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A Scaling Method for Priorities in Hierarchical Structures

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The purpose of this paper is to investigate a method of scaling ratios using the principal eigenvector of a positive pairwise comparison matrix. Consistency of the matrix data is defined and measured by an expression involving the average of the nonprincipal eigenvalues. We show that $\lambda_{max} = n$ is a necessary and sufficient condition for consistency. We also show that twice this measure is the variance in judgmental errors. A scale of numbers from 1 to 9 is introduced together with a discussion of how it compares with other scales. To illustrate the theory, it is then applied to some examples for which the answer is known, offering the opportunity for validating the approach. The discussion is then extended to multiple criterion decision making by formally introducing the notion of a hierarchy, investigating some properties of hierarchies, and applying the eigenvalue approach to scaling complex problems structured hierarchically to obtain a unidimensional composite vector for scaling the elements falling in any single level of the hierarchy. A brief discussion is also included regarding how the hierarchy serves as a useful tool for decomposing a large-scale problem, in order to make measurement possible despite the now-classical observation that the mind is limited to 7 ± 2 factors for simultaneous comparison.

1. INTRODUCTION

A fundamental problem of decision theory is how to derive weights for a set of activities according to importance. Importance is usually judged according to several criteria. Each criterion may be shared by some or by all the activities. The criteria may, for example, be objectives which the activities have been devised to fulfill. This is a process of multiple criterion decision making which we study here through a theory of measurement in a hierarchical structure.

The object is to use the weights which we call priorities, for example, to allocate a resource among the activities or simply implement the most important activities by rank if precise weights cannot be obtained. The problem then is to find the relative strength or priorities of each activity with respect to each objective and then compose the result obtained for each objective to obtain a single overall priority for all the activities. Frequently the objectives themselves must be prioritized or ranked in terms of yet another set of (higher-level) objectives. The priorities thus obtained are then used as weighting factors for the priorities just derived for the activities. In many applications we have noted that the process has to be continued by comparing the higher-level objectives in terms of still higher ones and so on up to a single overall objective. (The top level need not have a single element in which case one would have to assume rather than derive weights for the elements in that level.) The arrangement of the activities;

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first set of objectives, second set, and so on to the single element objective defines a hierarchical structure.

The paper is concerned with developing a method for scaling the weights of the elements in each level of the hierarchy with respect to an element (e.g. criterion or objective) of the next higher level. We construct a matrix of pairwise comparisons of the activities whose entries indicate the strength with which one element dominates another as far as the criterion with respect to which they are compared is concerned.

If, for example, the weights are w_i, $i = 1,..., n$, where n is the number of activities, then an entry a_{ij} is an estimate of w_i/w_j. This scaling formulation is translated into a largest eigenvalue problem. The Perron–Frobenius theory (Gantmacher, 1960) ensures the existence of a largest real positive eigenvalue for matrices with positive entries whose associated eigenvector is the vector of weights. This vector is normalized by having its entries sum to unity. It is unique.

Thus the activities in the lowest level have a vector of weights with respect to each criterion in the next level derived from a matrix of pairwise comparisons with respect to that criterion.

The weight vectors at any one level are combined as the columns of a matrix for that level. The weight matrix of a level is multiplied on the right by the weight matrix (or vector) of the next higher level. If the highest level of the hierarchy consists of a single objective, then these multiplications will result in a single vector of weights which will indicate the relative priority of the entities of the lowest level for accomplishing the highest objective of the hierarchy. If one decision is required, the option with the highest weight is selected; otherwise, the resources are distributed to the options in proportion to their weights in the final vector. Other optimization problems with constraints have been considered elsewhere.

Special emphasis is placed in this work on the integration of human judgments into decisions and on the measurement of the consistency of judgments. From a theoretical standpoint consistency is a necessary condition for representing a real-life problem with a scale; however, it is not sufficient. The actual validation of a derived scale in practice rests with statistical measures, with intuition, and with pragmatic justification of the results.

2. Ratio Scales from Reciprocal Pairwise Comparison Matrices

Suppose we wish to compare a set of n objects in pairs according to their relative weights (assumed to belong to a ratio scale). Denote the objects by $A_1,..., A_n$ and their weights by $w_1,..., w_n$. The pairwise comparisons may be represented by a matrix as follows:

$$A = \begin{array}{c|cccc} & A_1 & A_2 & \cdots & A_n \\ \hline A_1 & w_1/w_1 & w_1/w_2 & \cdots & w_1/w_n \\ A_2 & w_2/w_1 & w_2/w_2 & \cdots & w_2/w_n \\ \vdots & \vdots & \vdots & & \vdots \\ A_n & w_n/w_1 & w_n/w_2 & & w_n/w_n \end{array}$$

This matrix has positive entries everywhere and satisfies the reciprocal property $a_{ji} = 1/a_{ij}$. It is called a reciprocal matrix. We note that if we multiply this matrix by the transpose of the vector $w^T = (w_1, ..., w_n)$ we obtain the vector nw.

Our problem takes the form

$$Aw = nw.$$

We started out with the assumption that w was given. But if we only had A and wanted to recover w we would have to solve the system $(A - nI)w = 0$ in the unknown w. This has a nonzero solution if and only if n is an eigenvalue of A, i.e., it is a root of the characteristic equation of A. But A has unit rank since every row is a constant multiple of the first row. Thus all the eigenvalues λ_i, $i = 1, ..., n$, of A are zero except one. Also, it is known that

$$\sum_{i=1}^{n} \lambda_i = \text{tr}(A) \equiv \text{sum of the diagonal elements} = n.$$

Therefore only one of the λ_i, which we call $\lambda_{\max}$, equals n; and

$$\lambda_i = 0, \qquad \lambda_i \neq \lambda_{\max}.$$

The solution w of this problem is any column of A. These solutions differ by a multiplicative constant. However, it is desirable to have this solution normalized so that its components sum to unity. The result is a unique solution no matter which column is used. We have recovered the scale from the matrix of ratios.

The matrix A satisfies the "cardinal" consistency property $a_{ij}a_{jk} = a_{ik}$ and is called consistent. For example if we are given any row of A, we can determine the rest of the entries from this relation. This also holds for any set of n entries whose graph is a spanning cycle of the graph of the matrix.

Now suppose that we are dealing with a situation in which the scale is not known but we have estimates of the ratios in the matrix. In this case the cardinal consistency relation (elementwise dominance) above need not hold, nor need an ordinal relation of the form $A_i > A_j$, $A_j > A_k$ imply $A_i > A_k$ hold (where the A_i are rows of A).

As a realistic representation of the situation in preference comparisons, we wish to account for inconsistency in judgments because, despite their best efforts, people's feelings and preferences remain inconsistent and intransitive.

We know that in any matrix, small perturbations in the coefficients imply small perturbations in the eigenvalues. Thus the problem $Aw = nw$ becomes $A'w' = \lambda_{\max}w'$. We also know from the theorem of Perron–Frobenius that a matrix of positive entries has a real positive eigenvalue (of multiplicity 1) whose modulus exceeds those of all other eigenvalues. The corresponding eigenvector solution has nonnegative entries and when normalized it is unique. Some of the remaining eigenvalues may be complex.

Suppose then that we have a reciprocal matrix. What can we say about an overall estimate of inconsistency for both small and large perturbations of its entries? In other words how close is $\lambda_{\max}$ to n and w' to w? If they are not close, we may either revise the estimates in the matrix or take several matrices from which the solution vector w'

may be improved. Note that improving consistency does not mean getting an answer closer to the "real" life solution. It only means that the ratio estimates in the matrix, as a sample collection, are closer to being logically related than to being randomly chosen.

From here on we use $A = (a_{ij})$ for the estimated matrix and w for the eigenvector. There should be no confusion in dropping the primes.

It turns out that a reciprocal matrix A with positive entries is consistent if and only if $\lambda_{\max} = n$ (Theorem 1 below). With inconsistency $\lambda_{\max} > n$ always. One can also show that ordinal consistency is preserved, i.e., if $A_i \geqslant A_j$ (or $a_{ik} \geqslant a_{jk}$, $k = 1,..., n$) then $w_i \geqslant w_j$ (Theorem 2 below). We now establish $(\lambda_{\max} - n)/(n - 1)$ as a measure of the consistency or reliability of information by an individual to be of the form w_i/w_j. We assume that because of possible error the estimate has the form $w_i/w_j\, \epsilon_{ij}$ where $\epsilon_{ij} > 0$.

First we note that to study the sensitivity of the eigenvector to perturbations in a_{ij} we cannot make a precise statement about a perturbation $dw = (dw_1,..., dw_n)$ in the vector $w = (w_1,..., w_n)$ because everywhere we deal with w, it appears in the form of ratios w_i/w_j or with perturbations (mostly multiplicative) of this ratio. Thus, we cannot hope to obtain a simple measure of the absolute error in w.

From general considerations one can show that the larger the order of the matrix the less significant are small perturbations or a few large perturbations on the eigenvector. If the order of the matrix is small, the effect of a large array perturbation on the eigenvector can be relatively large. We may assume that when the consistency index shows that perturbations from consistency are large and hence the result is unreliable, the information available cannot be used to derive a reliable answer. If it is possible to improve the consistency to a point where its reliability indicated by the index is acceptable, i.e., the value of the index is small (as compared with its value from a randomly generated reciprocal matrix of the same order), we can carry out the following type of perturbation analysis.

The choice of perturbation most appropriate for describing the effect of inconsistency on the eigenvector depends on what is thought to be the psychological process which goes on in the individual. Mathematically, general perturbations in the ratios may be reduced to the multiplicative form mentioned above. Other perturbations of interest can be reduced to the general form $a_{ij} = (w_i/w_j)\, \epsilon_{ij}$. For example,

$$(w_i/w_j) + \alpha_{ij} = (w_i/w_j)(1 + (w_j/w_i)\, \alpha_{ij}).$$

Starting with the relation

$$\lambda_{\max} = \sum_{j=1}^{n} a_{ij}(w_j/w_i),$$

from the ith component of $Aw = \lambda_{\max} w$, we consider the two real-valued parameters $\lambda_{\max}$ and μ, the average of λ_i, $i \geqslant 2$ (even though they can occur as complex conjugate numbers),

$$\mu = -(1/(n-1)) \sum_{i=2}^{n} \lambda_i = (\lambda_{\max} - n)/(n - 1) \geqslant 0, \qquad \lambda_{\max} \equiv \lambda_1$$

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