[0,\ f_s - f_{max}]$, aliasing can be acceptable since the aliased part of the spectrum is not of interest. In **Fig. 2-18(b)**, over the band $[0,\ f_s - f_{max}]$ there is no aliasing and this portion can be used and processed further. For example, the portion with aliasing can be filtered out with a low-pass digital filter so as to preserve the band $[0,\ f_s - f_{max}]$. However, if $f_s \le f_{max}$, aliasing will extend over the entire bandwidth of the signal, rendering the sampled signal useless.

The replicas of the spectrum just touch each other if $f_s = 2f_{max}$, which still results in aliasing. The replicas of the spectrum do not overlap as long as $f_s > 2f_{max}$, as illustrated in **Fig. 2-18(c)**. The minimum sampling frequency to avoid aliasing completely is

$$f_s = 2f_{max} \tag{2.132}$$

and is known as the ***Nyquist sampling frequency***.

It is important to recognize that if the signal is not bandlimited, sampling always introduces aliasing. For some signals the aliasing error is always significant. For example, the Dirac delta function cannot be sampled; i.e., the discrete-time delta function (the Kronecker delta) cannot be obtained by sampling. Another function that cannot be sampled is the step function. However, for signals that are effectively bandlimited most of the energy is in a finite band and the aliasing error can be made small by choosing the sampling frequency to be sufficiently large.

One objective in the design of an ADC is to minimize the effect of aliasing. If the signal to be sampled is not bandlimited or its spectrum is unknown, aliasing is generally minimized by inserting an ***anti-aliasing filter*** (***AAF***), in front of the ADC, as was illustrated earlier in **Fig. 2-16**. The AAF is designed to remove all frequency components above $f_s/2$; i.e., the AAF is ideally a lowpass filter with a passband from 0 to $f_s/2$ Hz.

2-7.2 Sampling a complex-valued signal

In the above discussion on sampling, we assumed that the signal being sampled is a real-valued signal, having a one-sided bandwidth $W = f_{max}$. Recall from Section 2-8 that the bandwidth of complex-valued signals is double-sided. A complex-valued signal is bandlimited if there exists a band $[f_1, f_2]$ such that $X(f) = 0$ outside $[f_1, f_2]$. In particular, consider a complex-valued signal with double-sided bandwidth $W = f_2 - f_1$ (and where f_1 is negative), as was illustrated earlier in **Fig. 2-11**. Then, to completely avoid aliasing, the sampling frequency should be

$$f_s \ge W = f_2 - f_1 \tag{2.133}$$

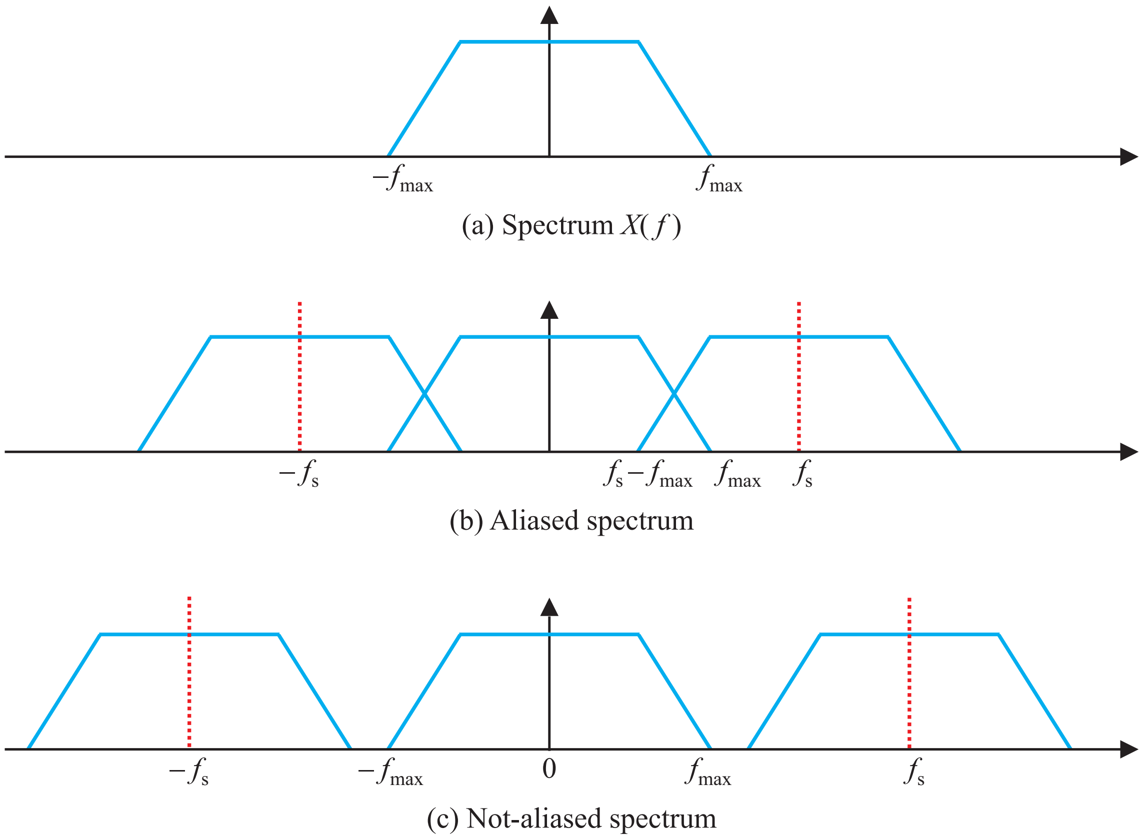

Figure 2-18: (a) Spectrum of a continuous-time real-valued signal, (b) spectrum of the discrete-time signal obtained after sampling the signal in (a) at $f_{\max} < f_s < 2f_{\max}$, and (c) spectrum of the discrete-time signal obtained after sampling the signal in (a) at $f_s \geq 2f_{\max}$.

or equivalently,

$$f_s + f_1 \geq f_2. \tag{2.134}$$

Figure 2-19 illustrates the spectrum of a sampled complex-valued signal, where the sampling frequency is chosen according to Eq. (2.134), so that aliasing is avoided.

Aliasing is only partial; i.e., there is a frequency band that is still useful, as long as

$$-f_s + f_2 < f_s + f_1. \tag{2.135}$$

In other words, so long as the sampling frequency is greater than one-half of the bandwidth of the signal,

$$f_s > (f_2 - f_1)/2 = W/2, \tag{2.136}$$

then there is no aliasing over the frequency band from $-f_s + f_2$ to $f_s + f_1$; aliasing is limited to the band from $f_s + f_1$ to f_2 (**Fig. 2-20**).

If there is aliasing, some components at negative frequencies appear "on top" of and indistinguishable from components at positive frequencies. The portion with aliasing can be filtered out with a low-pass digital filter that will preserve the band $[-f_s + f_2, f_s + f_1]$. However, if $f_s \leq W/2$ aliasing will extend over the entire bandwidth of the signal.

Sampling complex-valued signals is referred to as ***quadrature sampling***. Each of the real and imaginary components occupies only one-half of the bandwidth

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