Global Positioning System: The Mathematics of GPS Receivers

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Introduction

GPS satellite navigation, with small hand-held receivers, is widely used by military units, surveyors, sailors, utility companies, hikers, and pilots. Such units are even available in many rental cars. We will consider the mathematical aspects of three questions concerning satellite navigation.

How does a GPS receiver use satellite information to determine our position?

Why does the determined position change with each new computation, even though we are not moving?

What is done to improve the accuracy of these varying positions?

We will see that receivers use very simple mathematics, but that they use it in highly ingenious ways.

Being able to locate our position on the surface of the earth has always been important for commercial, scientific, and military reasons. The development of navigational methods has provided many mathematical challenges, which have been met and overcome by some of the best mathematicians of all time.

Navigation by means of celestial observation, spherical trigonometry, and hand computation had almost reached its present form by the time of Captain James Cook's 1779 voyage to the Hawaiian Islands. For the next 150 years these methods were used to determine our location on land or sea. In the 1940s electronic navigation began with the use of fixed, land-based, radio transmitters. The present-day *LOng RAnge Navigation* (LORAN-C) system uses sequenced chains of such transmitters.

The use of satellites in navigation became common in the 1970s, with the introduction of the *Navy Navigation Satellite System* (NAVSAT or TRANSIT). This system uses the Doppler shift in radio frequencies to determine lines of position and locations.

The Satellites

Almost all satellite navigation now uses the *Global Positioning System* (GPS). This system, operated by the United States Department of Defense, was developed in the 1980s and became fully operational in 1995. The system uses a constellation of satellites transmitting on radio frequencies, 1227.60 mHz and 1575.42 mHz.

The original design of the system provided for eighteen satellites, with three satellites in each of six orbits. Currently, there are four satellites in each orbit. In the basic plan, the six orbits are evenly spaced every 60° around the Earth, in planes that

FIGURE 1
The System of Satellites.

are inclined at 55° from the Equator. Orbits are circular, at a rather high altitude of 20,200 kilometers above the surface of the Earth, with periods of twelve hours. FIGURE 1 displays one configuration of the basic eighteen satellites. Although not drawn to scale, it gives the correct feeling that we are living inside a cage of orbiting satellites, several of which are "visible" from any point on the surface of the Earth at any given time.

Receivers

Current GPS receivers are electronic marvels. They are hand-held, run on small batteries, weigh as little as nine ounces, and can cost under $150. We can turn on a receiver at any point on or above the surface of the Earth and, within a few minutes, see a display showing our latitude, longitude, and altitude. The indicated surface position is usually accurate to within 100 meters, and the altitude is usually in error by no more than 160 meters.

How does a small radio receiver listen to a group of satellites, and then compute our position, with great accuracy? We start by noting exactly what sort of information is received from the satellites. Each satellite sends signals, on both of its frequencies, giving (i) its position and (ii) the exact times at which the signals were transmitted.

The receiver also picks up time signals from the satellites, and uses them to maintain its own clock. When a signal comes in from a satellite, the receiver records the difference, Δt, in the time at which the signal was transmitted and the time at which it was received. If the Earth had no atmosphere, the receiver could use the speed, c, of radio waves in a vacuum to compute our distance $d = c \cdot \Delta t$ from the known position of the satellite. This information would suffice to show that we are located at some point on a huge sphere of radius d, centered at the point from which the satellite transmitted. However, the layer of gasses surrounding the Earth slows down radio waves and, therefore, distorts the measurement of distance. Receivers can partially correct for this by allowing for the effect of mean atmospheric density and thickness. Information from several satellites is combined to give the coordinates—latitude, longitude, and altitude—of our position in any selected reference system.

Several factors restrict the accuracy of this process, including: (i) errors in the determined positions of the satellites; (ii) poor satellite positioning; (iii) limitations on the precision with which times and distances can be measured; and (iv) the varying density of Earth's atmosphere and the angles at which the radio signals pass through the atmosphere. Some of these difficulties are overcome by the use of an ingenious plan that provides the key to GPS technology. It is rather complicated to explore this method in the actual setting of positioning on the Earth: The distances are large, the time differences are small, and the geometry is all in three dimensions. Fortunately, we can capture most of the salient features of GPS receiver operation in a simple two-dimensional model.

A Simple Model

Suppose that you are standing somewhere in a circular lot, with a radius of 100 ft. The lot is paved, except for an irregularly-shaped gravel plot that surrounds you. The mean distance from your position to the edge of the gravel is 20 ft. Cars circle the lot on a road. To determine your position, messengers leave from cars on the road and walk straight toward you. When such a messenger arrives, he tells you where and at what time he left the road. You have a watch and know that all messengers walk at a rate of 5 ft/sec on pavement but slow down to 4 ft/sec on gravel. Our model is shown in FIGURE 2.

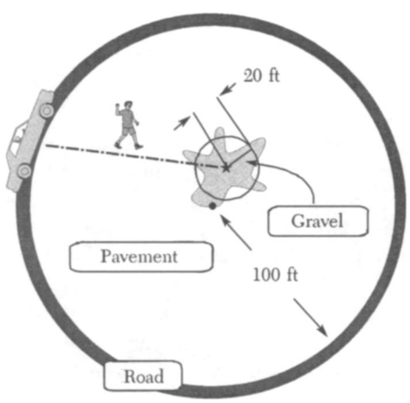

FIGURE 2
The Model.

Consider a rectangular coordinate system with its origin at the center of the lot. Distances will be measured to tenths of a foot, and time will be measured to tenths of a second. The location of a point on the road will be described by its angular distance from due north, measured in a clockwise direction.

At noon a messenger leaves a position 45° from north. When he arrives, your watch shows that it is 20.2 seconds after noon. Since you have no way to know the exact distance that he walked on the gravel, you assume that he covered the mean distance of 20 ft. At 4 ft/sec, this took him 5 sec. For the remaining 15.2 sec he walked on pavement, covering $5\frac{\text{ft}}{\text{sec}} \times 15.2 \text{ sec} = 76.0$ ft. Allowing for the assumed distance of 20 ft on the gravel, you know that you are located at some point on a circle of radius 96.0 ft, centered at the starting location of the messenger.

A second messenger leaves the road at a point 135° from north at 12:01 pm and walks to your position. On his arrival, your watch shows that it is 29.5 sec after he started. Assuming that he took 5 sec to walk 20 ft on the gravel, he walked 5 ft/sec 24.5 sec = 122.5 ft on the pavement. Hence, you are on a circle of radius 142.5 ft, centered at this messenger's point of departure.

The coordinates of the departure points for the two messengers are $P_1 = (100 \cdot \sin 45°, 100 \cdot \cos 45°)$ and $P_2 = (100 \cdot \sin 135°, 100 \cdot \cos 135°)$, respectively. Using our precision of one tenth of a foot, these are rounded to (70.7 70.7) and (70.7, − 70.7). Thus, the coordinates (x_0, y_0), of your position satisfy

$$\left\{ \begin{array}{l} (x_0 - 70.7)^2 + (y_0 - 70.7)^2 = 96.0^2 \\ (x_0 - 70.7)^2 + (y_0 + 70.7)^2 = 142.5^2 \end{array} \right\}.$$

The system has two solutions, (−20.0, 39.2) and (161.4, 39.2), rounded to tenths. Since the latter point is outside of the lot, you can conclude that you are located 20.0 ft west and 39.2 ft north of the center of the lot. The situation is shown in FIGURE 3.

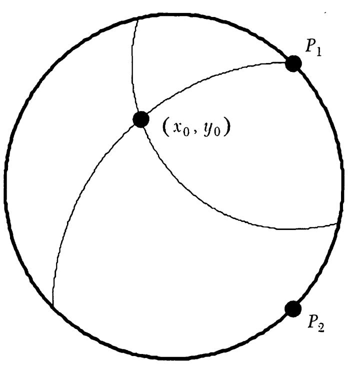

FIGURE 3
Two Messengers.

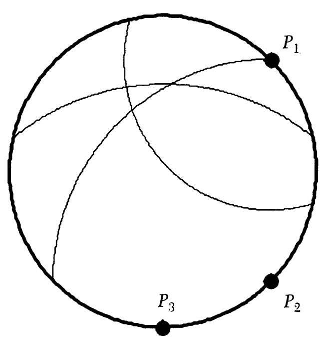

FIGURE 4
Three Messengers.

So far so good. Suppose that, just to be careful, you decide to check your position by having a third messenger leave the road at a point 180° from north and walk to your location. He leaves at 12:02 pm and, according to your watch, arrives 32.2 sec later. As before, you compute your distance from this departure point P_3. FIGURE 4 shows the result of adding information from the third messenger to your picture.

What has happened? *The most likely problem is that your watch does not agree with the times used at the departure points on the road.* Suppose that your watch runs steadily but has a fixed error of ε seconds, where a positive ε means that your watch is ahead of the road times and a negative ε means that your watch is behind the road times. If we let Δt be the time difference between departure and arrival, as shown on your watch, then the estimate for the distance traveled is

$$d(\Delta t, \varepsilon) = 20 \text{ ft} + (\Delta t \text{ sec} - \varepsilon \text{ sec} - 5 \text{ sec}) 5 \frac{\text{ft}}{\text{sec}}.$$

Thus, *the radius of each circle is in error by the same amount, $-5\,\varepsilon$ ft, and there must be a value of ε for which the three circles have a common point*. FIGURE 5 shows the effect of various watch errors.

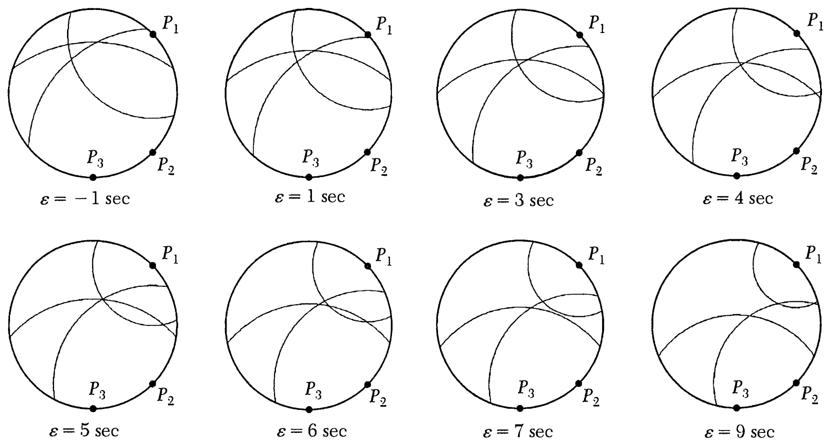

FIGURE 5
Effect of Watch Error.

It appears that your watch has an error of approximately 5 sec. The error and the coordinates of your position are a solution for the following system of equations:

$$\left\{\begin{array}{l}(x_0-70.7)^2+(y_0-70.7)^2=d(20.2,\varepsilon)^2\\(x_0-70.7)^2+(y_0+70.7)^2=d(29.5,\varepsilon)^2\\(x_0-0.0)^2+(y_0+100.0)^2=d(32.2,\varepsilon)^2\end{array}\right\}.$$

The system can be solved numerically, starting with seed values of 0 for ε and estimated coordinates of your position for x_0 and y_0. There is only one solution giving a location inside of our lot. Rounding this to our level of precision yields $(x_0, y_0, \varepsilon) = (10.9, 31.2, 4.9)$. You conclude that you are 10.9 ft east and 31.2 ft north of the center of the lot, and that your watch is 4.9 sec fast. You note the coordinates of your position, and discard the watch error, which is of no further interest to you.

As this example of our GPS model shows, you can use time difference information from three messengers to determine your position, relative to a coordinate system in the lot. *The only tools needed for this effort are a steady, but not necessarily accurate, watch and the ability to approximate the solution of a system of three equations in three unknowns.*

Back to the Satellites

Our "lot" is now the region inside of the satellite orbits (including the Earth), "cars on the road" are satellites, "messengers" are radio waves, and "gravel" is the Earth's atmosphere. We take the center of the Earth as the origin in our coordinate system. Working in three dimensions, we need information from four satellites. Call these S_1, S_2, S_3, and S_4; and suppose that S_i is located at (X_i, Y_i, Z_i) when it transmits a signal at time T_i. If the signals are received at times T_i', according to the clock in our receiver, we let $\Delta t_i = T_i' - T_i$, and let ε represent any error in our clock's time. The receiver allows for the mean effects of passage through the Earth's atmosphere and

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