2. *Verify* to some confidence that indeed the region was the desired one.
3. *Bound* the region accurately.

The outline the plan generation, scoring, and execution used in the system are described in the following paragraphs. The plans generated by the system are typically enhanced versions of plans like the telephone finder. Plan scoring proceeds as expected for such plans; allowances are made for the enhanced semantics of plan nodes. A "cost/confidence" scoring function is used, and various practical simplifications are made that do not affect the planning paradigm itself.

An Example Plan and Its Execution

The system's plans are enhanced plans, in the sense of Section 13.2.3. Actions can be *AND*, *OR* or *SEQUENCE* actions, and shared plan structure and loops are permitted. Loops that contain only internal, planning actions would never terminate. However, a loop with an OR node can terminate (has an exit) if one of the subactions of the OR is executable. A plan for locating a chair in an office scene is shown in Fig. 13.7. In Fig. 13.7, the acquire–validate–bound strategy is evident in the two SEQUENCE subgoals of the Find Chair main goal, which is an AND goal. The loop in the plan is evident, and makes sense here because often planning is done for information gathering, not for real world actions.

As noted in Section 13.2.3, an enhanced plan may not be completely specified. If it is to be executed one subgoal at a time (no parallelism is allowed), sequences of subactions must be determined for its AND and OR actions. In Garvey's planner, these sequences are determined initially on the basis of apriori information, but the partial results of actions are "fed back," so that dynamic rescoring and hence dynamic reordering of goal sequences is possible. For example, if one subgoal of an AND action fails, the AND action is abandoned. Thus this planner is to some degree incremental.

In execution, Fig. 13.7 might result in the sequence of actions depicted in Fig. 13.8. The acquisition phase of object location has the most alternatives, so plan generation effort is mainly spent there. Acquisition proceeds either directly or indirectly. Direct acquisition is the classification of input data gathered from a random sampling of a window in the image; the input data are rich enough to allow basic pattern recognition techniques to identify the source of individual pixels.

Indirect acquisition is the use of the location of other "objects" (really identified regions) in the scene to locate the desired region. The desired region might be found by "scanning" vertically or horizontally from the already identified region, for instance. The idea is a planning version of a common one (e.g., the geometric location networks of Section 10.3.2): use something already located to limit and direct search for something else.

Plan Generation

A plan such as Fig. 13.7 is "elaborated" from the basic Find Chair goal by recursively expanding goals. Some goals (such as to find a chair) are not directly executable; they need further elaboration. Elaboration continues until all the subgoals are executable. Executable subgoals are those that analyze the image, run filters and detectors over parts of it, and generate decisions about the presence or absence

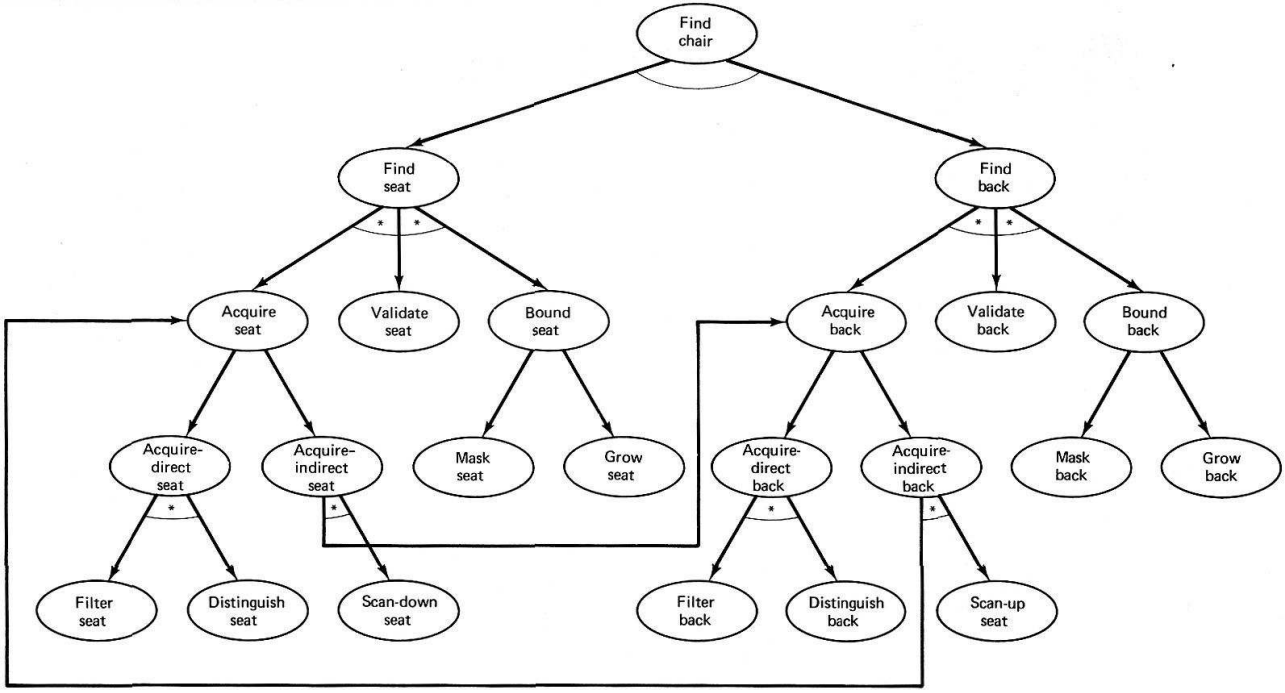


Fig. 13.7 An enhanced plan to locate a chair in an office scene. Untied multiple arcs denote OR actions, arcs tied together denote AND actions, those with *'s denote SEQUENCE actions. The loop in the plan has executable exits.

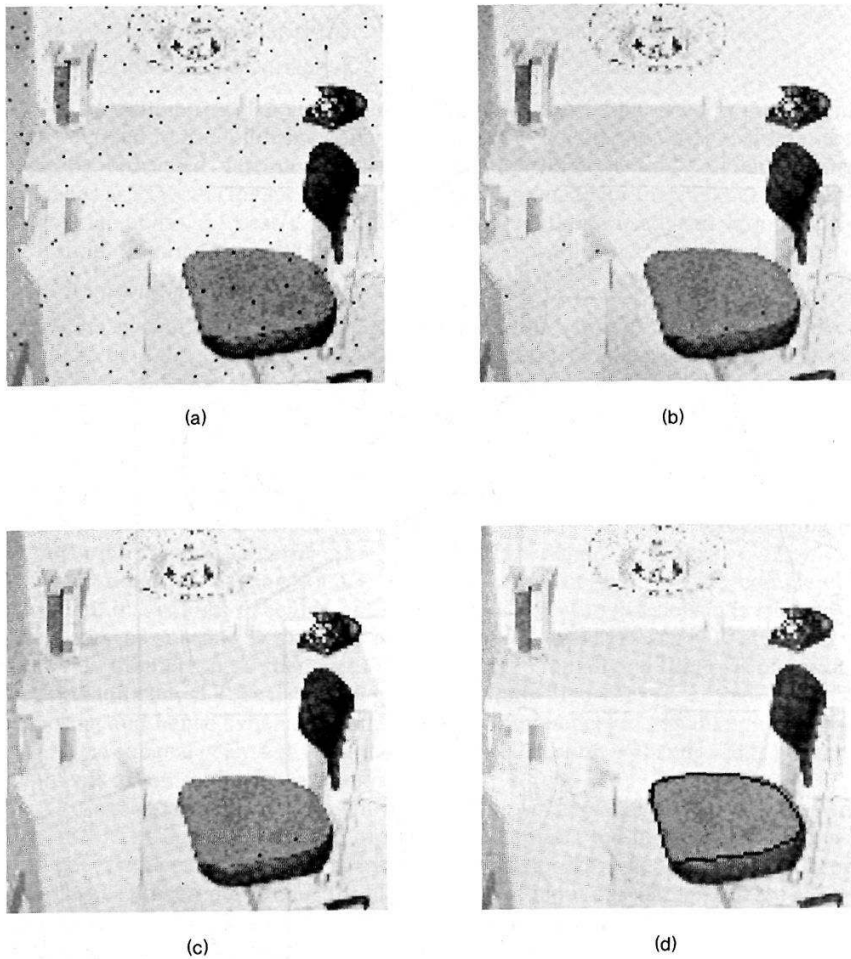
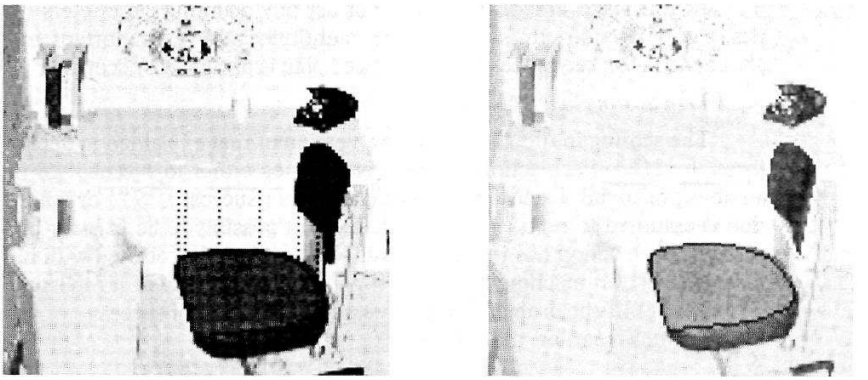


Fig. 13.8 The plan of Fig. 13.7 finds the most promising execution sequence for finding the chair in the scene of Fig. 13.6: find the seat first, then scan upwards from the seat looking for the back. Acquisition of the seat proceeds by sampling (a), followed by classification (b). The Validation procedure eliminates non-chair points (c), and the Bounding procedure produces the seat region (d). To find the back, scanning proceeds in the manner indicated by (e) (actually fewer points are examined in each scan). The back is acquired and bounded, leading to the final location of the chair regions (f).



(e) (f)

Fig. 13.8 (cont.)

of image phenomena. This straightforward elaboration is akin to macro expansion, and is not a very sophisticated planning mechanism (the program cannot criticize and manipulate the plan, only score it). A fully elaborated plan is presented for scoring and execution.

The elaboration process, or planner, has at its disposal several sorts of knowledge embodied as modules that can generate subgoals for a goal. Some are general (to find something, find all its parts); some are less general (a chair has a back and a seat); some are quite specific, being perhaps programs arising from an earlier interactive method-generation phase. The elaborator is guided by information stored about objects, for instance this about a tabletop:

OBJECT	PROPERTIES	RELATIONS
Table TOP	Hue:26–58	Supports Telephone 0.6
	Sat.: 0.23–0.32	Supports Book 0.4
	Bright.: 18–26	Occludes Wall 1
	Height: 26–28	
	Orient.: −7–7	

Here the orientation information indicates a vertical surface normal. The planner knows that it has a method of locating horizontal surfaces, and the plan elaborator can thus create a goal of direct acquisition by first locating a horizontal plane. The relational information allows for indirect acquisition plans. The elaborator puts direct and indirect alternatives under an *OR* node in the plan. Information not used for acquisition (height, color) may be used for validation.

Loops may occur in an elaborated plan because each newly generated goal is checked against goals already existing. Should it or an equivalent goal already exist, the existing goal is substituted for the newly generated one. Goals may thus have more than one ancestor, and may depend on one another.

At this stage, the planner does not use any planning parameters (cost, utilities, etc.); it is strictly symbolic. As mentioned above, important information about execution sequences in an enhanced plan is provided by scoring.

Plan Scoring and Execution

The scoring in the vision plan is a version of that explained in Sections 13.2.2 through 13.2.4. Each action in a plan is assumed either to succeed (S) in locating an object or to fail. Each action may report either success ("S") or failure. An action is assumed to report failure correctly, but possibly to be in error in reporting success. Each action has three "planning parameters" associated with it. They are C, its "cost" (in machine cycles), $P(\text{"}S\text{"})$ the probability of it reporting success, and $P(S|\text{"}S\text{"})$, the probability of success given a report of success.

As shown earlier, the product

$$P(S|\text{"}S\text{"})P(\text{"}S\text{"}) \tag{13.19}$$

is the probability that the action has correctly located an object and reported success. This product is called the "confidence" of the action. An action has structure as shown in Fig. 13.9.

The score of an action is computed as

$$\text{score} = \frac{\text{cost}}{\text{confidence}} \tag{13.20}$$

The planner thus must minimize the score.

The initial planning parameters of an executable action typically are determined by experimentation. The parameters of internal (AND, OR, SEQUENCE) actions by scoring methods alluded to in Sections 13.2.2, 13.2.3, and the Exercises (there are a few idiosyncratic ad hoc adjustments.).

It may bear repeating that planning, scoring, and execution are not separated temporally in this sytem. Scoring is used after the enhanced plan is generated to derive a simple plan (with ordered subgoals). Execution can affect the scores of nodes, and so execution can alternate with "replanning" (really rescoring resulting in a reordering). Recall the example of failure of an *AND* or *SEQUENCE* subgoal, which can immediately fail the entire goal. More generally, the entire goal and ultimately the plan may be rescored. For instance, the parameters of a successful action are modified by setting the cost of the executed action to 0 and its confidence to its second parameter, $P(S|\text{"}S\text{"})$.

Given a scored plan, execution is then easy; the execution program starts at the top goal of the plan, working its way down the best path as defined by the scores of nodes it encounters. When an executable subgoal is found (e.g. "look for a green region"), it is passed to an evaluation function that "runs" the action associated with the subgoal.

The subgoal is either achieved or not; in either case, information about its outcome is propagated back up the plan. Failure is easy; a failed subgoal of an AND or SEQUENCE goal fails the goal, and this failure is propagated. A failed subgoal of an OR goal is removed from the plan. The use of success information is more complex, involving the adjustment of confidences and planning parameters illustrated above.

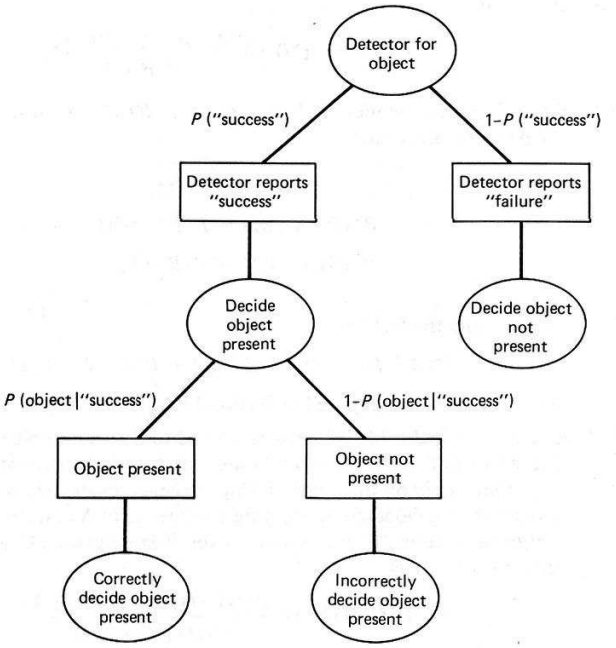


Fig. 13.9 This is the microstructure of a node ("action") of Garvey's planning system in terms of simple plans. Think of actions as being object detectors which announce "Found" or "Not Found." Garvey's planning parameters are P("Found") and P(Object is there|"Found"). Confidence in the action is their product; it is the probability of correctly detecting the object. All other outcomes are lumped together and not used for planning.

After the outcome of a goal is used to adjust the parameters of other goals, the plan is rescored and another cycle of execution performed. The execution can use knowledge about the image picked up along the way by prior execution. This is how results (such as acquired pixels) are passed to later processing stages (such as the validation process). Such a mechanism can even be used to remember successful subplans for later use.

EXERCISES

13.1 Complete the computation of outcome probabilities in the style of Section 13.2.2, using the assumptions given there. Check your work by showing (symbolically) that the probabilities of getting to the terminal actions ("goal states") of the plan sum to 1.

13.2 Assume in Section 13.2.2 that the results of the "table" and "telephone shape" detectors are not independent. Formulate your assumptions and compute the new outcome probabilities for Fig. 13.4.

13.3 Show that

$$P(A|(B \wedge C)) = \frac{P(B|(A \wedge C))P(A|C)}{P(B|C)}$$

13.4 B and C are independent if $P(B \wedge C) = P(B)\,P(C)$. Assuming that B and C are independent, show that

$$P(B|C) = P(B)$$
$$P((B \wedge C)|A) = P(B|A)P(C|A)$$
$$P(B|(A \wedge C)) = P(B|A)$$

13.5 Starting from the fact that

$$P(A \wedge B) = P(A \wedge B \wedge C) + P(A \wedge B \wedge (-C))$$

show how P_{15} was computed in Section 13.2.2.

13.6 A sequence $D(N)$ of N detectors is used to detect an object; the detectors either succeed or fail. Detector outputs are assumed independent of each other, being conditioned only on the object. Using previous results, show that the probability of an object being detected by applying a sequence of N detectors $D(N)$ is recursively rewritable in terms of the output of the first detector $D1$ and the remaining sequence $D(N-1)$ as

$$P(O|D(N)) = \frac{P(D1|O)P(O|D(N-1))}{P(D1|D(N-1))}$$

13.7 Consider scoring a plan containing an OR node (action). Presumably, each subgoal of the OR has an expected utility. The OR action is achieved as soon as one of the subgoals is achieved. Is it possible to order the subgoals for trial so as to maximize the expected utility of the plan? (This amounts to a unique "best" rewriting of the plan to make it a simple plan.)

13.8 Answer question 13.7 for an AND node; remember that the AND will fail as soon as any of its subgoals fails.

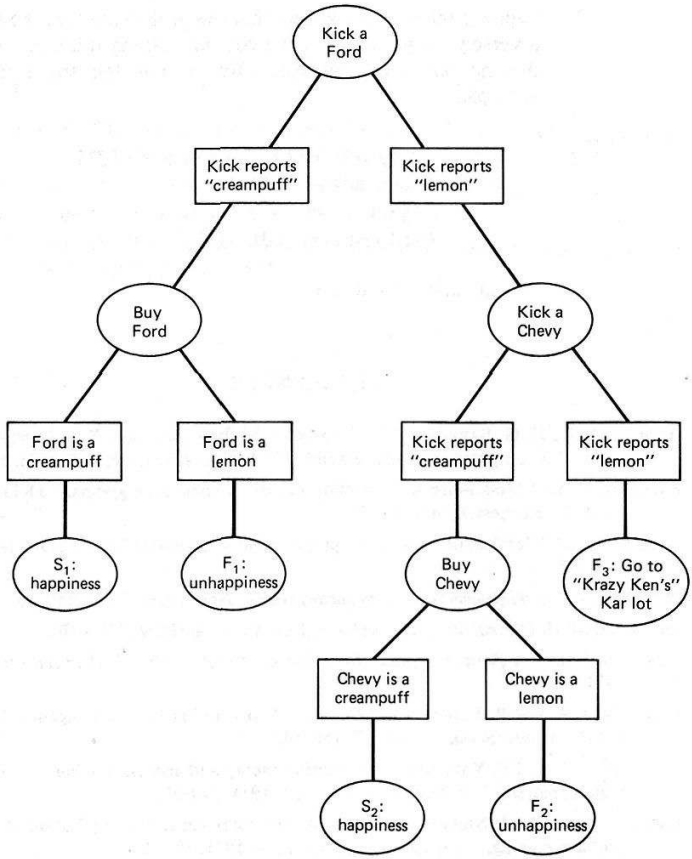
13.9 What can you say about how the cost/confidence ratio of Garvey's planner is related to the expected utility calculations of Section 13.2.2?

13.10 You are at Dandy Dan's used car lot. *Consumer Reports* says that the a priori probability that any car at Dandy Dan's is a lemon is high. You know, though, that to test a car you kick its tire. In fact, with probability:

$P(\text{"}C\text{"}|C)$: a kick correctly announces "creampuff" when the car actually is a creampuff
$P(\text{"}C\text{"}|L)$: a kick incorrectly announces "creampuff" when the car is actually a lemon
$P(L)$: the a priori probability that the car is a lemon

Your plan for dealing with Dandy Dan is shown below; give expressions for the probabilities of arriving at the nodes labeled S_1, S_2, F_1, F_2, and F_3. Give numeric answers using the following values

$$P(\text{"}C\text{"}|C) = 0.5,\ P(\text{"}C\text{"}|L) = 0.5,\ P(L) = 0.75$$



Ex. 13.10

13.11 Two bunches of bananas are in a room with a monkey and a box. One of the bunches is lying on the floor, the other is hanging from the ceiling. One of the bunches is made of wax. The box may be made of flimsy cardboard. Given that:

$P(\text{WH}) = 0.2$: probability that the hanging bananas are wax
$P(\text{WL}) = 0.8$: probability that the lying bananas are wax
$P(\text{C}) = 0.5$: probability that the box is cardboard
$U(\text{eat}) = 200$: utility of eating a bunch of bananas
$C(\text{walk}) = -10$: cost of walking a unit distance
$C(\text{push}) = -20$: cost of pushing the box a unit distance
$C(\text{climb}) = -20$: cost of climbing up on box

(a) Analyze two different plans for the monkey, showing all paths and calculations. Give criteria (based upon extra information not given here) that would allow the monkey to choose between these plans.

(b) Suppose the monkey knows that the probability that the box will collapse is inversely proportional to the cost of pushing the box a unit distance (and that he can sense this cost after pushing the box 1 unit distance). For example,

$$\begin{aligned} P(C) &= 1.0 - [C(\text{push}) \times 0.01] \\ P(C(\text{push}) &= 10) = 0.1 \\ P(C(\text{push}) &= 20) = 0.1 \\ P(C(\text{push}) &= 100) = 0.1 \end{aligned}$$

Repeat part(a) (in detail).

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Some Mathematical Tools

Appendix 1

A1.1 COORDINATE SYSTEMS

A1.1.1 Cartesian

The familiar two- and three-dimensional rectangular (Cartesian) coordinate systems are the most generally useful ones in describing geometry for computer vision. Most common is a right-handed three-dimensional system (Fig. A1.1.). The coordinates of a point are the perpendicular projections of its location onto the coordinate axes. The two-dimensional coordinate system divides two-dimensional space into quadrants, the three-dimensional system divides three-space into octants.

A1.1.2 Polar and Polar Space

Coordinate systems that measure locations partially in terms of angles are in many cases more natural than Cartesian coordinates. For instance, locations with respect

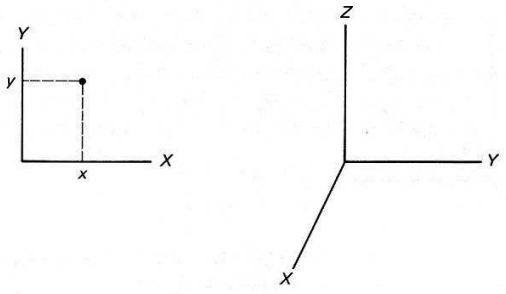

Fig. A1.1 Cartesian coordinate systems.

to the pan-tilt head of a camera or a robot arm may most naturally be described using angles. Two- and three-dimensional polar coordinate systems are shown in Fig. A1.2.

Cartesian Coordinates	*Polar Coordinates*
x	$\rho \cos \theta$
y	$\rho \sin \theta$
$(x^2 + y^2)^{1/2}$	ρ
$\tan^{-1}\left(\frac{y}{x}\right)$	θ

Cartesian Coordinates	*Polar Space Coordinates*
(x, y, z)	$(\rho \cos \xi, \rho \cos \eta, \rho \cos \zeta)$
$(x^2 + y^2 + z^2)^{1/2}$	ρ
$\cos^{-1}\left(\frac{x}{\rho}\right)$	ξ
$\cos^{-1}\left(\frac{y}{\rho}\right)$	η
$\cos^{-1}\left(\frac{z}{\rho}\right)$	ζ

In these coordinate systems, the Cartesian quadrants or octants in which points fall are often of interest because many trigonometric functions determine only an angle modulo $\pi/2$ or π (one or two quadrants) and more information is necessery to determine the quadrant. Familiar examples are the inverse angle functions (such as arctangent), whose results are ambiguous between two angles.

A1.1.3 Spherical and Cylindrical

The spherical and cylindrical systems are shown in Fig. A1.3.

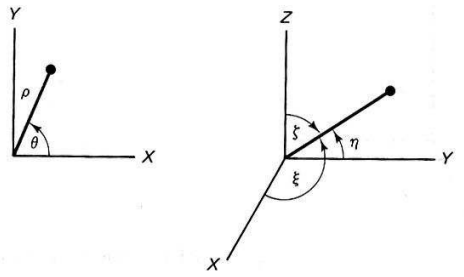

Fig. A1.2 Polar and polar space coordinate systems.

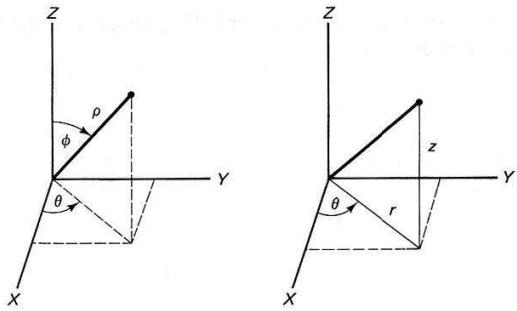

Fig. A1.3 Spherical and cylindrical coordinate systems.

Cartesian Coordinates	*Spherical Coordinates*
x	$\rho \sin \phi \cos \theta$
y	$\rho \sin \phi \sin \theta = x \tan \theta$
z	$\rho \cos \theta$
$(x^2 + y^2 + z^2)^{1/2}$	ρ
$\tan^{-1}\left(\frac{y}{x}\right)$	θ
$\cos^{-1}\left(\frac{z}{\rho}\right)$	ϕ

Cartesian Coordinates	*Cylindrical Coordinates*
x	$r \cos \theta$
y	$r \sin \theta$
z	z
$(x^2 + y^2)^{1/2}$	r
$\tan^{-1}\left(\frac{y}{x}\right)$	θ

A1.1.4 Homogeneous Coordinates

Homogeneous coordinates are a very useful tool in computer vision (and computer graphics) because they allow many important geometric transformations to be represented uniformly and elegantly (see Section A1.7). Homogeneous coordinates are redundant: a point in Cartesian n-space is represented by a line in homogeneous $(n + 1)$-space. Thus each (unique) Cartesian coordinate point corresponds to infinitely many homogeneous coordinates.

Cartesian Coordinates	*Homogeneous Coordinates*
(x, y, z)	(wx, wy, wz, w)
$\left(\frac{x}{w}, \frac{y}{w}, \frac{z}{w}\right)$	(x, y, z, w)

Here x, y, z, and w are real numbers, wx, wy, and wz are the products of the two reals, and x/w and so on are the indicated quotients.

A1.2. TRIGONOMETRY

A1.2.1 Plane Trigonometry

Referring to Fig. A1.4, define

$$\text{sine:} \quad \sin(A) \text{ (sometimes } \sin A) = \frac{a}{c}$$

$$\text{cosine:} \quad \cos(A) \text{ (or } \cos A) = \frac{b}{c}$$

$$\text{tangent:} \quad \tan(A) \text{ (or } \tan A) = \frac{a}{b}$$

The inverse functions arcsin, arccos, and arctan (also written $\sin^{-1}$, $\cos^{-1}$, $\tan^{-1}$) map a value into an angle. There are many useful trigonometric identities; some of the most common are the following.

$$\tan(x) = \frac{\sin(x)}{\cos(x)} = -\tan(-x)$$

$$\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y)$$

$$\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y)$$

$$\tan(x \pm y) = \frac{\tan(x) \mp \tan(y)}{1 \mp \tan(x)\tan(y)}$$

In any triangle with angles A, B, C opposite sides a, b, c, the Law of Sines holds:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

as does the Law of Cosines:

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$a = b\cos C + c\cos B$$

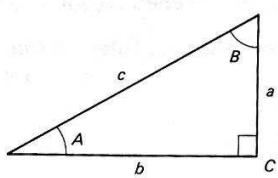

Fig. A1.4 Plane right triangle.

A1.2.2. Spherical Trigonometry

The sides of a spherical triangle (Fig. A1.5) are measured by the angle they subtend at the sphere center; its angles by the angle they subtend on the face of the sphere.

Some useful spherical trigonometric identities are the following.

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}$$

$$\cos a = \cos b \cos c + \sin b \sin c \cos A = \frac{\cos b \cos (c \pm \theta)}{\cos \theta}$$

$$\text{Where } \tan \theta = \tan b \cos A,$$

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a$$

A1.3. VECTORS

Vectors are both a notational convenience and a representation of a geometric concept. The familiar interpretation of a vector $\mathbf{v}$ as a directed line segment allows for a geometrical interpretation of many useful vector operations and properties. A more general notion of an n-dimensional vector $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ is that of an n-tuple abiding by mathematical laws of composition and transformation. A vector may be written horizontally (a row vector) or vertically (a column vector).

A point in n-space is characterized by its n coordinates, which are often written as a vector. A point at X, Y, Z coordinates x, y, and z is written as a vector $\mathbf{x}$ whose three components are (x, y, z). Such a vector may be visualized as a directed line segment, or arrow, with its tail at the origin of coordinates and its head at the point at (x, y, z). The same vector may represent instead the direction in which it points—toward the point (x, y, z) starting from the origin. An important type of direction vector is the normal vector, which is a vector in a direction perpendicular to a surface, plane, or line.

Vectors of equal dimension are equal if they are equal componentwise. Vectors may be multiplied by scalars. This corresponds to stretching or shrinking the vector arrow along its original direction.

$$\lambda \mathbf{x} = (\lambda x_1, \lambda x_2, \ldots, \lambda x_n)$$

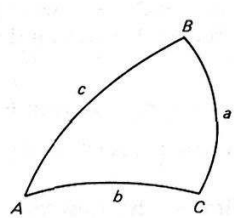

Fig. A1.5 Spherical triangle.

Vector addition and subtraction is defined componentwise, only between vectors of equal dimension. Geometrically, to add two vectors $\mathbf{x}$ and $\mathbf{y}$, put $\mathbf{y}$'s tail at $\mathbf{x}$'s head and the sum is the vector from $\mathbf{x}$'s tail to $\mathbf{y}$'s head. To subtract $\mathbf{y}$ from $\mathbf{x}$, put $\mathbf{y}$'s head at $\mathbf{x}$'s head; the difference is the vector from $\mathbf{x}$'s tail to $\mathbf{y}$'s tail.

$$\mathbf{x} \pm \mathbf{y} = (x_1 \pm y_1, x_2 \pm y_2, \ldots, x_n \pm y_n)$$

The length (or magnitude) of a vector is computed by an n-dimensional version of Euclidean distance.

$$|\mathbf{x}| = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2}$$

A vector of unit length is a unit vector. The unit vectors in the three usual Cartesian coordinate directions have special names.

$$\mathbf{i} = (1, 0, 0)$$
$$\mathbf{j} = (0, 1, 0)$$
$$\mathbf{k} = (0, 0, 1)$$

The inner (or scalar, or dot) product of two vectors is defined as follows.

$$\mathbf{x} \cdot \mathbf{y} = |\mathbf{x}||\mathbf{y}| \cos\theta = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

Here θ is the angle between the two vectors. The dot product of two nonzero numbers is 0 if and only if they are orthogonal (perpendicular). The projection of $\mathbf{x}$ onto $\mathbf{y}$ (the component of vector $\mathbf{x}$ in the direction $\mathbf{y}$) is

$$|\mathbf{x}| \cos\theta = \frac{\mathbf{x} \cdot \mathbf{y}}{|\mathbf{y}|}$$

Other identities of interest:

$$\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x}$$
$$\mathbf{x} \cdot (\mathbf{y} + \mathbf{z}) = \mathbf{x} \cdot \mathbf{y} + \mathbf{x} \cdot \mathbf{z}$$
$$\lambda(\mathbf{x} \cdot \mathbf{y}) = (\lambda \mathbf{x}) \cdot \mathbf{y} = \mathbf{x} \cdot (\lambda \mathbf{y})$$
$$\mathbf{x} \cdot \mathbf{x} = |\mathbf{x}|^2$$

The cross (or vector) product of two three-dimensional vectors is defined as follows.

$$\mathbf{x} \times \mathbf{y} = (x_2 y_3 - x_3 y_2, x_3 y_1 - x_1 y_3, x_1 y_2 - x_2 y_1)$$

Generally, the cross product of $\mathbf{x}$ and $\mathbf{y}$ is a vector perpendicular to both $\mathbf{x}$ and $\mathbf{y}$. The magnitude of the cross product depends on the angle θ between the two vectors.

$$|\mathbf{x} \times \mathbf{y}| = |\mathbf{x}||\mathbf{y}| \sin\theta$$

Thus the magnitude of the product is zero for two nonzero vectors if and only if they are parallel.

Vectors and matrices allow for the short formal expression of many symbolic

expressions. One such example is the formal determinant (Section A1.4) which expresses the definition of the cross product given above in a more easily remembered form.

$$\mathbf{x} \times \mathbf{y} = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}$$

Also,

$$\mathbf{x} \times \mathbf{y} = -\mathbf{y} \times \mathbf{x}$$
$$\mathbf{x} \times (\mathbf{y} \pm \mathbf{z}) = \mathbf{x} \times \mathbf{y} \pm \mathbf{x} \times \mathbf{z}$$
$$\lambda(\mathbf{x} \times \mathbf{y}) = \lambda\mathbf{x} \times \mathbf{y} = \mathbf{x} \times \lambda\mathbf{y}$$
$$\mathbf{i} \times \mathbf{j} = \mathbf{k}$$
$$\mathbf{j} \times \mathbf{k} = \mathbf{i}$$
$$\mathbf{k} \times \mathbf{i} = \mathbf{j}$$

The triple scalar product is $\mathbf{x} \cdot (\mathbf{y} \times \mathbf{z})$, and is equivalent to the value of the determinant

$$\det \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}$$

The triple vector product is

$$\mathbf{x} \times (\mathbf{y} \times \mathbf{z}) = (\mathbf{x} \cdot \mathbf{z})\mathbf{y} - (\mathbf{x} \cdot \mathbf{y})\mathbf{z}$$

A1.4. MATRICES

A matrix A is a two-dimensional array of elements; if it has m rows and n columns it is of dimension $m \times n$, and the element in the ith row and jth column may be named a_{ij}. If m or $n = 1$, a row matrix or column matrix results, which is often called a vector. There is considerable punning among scalar, vector and matrix representations and operations when the same dimensionality is involved (the 1×1 matrix may sometimes be treated as a scalar, for instance). Usually, this practice is harmless, but occasionally the difference is important.

A matrix is sometimes most naturally treated as a collection of vectors, and sometimes an $m \times n$ matrix M is written as

$$M = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n]$$

or

$$M = \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{b}_m \end{bmatrix}$$

where the **a**'s are column vectors and the **b**'s are row vectors.

Two matrices A and B are equal if their dimensionality is the same and they are equal elementwise. Like a vector, a matrix may be multiplied (elementwise) by a scalar. Matrix addition and subtraction proceeds elementwise between matrices of like dimensionality. For a scalar k and matrices A, B, and C of like dimensionality the following is true.

$$A = B \pm C \quad \text{if } a_{ij} = b_{ij} \pm c_{ij} \quad 1 \leqslant i \leqslant m, \quad 1 \leqslant j \leqslant n$$

Two matrices A and B are conformable for multiplication if the number of columns of A equals the number of rows of B. The product is defined as

$$C = AB \quad \text{where an element } c_{ij} \text{ is defined by } \quad c_{ij} = \sum_k a_{ik} b_{kj}$$

Thus each element of C is computed as an inner product of a row of A with a column of B. Matrix multiplication is associative but not commutative in general. The multiplicative identity in matrix algebra is called the identity matrix I. I is all zeros except that all elements in its main diagonal have value 1 ($a_{ij} = 1$ if $i = j$, else $a_{ij} = 0$). Sometimes the $n \times n$ identity matrix is written I_n.

The transpose of an $m \times n$ matrix A is the $n \times m$ matrix A^T such that the i,jth element of A is the j,ith element of A^T. If $A^T = A$, A is symmetric.

The inverse matrix of an $n \times n$ matrix A is written A^{-1}. If it exists, then

$$AA^{-1} = A^{-1}A = I$$

If its inverse does not exist, an $n \times n$ matrix is called singular.

With k and p scalars, and A, B, and C $m \times n$ matrices, the following are some laws of matrix algebra (operations are matrix operations):

$$\begin{aligned} A + B &= B + A \\ (A + B) + C &= A + (B + C) \\ k(A + B) &= kA + kB \\ (k + p)A &= kA + pA \\ AB &\neq BA \quad \text{in general} \\ (AB)C &= A(BC) \\ A(B + C) &= AB + AC \\ (A + B)C &= AC + BC \end{aligned}$$

$$A(kB) = k(AB) = (kA)B$$
$$I_m A = AI_n = A$$
$$(A + B^T) = A^T + B^T$$
$$(AB)^T = B^T A^T$$
$$(AB)^{-1} = B^{-1} A^{-1}$$

The determinant of an $n \times n$ matrix is an important quantity; among other things, a matrix with zero determinant is singular. Let A_{ij} be the $(n-1) \times (n-1)$ matrix resulting from deleting the ith row and jth column from an $n \times n$ matrix A. The determinant of a 1×1 matrix is the value of its single element. For $n > 1$,

$$\det A = \sum_{i=1}^{n} a_{ij} (-1)^{i+j} \det A_{ij}$$

for any j between 1 and n. Given the definition of determinant, the inverse of a matrix may be defined as

$$(a^{-1})_{ij} = \frac{(-1)^{i+j} \det A_{ji}}{\det A}$$

In practice, matrix inversion may be a difficult computational problem, but this important algorithm has received much attention, and robust and efficient methods exist in the literature, many of which may also be used to compute the determinant. Many of the matrices arising in computer vision have to do with geometric transformations, and have well-behaved inverses corresponding to the inverse transformations. Matrices of small dimensionality are usually quite computationally tractable.

Matrices are often used to denote linear transformations; if a row (column) matrix X of dimension n is post (pre) multiplied by an $n \times n$ matrix A, the result $X' = XA$ ($X' = AX$) is another row (column) matrix, each of whose elements is a linear combination of the elements of X, the weights being supplied by the values of A. By employing the common pun between row matrices and vectors, $\mathbf{x}' = \mathbf{x}A$ ($\mathbf{x}' = A\mathbf{x}$) is often written for a linear transformation of a vector $\mathbf{x}$.

An eigenvector of an $n \times n$ matrix A is a vector $\mathbf{v}$ such that for some scalar λ (called an eigenvalue),

$$\mathbf{v}A = \lambda \mathbf{v}$$

That is, the linear transformation A operates on $\mathbf{v}$ just as a scaling operation. A matrix has n eigenvalues, but in general they may be complex and of repeated values. The computation of eigenvalues and eigenvectors of matrices is another computational problem of major importance, with good algorithms for general matrices being complicated. The n eigenvalues are roots of the so-called characteristic polynomial resulting from setting a formal determinant to zero:

$$\det (A - \lambda I) = 0.$$

Eigenvalues of matrices up to 4 × 4 may be found in closed form by solving the characteristic equation exactly. Often, the matrices whose eigenvalues are of interest are symmetric, and luckily in this case the eigenvalues are all real. Many algorithms exist in the literature which compute eigenvalues and eigenvectors both for symmetric and general matrices.

A1.5. LINES

An infinite line may be represented by several methods, each with its own advantages and limitations. An example of a representation which is not often very useful is two planes that intersect to form the line. The representations below have proven generally useful.

A1.5.1 Two Points

A two-dimensional or three-dimensional line (throughout Appendix 1 this shorthand is used for "line in two-space" and "line in three-space"; similarly for "two (three) dimensional point") is determined by two points on it, **x1** and **x2**. This representation can serve as well for a half-line or a line segment. The two points can be kept as the rows of a (2 × n) matrix.

A1.5.2 Point and Direction

A two-dimensional or three-dimensional line (or half-line) is determined by a point **x** on it (its endpoint) and a direction vector **v** along it. This representation is essentially the same as that of Section A1.5.1, but the interpretation of the vectors is different.

A1.5.3 Slope and Intercept

A two-dimensional line can often be represented by the Y value b where the line intersects the Y axis, and the slope m of the line (the tangent of its inclination with the x axis). This representation fails for vertical lines (those with infinite slope). The representation is in the form of an equation making explicit the dependence of y on x:

$$y = mx + b$$

A similar representation may of course be based on the X intercept.

A1.5.4 Ratios

A two-dimensional or three-dimensional line may be represented as an equation of ratios arising from two points $\mathbf{x1} = (x_1, y_1, z_1)$ and $\mathbf{x2} = (x_2, y_2, z_2)$ on the line.

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

A1.5.5 Normal and Distance from Origin (Line Equation)

This representation for two-dimensional lines is elegant in that its parts have useful geometric significance which extends to planes (not to three-dimensional lines). The coefficients of the general two-dimensional linear equation represent a two-dimensional line and incidentally give its normal (perpendicular) vector and its (perpendicular) distance from the origin (Fig. A1.6).

From the ratio representation above, it is easy to derive (in two dimensions) that

$$(x - x_1) \sin\theta \quad - (y - y_1) \cos\theta = 0$$

so for

$$d = -(x_1 \sin\theta - y_1 \cos\theta),$$
$$x \sin\theta - y \cos\theta + d = 0$$

This equation has the form of a dot product with a formal homogeneous vector $(x, y, 1)$:

$$(x, y, 1) \cdot (\sin\theta, -\cos\theta, d) = 0$$

Here the two-dimensional vector $(\sin\theta, -\cos\theta)$ is perpendicular to the line (it is a unit normal vector, in fact), and d is the signed distance in the direction of the normal vector from the line to the origin. Multiplying both sides of the equation by a constant leaves the line invariant, but destroys the interpretation of d as the distance to the origin.

This form of line representation has several advantages besides the interpretations of its parameters. The parameters never go to infinity (this is useful in the Hough algorithm described in Chapter 4). The representation extends naturally to representing n-dimensional planes. Least squared error line fitting (Section A1.9) with this form of line equation (as opposed to slope-intercept) minimizes errors perpendicular to the line (as opposed to those perpendicular to one of the coordinate axes).

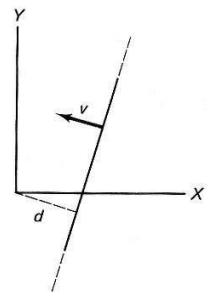

Fig. A1.6 Two-dimensional line with normal vector and distance to origin.

A1.5.6 Parametric

It is sometimes useful to be able mathematically to "walk along" a line by varying some parameter t. The basic parametric representation here follows from the two-point representation. If **x1** and **x2** are two particular points on the line, a general point on the line may be written as

$$\mathbf{x} = \mathbf{x1} + t(\mathbf{x2} - \mathbf{x1})$$

In matrix terms this is

$$\mathbf{x} = [t \quad 1]L$$

where L is the $2 \times n$ matrix whose first row is $(\mathbf{x2} - \mathbf{x1})$ and whose second is **x1**. Parametric representations based on points on the lines may be transformed by the geometric point transformations (Section A1.7).

A1.6. PLANES

The most common representation of planes is to use the coordinates of the plane equation. This representation is an extension of the line-equation representation of Section A1.5.5. The plane equation may be written

$$ax + by + cz + d = 0$$

which is in the form of a dot product $\mathbf{x} \cdot \mathbf{p} = 0$. Four numbers given by $\mathbf{p} = (a, b, c, d)$ characterize a plane, and any homogeneous point $\mathbf{x} = (x, y, z, w)$ satisfying the foregoing equation lies in the plane. In **p**, the first three numbers (a, b, c) form a normal vector to the plane. If this normal vector is made to be a unit vector by scaling **p**, then d is the signed distance to the origin from the plane. Thus the dot product of the plane coefficient vector and any point (in homogeneous coordinates) gives the distance of the point to the plane (Fig. A1.7).

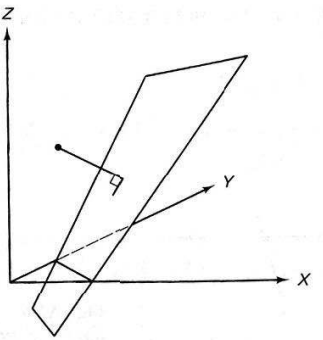

Fig. A1.7 Distance from a point to a plane.

Three noncollinear points **x1**, **x2**, **x3** determine a plane **p**. To find it, write

$$\begin{bmatrix} \mathbf{x1} & & & \\ \mathbf{x2} & & & \\ \mathbf{x3} & & & \\ 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{p} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

If the matrix containing the point vectors can be inverted, the desired vector **p** is thus proportional to the fourth column of the inverse.

Three planes **p1**, **p2**, **p3** may intersect in a point **x**. To find it, write

$$\mathbf{x} \begin{bmatrix} \mathbf{p1} & \mathbf{p2} & \mathbf{p3} & 0 \\ & & & 0 \\ & & & 0 \\ & & & 1 \end{bmatrix} = [0 \quad 0 \quad 0 \quad 1]$$

If the matrix containing the plane vectors can be inverted, the desired point **p** is given by the fourth row of the inverse. If the planes do not intersect in a point, the inverse does not exist.

A1.7 GEOMETRIC TRANSFORMATIONS

This section contains some results that are well known through their central place in the computer graphics literature, and illustrated in greater detail there. The idea is to use homogeneous coordinates to allow the writing of important transformations (including affine and projective) as linear transformations. The transformations of interest here map points or point sets onto other points or point sets. They include rotation, scaling, skewing, translation, and perspective distortion (point projection) (Fig. A1.8).

A point x in three-space is written as the homogeneous row four-vector (x, y, z, w), and postmultiplication by the following transformation matrices accomplishes point transformation. A set of m points may be represented as an $m \times 4$ matrix of row point vectors, and the matrix multiplication transforms all points at once.

A1.7.1 Rotation

Rotation is measured clockwise about the named axis while looking along the axis toward the origin.

Rotation by θ about the X axis:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

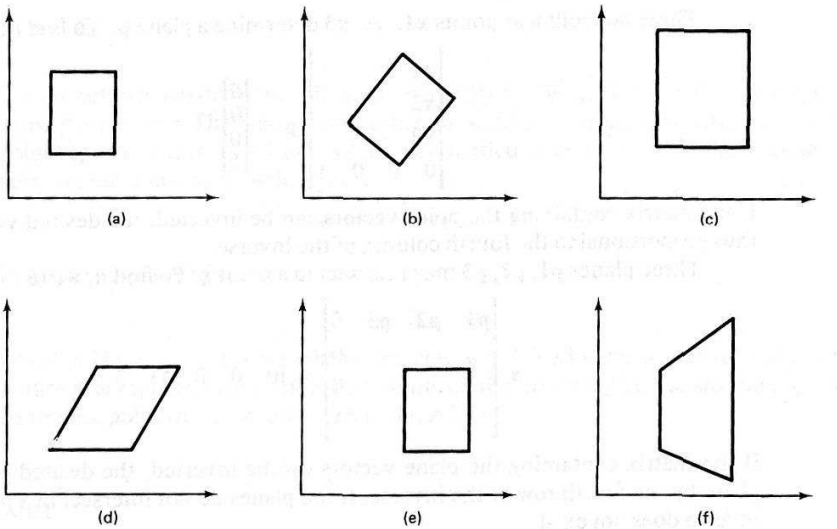

Fig. A1.8 Transformations: (a) original, (b) rotation, (c) scaling, (d) skewing, (e) translation, and (f) perspective.

Rotation by θ about the Y axis:

$$\begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotation by θ about the Z axis:

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

A1.7.2 Scaling

Scaling is stretching points out along the coordinate directions. Scaling can transform a cube to an arbitrary rectangular parallelepiped.

Scale by S_x, S_y, and S_z in the X, Y, and Z directions:

$$\begin{bmatrix} S_x & 0 & 0 & 0 \\ 0 & S_y & 0 & 0 \\ 0 & 0 & S_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

A1.7.3 Skewing

Skewing is a linear change in the coordinates of a point based on certain of its other coordinates. Skewing can transform a square into a parallelogram in a simple case:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ d & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

In general, skewing is quite powerful:

$$\begin{bmatrix} 1 & k & n & 0 \\ d & 1 & p & 0 \\ e & m & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotation is a composition of scaling and skewing (Section A1.7.7).

A1.7.4 Translation

Translate a point by (t, u, v):

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ t & u & v & 1 \end{bmatrix}$$

With a three-dimensional Cartesian point representation, this transformation is accomplished through vector addition, not matrix multiplication.

A1.7.5 Perspective

The properties of point projection, which model perspective distortion, were derived in Chapter 2. In this formulation the viewpoint is on the positive Z axis at $(0, 0, f, 1)$ looking toward the origin: f acts like a "focal length". The visible world is projected through the viewpoint onto the $Z = 0$ image plane (Fig. A1.9).

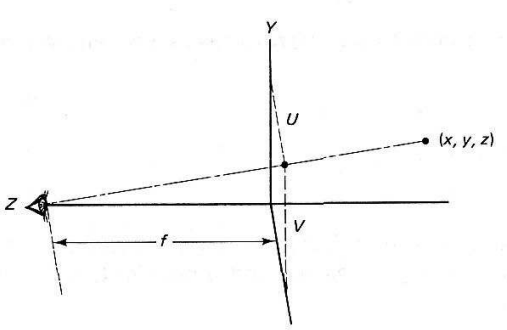

Fig. A1.9 Geometry of image formation.

Similar triangles arguments show that the image plane point for any world point (x, y, z) is given by

$$(U, V) = \left(\frac{fx}{f-z}, \frac{fy}{f-z} \right)$$

Using homogeneous coordinates, a "perspective distortion" transformation can be written which distorts three-dimensional space so that after orthographic projection onto the image plane, the result looks like that required above for perspective distortion. Roughly, the transformation shrinks the size of things as they get more distant in Z. Although the transformation is of course linear in homogeneous coordinates, the final step of changing to Cartesian coordinates by dividing through by the fourth vector element accomplishes the nonlinear shrinking necessary.

Perspective distortion (situation of Fig. A1.9):

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{-1}{f} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Perspective from a general viewpoint has nonzero elements in the entire fourth column, but this is just equivalent to a rotated coordinate system and the perspective distortion above (Section A1.7).

A1.7.6 Transforming Lines and Planes

Line and plane equations may be operated on by linear transformations, just as points can. Point-based parametric representations of lines and planes transform as do points, but the line and plane equation representations act differently. They have an elegant relation to the point transformation. If T is a transformation matrix (3×3 for two dimensions, 4×4 for three dimensions) as defined in Sections A1.7.1 to A1.7.5, then a point represented as a row vector is transformed as

$$\mathbf{x}' = \mathbf{x}T$$

and the linear equation (line or plane) when represented as a column vector $\mathbf{v}$ is transformed by

$$\mathbf{v}' = T^{-1}\mathbf{v}$$

A1.7.7 Summary

The 4×4 matrix formulation is a way to unify the representation and calculation of useful geometric transformations, rigid (rotation and translation), and nonrigid

(scaling and skewing), including the projective. The semantics of the matrix are summarized in Fig. A1.10.

Since the results of applying a transformation to a row vector is another row vector, transformations may be concatenated by repeated matrix multiplication. Such composition of transformations follows the rules of matrix algebra (it is associative but not commutative, for instance). The semantics of

$$\mathbf{x}' = \mathbf{x}ABC$$

is that $\mathbf{x}'$ is the vector resulting from applying transformation A to $\mathbf{x}$, then B to the transformed $\mathbf{x}$, then C to the twice-transformed $\mathbf{x}$. The single 4×4 matrix $D = ABC$ would do the same job. The inverses of geometric transformation matrices are just the matrices expressing the inverse transformations, and are easy to derive.

A1.8. CAMERA CALIBRATION AND INVERSE PERSPECTIVE

The aim of this section is to explore the correspondence between world and image points. A (half) line of sight in the world corresponds to each image point. Camera calibration permits prediction of where in the image a world point will appear. Inverse perspective transformation determines the line of sight corresponding to an image point. Given an inverse perspective transform and the knowledge that a visible point lies on a particular world plane (say the floor, or in a planar beam of light), then its precise three-dimensional coordinates may be found, since the line of sight generally intersects the world plane in just one point.

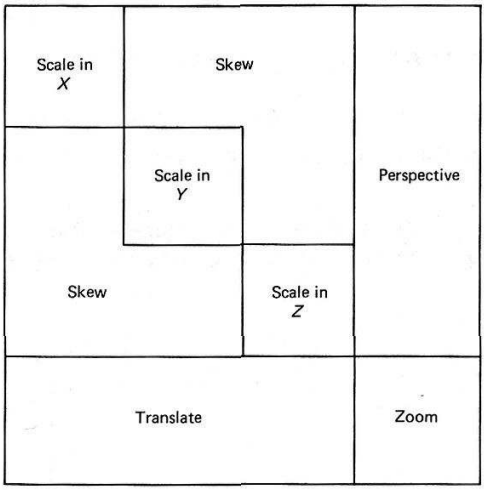

Fig. A1.10 The 4×4 homogeneous transformation matrix.

A1.8.1 Camera Calibration

This section is concerned with the "camera model"; the model takes the form of a 4×3 matrix mapping three-dimensional world points to two-dimensional image points. There are many ways to derive a camera model. The one given here is easy to state mathematically; in practice, a more general optimization technique such as hill climbing can be most effective in finding the camera parameters, since it can take advantage of any that are already known and can reflect dependencies between them.

Let the image plane coordinates be U and V; in homogeneous coordinates an image plane point is (u,v,t). Thus

$$U = \frac{u}{t}$$

$$V = \frac{v}{t}$$

Call the desired camera model matrix C, with elements C_{ij} and column four-vectors C_j. Then for any world point (x, y, z) a C is needed such that

$$(x, y, z, 1)C = (u, v, t)$$

So

$$u = (x, y, z, 1)C_1$$

$$v = (x, y, z, 1)C_2$$

$$t = (x, y, z, 1)C_3$$

Expanding the inner products and rewriting $u - Ut = 0$ and $v - Vt = 0$,

$$xC_{11} + yC_{21} + zC_{31} + C_{41} - UxC_{13} - UyC_{23} - UzC_{33} - UC_{43} = 0$$

$$xC_{12} + yC_{22} + zC_{32} + C_{42} - VxC_{13} - VyC_{23} - VzC_{33} - VC_{43} = 0$$

The overall scaling of C is irrelevant, thanks to the homogeneous formulation, so C_{43} may be arbitrarily set to 1. Then equations such as those above can be written in matrix form:

$$\begin{bmatrix} x^1 & y^1 & z^1 & 1 & 0 & 0 & 0 & 0 & -U^1x^1 & -U^1y^1 & -U^1z^1 \\ 0 & 0 & 0 & 0 & x^1 & y^1 & z^1 & 1 & -V^1x^1 & -V^1y^1 & -V^1z^1 \\ x^2 & y^2 & z^2 & 1 & \cdot & \cdot & \cdot & & & & \cdot \\ \cdot & & & & & & & & & & \cdot \\ \cdot & & & & & & & & & & \cdot \\ \cdot & & & & & & & & & & \cdot \\ 0 & 0 & 0 & 0 & x^n & y^n & z^n & 1 & -V^nx^n & -V^ny^n & -V^nz^n \end{bmatrix} \begin{bmatrix} C_{11} \\ C_{21} \\ \cdot \\ \cdot \\ \cdot \\ C_{34} \end{bmatrix} = \begin{bmatrix} U^1 \\ V^1 \\ \cdot \\ \cdot \\ \cdot \\ U^n \\ V^n \end{bmatrix}$$

Eleven such equations allow a solution for C. Two equations result for every association of an (x, y, z) point with a (U, V) point. Such an association must be established using visible objects of known location (often placed for the purpose). If more than 5½ such observations are used, a least-squared-error solution to the overdetermined system may be obtained by using a pseudo-inverse to solve the resulting matrix equation (Section A1.9).

A1.8.2 Inverse Perspective

Finding the world line corresponding to an image point relies on the fact that the perspective transformation matrix also affects the z component of a world point. This information is lost when the z component is projected away orthographically, but it encodes the relation between the focal point and the z position of the point. Varying this third component references points whose world positions vary in z but which project onto the same position in the image. The line can be parameterized by a variable p that formally occupies the position of that z coordinate in three-space that has no physical meaning in imaging.

Write the inverse perspective transform P^{-1} as

$$(x', y', p, 1)P^{-1} = (x', y', p, 1 + \frac{p}{f})$$

Rewriting this in the usual way gives these relations between the (x, y, z) points on the line.

$$(x, y, z, 1) = \left(\frac{fx'}{f+p}, \frac{fy'}{f+p}, \frac{fp'}{f+p}, 1\right)$$

Eliminating the parameter p between the expressions for z and x and those for z and y leaves

$$x = \frac{x'}{y'}\, y = \frac{-x'}{f}\,(z - f)$$

Thus x, y, and z are linearly related; as expected, all points on the inverse perspective transform of an image point lie in a line, and unsurprisingly both the viewpoint $(0, 0, f)$ and the image point $(x', y', 0)$ lie on it.

A camera matrix C determines the three-dimensional line that is the inverse perspective transform of any image point. Scale C so that $C_{43} = 1$, and let world points be written $\mathbf{x} = (x, y, z, 1)$ and image points $\mathbf{u} = (u, v, t)$. The actual image points are then

$$U = \frac{u}{t},\ V + \frac{v}{t}, \qquad \text{so } u = Ut, \quad v + Vt$$

Since

$$\begin{aligned} \mathbf{u} &= \mathbf{x}C, \\ u &= Ut = \mathbf{x}C_1 \\ v &= Vt = \mathbf{x}C_2 \\ t &= \mathbf{x}C_3 \end{aligned}$$

Substituting the expression for t into that for u and v gives

$$U\mathbf{x}C_3 = \mathbf{x}C_1$$
$$V\mathbf{x}C_3 = \mathbf{x}C_2$$

which may be written

$$\mathbf{x}(C_1 - UC_3) = 0$$
$$\mathbf{x}(C_2 - VC_3) = 0$$

These two equations are in the form of plane equations. For any U, V in the image and camera model C, there are determined two planes whose intersection gives the desired line. Writing the plane equations as

$$a_1x + b_1y + c_1z + d_1 = 0$$
$$a_2x + b_2y + c_2z + d_2 = 0$$

then

$$a_1 = C_{11} - C_{13}U \qquad a_2 = C_{12} - C_{13}V$$

and so on. The direction (λ, μ, ν) of the intersection of two planes is given by the cross product of their normal vectors, which may now be written as

$$(\lambda, \mu, \nu) = (a_1, b_1, c_1) \times (a_2, b_2, c_2)$$
$$= (b_1c_2 - b_2c_1,\ c_1a_2 - c_2a_1,\ a_1b_2 - a_2b_1)$$

Then if $\nu \neq 0$, for any particular z_0,

$$x_0 = \frac{b_1(c_2z_0 + d_2) - b_2(c_1z_0 - d_1)}{a_1b_2 - b_1a_2}$$

$$y_0 = \frac{a_2(c_1z_0 + d_1) - a_1(c_2z_0 - d_2)}{a_1b_2 - b_1a_2}$$

and the line may be written

$$\frac{x - x_0}{\lambda} = \frac{y - y_0}{\mu} = \frac{z - z_0}{\nu}$$

A1.9. LEAST-SQUARED-ERROR FITTING

The problem of fitting a simple functional model to a set of data points is a common one, and is the concern of this section. The subproblem of fitting a straight line to a set of (x, y) points ("linear regression") is the first topic. In computer vision, this line-fitting problem is encountered relatively often. Model-fitting methods try to find the "best" fit; that is, they minimize some error. Methods which yield closed-form, analytical solutions for such best fits are at issue here.

The relevant "error" to minimize is determined partly by assumptions of dependence between variables. If x is independent, the line may be represented as $y = mx + b$ and the error defined as the vertical displacement of a point from the line. Symmetrically, if x is dependent, horizontal error should be minimized. If neither variable is dependent, a reasonable error to minimize is the perpendicular distance from points to the line. In this case the line equation $ax + by + 1 = 0$ can be used with the method shown here, or the eigenvector approach of Section A1.9.2 may be used.

A1.9.1 Pseudo-Inverse Method

In fitting an $n \times 1$ observations matrix y by some linear model of p parameters, the prediction is that the linear model will approximate the actual data. Then

$$Y = XB + E$$

where X is an $n \times p$ formal independent variable matrix, B is a $p \times 1$ parameter matrix whose values are to be determined, and E represents the difference between the prediction and the actuality: it is an $n \times 1$ error matrix.

For example, to fit a straight line $y = mx + b$ to some data (x_i, y_i) points, form Y as a column matrix of the y_i.

$$X = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ 1 & x_3 \\ & \cdot \\ & \cdot \\ & \cdot \end{bmatrix}$$

$$B = \begin{bmatrix} b \\ m \end{bmatrix}$$

Now the task is to find the parameter B (above, the b and m that determine the straight line) that minimizes the error. The error is the sum of squared difference from the prediction, or the sum of the elements of E squared, or E^TE (if we do not mind conflating the one-element matrix with a scalar). The mathematically attractive properties of the squared-error definition are almost universally taken to compensate for whatever disadvantages it has over what is really meant by error (the absolute value is much harder to calculate with, for example).

To minimize the error, simply differentiate it with respect to the elements of B and set the derivative to 0. The second derivative is positive: this is indeed a minimum. These elementwise derivatives are written tersely in matrix form. First rewrite the error terms:

$$\begin{aligned} E^TE &= (Y - XB)^T(Y - XB) \\ &= Y^TY - B^TX^TY - Y^TXB + B^TX^TXB \\ &= Y^TY - 2B^TX^TY + B^TX^TXB \end{aligned}$$

(here, the combined terms were 1×1 matrices.) Now differentiate: setting the derivative to 0 yields

$$0 = X^T XB - X^T Y$$

and thus

$$B = (X^T X)^{-1} X^T Y = X^\dagger Y$$

where $X^\dagger$ is called the pseudo-inverse of X.

The pseudo-inverse method generalizes to fitting any parametrized model to data (Section A1.9.3). The model should be chosen with some care. For example, Fig. A1.11 shows a disturbing case in which the model above (minimize vertical errors) is used to fit a relatively vertical swarm of points. The "best fit" line in this case is not the intuitive one.

A1.9.2 Principal Axis Method

The principal axes and moments of a swarm of points determine the direction and amount of its dispersion in space. These concepts are familiar in physics as the principal axes and moments of inertia. If a swarm of (possibly weighted) points is translated so that its center of mass (average location) is at the origin, a symmetric matrix M may be easily calculated whose eigenvectors determine the best-fit line or plane in a least-squared-perpendicular-error sense, and whose eigenvalues tell how good the resulting fit is.

Given a set $\{\mathbf{x}^i\}$ row of vectors with weights w^i, define their "scatter matrix" to be the symmetric matrix M, where $\mathbf{x}^i = (x_1^i, x_2^i, x_3^i)$:

$$M = \sum_i \mathbf{x}^{iT} \mathbf{x}^i$$

$$M_{kp} = \sum_i x_k^i x_p^i \qquad 1 \leqslant k, p \leqslant 3$$

Define the dispersion of the $\mathbf{x}^i$ in a direction $\mathbf{v}$ (i.e., "dispersion around the plane whose normal is $\mathbf{v}$") to be the sum of weighted squared lengths of the $\mathbf{x}^i$ in the direction $\mathbf{v}$. This squared error E^2 is

$$E^2 = \sum_i w^i (\mathbf{x}^i \cdot \mathbf{v})^2 = \mathbf{v} \left(\sum_i w^i \mathbf{x}^{iT} \mathbf{x}^i\right) \mathbf{v}^T = \mathbf{v} M \mathbf{v}^T$$

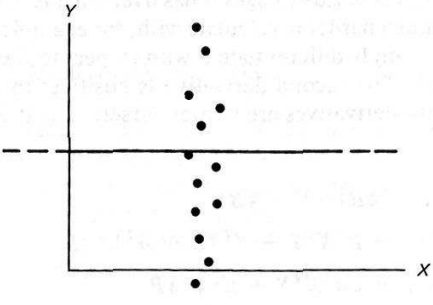

Fig. A1.11 A set of points and the "best fit" line minimizing error in Y.

To find the direction of minimum dispersion (the normal to the best-fit line or plane), note that the minimum of $\mathbf{v}M\mathbf{v}^T$ over all unit vectors $\mathbf{v}$ is the minimum eigenvalue λ_1 of M. If $\mathbf{v}_1$ is the corresponding eigenvector, the minimum dispersion is attained at $\mathbf{v} = \mathbf{v}_1$. The best fit line or plane of the points goes through the center of mass, which is at the origin; inverting the translation that brought the centroid to the origin yields the best fit line or plane for the original point swarm.

The eigenvectors correspond to dispersions in orthogonal directions, and the eigenvalues tell how much dispersion there is. Thus with a three-dimensional point swarm, two large eigenvalues and one small one indicate a planar swarm whose normal is the smallest eigenvector. Two small eigenvalues and one large one indicate a line in the direction of the normal to the "worst fit plane", or eigenvector of largest eigenvalue. (It can be proved that in fact this is the best-fit line in a least squared perpendicular error sense). Three equal eigenvalues indicate a "spherical" swarm.

A1.9.3 Fitting Curves by the Pseudo-Inverse Method

Given a function $f(\mathbf{x})$ whose value is known on n points $\mathbf{x}_1, \ldots, \mathbf{x}_n$, it may be useful is to fit it with a function $g(\mathbf{x})$ of m parameters $(b_1, \ldots, b_m)$. If the squared error at a point x_i is defined as

$$(e_i)^2 = [f(\mathbf{x}_i) - g(\mathbf{x}_i)]^2$$

a sequence of steps similar to that of Section A1.9.1 leads to setting a derivative to zero and obtaining

$$0 = G^T G\mathbf{b} - G^T\mathbf{f}$$

where $\mathbf{b}$ is the vector of parameters, $\mathbf{f}$ the vector of n values of $f(\mathbf{x})$, and

$$G = \frac{\partial \mathbf{g}}{\partial \mathbf{b}} = \begin{bmatrix} \dfrac{\partial g(x_1)}{\partial b_1} & \dfrac{\partial g(x_2)}{\partial b_2} & \cdots \\ \cdot & & \\ \cdot & & \\ \cdot & \cdots & \dfrac{\partial g(x_n)}{\partial b_m} \end{bmatrix}$$

As before, this yields

$$\mathbf{b} = (G^T G)^{-1} G^T\mathbf{f}$$

Explicit least-squares solutions for curves can have nonintuitive behavior. In particular, say that a general circle is represented

$$\mathcal{G}(x, y) = x^2 + y^2 + 2Dx + 2Ey + F$$

this yields values of D, E, and F which minimize

$$e^2 = \sum_{i=1}^{n} \mathcal{G}(x_i, y_i)^2$$

for n input points. The error term being minimized does not turn out to accord with our intuitive one. It gives the intuitive distance of a point to the curve, but weighted by a factor roughly proportional to the radius of the curve (probably not desirable). The best fit criterion thus favors curves with high average curvature, resulting in smaller circles than expected. In fitting ellipses, this error criterion favors more eccentric ones.

The most successful conic fitters abandon the luxury of a closed-form solution and go to iterative minimization techniques, in which the error measure is adjusted to compensate for the unwanted weighting, as follows.

$$e^2 = \sum_{i=1}^{n} \left(\frac{f(x_i, y_i)}{|\nabla f(x_i, y_i)|} \right)^2$$

A1.10 CONICS

The conic sections are useful because they provide closed two-dimensional curves, they occur in many images, and they are well-behaved and familiar polynomials of low degree. This section gives their equations in standard form, illustrates how the general conic equation may be put into standard form, and presents some sample specific results for characterizing ellipses.

All the standard form conics may be subjected to rotation, translation, and scaling to move them around on the plane. These operations on points affect the conic equation in a predictable way.

Circle: r = radius $\quad x^2 + y^2 = r^2$

Ellipse: a, b = major, minor axes $\quad \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

Parabola: $(p, 0)$ = focus, p = directrix $\quad y^2 = 4px$

Hyperbola: vertices $(\pm a, 0)$, asymptotes $y = \pm \left(\dfrac{b}{a} \right) x \quad \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

The general conic equation is

$$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0$$

This equation may be written formally as

$$(x \quad y \quad 1) \begin{bmatrix} A & B & D \\ B & C & E \\ D & E & F \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \mathbf{x} M \mathbf{x}^T = 0$$

Putting the general conic equation into one of the standard forms is a common analytic geometry exercise. The symmetric 3×3 matrix M may be diagonalized, thus eliminating the coefficients B, D, and E from the equation and reducing it to be close to standard form. The diagonalization amounts to a rigid motion that puts the conic in a symmetric position at the origin. The transformation is in fact the 3×3 matrix E whose rows are eigenvectors of M. Recall that if $\mathbf{v}$ is an eigenvector of M,

$$\mathbf{v} M = \lambda \mathbf{v}$$

Then if D is a diagonal matrix of the three eigenvalues, $\lambda_1, \lambda_2, \lambda_3$,

$$EM = DE$$

but then

$$EME^{-1} = DEE^{-1} = D$$

and M has been transformed by a similarity transformation into a diagonal matrix such that

$$\mathbf{x}D\mathbf{x}^T = 0$$

This general idea is of course related to the principal axis calculation given in Section A1.9.2, and extends to three-dimensional quadric surfaces such as the ellipsoid, cone, hyperbolic paraboloid, and so forth. The general result given above has particular consequences illustrated by the following facts about the ellipse. Given a general conic equation representing an ellipse, its center (x_c, y_c) is given by

$$x_c = \frac{BE - 2CD}{B^2 - 4AC}$$

$$y_c = \frac{2EA - BD}{B^2 - 4AC}$$

The orientation is

$$\theta = \tfrac{1}{2}\tan^{-1}\left(\frac{B}{A-C}\right)$$

The major and minor axes are

$$\frac{-2G}{(A + C) \pm [B^2 + (A - C)^2]^{1/2}}$$

where

$$G = F - (Ax_c^2 + B_{x_c y_c} + Cy_c^2)$$

A1.11 INTERPOLATION

Interpolation fits data by giving values between known data points. Usually, the interpolating function passes through each given data point. Many interpolation methods are known; one of the simplest is Lagrangean interpolation.

A1.11.1 One-Dimensional

Given $n + 1$ points (x_j, y_j), $x_0 < x_1 < \cdots < x_n$, the idea is to produce an nth-degree polynomial involving $n + 1$ so-called Lagrangean coefficients. It is

$$f(x) = \sum_{j=0}^{n} L_j(x) y_j$$

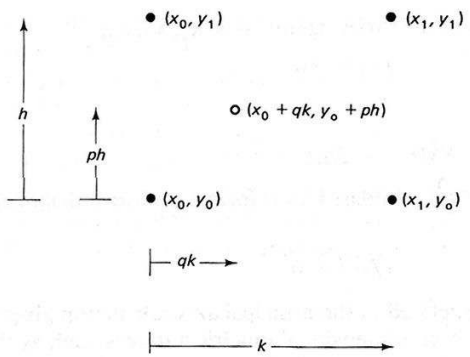

Fig. A1.12 Four point lagrangean interpolation on rectangular grid.

where $L_j(x)$ is the jth coefficient;

$$L_j(x) = \frac{(x-x_0)\ (x-x_1)\ \cdots\ (x-x_{j-1})\ (x-x_{j+1})\ \cdots\ (x-x_n)}{(x_j-x_0)\ (x_j-x_i)\ \cdots\ (x_j-x_{j-1})\ (x_j-x_{j+1})\ \cdots\ (x_j-x_n)}$$

Other interpolative schemes include divided differences, Hermite interpolation for use when function derivatives are also known, and splines. The use of a polynomial interpolation rule can always produce surprising results if the function being interpolated does not behave locally like a polynomial.

A1.11.2 Two-Dimensional

The four-point Lagrangean method is for the situation shown in Fig. A1.12. Let $f_{ij} = f(x_i, y_j)$. Then

$$f(x_0 + qk, y_0 + ph) = (1-p)\ (1-q)\ f_{00} + q(1-p)\ f_{10} + p(1-q)\ f_{01} + pqf_{11}$$

A1.12 THE FAST FOURIER TRANSFORM

The following routine computes the discrete Fourier transform of a one-dimensional complex array XIn of length $N = 2^{\log N}$ and produces the one-dimensional complex array XOut. It uses an array W of the N complex Nth roots of unity, computed as shown, and an array Bits containing a bit-reversal table of length N. N, LogN, W, and Bits are all global to the subroutine as written. If the logical variable Forward is TRUE, the FFT is performed; if Forward is FALSE, the inverse FFT is performed.

```
SUBROUTINE FFT(XIn, KOut, Forward)
GLOBAL W, Bits, N, LogN
LOGICAL Forward
COMPLEX XIn, Xout, W, A, B
INTEGER Bits
ARRAY(0:N) W, Bits, XIn, XOut
```

```
DO (I = 0, N - 1) XOut(I) = XIn(Bits(I))
JOff = N/2
JPnt = N/2
JBk = 2
IOFF = 1
DO (I = 1, LogN)
.    DO (IStart = 0, N - 1, JBk)
.    .    JWPnt = 0
.    .    DO (K = IStart, IStart + IOff - 1)
.    .    .    WHEN (Forward)
.    .    .    .    A = XOut(K + IOff) * W(JWPnt) + XOut(K)
.    .    .    .    B = XOut(K + IOff) * W(JWPnt + JOff) + XOut(K)
.    .    .    ...  FIN
.    .    .    ELSE
.    .    .    .    A = XOut (K + IOff) * CONJG(W(JWPnt)) + XOut(K)
.    .    .    .    B = XOut(K + IOff) * CONJG(W(JWPnt + JOff)) + XOu
.    .    .    ...  FIN
.    .    .    XOut(K) = A
.    .    .    XOut(K + IOff) = B
.    .    .    JWPnt = JWPnt + JPnt
.    .    ...  FIN
.    ...  FIN
.    JPnt = JPnt/2
.    IOff = JBk
.    JBk = JBk * 2
...  FIN
UNLESS (Forward)
.    DO (I = 0, N - 1) XOut(I) = XOut(I)/N
...  FIN
END
```

```
TO INIT-W
.    Pi = 3.14159265
.    DO (K = 0, N - 1)
.    .    Theta = 2 * Pi/N
.    .    W(K) = CMPLX(COS(Theta * K), SIN(Theta * K))
.    ...  FIN
...  FIN
```

```
TO BIT-REV
.    Bits(0) = 0
.    M = 1
.    DO (I = 0, LogN - 1)
.    .    DO (J = 0, M - 1)
.    .    .    Bits(J) = Bits(J) * 2
```

```
.    .    .    Bits(J + M) = Bits(J) + 1
.    .    ...  FIN
.    .    M = M * 2
.    ...  FIN
...  FIN
```

A1.13 THE ICOSAHEDRON

Geodesic dome constructions provide a useful way to partition the sphere (hence the three-dimensional directions) into relatively uniform patches. The resulting polyhedra look like those of Fig. A1.13.

The icosahedron has 12 vertices, 20 faces, and 30 edges. Let its center be at the origin of Cartesian coordinates and let each vertex be a unit distance from the center. Define

$$t, \text{ the golden ratio} = \frac{1+\sqrt{5}}{2}$$

$$a = \frac{\sqrt{t}}{5^{1/4}}$$

$$b = \frac{1}{(\sqrt{t}\ 5^{1/4})}$$

$$c = a + 2b = \frac{1}{b}$$

$$d = a + b = \frac{t^{3/2}}{5^{1/4}}$$

$$A = \text{angle subtended by edge at origin} = \arccos\left(\frac{\sqrt{5}}{5}\right)$$

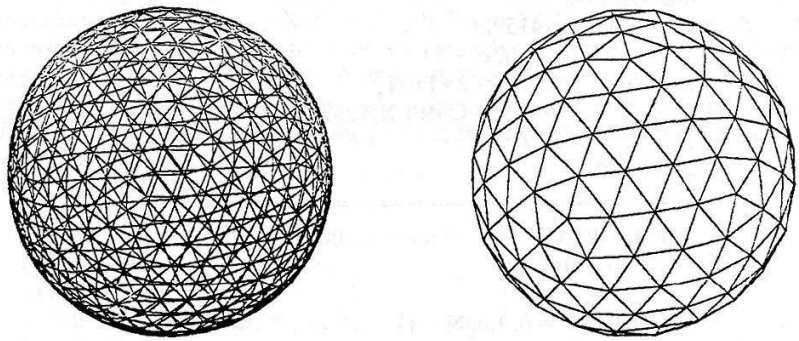

Fig. A1.13 Multifaceted polyhedra from the icosahedron.

Then

angle between radius and an edge $= b = \arccos(b)$
edge length $= 2b$
distance from origin to center of edge $= a$
distance from origin to center of face $= \frac{ta}{\sqrt{3}}$

The 12 vertices may be placed at

$$(0, \pm a, \pm b)$$
$$(\pm b, 0, \pm a)$$
$$(\pm a, \pm b, 0)$$

Then midpoints of the 20 faces are given by

$$\tfrac{1}{3}(\pm d, \pm d, \pm d)$$
$$\tfrac{1}{3}(0, \pm a, \pm c)$$
$$\tfrac{1}{3}(\pm c, 0, \pm a)$$
$$\tfrac{1}{3}(\pm a, \pm c, 0)$$

To subdivide icosahedral faces further, several methods suggest themselves, the simplest being to divide each edge into n equal lengths and then construct n^2 congruent equilateral triangles on each face, pushing them out to the radius of the sphere for their final position. (There are better methods than this if more uniform face sizes are desired.)

A1.14 ROOT FINDING

Since polynomials of fifth and higher degree are not soluble in closed form, numerical (approximate) solutions are useful for them as well as for nonpolynomial functions. The Newton–Raphson method produces successive approximations to a real root of a differentiable function of one variable.

$$x^{i+1} = x^i - \frac{f(x^i)}{f'(x^i)}$$

Here x^i is the ith approximation to the root, and $f(x^i)$ and $f'(x^i)$ are the function and its derivative evaluated at x^i. The new approximation to the root is x^{i+1}. The successive generation of approximations can stop when they converge to a single value. The convergence to a root is governed by the choice of initial approximation to the root and by the behavior of the function in the vicinity of the root. For instance, several roots close together can cause problems.

The one-dimensional form of this method extends in a natural way to solving systems of simultaneous nonlinear equations. Given n functions F_i, each of n parameters, the problem is to find the set of parameters that drives all the functions to zero. Write the parameter vector $\mathbf{x}$.

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix}$$

Form the function column vector **F** such that

$$\mathbf{F}(\mathbf{x}) = \begin{bmatrix} F_1(\mathbf{x}) \\ F_2(\mathbf{x}) \\ \cdot \\ \cdot \\ \cdot \\ F_n(\mathbf{x}) \end{bmatrix}$$

The Jacobean matrix J is defined as

$$J = \begin{bmatrix} \dfrac{\partial F_1}{\partial x_1} & \dfrac{\partial F_1}{\partial x_2} & \cdots & \dfrac{\partial F_1}{\partial x_n} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ \dfrac{\partial F_n}{\partial x_1} & & \cdots & \dfrac{\partial F_n}{\partial x_n} \end{bmatrix}$$

Then the extension of the Newton–Raphson formula is

$$\mathbf{x}^{i+1} = \mathbf{x}^i - J^{-1}(\mathbf{x}^i)F(\mathbf{x}^i)$$

which requires one matrix inversion per iteration.

EXERCISES

A1.1 **x** and **y** are two two-dimensional vectors placed tail to tail. Prove that the area of the triangle they define is $|\mathbf{x} \times \mathbf{y}|/2$.

A1.2 Show that points **q** in a plane defined by the three points **x**, **y**, and **z** are given by

$$\boldsymbol{q} \cdot \Big[(\mathbf{y}-\mathbf{x}) \times (\mathbf{z}-\mathbf{x})\Big] = \mathbf{x} \cdot (\mathbf{y} \times \mathbf{z})$$

A1.3 Verify that the vector triple product may be written as claimed in its definition.

A1.4 Given an arctangent routine, write an arcsine routine.

A1.5 Show that the closed form for the inverse of a 2×2 A matrix is

$$\frac{1}{\det A}\begin{bmatrix} a_{22} & -a_{21} \\ -a_{12} & a_{11} \end{bmatrix}$$

A1.6 Prove by trigonometry that the matrix transformations for rotation are correct.

A1.7 What geometric transformation is accomplished when a_{44} of a geometric transformation matrix A varies from unity?

A1.8 Establish conversions between the given line representations.

A1.9 Write a geometric transform to mirror points about a given plane.

A1.10 What is the line-equation representation of a line $L1$ through a point $\mathbf{x}$ and perpendicular to a line $L2$ (similarly represented)? Parallel to $L2$?

A1.11 Derive the ellipse results given in Section A1.10.

A1.12 Explicitly derive the values of D, E, and F minimizing the error term

$$\sum_{i=1}^{n} [f(x_i, y_i)]^2$$

in the general equation for a circle

$$x^2 + y^2 + 2Dx + 2Ey + F = 0$$

A1.13 Show that if points and lines are transformed as shown in Section A1.7.6, the transformed points indeed lie on the transformed lines.

A1.14 Explicitly derive the least-squared-error solution for lines represented as $ax + by + 1 = 0$.

A1.15 If three planes intersect in a point, is the inverse of

$$\begin{bmatrix} \mathbf{p1} & \mathbf{p2} & \mathbf{p3} & 0 \\ & & & 0 \\ & & & 0 \\ & & & 1 \end{bmatrix}$$

guaranteed to exist?

A1.16 What is the angle between two three-space lines?

A1.17 In two dimensions, show that two lines $\mathbf{u}$ and $\mathbf{v}$ intersect at a point $\mathbf{x}$ given by $\mathbf{x} = \mathbf{u} \times \mathbf{v}$.

A1.18 How can you tell if two line segments (defined by their end points) intersect in the plane?

A1.19 Find a 4×4 matrix that transforms an arbitrary direction (or point) to lie on the Z axis.

A1.20 Derive a parametric representation for planes based on three points lying in the plane.

A1.21 Devise a scheme for interpolation on a triangular grid.

A1.22 What does the homogeneous point $(x, y, z, 0)$ represent?

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Advanced Control Mechanisms

Appendix 2

This appendix is concerned with specific control mechanisms that are provided by programming languages or that may be implemented on top of existing languages as aids to doing computer vision. The treatment here is brief; our aim is to expose the reader to several ideas for control of computer programs that have been developed in the artificial intelligence context, and to indicate how they relate to the main computational goals of computer vision.

A2.1 STANDARD CONTROL STRUCTURES

For completeness, we mention the control mechanisms that are provided as a matter of course by conventional research programming languages, such as Pascal, Algol, POP-2, SAIL, and PL/1. The influential language LISP, which provides a base language for many of the most advanced control mechanisms in computer vision, ironically is itself missing (in its pure form) a substantial number of these more standard constructs. Another common language missing some standard control mechanisms is SNOBOL. These standard constructions are so basic to the current conception of a serial von Neumann computer that they are often realized in the instruction set of the machine. In this sense we are almost talking here of computer hardware.

The standard mechanisms are the following:

1. *Sequence.* Advance the program counter to the next intruction.
2. *Branch instruction.* Go to a specific address.
3. *Conditional branch.* Go to a specific address if a condition is true, otherwise, go to the next instruction.
4. *Iteration.* Repeat a sequence of instructions until a condition is met.

5. *Subroutines.* Go to a certain location; execute a set of instructions using a set of supplied parameters; then return to the next instruction after the subroutine call.

All the standard control structures should be in the toolkit of a programmer. They will be used, together with the data structures and data types supplied in the working language, to implement other control mechanisms. The remainder of this appendix deals with "nonstandard" control mechanisms; those not typically provided in commercial programming languages and which have no close correlates in primitive machine instructions. Nonstandard control mechanisms, although not at all domain-specific, have developed to meet needs that are not the "lowest common denominator" of computer programming. They impose their own view of problem decomposition just as do the standard structures.

Less standard mechanisms are *recursion* and *co-routining.* Co-routining can be thought of as a form of recursion.

A2.1.1 Recursion

Recursion obeys all the constraints of subroutining, except that a routine may call upon "itself." The user sees no difference between recursive and nonrecursive subroutines, but internally recursion requires slightly more bookkeeping to be performed in the language software, since typically the hardware of a computer does not extend to managing recursion (although some machines have instructions that are quite useful here).

A typical use of a recursive control paradigm in computer vision might be:

```
To Understand-Scene (X);
(
If Immediately-Apparent(X)
then Report-Understanding-Of(X);
else
   (SimplerParts ← Decompose(X);
   ForEach Part in SimplerParts
      Understand-Scene(Part);
   )
[ )
```

Recursion is an elegant way to specify many important algorithms (such as tree traversals), but in a way it has no conceptual differences from subroutining. A routine is broken up into subroutines (some of which may involve smaller versions of the original task); these are attacked sequentially, and they must finish before they return control to the routine that invokes them.

A2.1.2 Co-Routining

Co-routines are simply programs that can call (invoke) each other. Most high-level languages do not directly provide co-routines, and thus they are a nonstandard control structure. However, co-routining is a fundamental concept [Knuth 1973]

and serves here as a bridge between standard and nonstandard control mechanisms.

Subroutines and their calling programs have a "slave–master" aspect: control is always returned to the master calling program after the subroutine has carried out its job. This mechanism not only leads to efficiencies by reducing the amount of executable code, but is considered to be so useful that it is built into the instruction set of most computers. The pervasiveness of subroutining has subtle effects on the approach to problem decomposition, encouraging a hierarchical subproblem structure. The co-routine relationship is more egalitarian than the subroutine relationship. If co-routine A needs the services of co-routine B, it can call B, and (here is the difference) conversely, B can call A if B needs A's services.

Here is a simple (sounding) problem [Floyd 1979]: "Read lines of text, until a completely blank line is found. Eliminate redundant blanks between the words. Print the text, 30 characters to a line, without breaking words between lines." This problem is hard to program elegantly in most languages because the iterations involved do not nest well (try it!). However, an elegant solution exists if the job is decomposed into three co-routines, calling each other to perform input, formatting, and output of a character stream.

A useful paradigm for problem solving, besides the strictly hierarchical, is that of a "heterarchical" community of experts, each performing a job and when necessary calling on other experts. A heterarchy can be implemented by co-routines. Many of the nonstandard mechanisms discussed below are in the spirit of co-routines.

A2.2 INHERENTLY SEQUENTIAL MECHANISMS

A2.2.1 Automatic Backtracking

The PLANNER language [Hewitt 1972] implicitly implemented the feature of "automatic backtracking." The advisability of uniformly using this technique, which is equivalent to depth-first search, was questioned by those who wished to give the programmer greater freedom to choose which task to activate next [Sussman and McDermott 1972].

A basic backtracking discipline may be provided by recursive calls, in which a return to a higher level is a "backtrack." The features of automatic backtracking are predicated on an ability to save and reinstate the computational state of a process automatically, without explicit specification by the programmer.

Automatic backtracking has its problems. One basic problem occurs in systems that perform inferences while following a particular line of reasoning which may ultimately be unsuccessful. The problem is that along the way, perhaps many perfectly valid and useful computations were performed and many facts were added to the internal model. Mixed in with these, of course, are wrong deductions which ultimately cause the line of reasoning to fail. The problem: After having restored control to a higher decision point after a failure is noticed, how is the system

to know which deductions were valid and which invalid? One expensive way suggested by automatic backtracking is to keep track of all hypotheses that contributed to deriving each fact. Then one can remove all results of failed deduction paths. This is generally the wrong thing to do; modern trends have abandoned the automatic backtracking idea and allow the programmer some control over what is restored upon failure-driven backtracking. Typically, a compromise is implemented in which the programmer may mark certain hypotheses for deletion upon backtracking.

A2.2.2 Context Switching

Context switching is a general term that is used to mean switching of general process state (a control primitive) or switching a data base context (a data access primitive). The two ideas are not independent, because it could be confusing for a process to put itself to sleep and be reawakened in a totally different data context.

Backtracking is one use of general control context switching. The most general capability is a "general GO TO." A regular GO TO allows one to go only to a particular location defined in a static program. After the GO TO, all bindings and returnpoints are still determined by the current state of processing. In contrast, a general GO TO allows a transfer not only across program "space," but through program "time" as well. Just as a regular GO TO can go to a predefined program label, a general GO TO can go to a "tag" which is created to save the entire state of a process. To GO TO such a tag is to go back in time and recreate the local binding, access, control, and process state of the process that made the tag.

A good example of the use of such power is given in a problem-solving program that constructs complex structures of blocks [Fahlman 1974].

A2.3 SEQUENTIAL OR PARALLEL MECHANISMS

Some language constructs explicity designate parallel computing. They may actually reflect a parallel computing environment, but more often they control a simulated version in which several control paths are maintained and multi-processed under system control. Examples here are module and message primitives given below and statements such as the CO-BEGIN, CO-END pairs which can bracket notionally parallel blocks of code in some Algol-like language extensions.

A2.3.1 Modules and Messages

Modules and messages form a useful, versatile control paradigm that is relatively noncommittal. That is, it forces no particular problem decomposition or methodological style on its user, as does a pure subroutine paradigm, for example. Message passing is a general and elegant model of control which can be used to subsume others, such as subroutining, recursion, co-routining, and parallelism [Feldman 1979].

There are many antecedents to the mechanism of modules communicating by messages described here. They include [Feldman and Sproull 1971; Hewitt and

Smith 1975; Goldberg and Kay 1976; Birtwhistle et al. 1973]. In the formulation presented by Hewitt, the message-passing paradigm can be extended down into the lowest level of machine architecture. The construction outlined here [Feldman 1979] is more moderate, since in it the base programming language may be used with its full power, and itself is not module and message based.

A program is made up of *modules.* A module is a piece of code with associated local data. The crucial point is that the internal state of a module (e.g. its data) is not accessible to other modules. Within a module, the base programming language, such as Algol, may be used to its full power (subroutine calls, recursion, iteration, and so forth are allowed). However, modules may not in any sense "call upon" each other. Modules communicate only by means of *messages.* A module may send a message to another module; the message may be a request for service, an informational message, a signal, or whatever. The module to whom the message is sent may, when it is ready, receive the message and process it, and may then itself send messages either to the original module, or indeed to any combination of other modules.

The module–message paradigm has several advantages over subroutine (or co-routine) calls.

1. If subroutines are in different languages, the subroutine call mechanisms must be made compatible.
2. Any sophisticated lockout mechanism for resource access requires the internal coding of queues equivalent to that which a message switcher provides.
3. A subroutine that tries to execute a locked subroutine is unable to proceed with other computation.
4. Having a resource always allocated by a single controlling module greatly simplifies all the common exclusion problems.
5. For inherently distributed resources, message communication is natural. Module-valued slots provide a very flexible but safe discipline for control transfers.

Another view of messages is as a generalization of parameter lists in subroutine or coroutine calls. The idea of explicitly naming parameters is common in assembly languages, where the total number of parameters to a routine may be very large. More important, the message discipline presents to a module a collection of suggested parameters rather than automatically filling in the values of parameters. This leads naturally to the use of semantic checks on the consistency of parameters and to the use of default values for unspecified ones, which can be a substantial improvement on type checking. The use of return messages allows multiple-valued functions; an answer message may have several slots. Messages solve the so-called "uniform reference problem"—one need not be concerned with whether an answer (say an array element) is computed by a procedure or a table.

There is yet another useful view of messages. One can view a message as a partially specified relation (or pattern), with some slot values filled in and some unbound. This is common in relational data bases [Astrahan et al. 1976] and artificial intelligence languages [Bobrow and Raphael 1974]. In this view, a mes-

sage is a task specification with some recipient and some complaint departments to talk to about it. Various modules can attempt to satisfy or contract out parts of the task of filling in the remaining slots. A module may handle messages containing slots unknown to it. This allows several modules to work together on a task while maintaining locality. For example, an executive module could route messages (on the basis of a few slots that it understands) to modules that deal with special aspects of a problem using different slots in the message.

There is no apparent conflict between these varying views of messages. It is too early in their development to be sure, but the combined power of these paradigms seems to provide a qualitative improvement in our ability to develop vision programs.

A2.3.2 Priority Job Queue

In any system of independent processes on a serial computer, there must be a mechanism for scheduling activation. One general mechanism for accomplishing scheduling is the priority job queue. Priority queues are a well-known abstraction [Aho et al. 1974]. Informally, a priority job queue is just an ordered list of processes to be activated. A monitor program is responsible for dequeueing processes and executing them; processes do not give control directly to other processes, but only to the monitor. The only way for a process to initiate another is to enqueue it in the job queue. It is easiest to implement a priority job queue if processes are definable entities in the programming language being used; in other words, programs should be manipulable datatypes. This is possible in LISP and POP-2, for example.

If a process needs another job performed by another process, it enqueues the sub job on the job queue and *suspends* itself (it is *deactivated*, or put to sleep). The sub job, when it is dequeued and executed by the monitor, must explicitly enqueue the "calling" process if a subroutining effect is desired. Thus along with usual arguments telling a job what data to work on, a job queue discipline implies passing of control information.

Job queues are a general implementational technique useful for simulating other types of control mechanisms, such as active knowledge (Chapter 12). Also, a job queue can be used to switch between jobs which are notionally executing in parallel, as is common in multiprocessing systems. In this case sufficient information must be maintained to start the job at arbitrary points in its execution.

An example of a priority job queue is a program [Ballard 1978] that locates ribs in chest radiographs. The program maintains a relational model of the ribcage including geometric and procedural knowledge. Uninstantiated model nodes corresponding to ribs might be called hypotheses that those ribs exist. Associated with each hypothesis is a set of procedures that may, under various conditions, be used to verify it (i.e., to find a rib). Procedures carry information about preconditions that must be true in order that they may be executed, and about how to compute estimates of their utility once executed. These descriptive components allow an executive program to rank the procedures by expected usefulness at a given time.

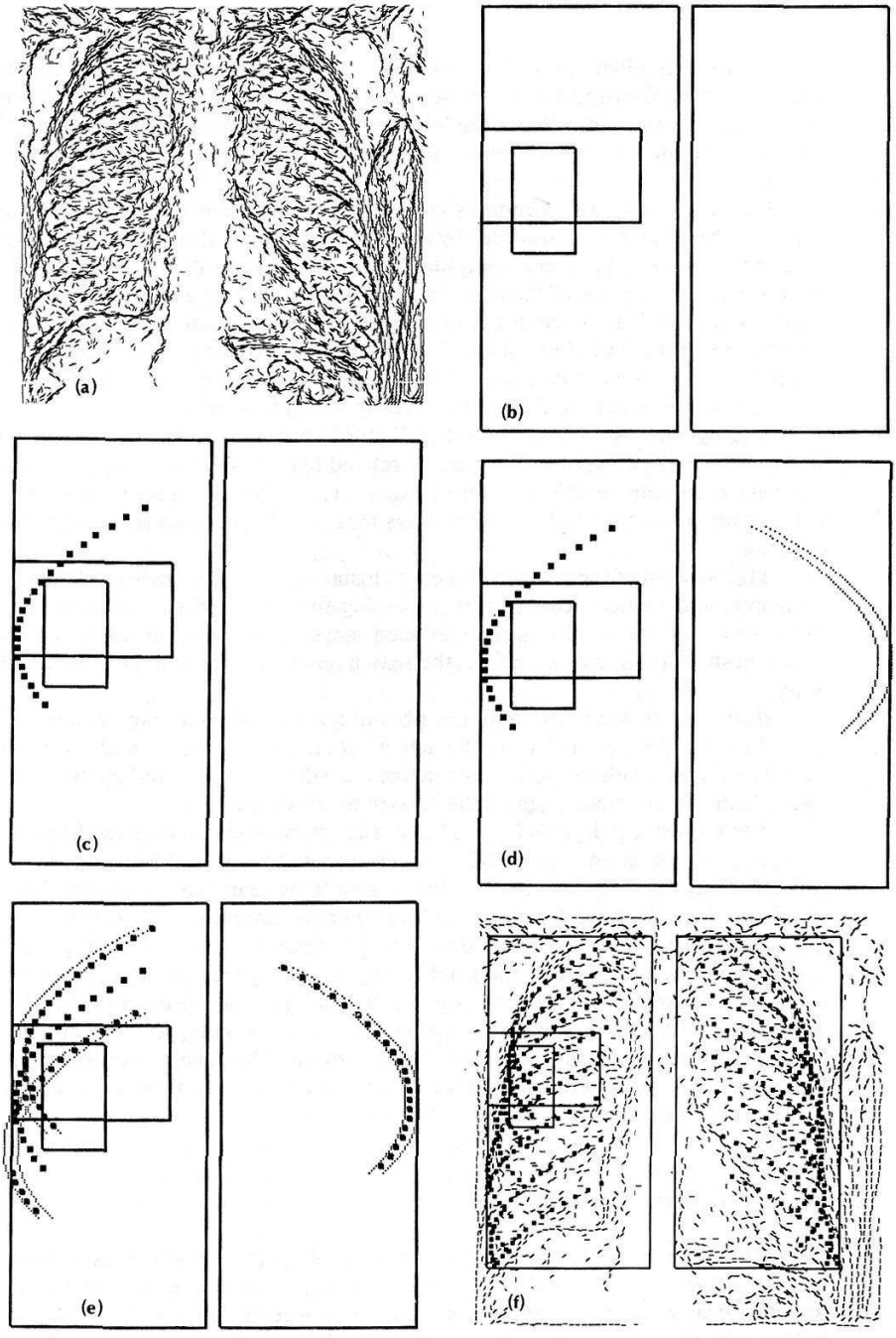

Fig. A2.1 The rib-finding process in action (see text).

There is an initial action that is likely to succeed (locating a particular rib that is usually obvious in the x-ray). In heterarchical fashion, further actions use the results of previous actions. Once the initial rib has been found, its neighbors (both above and below and directly across the body midline) become eligible for consideration.

Eligible rib-finding procedures correspond to short-term plans; they are all put on a job queue to be considered by an *executive* program that must compute the expected utility of expending computational energy on verifying one of the hypotheses by running one of the jobs. The executive computes a priority on the jobs based on how likely they are to succeed, using the utility functions and parameters associated with the individual nodes in the rib model (the individual hypotheses) and the current state of knowledge. The executive not only picks a hypothesis but also the procedure that should be able to verify it with least effort.

The hypothesis is either "verified," "not-verified," or "some evidence is found." Verifying a hypothesis results in related hypotheses (about the neighboring ribs) becoming eligible for consideration. The information found during the verification process is used in several ways that can affect the utility of other procedures.

The position of the rib with respect to instantiated neighbors is used to adjust horizontal and vertical scale factors governing the predicted size of the ribcage. The position of the rib affects the predicted range of locations for other unfound ribs. The shape of the rib also affects the search region for uninstantiated rib neighbors.

If some evidence is found for the rib, but not enough to warrant an instantiation, the rib hypothesis is left on the active list and the rib model node is not instantiated. Rib hypotheses left on the active list will be reconsidered by the executive, which may try them again on the basis of new evidence.

The sequence of figures (Fig. A2.1, p. 503) shows a few steps in the finding of ribs using this program. Figure A2.1a shows the input data. A2.1b shows rectangles enclosing the lung field and the initial area to be searched for a particular rib which is usually findable. Only one rib-finding procedure is applicable for ribs with no neighbors found, so it is invoked and the rib shown by dark boxes in Fig. A2.1b is found. Predicted locations for neighboring ribs are generated and are used in order by the executive which invokes the rib-finding procedures in order of expected utility (A2.1c-e). Predicted locations are shown by dots, actual locations by crosses; in Fig. A2.1f, all modelled ribs are found. The type of procedure that found the rib is denoted by the symbol used to draw in the rib. Figure A2.1f shows the final rib borders superimposed on the data.

A2.3.3 Pattern Directed Invocation

Considerable attention has been focused recently on pattern directed systems (see, e.g., [Waterman and Hayes-Roth 1978]). Another common example of a pattern directed system is the production system, discussed in Section 12.3. The idea behind a pattern directed system is that a procedure will be activated not when its

name is invoked, but when a key situation occurs. These systems have in common that their activity is guided by the appearance of "patterns" of data in either input or memory. Broadly construed, all data forms patterns, and hence patterns guide any computation. This section is concerned with a definition of patterns as something very much smaller than the entire data set, together with the specification of control mechanisms that make use of them.

Pattern directed systems have three components.

1. A data structure or data base containing modifiable items whose structure may be defined in terms of patterns
2. Pattern-directed modules that match patterns in the data structure
3. A controlling executive that selects modules that match patterns and activates them

A popular name for a pattern-directed procedure is a *demon.* Demons were named originally by Selfridge [Selfridge 1959]. They are used successfully in many AI programs, notably in a natural language understanding system [Charniak 1972]. Generally, a demon is a program which is associated with a *pattern* that describes part of the knowledge base (usually the pattern is closely related to the form of "items" in a data base). When a part of the knowledge base matching the pattern is added, modified, or deleted, the demon runs "automatically." It is as if the demon were constantly watching the data base waiting for information associated with certain patterns to change. Of course, in most implementations on conventional computers, demons are not always actively watching. Equivalent behavior is simulated by having the demons register their interests with the system routines that access the data base. Then upon access, the system can check for demon activation conditions and arrange for the interested demons to be run when the data base changes.

Advanced languages that support a sophisticated data base often provide demon facilities, which are variously known as if-added and if-removed procedures, antecedent theorems, traps, or triggers.

A2.3.4 Blackboard Systems

In artificial intelligence literature, a "blackboard" is a special kind of globally accessible data base. The term first became prominent in the context of a large pattern directed system to understand human speech [Erman and Lesser 1975; Erman et al. 1980]. More recently, blackboards have been used as a vision control system [Hanson and Riseman 1978]. Blackboards often have mechanisms associated with them for invoking demons and synchronizing their activities. One can appreciate that programming with demons can be difficult. Since general patterns are being used, one can never be sure exactly when a pattern directed procedure will be activated; often they can be activated in incorrect or bizarre sequences not anticipated by their designer. Blackboards attempt to alleviate this uncertainly by controlling the matching process in two ways:

1. Blackboards represent the current part of the model that is being associated with image data;

2. Blackboards incorporate rules that determine which specialized subsystems of demons are likely to be needed for the current job. This structuring of the data base of procedures increases efficiency and loosely corresponds to a "mental set."

These two ideas are illustrated by Figs. A2.2 and A2.3 [Hanson and Riseman 1978]. Figure A2.2 shows the concept of a blackboard as a repository for only model-image bindings. Figure A2.3 shows transformations between model entities that are used to select appropriate groups of demons.

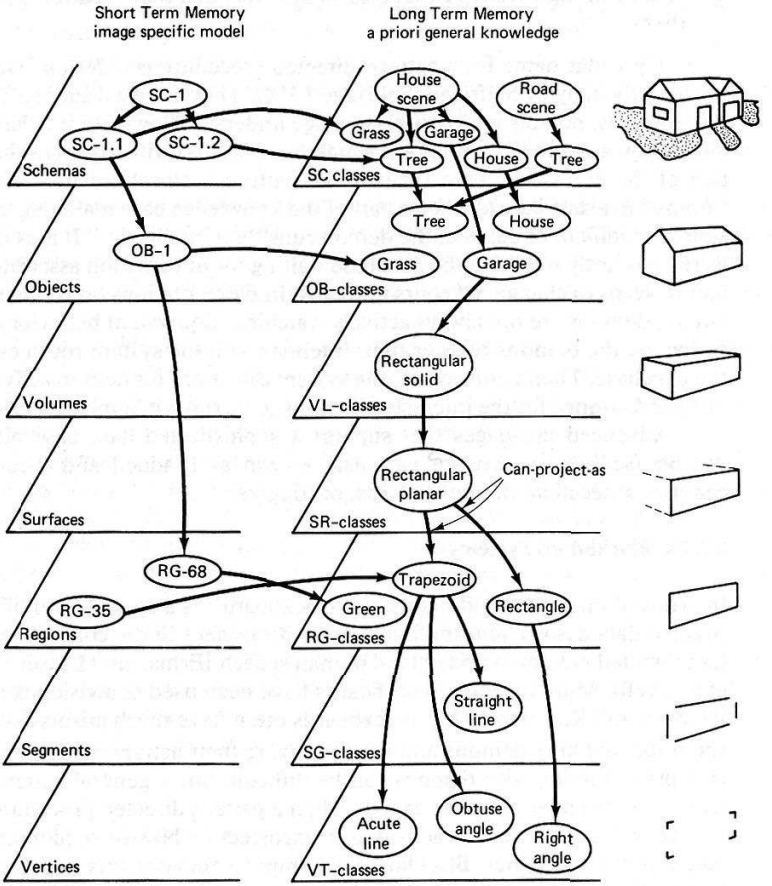

Fig. A2.2 An implementation of the blackboard concept. Here the blackboard is called Short Term Memory; it holds a partial interpretation of a specific image.

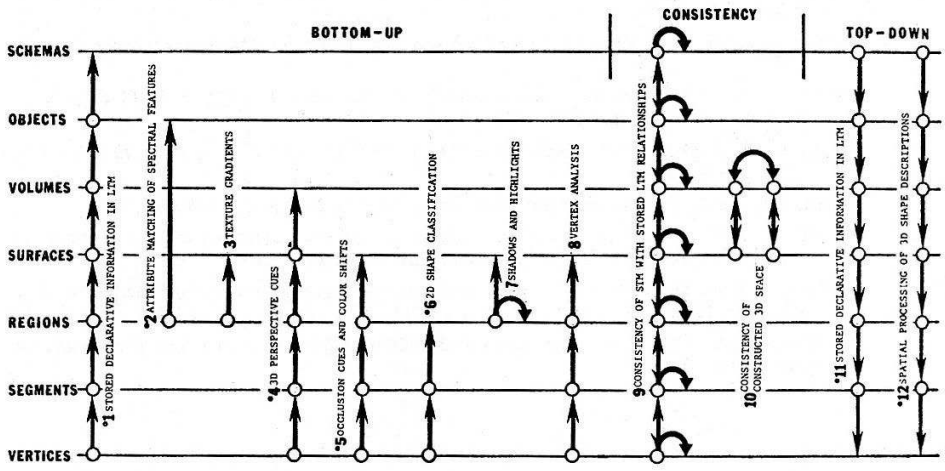

Fig. A2.3 Paths for hypothesis flow, showing transformations between model entities and the sorts of knowledge needed for the transformations.

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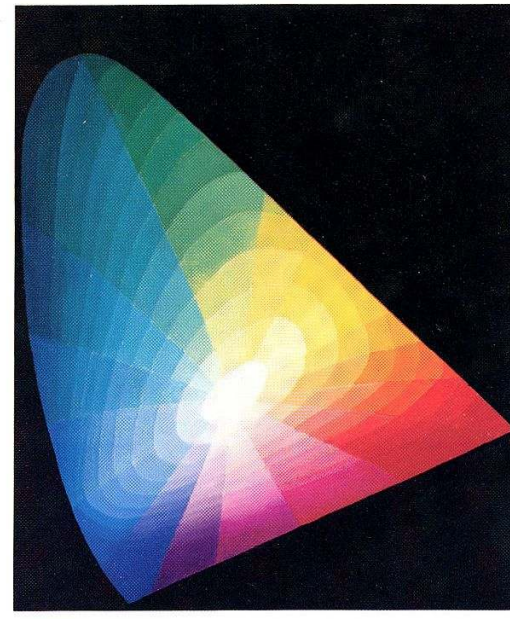
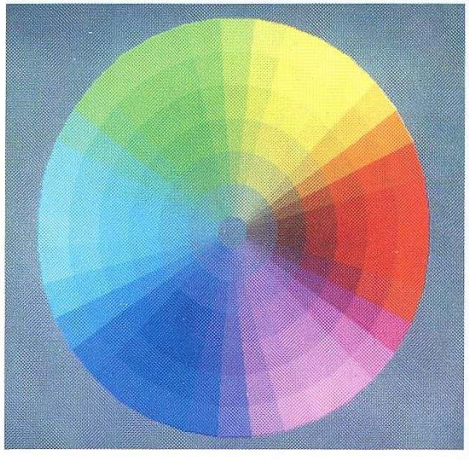
FIG. 2-7a



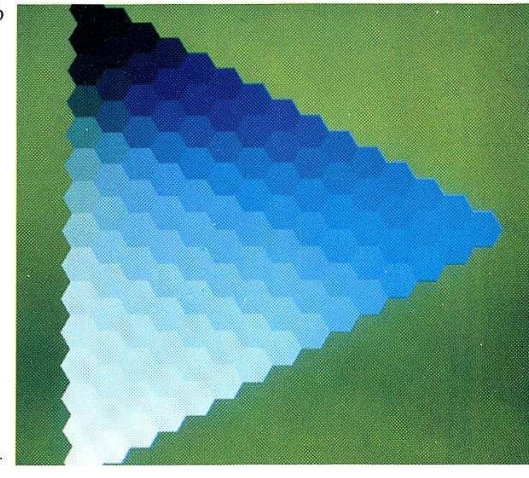
A painting by Louis Condax; courtesy of Eastman Kodak Company and the Optical Society of America.

FIG. 2-8a



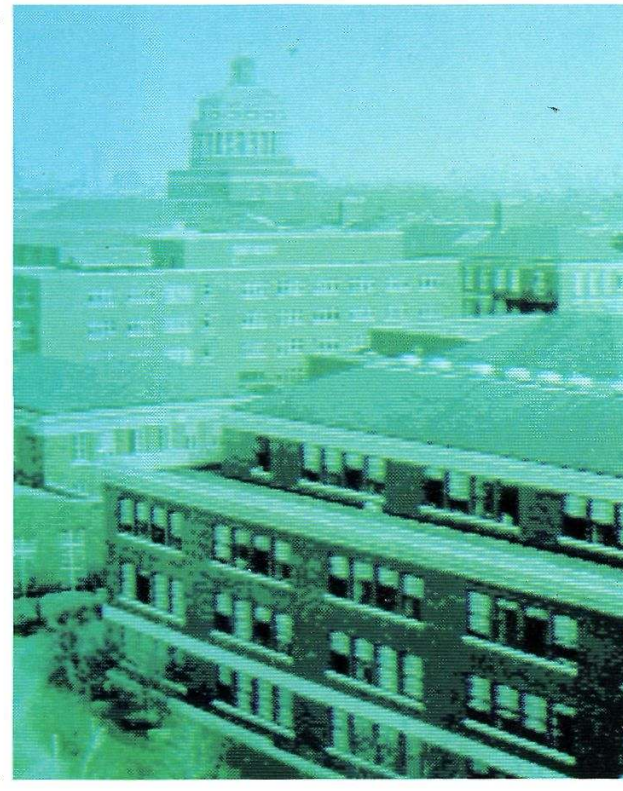
Courtesy of D. Greenberg and G. Joblove, Cornell Program of Computer Graphics.

FIG. 2-8b



Courtesy of Tom Check.

FIG. 5-4a



Courtesy of Sam Kapilivsky.

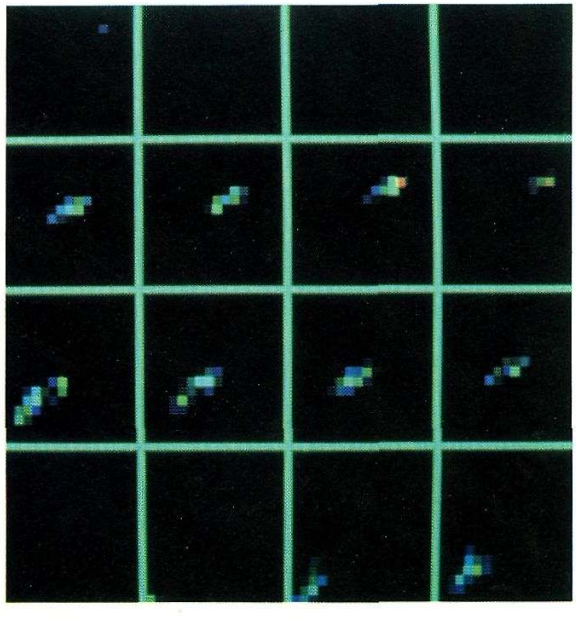


FIG. 5-4b

Courtesy of Sam Kapilivsky.

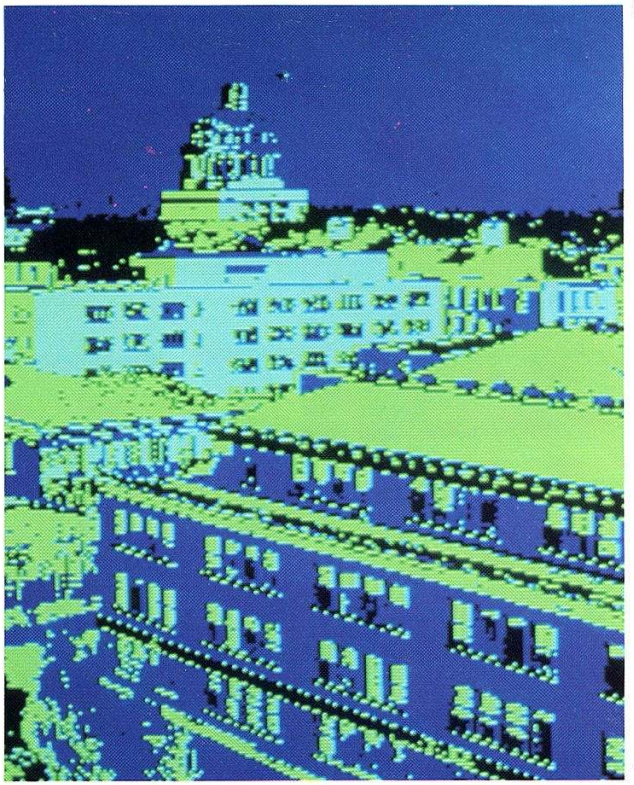
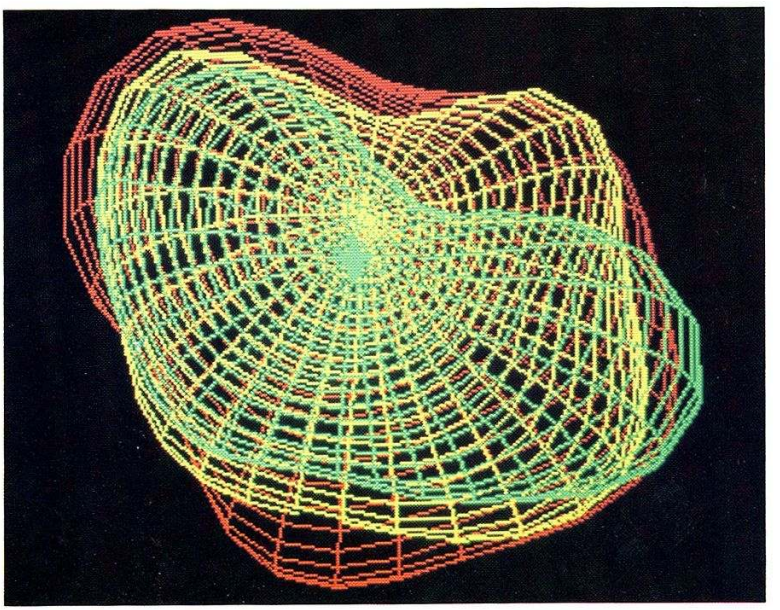


FIG. 5-4c

Courtesy of Sam Kapilivsky.

FIG. 9-10



Courtesy of Robert Schudy.

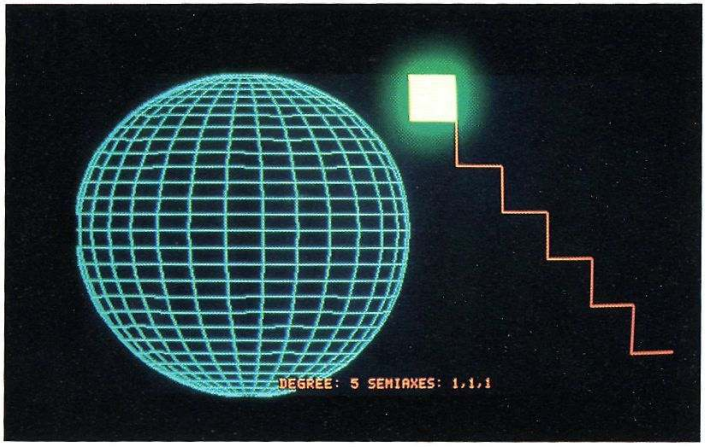
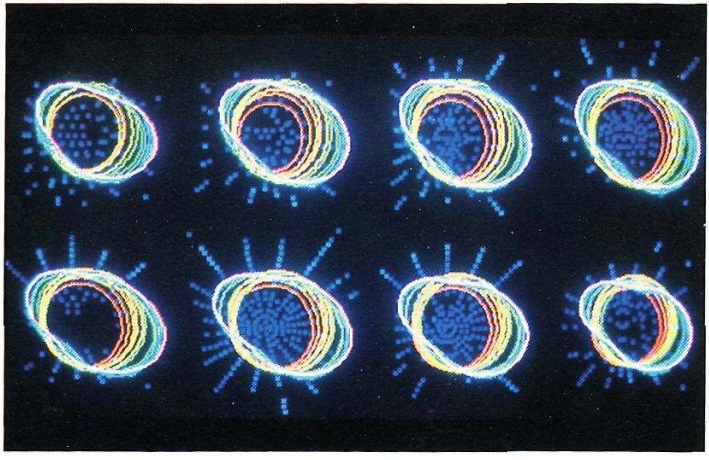


FIG. 11-3a

Courtesy of Robert Schudy.

FIG. 11-3b



Courtesy of Robert Schudy.

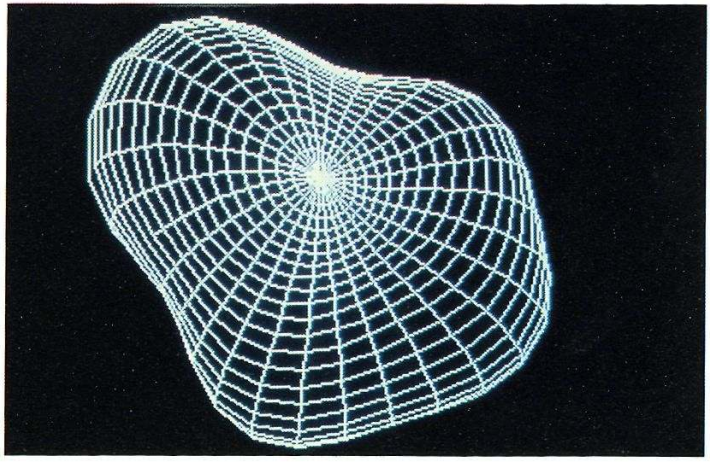


FIG. 11-3c

Courtesy of Robert Schudy.

COMPUTER VISION

DANA H. BALLARD ▪ CHRISTOPHER M. BROWN

What information about scenes can be extracted from an image using only basic assumptions about physics and optics?

How are images segmented into meaningful parts?

At what stage must domain-dependent, prior knowledge about the world be incorporated into the understanding process?

How are world models and conceptual knowledge represented and used?

These and many other questions, inherent in this relatively new and fast-growing field, are explored and answered in **Computer Vision** by **Dana H. Ballard** and **Christopher M. Brown**. The authors assemble crucial material from many disciplines including artificial intelligence, psychology, computer graphics, and image processing to form a practical text and reference for anyone involved in building vision systems.

Ballard and Brown write in their preface, "**Computer Vision** has a strong artificial intelligence flavor, and we hope this will provoke thought. The text shows how both intrinsic image information and internal models of the world are important in successful vision systems."

Divided into four parts, **Computer Vision** offers descriptions of objects at four levels of abstraction:

- Generalized images—images and image-like entities;
- Segmented images—images organized into sub-images that are likely to correspond to "interesting objects";
- Geometrical structures—quantitative models of image and world structures;
- Relational structures—complex symbolic descriptions of image and world structures.

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