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Dependence of inertial measurements of distance on accelerometer noise

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Abstract
An investigation is made into the errors in estimated position that are caused by noise and drift effects in stationary accelerometers. An analytical study is made into the effects of biases in the accelerometer data and the effects of changing the cut-off frequency in the anti-aliasing filter. The root mean square errors in position are calculated as a function of time and sampling frequency. A comparison is made between the theoretical results and experimental data taken from two commercial accelerometers. Recommendations are made regarding the calibration of accelerometers prior to their use in practical situations.

Keywords: accelerometers, noise, micro-electro-mechanical systems, inertial navigation systems

1. Introduction

Accelerometers are widely used in many applications to determine position. These devices can be used either on their own or in combination with other navigation equipment, for example gyroscopes [1, 2] or velocity meters [3]. Application areas are numerous varying from measurement of forces on a car that is turning or accelerating [4] to the 'smart pen' which can store what it writes for the future [5]. Another application is the investigation of structures under impact load [6]. The authors have been looking, in particular, at the application of an accelerometer-only inertial navigation system (INS) to various desktop applications, for example its use as a computer mouse.

The basic principle of the accelerometer as an inertial sensor is very straightforward: the accelerometer measures acceleration and displacement is determined by double integrating the data. The integration could be carried out using analogue methods [7, 8] or it could be performed numerically after the data have been digitized [9].

However, there is the problem of measurement noise and drift [10–12]. It is shown in [1] that the standard deviation of the measured position due to acceleration noise, in the absence of drift and initialization errors, increases as $t^{1.5}$ where t is the integration time. This result is derived by using the continuous Kalman filter. In [13], it is suggested that the standard deviation of the error in position increases as t. In [14], it is assumed that if ε represents the accelerometer error, then the measured position would have an error that is proportional to εt^2. This last assumption would only be true if the error concerned were a bias rather than white noise. What this prior work demonstrates is a lack of consensus regarding how noise affects the rms errors in the estimated displacement.

The accelerometer data has already been filtered by an in-built anti-aliasing filter. It is found that further filtering reduces the absolute value of the error in position, but there is still a tendency for the variation in the positional error to increase with time. Another problem with this additional filtering of the input to the accelerometer is that one would be reducing the bandwidth of measurable accelerations. As examples, acceleration data taken from two commercially available accelerometers are shown in figures 1(a) and (b). The accelerometers are both at rest on an optical bench. The sampling frequency is 3 kHz. For each accelerometer, the data have been filtered using a moving average of 1000 samples so that the effects of drift can be brought out. It can be seen that there is noise and drift in the data, for both accelerometers, which will contribute to errors in the estimated position.

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For the ADXL250 [15] accelerometer, figure 1(a), the main contribution to the output from the accelerometer is noise with a relatively small amount of drift. For the Crossbow CXL01F3 accelerometer [16], figure 1(b), there is less noise than for the ADXL but the contribution from drift effects is more significant. The question to be asked is how the aforementioned increase in positional error depends on the parameters of the accelerometer, the sampling frequency, the filter parameters and the level of noise.

The aim of this paper is to provide a theoretical study of the errors caused in the measurements of position by the noise in the accelerometer. This investigation would be of use in deciding whether to use a particular accelerometer in a particular application. We shall be using as our model a stationary accelerometer so that any errors are due to the noise and not due to any contributions from motion of the accelerometer. In this way, the errors in the estimated displacement arising from the double integration of noise are isolated from the corresponding errors from specific acceleration signals.

Firstly, an expression will be derived for the rms errors as a function of time, for the case of ideal double integration of coloured noise. The specific cases of white noise and filtering using an analogue single-pole filter are discussed subsequently.

In particular, we address the following two questions.

(i) Given a set value of the sampling frequency, how do the root mean square (rms) errors in position vary with integration time?
(ii) Given a set value of the integration time and bandwidth, how do the rms errors vary with sampling frequency?

The theoretical work will be assessed by comparison with the analysis of experimental data taken for an accelerometer on an optical bench, where intrinsic vibration amplitude is minimal and well below the noise amplitude of the accelerometer that is used.

2. Theory

2.1. Introduction

A theoretical analysis is now made of the dependence with integration time and sampling frequency of the rms error of the measured displacement from accelerometer measurements. In section 2.2, the case when the measurements are represented by a dc bias is described. In section 2.3, the double integration of coloured noise is first described and the particular cases of white noise and noise filtered by a single-pole filter will be analysed.

2.2. Double integration of a dc bias

We first look at the case where a dc signal is being double integrated. If the acceleration is a constant, A, then the estimated displacement assuming zero initial displacement and velocity is given by

$$s(T) = \tfrac{1}{2}AT^2. \quad (1)$$

As A is constant, then the rms value of A is equal to A and the rms value of displacement is hence given by

$$\mathrm{RMS}(s(T)) = \tfrac{1}{2}AT^2. \quad (2)$$

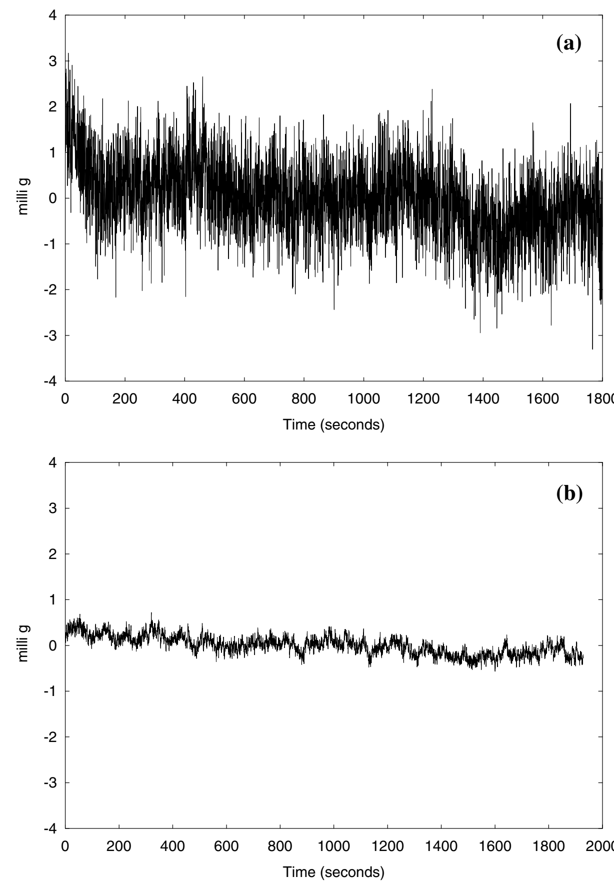

Figure 1. Output signals from (a) ADXL250 and (b) Crossbow CXL01F3 accelerometers.

It can be seen from equation (2) that

(i) $\mathrm{RMS}(s(T))$ varies as T^2 and
(ii) for a particular time, $\mathrm{RMS}(s(T))$ for a dc signal is independent of the sampling frequency.

2.3. Rms error in double integration of coloured noise from a stationary accelerometer

In this section, an expression for the rms errors in position as a function of time, arising from double integration, are derived for the two cases of white noise and coloured noise formed by passing white noise through a single-pole filter.

Let the data segment be T seconds long, the number of samples be N and the sampling frequency be f_s Hz.

Now the time between samples is $1/f_s$ seconds, hence, for N samples,

$$T = \frac{N}{f_s}. \quad (3)$$

Let σ_d be the standard deviation of noise for each data point. We model the data as coloured noise with no 'signal' component, i.e. an absolutely stationary horizontal accelerometer.

Let $\{a[n]\}$ represent the noisy acceleration measurements, with $a[n]$ signifying the acceleration at sample point n. It is assumed that the data are stationary. In this case, we can define the pth lag of the autocorrelation function as

$$r[p] = E[a[i] \cdot a[i+p]] \quad (4)$$

where $E[\cdot]$ signifies expectation.

Now the dc value of the acceleration at the Nth sample point is given by an average of the noise values over the first N sample points:

$$D_N = \frac{1}{N}\sum_{i=1}^{N} a[i]. \tag{5}$$

This summation approaches zero as the number of samples becomes infinite:

$$\lim_{N\to\infty} D_N = 0. \tag{6}$$

However, for a finite number of samples, D_N will in general be non-zero.

It is now assumed that the underlying acceleration is a constant over the whole data interval. The average of the acceleration measurements over N samples, equation (5), is taken as the estimate of this constant acceleration. Thus, the displacement at time T can be determined from the following analytical double integration:

$$s(T) = \int_0^T \int_0^t D_N \,\mathrm{d}t'\,\mathrm{d}t = \tfrac{1}{2} D_N T^2 \tag{7}$$

where it is assumed that D_N is a constant over the time interval T and that the initial velocity and displacement of the accelerometer are zero, consistent with the assumption that the accelerometer is stationary. It should be pointed out that, in practice, the data would be numerically integrated from sample to sample. Hence, equation (7) is a simplified approximation to the estimate of displacement found in practice.

As the accelerometer is stationary, $s(T)$ in equation (7) can be considered to be an error in the measured displacement.

Now D_N will depend on the particular sequence of noise values up to time T. Taking rms values of both sides of equation (7),

$$\mathrm{RMS}(s(T)) = \tfrac{1}{2} T^2 \mathrm{RMS}(D_N). \tag{8}$$

From equation (5), the expectation value of the square of the mean of the acceleration is given by

$$E[(D_N)^2] = \frac{1}{N^2} E[(a[1] + a[2] + \cdots + a[N]) \times (a[1] + a[2] + \cdots a[N])]. \tag{9}$$

Expanding the brackets, using the symmetry condition

$$E[a[i]\cdot a[j]] = E[a[j]\cdot a[i]] \tag{10}$$

and using the stationary property, equation (4), we may rewrite equation (9) as

$$E[(D_N)^2] = \frac{C[0] + C[1] + C[2] + C[3] + \cdots + C[N-1]}{N^2} \tag{11}$$

where

$$C[0] = Nr[0] \tag{12}$$

$$C[1] = 2(N-1)r[1] \tag{13}$$

$$C[2] = 2(N-2)r[2]\ldots \tag{14}$$

$$C[N-1] = 2r[N-1]. \tag{15}$$

Substituting for $C[0], C[1], \ldots, C[N-1]$ into equation (11) the following expression can be derived for the mean square value of correlated noise:

$$E[(D_N)^2] = R_1 + R_2 \tag{16}$$

where

$$R_1 = \frac{r[0]}{N} \tag{17}$$

and

$$R_2 = \frac{2}{N^2}\sum_{j=1}^{N-1} r[j](N-j). \tag{18}$$

R_1 is the variance of the acceleration in the absence of correlations between samples which would be $E[(D_N)^2]$ for white noise. R_2 is the contribution from the correlations between samples of the noise.

In the derivation of equation (16), the effects of bias, for example from the acceleration due to gravity or dc offset, have been ignored.

2.3.1. White noise. For the white noise model, it is assumed that the noise is uncorrelated from sample value to sample value. In this case R_2 in equation (16) is zero. Hence, from equations (16) and (17),

$$E[(D_N)^2] = \frac{r[0]}{N}. \tag{19}$$

Let σ_d be the standard deviation of the noise, so that

$$r[0] = \sigma_d^2. \tag{20}$$

Taking square roots of both sides of equation (19) and substituting for $r[0]$ from equation (20),

$$\mathrm{RMS}(D_N) = \frac{\sigma_d}{\sqrt{N}}. \tag{21}$$

Substituting for $\mathrm{RMS}(D_N)$ from equation (21) into equation (8), we obtain the following expression for the rms errors in position:

$$\mathrm{RMS}(s(T)) = \frac{1}{2} T^2 \frac{\sigma_d}{\sqrt{N}}. \tag{22}$$

Substituting for N from equation (3) above,

$$\mathrm{RMS}(s(T)) = \frac{1}{2} T^2 \frac{\sigma_d}{\sqrt{T f_s}} = \frac{1}{2}\frac{\sigma_d}{\sqrt{f_s}} T^{1.5}. \tag{23}$$

Hence, for a fixed sampling frequency, the rms error in estimated displacement increases as $T^{1.5}$.

It is also of interest to investigate the effect of increasing the sampling frequency on $\mathrm{RMS}(s(T))$ keeping the integration time, T, constant. Intuitively, we would expect $\mathrm{RMS}(s(T))$ to go to zero, as we are averaging over more samples N (see equation (21)); note that this is a consideration only for discrete, rather than continuous, processes.

From equation (23),

$$\mathrm{RMS}(s(T)) = \frac{C}{\sqrt{f_s}} \tag{24}$$

where $C = 0.5\sigma_d T^{1.5}$ is, in this case, a constant. It should be noted that equation (24) is appropriate for the simplified model used in equation (7). This result will be tested later when experimental data are analysed.

Hence, equation (24) predicts that if we keep T constant but change f_s then RMS($s(T)$) is proportional to the square root of the inverse of the sampling frequency.

To summarize, for the case of white noise,

(i) for a particular sampling frequency, RMS($s(T)$) varies with integration time as $T^{1.5}$;
(ii) for a particular integration time, RMS($s(T)$) varies as the square root of the inverse of the sampling frequency.

Equation (23) is in good agreement with the result in [1], where it is shown that, in the absence of initialization and drift errors, the rms error in estimated position from an accelerometer is given by

$$\mathrm{RMS}(s(T)) = \frac{1}{\sqrt{3}}\sqrt{R_v}T^{1.5} \qquad (25)$$

where R_v is the variance of continuous noise. This expression has been derived using state-space analysis. In [17], it is shown that R_v is related to the variance, σ_d^2, of discrete noise by

$$R_v = \frac{\sigma_d^2}{f_s}. \qquad (26)$$

Substituting for R_v from equation (26) into equation (25), we find that

$$\mathrm{RMS}(s(T)) = \frac{1}{\sqrt{3}}\frac{\sigma_d}{\sqrt{f_s}}T^{1.5}. \qquad (27)$$

Apart from the constant factor pre-multiplying the expressions, equations (23) and (27) are in agreement with each other. Equation (27) can also be derived by considering the double integration of acceleration as an integrated Wiener process [18]. In the approach used in [1], double integration is carried out continuously up to the time of interest. In the simplified approach used in this paper, the rms acceleration at the time point of interest is found first. Then, an analytical double integration is carried out, assuming that this acceleration is a constant over the interval of integration, to obtain an estimate of the displacement. Unlike the approach in [1], this latter analysis is retrospective in nature leading to a different constant prefactor in equations (23) and (27).

The advantage of the analysis presented in this section is that it is easier to understand physically the factors that have lead to the dependence of the rms error in displacement on both the sampling frequency, f_s, and time, T, in equation (23).

2.3.2. Noise filtered with a single-pole filter. The accelerometer data will, in practice, be filtered prior to processing. An anti-aliasing filter will have a finite cut-off frequency and, even after conversion to digital form, it may be required to filter the digital signal further prior to double integration.

Equations (16)–(18) apply to the general case where no particular filter is specified. In this discussion, we model the anti-aliasing filter as a single-pole filter, with frequency response

$$H(\mathrm{j}\omega) = \frac{\omega_c}{\omega_c + \mathrm{j}\omega} \qquad (28)$$

where $\omega_c = 2\pi f_c$ is the 3 dB cut-off frequency in rad s^{-1}. This type of filter is built into the two accelerometers under study.

Using Parseval's theorem, if noise with power spectral density $\frac{1}{2}\sigma_c^2$ cm^2 s^{-4} Hz^{-1} is input to the single-pole filter with frequency response given by equation (28), then the energy of the output signal is given by

$$E_{out} = \frac{\sigma_c^2}{2}\frac{1}{\pi}\int_0^\infty |H(\mathrm{j}\omega)|^2\,\mathrm{d}\omega = \frac{\sigma_c^2}{2\pi}\int_0^\infty \frac{\omega_c^2}{\omega_c^2+\omega^2}\,\mathrm{d}\omega. \qquad (29)$$

Hence

$$E_{out} = \frac{\sigma_c^2\omega_c^2}{2\pi}\frac{1}{\omega_c}\left[\tan^{-1}\left(\frac{\omega}{\omega_c}\right)\right]_0^\infty = \sigma_c^2\frac{\omega_c}{4}. \qquad (30)$$

Substituting $\omega_c = 2\pi f_c$ into equation (30) and taking square roots of both sides of this equation, it can be shown that the rms value, σ_f, of the filtered noise is given by

$$\sigma_f = \sqrt{E_{out}} = \left[\frac{\pi f_c}{2}\right]^{0.5}\sigma_c. \qquad (31)$$

In the appendix, it is shown that the rms error in displacement using the filter model in equation (28) is given by

$$\mathrm{RMS}(s(T)) = \frac{T^2}{2}\left[\frac{\alpha\sigma_c^2 f_s}{2N^2} \times \left\{\frac{\frac{N}{2}(1-\mathrm{e}^{-2\alpha}) - \mathrm{e}^{-\alpha} + \mathrm{e}^{-(N+1)\alpha}}{(1-\mathrm{e}^{-\alpha})^2}\right\}\right]^{0.5} \qquad (32)$$

where $N = Tf_s$ is the number of samples processed and

$$\alpha = \frac{2\pi f_c}{f_s}. \qquad (33)$$

It should be noted that a similar analysis can be made by using the continuous-time version of equations (16)–(18):

$$E[d(T)^2] = \frac{2}{T^2}\int_0^T (T-t)r(t)\,\mathrm{d}t$$

where $r(t)$ is given by equation (A.4) and d(T) is the time-averaged filtered accelerometer signal, analogous to D_N in equation (16). Further details are contained in [19].

It is of interest to investigate the time dependence of the rms errors for doubly integrated filtered noise for small and large integration times.

Small time approximation. For small enough α, the following approximations can be made:

$$\mathrm{e}^{-\alpha} \approx 1 - \alpha + \frac{\alpha^2}{2} \qquad (34)$$

$$\mathrm{e}^{-2\alpha} \approx 1 - 2\alpha + 2\alpha^2. \qquad (35)$$

These approximations would be valid for $f_c \ll f_s$. Let the number of samples N be small enough so that the last two terms in the numerator of equation (32) cannot be neglected.

In addition, we make the approximation valid for small enough α and N:

$$\mathrm{e}^{-(N+1)\alpha} = 1 - (N+1)\alpha + \frac{(N+1)^2}{2}\alpha^2. \qquad (36)$$

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