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Communication Systems

AN INTRODUCTION TO SIGNALS AND NOISE IN ELECTRICAL COMMUNICATION

SECOND EDITION

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COMMUNICATION SYSTEMS
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To the memory of my father,
ALBIN JOHN CARLSON

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9

INFORMATION THEORY AND COMMUNICATION SYSTEMS

The past several chapters have dealt with electrical communication primarily in terms of signals, both desired and undesired. We have devised signal models, examined the effects of networks on signals, and analyzed modulation as a means of signal transmission. Although many rewards and much insight have been gained by this approach, signal theory alone is not sufficient for a complete understanding of our subject matter, particularly when it comes to the design of new and improved systems. What is needed is a more encompassing view of the communication process, a broader perspective leading to basic principles for system design and comparison — in short, a general theory of communication.

Prior to the 1940s a few steps were taken toward such a theory in the telegraphy investigations of Nyquist and Hartley. But then, shortly after World War II, Claude Shannon (1948) and Norbert Wiener (1949) set forth new concepts that had and continue to have major impact. Taken together, the ideas of Wiener and Shannon established the foundation of modern (statistical) *communication theory*. Both men were concerned with extracting information from a background of noise, and both applied statistical concepts to the problem.† There were, however, differences in emphasis.

† Both men are also famous for other accomplishments. Wiener founded the subject known as cybernetics, and Shannon first pointed out the relationship of Boolean algebra to switching-circuit design — in his master's thesis!

Wiener treated the case where the information-bearing signals are beyond the designer's control, in whole or part, all the processing being at the receiving end. The problem then can be stated in this fashion: Given the set of possible signals, not of our choosing, plus the inevitable noise, how do we make the best estimate of the present and future values of the signal being received? Optimum solutions to this and similar problems are sought in the discipline known as *detection theory*.

Shannon's work is more nearly akin to what we think of as communication, where signal processing can take place at both transmitter and receiver. Shannon posed this problem: given the set of possible messages a source may produce, not of our choosing, how shall the messages be represented so as best to convey the information over a given system with its inherent physical limitations? To handle this problem in quite general terms it is necessary to concentrate more on the *information* per se than on the signals, and Shannon's approach was soon rechristened *information theory*.

Information theory is a mathematical subject dealing with three basic concepts: the measure of information, the capacity of a communication channel to transfer information, and coding as a means of utilizing channels at full capacity. These concepts are tied together in what can be called the fundamental theorem of information theory, as follows.

> Given an information source and a communication channel, there exists a coding technique such that the information can be transmitted over the channel at any rate less than the channel capacity and with arbitrarily small frequency of errors despite the presence of noise.

The surprising, almost astonishing aspect of this theorem is *error-free* transmission on a *noisy* channel, a condition achieved through the use of coding. In essence, coding is used to match the source and channel for maximum reliable information transfer, roughly analogous to impedance matching for maximum power transfer.

But the study of coding is, by and large, tangential to our inmediate aims. Thus, with some reluctance, we limit this chapter primarily to the concepts of information measure and channel capacity, with emphasis on the latter. By so doing we shall eventually arrive at answers to these significant questions:

1 Precisely how do the fundamental physical limitations (i.e., bandwidth and noise) restrict information transmission?

2 Is there such a thing as an *ideal* communication system, and, if so, what are its characteristics?

3 How well do existing communication systems measure up to the ideal, and how can their performance be improved?

Answers to these questions are certainly germane to electrical communication.

They will be explored in some detail at the close of the chapter. But we must begin with information theory.

9.1 INFORMATION MEASURE: ENTROPY

The crux of information theory is the measure of information. Here we are using *information* as a technical term, not to be confused with its more conventional interpretations. In particular, the information of information theory has little to do with knowledge or meaning, concepts which defy precise definition, to say nothing of quantitative measurement. In the context of communication, information is simply that which is produced by the source for transfer to the user. This implies that before transmission, the information was not available at the destination; otherwise the transfer would be zero. Pursuing this line of reasoning, consider the following somewhat contrived situation.

A man is planning a trip to Chicago. To determine what clothes he should pack, he telephones the Chicago weather bureau and receives one of the following forecasts:

The sun will rise.
It will rain.
There will be a tornado.

Clearly, the amount of information gained from these messages is quite different. The first contains virtually no information, since we are reasonably sure in advance that the sun will rise; there is no uncertainty about this, and the call has been wasted. But the forecast of rain does provide information not previously available to the traveler, for rain is not an everyday occurrence. The third forecast contains even more information, tornadoes being relatively rare and unexpected events.

Note that the messages have been listed in order of decreasing likelihood and increasing information. The less likely the message, the more information it conveys to the user. We are thus inclined to say that information measure is related to *uncertainty*, the uncertainty of the user as to what the message will be. Moreover, the amount of information depends only on the message uncertainty, rather than its actual content or possible interpretations. Had the Chicago weather forecast been "The sun will rain tornadoes," it would convey information, being quite unlikely, but not much meaning.

Alternately, going to the transmitting end of a communication system, information measure is an indication of the *freedom of choice* exercised by the source in selecting a message. If the source can freely choose from many different messages, the user is highly uncertain as to which message will be selected. But if there is no choice at all, only one possible message, there is no uncertainty and hence no information.

Whether one prefers the uncertainty viewpoint or the freedom-of-choice interpretation, it is evident that the measure of information involves *probabilities*. Messages of high probability, indicating little uncertainty on the part of the user or little choice on the part of the source, convey a small amount of information, and vice versa. This notion is formalized by defining self-information in terms of probability.

Self-Information

Consider a source that produces various messages. Let one of the messages be designated A, and let P_A be the probability that A is selected for transmission. Consistent with our discussion above, we write the self-information associated with A as

$$\mathcal{I}_A = f(P_A)$$

where the function $f(\ )$ is to be determined. As a step toward finding $f(\ )$, intuitive reasoning suggests that the following requirements be imposed:

$$f(P_A) \geq 0 \qquad \text{where } 0 \leq P_A \leq 1 \tag{1}$$

$$\lim_{P_A \to 1} f(P_A) = 0 \tag{2}$$

$$f(P_A) > f(P_B) \qquad \text{for } P_A < P_B \tag{3}$$

The student should have little trouble interpreting these requirements.

Many functions satisfy Eqs. (1) to (3). The final and deciding factor comes from considering the transmission of *independent* messages. When message A is delivered, the user receives $\mathcal{I}_A$ units of information. If a second message B is also delivered, the total information received should be the sum of the self-informations, $\mathcal{I}_A + \mathcal{I}_B$. This summation rule is readily appreciated if we think of A and B as coming from different sources. But suppose both messages come from the same source; we can then speak of the compound message $C = AB$. If A and B are statistically independent, $P_C = P_A P_B$ and $\mathcal{I}_C = f(P_A P_B)$. But the received information is still $\mathcal{I}_C = \mathcal{I}_A + \mathcal{I}_B = f(P_A) + f(P_B)$ and therefore

$$f(P_A P_B) = f(P_A) + f(P_B) \tag{4}$$

which is our final requirement for $f(\ )$.

There is one and only one function† satisfying the conditions (1) to (4), namely, the *logarithmic function* $f(\ ) = -\log_b(\ )$, where b is the logarithmic base. Thus self-information is defined as

$$\mathcal{I}_A \triangleq -\log_b P_A = \log_b \frac{1}{P_A} \tag{5}$$

† See Ash (1965, chap. 1) for proof.

where b is unspecified for the moment. The minus sign in $-\log_b P_A$ is perhaps disturbing at first glance. But, since probabilities are bounded by $0 \leq P_A \leq 1$, the negative of the logarithm is positive, as desired. The alternate form $\log_b (1/P_A)$ helps avoid confusion on this score, and will be used throughout.

Specifying the logarithmic base b is equivalent to selecting the *unit* of information. While common or natural logarithms ($b = 10$ or $b = e$) seem obvious candidates, the standard convention of information theory is to take $b = 2$. The corresponding unit of information is termed the *bit*, a contraction for *binary digit* suggested by J. W. Tukey. Thus

$$\mathscr{I}_A = \log_2 \frac{1}{P_A} \qquad \text{bits}$$

The reasoning behind this rather strange convention goes like this. Information is a measure of choice exercised by the source; the simplest possible choice is that between two equiprobable messages, i.e., an unbiased binary choice. The information unit is therefore normalized to this lowest-order situation, and 1 bit of information is the amount required or conveyed by the choice between two equally likely possibilities, i.e., if $P_A = P_B = 1/2$, then $\mathscr{I}_A = \mathscr{I}_B = \log_2 2 = 1$ bit.

Binary *digits* enter the picture simply because any two things can be represented by the two binary digits **0** and **1**. Note, however, that one binary digit may convey more or less than 1 bit of information, depending on the probabilities. To prevent misinterpretation, binary digits as message elements are called *binits* in this chapter.

Since tables of base 2 logarithms are relatively uncommon, the following conversion relationship is needed:

$$\log_2 v = \log_2 10 \log_{10} v \approx 3.32 \log_{10} v \qquad (6)$$

Thus, if $P_A = 1/10$, $\mathscr{I}_A = 3.32 \log_{10} 10 = 3.32$ bits. In the remainder of this chapter, all logarithms will be base 2 unless otherwise indicated.

Example 9.1 The Information in a Picture

It has often been said that one picture is worth a thousand words. With a little stretching, information measure supports this old saying.

For analysis we decompose the picture into a number of discrete dots, or elements, each element having a brightness level ranging in steps from black to white. The standard television image, for instance, has about $500 \times 600 = 3 \times 10^5$ elements and eight easily distinguishable levels. Hence, there are $8 \times 8 \times \ldots = 8^{3 \times 10^5}$ possible pictures, each with probability $P = 8^{-(3 \times 10^5)}$ if selected at random. Therefore

$$\mathscr{I} = \log 8^{3 \times 10^5} = 3 \times 10^5 \log 8 \approx 10^6 \text{ bits}$$

Alternately, assuming the levels to be equally likely, the information per element is $\log 8 = 3$ bits, for a total of $3 \times 10^5 \times 3 \approx 10^6$ bits, as before.

But what about the thousand words? Suppose, for the sake of argument, that a vocabulary consists of 100,000 equally likely words. The probability of any one word is then $P = 10^{-5}$, so the information contained in 1,000 words is

$$\mathscr{I} = 1{,}000 \log 10^5 = 10^3 \times 3.32 \log_{10} 10^5 \approx 2 \times 10^4 \text{ bits}$$

or substantially less than the information in one picture.

The validity of the above assumptions is of course open to question; the point of this example is the method, not the results. ////

Entropy and Information Rate

Self-information is defined in terms of the individual messages or symbols a source may produce. It is not, however, a useful description of the source relative to communication. A communication system is not designed around a particular message but rather all possible messages, i.e., what the source could produce as distinguished from what it does produce on a given occasion. Thus, although the instantaneous information flow from a source may be erratic, one must describe the source in terms of the *average information* produced. This average information is called the source *entropy*.

For a discrete source whose symbols are *statistically independent*, the entropy expression is easily formulated. Let m be the number of different symbols, i.e., an alphabet of size m. When the jth symbol is transmitted, it conveys $\mathscr{I}_j = \log (1/P_j)$ bits of information. In a long message of $N \gg 1$ symbols, the jth symbol occurs about NP_j times, and the total information in the message is approximately

$$NP_1\mathscr{I}_1 + NP_2\mathscr{I}_2 + \cdots + NP_m\mathscr{I}_m = \sum_{j=1}^{m} NP_j\mathscr{I}_j \qquad \text{bits}$$

which, when divided by N, yields the average information per symbol. We therefore define the entropy of a discrete source as

$$\mathscr{H} \triangleq \sum_{j=1}^{m} P_j \mathscr{I}_j = \sum_{j=1}^{m} P_j \log \frac{1}{P_j} \qquad \text{bits/symbol} \qquad (7)$$

It should be observed that Eq. (7) is an ensemble average. If the source is nonstationary, the symbol probabilities may change with time and the entropy is not very meaningful. We shall henceforth assume that information sources are *ergodic*, so that time and ensemble averages are identical.

The name *entropy* and its symbol $\mathscr{H}$ are borrowed from a similar equation in statistical mechanics. Because of the mathematical similarity, various attempts have been made to relate communication entropy with thermodynamic entropy.† However, the attempted relationships seem to cause more confusion than illumination, and it is perhaps wiser to treat the two entropies as different things with the same name. For this reason the alternate designation *comentropy* has been suggested for communication entropy.

But what is the meaning of communication entropy as written in Eq. (7)? Simply this: although one cannot say which symbol the source will produce next, on the average we expect to get $\mathscr{H}$ bits of information per symbol or $N\mathscr{H}$ bits in a message of N symbols, if N is large.

For a fixed alphabet size (fixed m) the entropy of a discrete source depends on the symbol probabilities but is bounded by

$$0 \leq \mathscr{H} \leq \log m \tag{8}$$

These extreme limits are readily interpreted and warrant further discussion. The lower limit, $\mathscr{H} = 0$, implies that the source delivers no information (on the average), and hence there is no uncertainty about the message. We would expect this to correspond to a source that continually produces the same symbol; i.e., all symbol probabilities are zero, save for one symbol having $P = 1$. It is easily shown that $\mathscr{H} = 0$ in this case. At the other extreme, the maximum entropy must correspond to maximum uncertainty or maximum freedom of choice. This implies that all symbols are equally likely; there is no bias, no preferred symbol. A little further thought reveals that the symbol probabilities must be the same; i.e., $\mathscr{H} = \mathscr{H}_{max} = \log m$ when all $P_j = 1/m$, which has particular significance for our later work.

The variation of $\mathscr{H}$ between the limits of Eq. (8) is best illustrated by considering a binary source ($m = 2$). The symbol probabilities are then related and can be written as P and $1 - P$. Thus

$$\mathscr{H} = P \log \frac{1}{P} + (1 - P) \log \frac{1}{1 - P} \tag{9}$$

which is plotted versus P in Fig. 9.1. Note the rather broad maximum centered at $P = 0.5$, the equally likely case, where $\mathscr{H} = \log 2 = 1$ bit.

Bringing the time element into the picture, suppose two sources have equal entropies but one is "faster" than the other, producing more symbols per unit time. In a given period, more information must be transferred from the faster source than from the slower, which obviously places greater demands on the communication system. Thus, for our purposes, the description of a source is not its entropy alone but its *entropy rate*, or information rate, in bits per second. The entropy rate of a discrete source is simply defined as

† See Brillouin (1956).

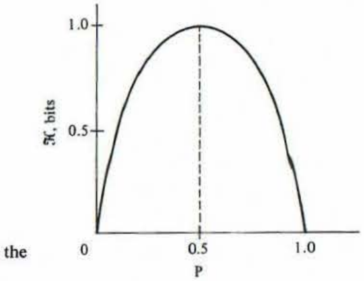

FIGURE 9.1
Entropy of a binary source versus the probability of one symbol.

$$\mathscr{R} \triangleq \frac{\mathscr{H}}{\bar{\tau}} \qquad \text{bits/s} \tag{10}$$

where $\bar{\tau}$ is the average symbol duration, namely,

$$\bar{\tau} = \sum_{j=1}^{m} P_j \tau_j \tag{11}$$

so $1/\bar{\tau}$ equals the average number of symbols per unit time.

EXERCISE 9.1 Calculate the entropy rate of a telegraph source having $P_{dot} = 2/3$, $\tau_{dot} = 0.2$ s, $P_{dash} = 1/3$, $\tau_{dash} = 0.4$ s. *Ans.*: $\mathscr{R} = 3.44$ bits/s.

Example 9.2 Entropy and PCM

As an example of entropy applied to our earlier studies, consider a PCM system whose input is the continuous signal $x(t)$ bandlimited in $W = 50$ Hz. Suppose $x(t)$ is sampled at the Nyquist rate $f_s = 2W$, and let there be four quantum levels such that the quantized values have probabilities $1/2$, $1/4$, $1/8$, and $1/8$. Identifying each possible quantized value as a "symbol," the output of the quantizer then looks like a discrete source with $m = 4$ and

$$\mathscr{H} = \tfrac{1}{2}\log 2 + \tfrac{1}{4}\log 4 + \tfrac{1}{8}\log 8 + \tfrac{1}{8}\log 8$$
$$= 1.75 \text{ bits/symbol}$$

Since the symbol rate is $1/\bar{\tau} = f_s = 100$,

$$\mathscr{R} = 100 \times 1.75 = 175 \qquad \text{bits/s}$$

Thus, it should be possible to represent this same information by equiprobable binary digits (binits) generated at a rate of 175 binits/s.

To check this idea, suppose that the system is in fact *binary* PCM with the quantized samples transmitted as coded binary pulses. Labeling the quantum levels by the code numbers 0, 1, 2, and 3, let us assume the following coding:

Code number	Probability	Binary code
0	$\frac{1}{2}$	00
1	$\frac{1}{4}$	01
2	$\frac{1}{8}$	10
3	$\frac{1}{8}$	11

Since there are two binary digits for each sample, the PCM signal has 200 pulses per second. Now we argued that 175 binary digits per second should be sufficient; yet 200 per second is required with this code.

The anomaly is quickly resolved by noting that the binary digits are not equally likely; in fact, $P_0 = 11/16$ and $P_1 = 5/16$, as the reader can verify. Clearly, the suggested code is not optimum. On the other hand, it is simple and reasonably efficient.

Pressing onward, the binit rate and probabilities given above suggest that the information rate at the encoder output is

$$\begin{aligned}\mathscr{R} &= 200(\tfrac{11}{16} \log \tfrac{16}{11} + \tfrac{5}{16} \log \tfrac{16}{5}) \\ &= 200 \times 0.897 = 179 \text{ bits/s}\end{aligned}$$

Again something is wrong, for the information rate into the encoder is only 175 bits/s. Surely direct encoding does not *add* information!

To explain the discrepancy we must recall that the entropy equation (7) is based on statistically *independent* symbols. And, while it may be true that successive quantized samples are independent, the successive binary pulses are not, because we have encoded in groups of two. For the case of dependent symbols, we must modify the measure of information and introduce conditional entropy. ////

Conditional Entropy and Redundancy

Discrete sources are often constrained by certain rules which limit the choice in selecting successive symbols. The resulting *intersymbol influence* reduces uncertainty and thereby reduces the amount of information produced. We account for this effect by using conditional probabilties and conditional entropy.

Written text, being governed by rules of spelling and grammar, is a good example of intersymbol influence. On a relative-frequency basis, the probability of U in printed English is $P_U = 0.02$; but if the previous letter is Q, the conditional probability of U given Q is $P(U|Q) \approx 1$, whereas in contrast $P(U|W) < 0.001$. The influence may well extend over several symbols, phrases, or even complete sentences, as illustrated by the fact that in most textbooks, including this one, $P(\text{THAT}|\text{IT CAN BE SHOWN}) \approx 1$.

The expression for conditional entropy therefore is formulated by considering the entire past history of the source—more precisely, all possible past histories. Thus, if j represents the next symbol (or group of symbols) and i represents the preceding sequence, the information conveyed by j given i is $\log [1/P(j|i)]$. Averaging over all j's and i's gives the conditional entropy

$$\mathscr{H}_c = \sum_i \sum_j P_i P(j|i) \log \frac{1}{P(j|i)} \tag{12}$$

In general $\mathscr{H}_c \leq \mathscr{H}$; the equality applies only when the symbols are independent and $P(j|i) = P_j$.

A source producing dependent symbols is said to be *redundant*, meaning that symbols are generated which are not absolutely essential to convey the information. (Is it really necessary to explicitly write the U following every Q in English?) The redundancy of English text is estimated to be roughly 50 percent. This implies that in the long run, half the symbols are unnecessary: yu shld babl t read ths evntho sevrl ltrs r msng. The reader may wish to ponder the observation that without redundancy, abbreviation would be impossible, and any two-dimensional array of letters would form a valid crossword puzzle.

For very long passages of printed English, the conditional entropy may be as low as 0.5 to 1.0 bits/symbol because of contextual inferences. Thus, with suitable coding, printed English theoretically could be transmitted in binary form with an average of one binary digit per symbol. Contrast this with existing teletype systems that use five binary digits per character.

From the viewpoint of efficient communication, redundancy in a message is undesirable; the same information could be sent with fewer nonredundant (independent) symbols. Thus, coding to reduce intersymbol influence is a method of improving efficiency. On the other hand, redundancy is a definite aid in resolving ambiguities if the message is received with *errors*, a not uncommon phenomenon in telegraphy, for example. Indeed, coding for error protection is based on the insertion of redundant symbols.

Optimum transmission therefore entails coding to reduce the inefficient redundancy of the message, plus coding to add "efficient" redundancy for error control. Much has been done in the area of error-detecting and error-correcting codes, but coding to reduce message redundancy is far more difficult, and relatively little has been accomplished in this direction. (In retrospect, the bandwidth-reduction feature of differential PCM relies on the redundancy of the input signal.)

Continuous Information Sources

Having discussed discrete sources at some length, the next logical step would be the definition of entropy for *continuous* sources, sources whose messages are continuously varying functions of time. Such a definition is possible but will not be presented here. For one reason, the mathematics gets rather complicated and tends to obscure physical interpretation; for another, the entropy of a continuous source turns out to be a relative measure instead of an absolute measure of information.

Fortunately, the goals of this chapter can be achieved by sticking to the discrete formulation, and most of our conclusions will apply to continuous sources with minor modification. But more important, we shall find that because of the fundamental physical limitations, communication is inherently a *discrete process* regardless of the source. This striking conclusion is one of Shannon's principal contributions to the theory of communication, but it was noted by Hartley as far back as 1928.

9.2 CHANNEL CAPACITY AND DISCRETE CHANNELS

We have seen that it is often convenient to treat the terminal equipment of a communication system as being perfect (noise-free, distortionless, etc.) and think of all undesired effects as taking place in the channel. The communication channel is therefore an abstraction, a model representing the vehicle of transmission plus all phenomena that tend to restrict transmission. The fact that there are fundamental physical limitations to information transfer by electrical means leads to the notion of *channel capacity.*

Just as entropy rate measures the amount of information produced by a source in a given time, capacity is a measure of the amount of information a channel can transfer per unit time. Channel capacity is symbolized by $\mathscr{C}$, and its units are bits per second. Restating the fundamental theorem in terms of $\mathscr{R}$ and $\mathscr{C}$, we have:

> Given a channel of capacity $\mathscr{C}$ and a source having entropy rate $\mathscr{R}$, then if $\mathscr{R} \leq \mathscr{C}$, there exists a coding technique such that the output of the source can be transmitted over the channel with an arbitrarily small frequency of errors, despite the presence of noise. If $\mathscr{R} > \mathscr{C}$, it is not possible to transmit without errors.

Although we shall attempt to make the theorem plausible, complete proof involves a great deal of coding theory and is omitted here. Instead we shall concentrate on aspects more pertinent to electrical communication.

Channel Capacity

The fundamental theorem implicitly defines channel capacity as the maximum rate at which the channel supplies reliable information to the destination. With this interpretation in mind we can formulate a general expression for capacity by means of the following argument.

Consider all the different messages of length T a source might produce. If the channel is noisy, it will be difficult to decide at the receiver which particular message was intended and the goal of information transfer is partially defeated. But suppose we restrict the messages to only those that are "very different" from each other, such that the received message can be correctly identified with sufficiently small probability of error. Let $M(T)$ be the number of these very different messages of length T.

Now, insofar as the destination or user is concerned, the source-plus-channel combination may be regarded as a new source generating messages at the receiving end. With the above message restriction, this equivalent source is discrete and has an alphabet of size $M(T)$. Correspondingly, the maximum entropy produced by the equivalent source is $\log M(T)$, and the maximum entropy rate at the destination is $(1/T) \log M(T)$. Hence, letting $T \rightarrow \infty$ to ensure generality,

$$\mathscr{C} = \lim_{T\rightarrow\infty} \frac{1}{T} \log M(T) \qquad \text{bits/s} \tag{1}$$

an alternate definition for channel capacity. The following discussion of discrete channels shows that Eq. (1) is an intuitively meaningful definition.

Discrete Noiseless Channels

A discrete channel is one that transmits information by successively assuming various disjoint electrical states—voltage levels, instantaneous frequency, etc. Let μ be the number of possible states and r the signaling rate in states per unit time. If the signal-to-noise ratio is sufficiently large, the error probability at the receiver can be extremely small, so small that to all intents and purposes the channel is deemed to be noiseless. Under this assumption, any sequence of symbols will be correctly identified and the capacity calculation is straightforward.

A received message of length T will consist of rT symbols, each symbol being one of the μ possible states. The number of different messages is thus $M(T) = \mu^{rT}$ and hence

$$\begin{aligned}\mathscr{C} &= \lim_{T\rightarrow\infty} \frac{1}{T} \log \mu^{rT} = \lim_{T\rightarrow\infty} \frac{rT}{T} \log \mu \\ &= r \log \mu \qquad \text{bits/s}\end{aligned} \tag{2}$$

The capacity of a noiseless discrete channel is therefore proportional to the signaling rate and the logarithm of the number of states. For a binary channel ($\mu = 2$) the capacity is numerically equal to the signaling rate, that is, $\mathscr{C} = r$.

According to Eq. (2), one can double channel capacity by doubling the signaling speed (which is certainly reasonable) or by *squaring* the number of states (somewhat more subtle to appreciate). In regard to the latter, suppose two identical but independent channels are operated in parallel so their combined capacity is clearly $2(r \log \mu) = 2r \log \mu = r \log \mu^2$. At the output we can say that we are receiving $2r$ symbols per second, each symbol being drawn from an alphabet of size μ; or the output can be viewed as r *compound* symbols per second, each being drawn from an alphabet of size $\mu \times \mu = \mu^2$.

The following example illustrates a coding technique that achieves $\mathcal{R} = \mathcal{C}$ on a noiseless discrete channel.

Example 9.3

Consider the PCM source in Example 9.2. In theory a noiseless binary channel will suffice to convey the information if $r \geq 175$ binits/s. To minimize $\mathcal{R}$ we cannot use the previous code because it requires two binary digits per source symbol and $r = 200$. A more efficient code is as follows.

Code number	Probability	Binary code
0	½	**0**
1	¼	**10**
2	⅛	**110**
3	⅛	**111**

With this code, a message containing $N \gg 1$ source symbols requires the transmission of $N/2 + 2(N/4) + 3(N/8) + 3(N/8) = 1.75N$ channel symbols, that is, 1.75 binary digits per source symbol. The required signaling rate is then $r = 1.75/\bar{\tau} = 175$, and we have achieved transmission at $\mathcal{R} = \mathcal{C}$; the encoding has produced a perfect match between source and channel.

Although this example is admittedly a special case, there are two general hints about efficient encoding to be gained from it. First, the code is such that the channel symbols **0** and **1** are equally likely and statistically independent. (The skeptical reader should verify this.) Second, the source symbol with the highest probability is assigned the shortest word code, and so forth down the line to the least probable symbol, which gets the longest code. Systematic procedures for devising such codes are described in the literature.†

Somewhat parenthetically we might note that assigning shorter code words to the more probable symbols is just common sense. Over a century ago, long before Shannon, Samuel Morse constructed his telegraph code using this very principle,

† E.g., Gallager (1968, chap. 3) or Thomas (1969, chap. 8).

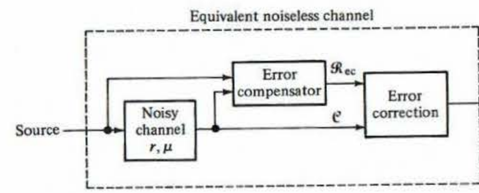

FIGURE 9.2

representing the letter E by a single dot, etc. Lacking the necessary data, Morse estimated letter frequencies by counting the distribution of type in a printer's font

me
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Discrete Channels with Noise

When channel noise cannot be ignored, the capacity is less than $r \log \mu$ because of the errors. We calculate the capacity reduction by thinking of a fictitious *error compensator*, Fig. 9.2, which examines the channel input and output and tells us what corrections should be made. Let the channel be operating at its maximum of $\mathcal{C}$ bits/s, and let the information rate supplied by the compensator be $\mathcal{R}_{ec}$ bits/s. Then, since the noisy channel plus compensator is equivalent to the same channel without noise, the net information rate over the noisy channel is $r \log \mu$ minus the information rate from the compensator, that is,

$$\mathcal{C} = r \log \mu - \mathcal{R}_{ec} \tag{3}$$

If $\mathcal{R}_{ec} < r \log \mu$, the fundamental theorem asserts that it is possible to get a nonzero rate of virtually errorless information at the channel output.

EXERCISE 9.2 The so-called *binary symmetric channel* (BSC) is one in which both states have the same error probability, say p. Show that $\mathcal{R}_{ec} = r\{p \log (1/p) + (1-p) \log [1/(1-p)]\}$ and hence

$$\mathcal{C} = r[1 + p \log p + (1-p) \log (1-p)] \tag{4}$$

Note that $\mathcal{C} = 0$ if $p = ½$. Why? (*Hint*: The error compensator produces two possible messages, "Error" and "No error.")

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Coding for the Binary Symmetric Channel ★

As part of his proof of the fundamental theorem, Shannon demonstrated the possibility of nearly errorless transmission at $\mathcal{R} \leq \mathcal{C}$ on a BSC. His demonstration, outlined below, involves selecting code words at *random*, and it met with criticism at first. Since then, the method has been acknowledged as one of great insight.

Let the channel have $r = 1$ for convenience and let $q = 1 - p$, so, from Eq. (4),

$$\mathcal{C} = 1 + p \log p + q \log q \tag{5}$$

Let the source have m equiprobable messages each of which is represented by a code word having N binary digits. This requires that $\bar{\tau} = N$ and hence

$$\mathcal{R} = \frac{\log m}{N} \tag{6}$$

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,ce $\mathcal{H} = \mathcal{H}_{\max}$.

Owing to noise, the received code words will have errors and, from our study of the binomial distribution, the expected number of errors per word is $\bar{n} = Np$. But this does not necessarily mean that the decoded message will be wrong for, if $m \ll 2^N$, we can select the m code words so that they are "very different" from each other and correctly recognized despite errors. By way of illustration, suppose $m = 2$ and $N = 3$; there are $2^3 = 8$ possible three-digit binary code words but we only need $m = 2$ of them. If the selected words are **000** and **111**, a single error is easily corrected automatically using majority rule, e.g., decode **001** as **000** and **101** as **111**. Generalizing, if one selects the code words in such a way that they differ from each other by at least $\bar{n} + 1$ digits, then the probability of decoding errors becomes negligibly small providing $N \gg m$.

Shannon, however, said that the code words could be randomly selected when $N \gg 1$ and $p < 1/2$. (If $p > 1/2$, we simply interchange the **1**s and **0**s at the receiver!) Under these conditions he showed that the probability of a decoding error is†

$$P_e \approx \frac{m}{2^N} p^{-pN} q^{-qN} \sqrt{\frac{pN}{2\pi q}} \tag{7}$$

which approximates the probability that any two or more words differ by Np digits or less. Then, multiplying Eq. (5) by $-N$ yields $2^{-N\mathcal{C}} = 2^{-N} p^{-pN} q^{-qN}$ (a very clever manipulation) so

$$P_e \approx m 2^{-N\mathcal{C}} \sqrt{\frac{pN}{2\pi q}} \tag{8}$$

† See Thomas (1969, chap. 8).

Therefore, imposing the condition

$$m \leq \frac{2^{N\mathcal{C}}}{N^\alpha} \qquad \alpha > 1/2 \tag{9}$$

and letting $N \to \infty$, we have

$$P_e \leq \lim_{N\to\infty} \frac{1}{N^{\alpha - 1/2}} \sqrt{\frac{p}{2\pi q}} = 0 \tag{10}$$

Finally, inserting Eq. (9) in Eq. (6),

$$\mathcal{R} \leq \lim_{N\to\infty} \frac{1}{N}\left[N\mathcal{C} - \left(\frac{\alpha}{N}\right) \log N\right] = \mathcal{C} \tag{11}$$

which proves error-free transmission at $\mathcal{R} \leq \mathcal{C}$, at least in the limit as $N \to \infty$.

More practically speaking, it seems reasonable to infer that one can come arbitrarily close to this situation with carefully chosen code words and large but finite N. The techniques of practical error-control coding are examined in Chap. 10. For now, we turn our attention to the case of continuous channels.

9.3 CONTINUOUS CHANNELS

A continuous channel is one in which messages are represented as waveforms, i.e., continuous functions of time, and the relevant parameters are bandwidth B and signal-to-noise ratio S/N. For simplicity, we will deal only with a baseband channel whose frequency response has been equalized to be flat over $|f| \leq B$. Although the channel is continuous and has noise, it is informative to develop an intuitive relationship to r and μ, the parameters of a discrete noiseless channel.

Relative to the signaling rate r, it has already been shown in Sect. 4.5 that $r \leq 2B$ where the equality entails using sinc pulses. As to the equivalent number of channel "states" (i.e., voltage levels, etc.), there is no inherent limitation on μ in absence of noise; voltage levels having arbitrarily small spacing still could be distinguished at the output. The value of μ on a noisy channel is estimated in terms of the signal-to-noise ratio in the following manner.

Let the average signal power and noise power at the channel output be S and N, respectively, so the total received power is $S + N$, and the rms output voltage is $\sqrt{S + N}$. Because of noise corruption one can never exactly identify the intended signal voltage. But it can be identified wich reasonably low probability of error if the voltage levels are separated by an amount equal to or exceeding the rms noise voltage σ. Thus, at the receiving end we have voltage levels *spaced* by $\sigma = \sqrt{N}$ and an rms

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voltage *range* of $\sqrt{S+N}$. The maximum number of channel states is therefore approximately

$$\mu = \frac{\sqrt{S+N}}{\sqrt{N}} = \left(1 + \frac{S}{N}\right)^{1/2}$$

For example, a binary channel would require $(1 + S/N)^{1/2} = 2$, or $S/N = 3$; with gaussian noise, the corresponding error probability is about 0.05.

Combining these values for r and μ yields

$$\mathscr{C} = r \log \mu = 2B \log \left(1 + \frac{S}{N}\right)^{1/2}$$

Hence,

$$\mathscr{C} = B \log \left(1 + \frac{S}{N}\right) \quad \text{bits/s} \tag{1}$$

This famous equation is called the *Hartley-Shannon law*. (Hartley did the preliminary spadework and Shannon derived it with rigor.) It is written in terms of parameters that apply equally well to discrete or continuous channels, suggesting that the capacity of a continuous channel is $B \log (1 + S/N)$. In fact, Shannon's derivation was based on the continuous case and will be discussed shortly.

The Hartley-Shannon law, coupled with the fundamental theorem, has two important implications for communication engineers. First, it tells us the absolute best that can be done in the way of reliable information transmission, given the channel parameters. Second, for a specified information rate, it says we can reduce signal power providing we increase the bandwidth an appropriate amount, and vice versa.

The exchange of bandwidth for power or signal-to-noise ratio is not new to us, for we have noted the effect in wideband noise reduction systems such as FM and PCM. But the Hartley-Shannon law specifies the *optimum* possible exchange and further implies that bandwidth *compression is possible*. To illustrate, suppose it is desired to transmit digital data at a rate of 30,000 bits/s. According to the theory, we could use a channel having $B = 30$ kHz and $S/N = 1$, since

$$\mathscr{C} = 30 \times 10^3 \log (1 + 1) = 3 \times 10^4 \text{ bits/s}$$

Alternately, the bandwidth can be reduced to $B = 3$ kHz if the power is increased by a factor of 1,000, that is, $S/N = 10^3 = 30$ dB. (Note the handy approximation $10^3 \approx 2^{10}$.) Incidentally, the latter parameters are typical of standard voice telephone circuits; but when used for digital signals, the data rate on such channels is normally 4,800 bits/s or less, indicating considerable room for improvement.

Some of the above matters are further pursued in Sect. 9.4 after we examine the continuous channel more closely. Since practically all communication systems are capable of handling continuous signals and all systems have noise, the noisy continuous channel merits detailed investigation. As a preliminary, we will describe the characteristics of an ideal communication system with a continuous channel.

EXERCISE 9.3 If $(S/N) \gg 1$, show that minimum time required to transmit K binary digits is

$$T_{\min} \approx \frac{3K}{B(S/N)_{\text{dB}}} \tag{2}$$

(*Hint*: See Eq. (6), Sect. 9.1.)

Ideal Communication Systems

A communication system capable of transmitting without errors at a rate of $B \log (1 + S/N)$ bits/s, where B and S/N are the channel parameters, is called an *ideal* system. While no practical system is or can be ideal, it is possible to visualize systems whose performance approaches that of the ideal. As an aid to the design of such systems, let us examine the characteristics of a nearly ideal system.

To begin with, the number of different signals (or messages or symbols) of length T is $M = (1 + S/N)^{BT}$, and the maximum information rate is $\mathscr{R} = (1/T) \log M$. The channel signals are chosen such that they can be identified at the receiver with very small probability of error. Shannon has shown that if the signals themselves are randomly selected sample functions of gaussian white noise and if $2BT \gg 1$, then this condition is closely approximated.

The information produced by a source is conveyed over the system in the following fashion. The source output is observed for T seconds, and the message is represented (encoded) as one of the noiselike channel signals which is then transmitted; thus, the information is encoded in *blocks* of length T. (This implies that the number of possible source messages of length T is not greater than M.) At the output, the received signal plus noise is compared with stored copies of the channel signals.† The one that best matches the signal plus noise is presumed to be the signal actually transmitted, and the corresponding message is decoded. A total time delay of $2T$ is therefore required for the encoding and decoding operations.

Throughout the above description it has been tacitly assumed that $T \rightarrow \infty$,

† Appendix A covers implementation of this comparison process under the heading Detection Theory.

for only in this limit are all the conditions satisfied so that $\mathscr{R} = B \log (1 + S/N)$. Thus, the characteristics of an ideal system are as follows:

1 The information rate approaches $B \log (1 + S/N)$.
2 The frequency of errors approaches zero.
3 The statistical properties of the transmitted signal approach those of bandlimited gaussian white noise.
4 The coding time delay increases indefinitely.

Additionally, as will be shown, the system has a sharp threshold effect.

Of course, if the channel bandwidth is large enough, we can still have $2BT \gg 1$ with reasonable values of T. However, the design of nearly ideal systems is no trivial matter, for one must balance off coding delay and signal selection against reliability. Accurate signal identification argues for large T. But as T is increased, M must increase exponentially to maintain constant information rate. Rice (1950) has shown that to achieve $\mathscr{R} = 0.96\mathscr{C}$ with $S/N = 10$ and an error probability of 10^{-5}, the number of channel signals required is $M = 2^{10,000}$!

Clearly, efficient *block coding* with its extravagant number of channel signals is just as impractical as the infinite coding delay of an ideal system. Alternate schemes using *sequential coding* are nearly as efficient and require relatively simple equipment.

Signal-Space Description of Communication ★

Not so long ago imaginary numbers were playthings of pure mathematics, deemed to have no practical value. But physicists and electrical engineers have since assigned a useful interpretation to $\sqrt{-1}$, and it is now almost unthinkable to discuss signal analysis, electromagnetic waves, or system theory without the aid of this tool. Similarly, the once esoteric geometry of multidimensional spaces (*hyperspace*) was given new meaning by Shannon (1949) for the study of communication on continuous channels, reducing an otherwise intractable problem to more familiar terms. This section outlines his derivation of the Hartley-Shannon law using concepts of signal space.

Consider a continuous baseband channel of bandwidth B so that, of necessity, all signals at the receiving end are bandlimited in B. Borrowing a page from sampling theory—namely Eq. (15), Sect. 8.1—any of the information signals can be written as

$$x(t) = \sum_k x_k \operatorname{sinc}(2Bt - k) \qquad (3)$$

where

$$x_k \triangleq x(kT_s) \qquad T_s = \frac{1}{2B}$$

We assume that these signals are drawn from an ergodic ensemble with an average-power constraint so

$$S = \overline{x^2} = \overline{x_k^2} \qquad (4)$$

as follows from Eq. (12), Sect. (4.5).

Now temporarily suppose that $x(t)$ is essentially zero outside a "long" time interval of duration T; then it is completely described by $D = T/T_s = 2BT$ sample values, $x_1, x_2, \ldots, x_D$. (True, a bandlimited signal cannot be simultaneously timelimited, and we will eventually let $T \to \infty$ to compensate for this.) The fact that D numbers uniquely specify $x(t)$ leads to the notion of *signal space*, a D-dimensional space in which $x(t)$ is represented by a *vector* (or D-tuple)

$$\mathbf{x} = (x_1, x_2, \ldots, x_D) \qquad D = 2BT \qquad (5)$$

The vector starts at the origin and terminates at a point whose coordinates are $x_1, x_2, \ldots, x_D$.

D-dimensional space is like ordinary space save that it has D mutually perpendicular axes. And though we cannot construct more than three such axes in our three-dimensional world,† we can deal logically and mathematically with spaces of higher dimensionality. In particular, signal space is *euclidean*, i.e., the square of the distance from the origin to any point is the sum of the squares of the coordinates. The magnitude or norm squared of a signal vector is therefore

$$\|\mathbf{x}\|^2 = x_1^2 + x_2^2 + \cdots + x_D^2 = \sum_{k=1}^{D} x_k^2 \qquad (6)$$

If $T \gg 1/2B$, which quantifies our meaning of a "long" time interval, then $D \gg 1$ and

$$\frac{1}{D}\sum_{k=1}^{D} x_k^2 \approx \overline{x_k^2} \qquad (7)$$

Therefore, combining Eqs. (4), (6), and (7)

$$\|\mathbf{x}\| = \sqrt{DS} = \sqrt{2BTS} \qquad (8)$$

so the length of all signal vectors is proportional to the square root of the average signal power.

If the tip of $\mathbf{x}$ is swept through all possible positions, the surface generated thereby is a *hypersphere* of radius $\|\mathbf{x}\|$ and all possible signal vectors terminate at the surface of this hypersphere. The "volume" enclosed by such a sphere is‡

$$\mathscr{V}_D = K_D \|\mathbf{x}\|^D \qquad (9)$$

† Abbott (1950) is an amusing discourse on the mysteries of three-dimensional space as viewed by a native of Flatland, a two-dimensional space.
‡ Sommerville (1929).

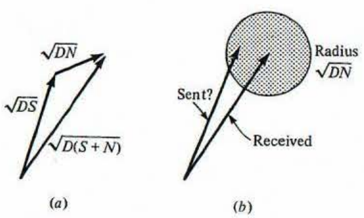

FIGURE 9.3
Signal-space representations. (*a*) Signal and noise vectors; (*b*) uncertainty sphere due to noise.

where the constant K_D does not particularly concern us here. A curious consequence of Eq. (9) is that most of the volume of a hypersphere of high dimensionality ($D \gg 1$) is concentrated at the surface. To illustrate, the relative volume between $\|x\|/2$ and $\|x\|$ is $1 - 2^{-D}$, so if $D = 3$ (a conventional sphere), 87.5 percent of the volume is in the outer "half"; if $D = 100$, the relative volume of the outer portion is approximately $1 - 10^{-30}$. This *volume-concentration* effect proves to be useful in our development, for the dimensionality of typical signal spaces is indeed large. For example, a 3-minute telephone call with $B = 4$ kHz has $D \approx 10^6$.

Our vector description of the channel signals also applies to the *noise* providing it is gaussian white noise from an ergodic source, bandlimited in B. With this condition, sample values spaced by $1/2B$ are uncorrelated and statistically independent. The noise energy in time T is then very nearly NT, N being the average noise power. Hence, the noise is represented in signal space by a vector of length $\sqrt{DN}$, and all possible noise signals are contained within a sphere of that radius. Because the noise is *random*, it might seem that the noise sphere should be "fuzzy"; i.e., a particular sample function of length T may have an energy quite different from NT. But if the dimensionality is high, volume concentration indicates that the noise sphere is quite sharply defined, like a Ping-Pong ball rather than a cloud of gas.

Consider now the state of affairs at the channel output, where we have the desired signal contaminated by noise. Under the usual assumption that signal and noise are independent, their average powers add, and the received signal plus noise vector has length $\sqrt{D(S + N)}$. Figure 9.3*a* shows the geometric interpretation of the transmitted signal, added noise, and signal plus noise. The transmitted signal is seen to lie within a sphere of radius $\sqrt{DN}$ at the tip of the signal-plus-noise vector (Fig. 9.3*b*), and this noise sphere indicates the uncertainty of the receiver as to which signal was intended.

If the possible transmitted signals are known in advance at the receiver, and if the sphere of uncertainty contains the tip of one and only one of the possible signal vectors, then the intended signal can be *exactly determined* despite the noise. Thus, suppose we put into the hypersphere of radius $\sqrt{D(S + N)}$ a large number of non-

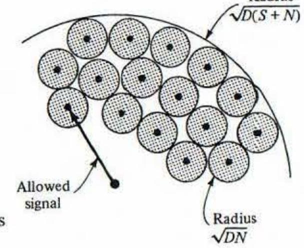

FIGURE 9.4
Signal vectors for virtually errorless transmission.

overlapping noise spheres of radius $\sqrt{DN}$ and then send only those signals corresponding to the center points of the noise spheres, Fig. 9.4. When transmitted signals are selected in this fashion, it is possible to convey information over a noisy continuous channel with vanishingly small error probability.

How many little noise spheres can be packed into the big signal-plus-noise sphere without overlapping? The calculation is important, for it tells us $M = M(T)$, the number of "very different" signals (messages) of length T that can be correctly identified at the channel output, from which we then can find the channel capacity. Clearly, M does not exceed the volume of the big sphere divided by the volume of one of the little spheres, i.e., using Eq. (9),

$$M \leq \frac{K_D[\sqrt{D(S + N)}]^D}{K_D[\sqrt{DN}]^D} = \left(1 + \frac{S}{N}\right)^{D/2} \qquad (10)$$

Note that M is finite for all but truly noiseless channels. Moreover, because a real channel has noise and M is finite, communication over a continuous channel is inherently a *discrete* process. Setting $D = 2BT$ and inserting M into Eq. (1), Sect. 9.2, gives

$$\mathscr{C} \leq \lim_{T \to \infty} \frac{1}{T} \log \left(1 + \frac{S}{N}\right)^{BT} = B \log \left(1 + \frac{S}{N}\right) \qquad (11)$$

which is an upper bound on the capacity of a continuous channel.

To show that information can actually be transmitted at $\mathscr{R} \leq B \log (1 + S/N)$ with negligible errors, Shannon proposed selecting the M waveforms at *random*. If a particular waveform or signal is sent and results in the received signal-plus-noise vector diagramed in Fig. 9.5, there will be no confusion and no "decoding" error providing all other $M - 1$ signal vectors fall outside the lens-shaped volume indicated. Hence, since the signals are randomly chosen, the error probability P_e equals $(M - 1)$ times the ratio of the lens-shaped volume to the volume of the signal sphere.

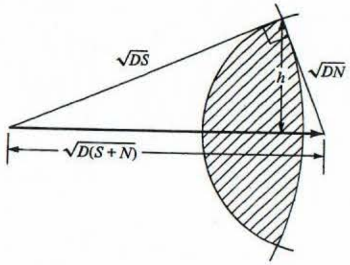

FIGURE 9.5

The volume of a D-dimensional lens is hard to calculate, but it is clearly less than the volume of a sphere with radius h where, from simple geometry, $h = \sqrt{DSN/(S+N)}$. Therefore

$$P_e \le (M-1)\,\frac{K_D h^D}{K_D \|x\|^D} = (M-1)\left(\frac{N}{S+N}\right)^{D/2} \tag{12}$$

so P_e can be made as small as desired if

$$M - 1 \le \left(1 + \frac{S}{N}\right)^{BT} P_e \tag{13}$$

similar to Eq. (9), Sect. 9.2. Finally, taking the logarithm of Eq. (13), we have

$$\frac{1}{T}\log(M-1) \le B \log\left(1 + \frac{S}{N}\right) - \frac{1}{T}\log\frac{1}{P_e}$$

Hence, M may be chosen such that $\mathscr{R} = (1/T)\log M$ approaches arbitrarily close to $\mathscr{C} = B\log(1 + S/N)$ in the limit as $T \to \infty$, Furthermore, since Eq. (12) assumes randomly selected signals, there must be some specific sets of M signals that yield an even lower error probability.

Threshold Effect and Wideband Modulation ★

It is important to observe that attaining information transmission at a rate of $B\log(1 + S/N)$ bits/s requires $D = 2BT \gg 1$ and $M(T) = (1 + S/N)^{BT}$; in other words, the noise spheres must be packed as closely as possible without overlapping. Now suppose the signal-to-noise ratio drops slightly below the design value. The noise spheres will then overlap and the receiver will make frequent decoding errors. Hence there is a sharp *threshold effect* in that a small increase of noise power (or a decrease of signal power) produces a large increase in the probability of error. As a result of these errors the information is lost.

Rather surprisingly, this same explanation holds for the threshold effect in analog message transmission using wideband modulation, e.g., FM and PCM. The signal-space description of modulation is a *one-to-one mapping* of the space of message vectors into the space of modulated signal vectors; if the modulation is wideband ($B_T \gg W$), the mapping is between spaces of different dimensionality. But a well-known theorem of topology says that any one-to-one mapping between spaces of different dimensionality must be *discontinuous* in that a continuous path in one space maps into a broken path in the other. Hence, adjacent vectors in the modulated signal space do not necessarily represent adjacent vectors in the message space, and a slight overlapping of the noise spheres may cause the receiver to "demodulate" a message totally unlike the one that was sent. Accordingly, we conclude that *mutilation and threshold effect are inevitable in all types of wideband modulation.*

9.4 SYSTEM COMPARISONS

The Hartley-Shannon law applies to a restricted class of channels, namely, continuous channels having an average power constraint and additive gaussian white noise. But this description fits many practical communication systems to a reasonable degree, so the hypothetical ideal system that delivers information at a rate $\mathscr{C} = B\log(1 + S/N)$ is the generally accepted standard for system comparisons. In this section we re-examine various existing systems in the light of information theory and see how they measure up against the ideal.

One aspect of particular interest is wideband noise reduction. Hence, as a preliminary, we investigate the exchange of bandwidth for signal-to-noise ratio (or transmitted power) implied by the Hartley-Shannon law, for this is the optimum bandwidth-power exchange.

Optimum Bandwidth-Power Exchange

Suppose it is desired to transmit a signal, bandlimited in W, such that the output signal-to-noise ratio at the destination is $(S/N)_D$. No matter how the transmission is accomplished, the information rate at the output can be no greater that $\mathscr{R}_{max} = W\log[1 + (S/N)_D]$. Further suppose that an ideal system is available for this purpose and that the channel (transmission) bandwidth is B_T, the noise power density is η, and the average signal power at the receiver is S_R. In other words, the channel capacity is $\mathscr{C} = B_T\log[1 + (S/N)_R]$, where $(S/N)_R = S_R/\eta B_T$. These factors are summarized in Fig. 9.6.

If information is neither destroyed not accumulated in the receiver, the output

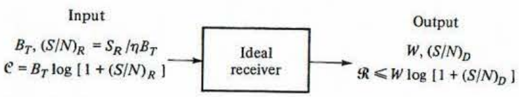

FIGURE 9.6

rate must equal the information rate on the channel. Assuming the system is operating at capacity and the output rate is maximum, then $\mathcal{R}_{\max} = \mathcal{C}$, so

$$B_T \log \left[1 + \left(\frac{S}{N}\right)_R\right] = W \log \left[1 + \left(\frac{S}{N}\right)_D\right]$$

Solving for $\left(\frac{S}{N}\right)_D$ yields

$$\left(\frac{S}{N}\right)_D = \left[1 + \left(\frac{S}{N}\right)_R\right]^{B_T/W} - 1 \qquad (1)$$

which shows that the optimum exchange of bandwidth for power is *exponential*. To emphasize this relation, note that $(S/N)_D \approx (S/N)_R^{B_T/W}$ if the signal-to-noise ratios are large. The exponential trade-off is realized by an ideal system operating to its fullest capacity.

However, with fixed channel *noise density*, as distinguished from fixed noise power, Eq. (1) does not tell the exact story. For as channel bandwidth is increased, the noise power $N_R = \eta B_T$ is likewise increased, and $(S/N)_R$ decreases. A more equitable basis for comparison is obtained by rewriting Eq. (1) in terms of the normalized parameters

$$\gamma = \frac{S_R}{\eta W} \qquad \text{and} \qquad \mathcal{B} = \frac{B_T}{W}$$

used in previous chapters.

The signal to-noise ratio at the receiver input is then $(S/N)_R = (S_R/\eta W)(W/B_T) = \gamma/\mathcal{B}$, and Eq. (1) becomes

$$\left(\frac{S}{N}\right)_D = \left(1 + \frac{\gamma}{\mathcal{B}}\right)^{\mathcal{B}} - 1 \qquad (2)$$

$$\approx \left(\frac{\gamma}{\mathcal{B}}\right)^{\mathcal{B}} \qquad \frac{\gamma}{\mathcal{B}} \gg 1$$

Thus, while the exchange is not strictly exponential it is very nearly so for large signal-to-noise ratios. This means that doubling the transmission bandwidth of an ideal system squares (approximately) the output signal-to-noise ratio. Alternately,

since γ is proportional to S_T, the transmitted power can be reduced to about the square root of its original value without reducing $(S/N)_D$ if bandwidth is increased by a factor of 2. As demonstrated shortly, this exchange is considerably better than that of most existing systems.

Equation (2) also shows what is involved in bandwidth *compression*—transmitting a signal of bandwidth W over a channel of bandwidth $B_T < W$, so that $\mathcal{B} < 1$. Inserting typical values, one finds that such compression is exceedingly costly in terms of transmitted power. To illustrate, suppose we want $(S/N)_D = 10^4$ and suppose that $\eta W = 10^{-6}$. Transmission at baseband ($\mathcal{B} = 1$) requires $S_R = \eta W (S/N)_D = 10$ mW. But to compress bandwidth by a factor of $\frac{1}{2}$ we need $\gamma = \mathcal{B}[1 + (S/N)_D]^{1/\mathcal{B}} - \mathcal{B} \approx \frac{1}{2}(10^4)^2 = 5 \times 10^7$, or $S_R = 50$ W, which is 5,000 times the baseband power. Similarly, for $\mathcal{B} = \frac{1}{10}$, the power requirement is a colossal 10^{33} W!

As a general conclusion we can say that even the optimum bandwidth-power exchange is practical in one direction only, the direction of increasing bandwidth and decreasing power.

Analog Signal Transmission

It is a difficult matter to assess the information rate of analog signals: voice and music waveforms, television video, etc. Furthermore, communication systems designed for such signals have as their goal reasonably faithful reproduction of the signals themselves, with a minimum of noise and distortion. This goal is not quite the same thing as reliable information transfer in the sense of information theory; i.e., the communication engineer may be more concerned with transmission bandwidth, threshold power requirements, and signal-to-noise ratios than he is with channel capacity and its utilization.

Nonetheless, information theory does have something to say in regard to analog signal transmission. Specifically, it tells us the best signal-to-noise ratio that can be obtained with given channel parameters; it tells us the minimum power required to achieve a specified signal-to-noise ratio, as a function of bandwidth; and it indicates the optimum possible exchange of bandwidth for power. Therefore, let us compare the performance of existing systems with that of an ideal system as described by Eq. (2).

Table 9.1 summarizes many of our earlier results. It is assumed that the signal-to-noise ratios are large, all systems are above threshold, and the message is normalized so that $\overline{x^2} = \frac{1}{2}$. The values for the pulsed systems assume $f_s = 2W$, etc., and would not be achieved in practice. Clearly, none of the practical systems exhibit the output improvement that can be had in an ideal system by increasing *either* γ (power) or $\mathcal{B}$ (bandwidth). PCM does have an exponential bandwidth dependence, but, once above threshold, increasing transmitted power yields no further improvement of

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Table 9.1 COMPARISON OF ANALOG MESSAGE TRANSMISSION SYSTEMS

System	$(S/N)_D$
AM	$\gamma/3$
SSB	γ
PDM	$\mathscr{B}\gamma/4$
PPM	$\mathscr{B}^2\gamma/16$
WBFM	$3\mathscr{B}^2\gamma/8$
DM†	$3\mathscr{B}^3/\pi^2$
PCM	$3\mu^{2\mathscr{B}}/2$
Ideal	$(\gamma/\mathscr{B})^{\mathscr{B}}$

† Assuming $f_0 = W$.

$(S/N)_D$; its value is determined by the quantization. It also might be noticed that SSB is just as good as an ideal system having $\mathscr{B} = 1$. Of course the SSB bandwidth ratio is fixed at $\mathscr{B} = 1$, so there is no possibility of wideband noise reduction.

Since, in general, the output signal-to-noise ratio depends on both γ and $\mathscr{B}$, it is difficult to give a complete graphical display of the relations in Table 9.1. As an alternate we can plot $(S/N)_D$ as a function of γ for typical bandwidth ratios (Fig. 9.7) or plot the value of γ required for a specified $(S/N)_D$ as a function of $\mathscr{B}$ (Fig. 9.8).

Figure 9.7 repeats some of the curves from Chaps. 7 and 8, with the addition of a curve for an ideal system having $\mathscr{B} = 6$. Also shown are the threshold points of the practical systems— which is precisely the reason for the figure. It can be seen that

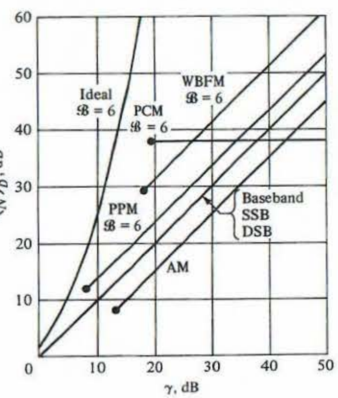

FIGURE 9.7
Postdetection S/N versus $\gamma = S_R/\eta W$.

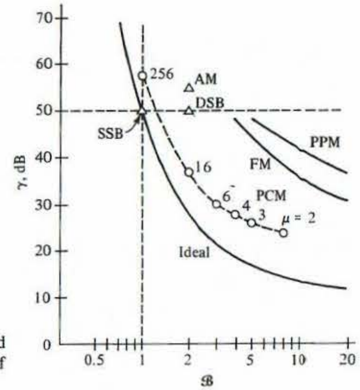

FIGURE 9.8
Normalized power $\gamma = S_R/\eta W$ required for $(S/N)_D = 50$ dB as a function of bandwidth ratio $\mathscr{B} = B_T/W$.

practical wideband noise-reduction systems, e.g., FM, PPM, and PCM, fall short of ideal performance primarily because of threshold limitations. For example, FM and PCM with $\mathscr{B} = 6$ have threshold points offset horizontally by about 8 dB from an ideal system with the same bandwidth ratio. The PPM threshold is much closer but occurs at too low a value of $(S/N)_D$ to be useful for analog signals.

As to the exchange of bandwidth for power, Fig. 9.8 shows the minimum value of γ needed for $(S/N)_D = 50$ dB as a function of bandwidth ratio. For this relatively high output S/N we see that PCM does considerably better than FM or PPM but requires 6 to 8 dB more power than an ideal system. (Just how the PCM curve was obtained is discussed under the next heading.) We might also point out the sharp increase in γ for the ideal system when $\mathscr{B} < 1$, echoing our earlier observation about bandwidth compression.

In summary; at high signal-to-noise ratios FM and PCM give the best wideband performance, PCM being somewhat better. From the power-bandwidth viewpoint, all practical wideband systems are an order of magnitude below the ideal. At low signal-to-noise ratios, only SSB and DSB are useful, having no threshold effect.

The Channel Capacity of PCM

Of the various systems we have discussed, PCM is the most amenable to direct analysis in terms of information theory. This is because the transmitted signal is discrete, even though it represents an analog signal, and the information rate can be readily calculated. Therefore, let us find the channel capacity of PCM and make a direct comparison with the Hartley-Shannon law.

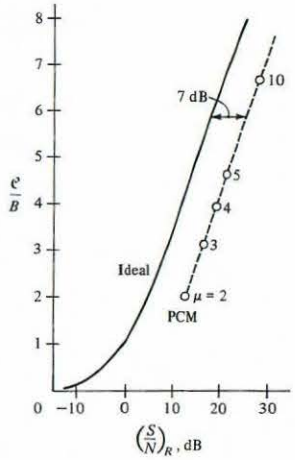

FIGURE 9.9
Channel capacity per unit bandwidth for PCM (with $P_e \approx 10^{-4}$) compared to an ideal system.

Consider a baseband PCM system having transmission bandwidth B, μ equally spaced coded pulse amplitudes, and channel signal-to-noise ratio $(S/N)_R$. Since the entropy of the digital signal is $\mathscr{H} \leq \log \mu$ and the signaling rate is $r \leq 2B$, the information rate on the channel is $\mathscr{R} \leq 2B \log \mu$. Hence,

$$\mathscr{C} = \mathscr{R}_{\max} = 2B \log \mu = B \log \mu^2$$

providing decoding errors can be ignored. But we previously determined that decoding errors are ignorable if the system is above threshold, which requires that

$$\mu^2 \leq 1 + \frac{1}{5}\left(\frac{S}{N}\right)_R \tag{3}$$

Therefore, just above threshold,

$$\mathscr{C} = B \log \left[1 + \frac{1}{5}\left(\frac{S}{N}\right)_R\right] \tag{4}$$

so, if $(S/N)_R \gg 5$, $\mathscr{C} \approx B \log [(S/N)_R/5]$ or

$$\mathscr{C} \approx \mathscr{C}_{\text{ideal}} - B \log_2 5 \tag{5}$$

Based on Eq. (4), Fig. 9.9 plots $\mathscr{C}/B$ versus $(S/N)_R$. The corresponding curve for an ideal system is also given by way of comparison. Viewed in this light, PCM is seen to require about 7 dB more power than an ideal system. However, it is well to bear in mind that an ideal system would have vanishingly small error probability, whereas the PCM curve is for $P_e \approx 10^{-4}$. The reason why PCM compares as favorably as it does to an ideal system stems from the earlier conclusion that because of channel noise, electrical communication is inherently a discrete process. PCM design recognizes and accepts this fact; the transmitted PCM signal, being discrete, is better suited to the noisy channel than uncoded continuous signals.

Communication Efficiency ★

Finally, let us examine minimum power requirements in terms of information rate. This viewpoint is particularly relevant to long-range systems not having a bandwidth constraint—e.g., space communication systems—and leads to the so-called communication efficiency. As before, our reference is the ideal system with

$$\mathscr{C} = B_T \log \left(1 + \frac{S_R}{\eta B_T}\right)$$

$$= \frac{S_R}{\eta} \log \left(1 + \frac{S_R}{\eta B_T}\right)^{\eta B_T/S_R} \tag{6}$$

which has been rewritten to show that if S_R and η are fixed, $\mathscr{C}$ is maximized by taking $B_T \to \infty$. Therefore, using the fact that $\lim_{v \to 0} (1 + v)^{1/v} = e$, the maximum information rate on an ideal system is

$$\mathscr{R}_{\max} = \lim_{B_T \to \infty} \mathscr{C} = \frac{S_R}{\eta} \log_2 e = 1.44 \frac{S_R}{\eta} \tag{7a}$$

or, for a specified information rate $\mathscr{R}$,

$$S_{R_{\min}} = 0.693\, \eta \mathscr{R} \tag{7b}$$

both of which require a coding technique such that the transmission bandwidth approaches infinity. And at the same time $(S/N)_R = S_R/\eta B_T$ goes to zero!

Now consider any system having an information rate $\mathscr{R}$ and received power S_R. Its communication efficiency may be defined as $\mathscr{E} = \mathscr{R}/\mathscr{R}_{\max} = S_{R_{\min}}/S_R$ or, using Eq. (7),

$$\mathscr{E} \triangleq 0.693 \frac{\eta \mathscr{R}}{S_R} \tag{8}$$

so if $\mathscr{E} = 0.2$, for instance, the system requires five times as much power as an ideal system operating with $B_T \to \infty$. To underscore the meaning of $\mathscr{E}$, let $\eta = k\mathscr{T}_N$

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where k is the Boltzmann constant and $\mathcal{T}_N$ the system noise temperature; if $\mathcal{L}$ is the transmission loss, then the *transmitted* power required is

$$S_T = \mathcal{L} S_R = 0.693k \frac{\mathcal{L}\mathcal{T}_N \mathcal{R}}{\mathcal{E}}$$

$$\approx 10^{-23} \frac{\mathcal{L}\mathcal{T}_N \mathcal{R}}{\mathcal{E}} \qquad (9)$$

Clearly, minimizing S_T entails maximizing $\mathcal{E}$.

Relative to the design of efficient practical systems, we infer from Eq. (7) that $\mathcal{E}$ is maximized if B_T is made as large as possible and $(S/N)_R$ as small as possible. We can get large B_T with *wideband modulation* techniques, but the inevitable threshold effect prohibits very small values of $(S/N)_R$ and thereby precludes operation at highest efficiency. This point is further demonstrated by again examining PCM.

Assuming the PCM information rate to be $\mathcal{R} = \mathcal{C}$, with $\mathcal{C}$ given by Eq. (4), the efficiency is

$$\mathcal{E} = 0.693 \frac{\eta}{S_R} B_T \log\left[1 + \frac{1}{5}\left(\frac{S}{N_R}\right)\right]$$

$$= \frac{0.693}{(S/N)_R} \log\left[1 + \frac{1}{5}\left(\frac{S}{N}\right)_R\right]$$

However, from Eq. (3), the threshold condition is $(S/N)_R \geq 5(\mu^2 - 1)$, so we cannot take $(S/N)_R$ arbitrarily small. The best we can do is $\mu = 2$ (*binary* PCM), which gives the largest transmission bandwidth, the smallest channel signal-to-noise ratio, and the highest efficiency. Several times in the past we have suspected that binary PCM is superior to PCM with $\mu > 2$; that suspicion is now confirmed from the power viewpoint. For $\mu = 2$, $(S/N)_R \geq 15$, and

$$\mathcal{E} = \frac{0.693}{15} \log 4 = 0.0924$$

Therefore, the efficiency is in the neighborhood of 9 percent. Such a low efficiency may seem discouraging at first. Putting the matter in proper perspective, it is better to say that binary PCM requires about 10 dB more power than an *ideal* system with *infinite* bandwidth.

Turning to analog modulation, we are faced with the problem of estimating the information rate $\mathcal{R}$ of an analog signal.† A crude but simple expedient is to take the upper bound $\mathcal{R} \leq W \log [1 + (S/N)_D]$, where $(S/N)_D$ and W are the signal parameters after demodulation. This gives an *upper bound* on the efficiency, namely,

$$\mathcal{E} \leq \frac{0.693}{\gamma} \log\left[1 + \left(\frac{S}{N}\right)_D\right] \qquad (10)$$

† Sanders (1960) gives a more complete discussion.

For suppressed-carrier linear modulation (SSB and DSB) we have $(S/N)_D = \gamma$, and such systems are normally operated with signal-to-noise ratios of 30 to 40 dB. The corresponding efficiency is substantially less than 1 percent, not unexpected since the transmission bandwidth is relatively small. On the other hand, wideband analog modulation should prove to be better, and in fact it is.

For FM (without deemphasis) it can be shown that there is an optimum deviation ratio, $\Delta_{\text{opt}} \approx 2$, giving $\mathcal{E} < 0.1$; with $\Delta < 2$ the loss of wideband noise reduction decreases efficiency, while for $\Delta > 2$ the higher threshold level counterbalances the S/N improvement and again decreases efficiency. Clearly, the threshold-extension techniques mentioned in Sect. 7.5 would be of benefit here since they lower the value of γ_{th}. In theory, FM with frequency-compressive feedback in the receiver (FMFB) is capable of achieving 50 percent maximum efficiency; in practice, efficiencies of 25 to 30 percent are possible. PPM appears to have efficiencies comparable to FMFB thanks to its low threshold level.

Example 9.4

To illustrate the implications of Eq. (9), suppose it is desired to transmit a still picture from Mars to earth with $S_T = 10$ W, $\mathcal{L} = 200$ dB, and $\mathcal{T}_N = 50°$K. If the system has 100 percent efficiency ($\mathcal{E} = 1$), the maximum information rate is

$$\mathcal{R} = 10^{23} \frac{\mathcal{E} S_T}{\mathcal{L}\mathcal{T}_N} = 200 \text{ bits/s}$$

Assuming that the photograph is quantized into $200 \times 200 = 4 \times 10^4$ elements, each element having one of 64 possible brightness levels, the total information to be transferred is $\mathcal{I} = 24 \times 10^4$ bits (see Example 9.1). Therefore, the total transmission time T must be

$$T = \frac{\mathcal{I}}{\mathcal{R}} = \frac{24 \times 10^4}{200} = 1{,}200 \text{ s} = 20 \text{ min}$$

The Mariner IV mission, using a less efficient but practical system, required 8 hours to transmit each picture. ////

9.5 PROBLEMS

9.1 (Sect. 9.1) State in words and interpret the requirements on $f(\ )$ in Eqs. (1) to (3).

9.2 (Sect. 9.1) A card is selected at random from a deck of 52. You are told it is a heart. How much information (in bits) have you received? How much more information is needed to completely specify the card? *Ans.*: 2 bits, 3.7 bits.

9.3 (Sect. 9.1) Calculate the amount of information needed to open a lock whose combination consists of three numbers, each ranging from 00 to 99.

9.4 (Sect. 9.1) A source produces six symbols with probabilities $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, $\frac{1}{32}$, and $\frac{1}{32}$. Find the entropy $\mathscr{H}$.

9.5 (Sect. 9.1) A source has an alphabet of size m. One symbol has probability ϵ while the other symbols are equally likely. Find $\mathscr{H}$ in terms of m and ϵ.

9.6★ (Sect. 9.1) Show that $\sum_{j=1}^{m} P_j \log_2 (1/mP_j) = \mathscr{H} - \log_2 m$ and use this, together with the fact that $\ln v \leq v - 1$, to prove that $\mathscr{H} \leq \log_2 m$.

9.7★ (Sect. 9.1) Suppose you are given nine pennies, eight of which are good. The remaining coin is counterfeit and weighs more or less than a good one. Given an uncalibrated balancing scale, how many weighings are necessary in theory to locate the bad coin and determine whether it is light or heavy? (*Hint*: The scale has three possible positions, balanced and unbalanced on one side or the other. The average information per weighing, i.e., the entropy, is maximized if each of these positions is equally likely. Although it is not required, you may wish to devise the actual weighing procedure.)

9.8 (Sect. 9.1) A certain data source has eight symbols that are produced in blocks of three at a rate of 1,000 blocks per second. The first symbol in each block is always the same (presumably for synchronization); the remaining two places are filled by any of the eight symbols with equal probability. What is the entropy rate $\mathscr{R}$? *Ans.*: 6,000 bits/s.

9.9 (Sect. 9.1) A certain data source has 16 possible and equiprobable symbols, each 1 millisecond (ms) long. The symbols are sent in blocks of 15, separated by a 5-ms synchronization pulse. Calculate $\mathscr{R}$.

9.10 (Sect. 9.1) A binary data source has $P_0 = \frac{3}{8}$, $P_1 = \frac{5}{8}$, and intersymbol influence extending over groups of two successive symbols such that $P(1|0) = \frac{3}{4}$ and $P(0|1) = \frac{1}{16}$. Calculate the conditional entropy $\mathscr{H}_c$ and compare with $\mathscr{H}$.

9.11 (Sect. 9.1) Analogous to Eq. (7) the entropy of a *continuous signal* $x(t)$ can be defined as

$$\mathscr{H}(x) = -\int_{-\infty}^{\infty} p(x) \log p(x)\, dx$$

where $p(x)$ is the probability density function. Show that this is a relative rather than absolute measure of information by considering $\mathscr{H}(x)$ and $\mathscr{H}(y)$ when $y(t) = Kx(t)$ and $x(t)$ is uniformly distributed over $[-1, 1]$.

9.12 (Sect. 9.2) Verify that $P(1) = P(0) = \frac{1}{2}$ in Example 9.3.

9.13 (Sect. 9.2) A noiseless discrete channel is to convey information at a rate of 900 bits/s. Determine the minimum number of channel states required if $r = 200$ or 1,000. *Ans*: 23, 2.

9.14 (Sect. 9.2) A discrete source produces the symbols A and B with $P_A = \frac{3}{4}$ and $P_B = \frac{1}{4}$ at a rate of 100 symbols/s. In an attempt to match the source to a noiseless binary channel, the symbols are grouped in blocks of two and encoded as follows:

Grouped symbol	Binary code
AA	1
AB	01
BA	001
BB	000

By calculating the source entropy rate and the channel symbol rate, show that this code is not optimum but is reasonably efficient.

9.15★ (Sect. 9.2) Consider a binary channel with noise that affects the channel symbols in blocks of three such that a block is received without errors or there is exactly one error in the first, second, or third digit. These four possibilities are equiprobable.
(*a*) Show that $\mathscr{C} = r/3$.
(*b*) Devise a code that yields error-free transmission at $\mathscr{R} = \mathscr{C}$.

9.16 (Sect. 9.2) The per-digit error and no-error probabilities for a *nonsymmetric binary channel* are indicated schematically in Fig. P9.1, i.e., P(0 received | 0 sent) $= \alpha$, etc. Show that

$$\mathscr{R}_{ec} = r\left(\frac{1-\alpha+\beta}{2}\log\frac{2}{1-\alpha+\beta} + \frac{1+\alpha-\beta}{2}\log\frac{2}{1+\alpha-\beta}\right)$$

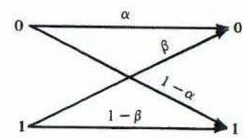

FIGURE P9.1.

9.17 (Sect. 9.2) The channel in Prob. 9.16 is said to be *useless* if $\alpha = \beta$. Justify this mathematically and intuitively.

9.18 (Sect. 9.3) A noiseless discrete channel has $r = 10^5$. Investigate the possibility of replacing it with a continuous channel having $B = 8$ kHz and $S/N = 31$.

9.19 (Sect. 9.3) Find the minimum time required to transmit 600 *decimal* digits on the continuous channel in Prob. 9.18. *Ans.*: 0.05 s.

9.20 (Sect. 9.3) A certain communication system uses RF pulses with four possible amplitude levels. If an information rate of 10^6 bits/s is desired, what is the minimum practical value for the carrier frequency f_c? (*Hint*: Recall the fractional bandwidth constraints.)

9.21 (Sect. 9.3) Estimate $M(T)$ for a typical telephone channel with $B = 3$ kHz, $S/N = 30$ dB, and $T = 3$ min.

9.22★ (Sect. 9.3) Consider the set of channel signals $x_i(t) = \text{sinc}\,(2Bt - i), i = 0, \pm 1, \pm 2, \ldots$. Discuss the signal-space interpretation and determine the upper limit on $|i|$ for a space of D dimensions.

9.23★ (Sect. 9.3) A continuous channel has $S/N = 63$ and $B = 1$ kHz. Obtain numerical bounds on M and T such that $\mathscr{R} = 5{,}000$ with $P_e \leq 10^{-3} \approx 2^{-10}$.

9.24 (Sect. 9.4) An engineer claims to have designed a communication system giving $(S/N)_D = 10^6$ when $(S/N)_R = 3$ and $B_T = 10$ W. Do you believe him? Explain.

9.25 (Sect. 9.4) An ideal system has $\mathscr{B} = 4$ and $(S/N)_D = 40$ dB. What is the new value of $(S/N)_D$ if B_T is tripled while all other parameters are fixed? *Ans.*: 72 dB.

9.26 (Sect. 9.4) Repeat Prob. 9.25 with B_T changed to $W/2$ instead of being tripled.

9.27 (Sect. 9.4) A communication system has $B_T = 5$ kHz, $\mathscr{L} = 30$ dB, and $\eta = 10^{-7}$ W/Hz. Find the theoretical minimum value for S_T to yield $(S/N)_D = 50$ dB when $W = 1$ kHz.

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9.28★(Sect. 9.4) Consider the efficiency of an ideal system with finite B_T.

(*a*) Show that $\mathscr{E} = 0.693(\mathscr{R}/B_T)/(2^{\mathscr{R}/B_T} - 1)$.

(*b*) Sketch $\mathscr{E}$ versus $B_T/\mathscr{R}$ and discuss the implications of this curve.

9.29★(Sect. 9.4) Using Eq. (10) and Carson's rule, obtain an upper bound on $\mathscr{E}$ in terms of Δ for an FM system (without deemphasis) operating just above threshold.

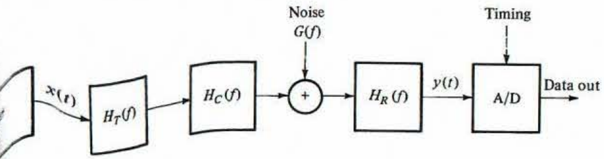
9.30★(Sect. 9.4) Suppose it is desired to have live voice transmission from Mars to earth via binary PCM with $f_s = 8$ kHz and $\mathcal{Q} = 64$. Estimate the power requirement at the Mars transmitter by inserting typical values into Eq. (9).

10

DIGITAL DATA SYSTEMS

Until about 1950, analog signal transmission was the primary stock-in-trade of the communications industry, save, of course, for teletype and telegraph. But that was before the advent of automation and the computer revolution. Today, enormous quantities of digital data are being generated in government, commerce, and science. Furthermore, the need to eliminate manual handling and human error has led to the concept of integrated data processing systems. These developments, coupled with a growing trend toward decentralization, have made the transmission and distribution of digital signals a major task of electrical communication.

While according to information theory, highly efficient and virtually errorless data transmission is possible, the vast majority of applications simply do not warrant the cost and complexity demanded by a near-ideal communication system. This is not to say that information theory has no place in the study of data transmission but rather that it may be better to do the job now, in less than optimum fashion, than wait until technological breakthroughs permit a more sophisticated solution. Consequently the design of a practical data system is usually quite pragmatic, based on two elementary considerations: the gross source rate in digits per second, as distinguished from the entropy rate in bits per second; and the desired transmission reliability, i.e., the tolerated error rate. Before going on, it is perhaps helpful to get a feeling for typical values of these parameters.



...em.

...log signals have a wide range of bandwidths, so digital signals have a ...it rates,† depending on the source and application. The output of ...d devices, such as the teletypewriter, is seldom more than 100 bits/s. ...may be even lower. At the other extreme, a PCM color TV signal ...100 megabits/s. Intermediate rates of 1 to 100 kilobits/s are attained ...nical and electromagnetic data readers. Currently, the best wideband ...ced capabilities of several hundred megabits per second. But most ...ion is via standard telephone voice circuits with top speeds of 1,000

...aling speed, the question of acceptable error rate is rather nebulous. ...s are of grave concern in computerized banking systems (especially ...ance is adversely affected) whereas requirements can be quite relaxed ...ering data from measuring instruments having 10 percent accuracy. ...an error probability of 10^{-5} is probably intolerable; for the latter ...10 would be an extravagant waste. Nonetheless, error probabilities ...are representative and suitable to many applications. We shall take ...del line for comparison purposes, bearing in mind its arbitrary nature. ...surveys digital data systems, their problems, and design principles. ...ft off in Sect. 4.5 with baseband design techniques, many of which ...digital modulation systems. An introduction to error-control ...pter.

...D SYSTEM DESIGN

...nts of a baseband data system are laid out in Fig. 10.1. A data ...habet of μ possible digits (or symbols) delivers r digits/s to a pulse ...the analog signal

$$x(t) = \sum_k a_k p_S\left(t - \frac{k}{r}\right) \qquad p_S(0) = 1 \tag{1}$$

† ...use the terms *bit* and *binary digit* interchangeably in this

where $p_S(t)$ is the normalized pulse shape and the a_k represent the source digits. After passage through a transmitting filter $H_T(f)$ and the channel $H_C(f)$ the signal is corrupted by additive noise, so the signal plus noise at the output of the receiving filter $H_R(f)$ is

$$y(t) = \sum_k A_k p\left(t - t_d - \frac{k}{r}\right) + n(t) \tag{2}$$

where t_d is the total time delay and K_R the amplification, i.e.,

$$A_k = K_R a_k \qquad p(0) = 1$$

The A/D converter operates on $y(t)$ by sampling it at the optimum times

$$t_m = \frac{m}{r} + t_d$$

giving

$$y(t_m) = A_m + \sum_{k \neq m} A_k p\left(\frac{m-k}{r}\right) + n(t_m) \tag{3}$$

of which the first term represents the mth digit while the other two terms are *intersymbol interference* (ISI) and *noise*, respectively. Finally, the digital message (plus errors) is regenerated by comparing $y(t_m)$ with a set of threshold levels.

Equation (3) brings out two aspects of baseband system design for reliable data transmission, namely, minimizing the noise and eliminating or minimizing ISI. Additionally, if there is a bandwidth constraint, it may be desired to maximize the signaling rate r for a given channel bandwidth B or minimize the bandwidth required for a given rate. All these aspects are considered here. Unless otherwise stated, the noise is presumed to be zero-mean gaussian with spectral density $G(f) = \eta/2$. The source is always taken to be ergodic with equiprobable digits.

Ideal Baseband System

In Sect. 4.5 we assumed the channel was distortionless with bandwidth B, i.e., $H_C(f) = K_R e^{-j\omega t_d}$ when time delay is included. Accordingly, we took the bandlimited pulse shape $p_S(t) = \text{sinc}\, rt$ with $r = B/2$ and let $H_R(f)$ be an ideal LPF. Therefore, the signaling rate is maximum

$$r = 2B \tag{4}$$

and there is no intersymbol interference since

$$p(t) = \text{sinc}\, rt \qquad p\left(\frac{m-k}{r}\right) = \begin{cases} 1 & m = k \\ 0 & m \neq k \end{cases} \tag{5}$$

We also assumed a polar signal with level spacing $2A$, i.e.,

$$A_k = \begin{cases} 0, \pm 2A, \pm 4A, \ldots, \pm(\mu - 1)A & \mu \text{ odd} \\ \pm A, \pm 3A, \ldots, \pm(\mu - 1)A & \mu \text{ even} \end{cases} \tag{6}$$

so the received signal power is

$$S_R = \overline{A^2} = \frac{(\mu^2 - 1)A^2}{3} \tag{7}$$

It was then shown that, with zero-mean gaussian noise, the optimum threshold levels are midway between the values of A_k and

$$P_e = 2\left(1 - \frac{1}{\mu}\right) Q\left(\frac{A}{\sigma}\right) \tag{8}$$

where σ is the rms value of n so

$$\sigma^2 = \overline{n^2} = N = \eta B = \frac{\eta r}{2}$$

Thus

$$\left(\frac{A}{\sigma}\right)^2 = \frac{3}{\mu^2 - 1}\frac{S_R}{\eta B} = \frac{3}{\mu^2 - 1}\frac{S}{N} = \frac{6}{\mu^2 - 1}\rho \tag{9}$$

where

$$\rho \triangleq \frac{S_R}{\eta r} \tag{10}$$

This system will be called the *ideal* baseband system, and its performance serves as a benchmark for system comparisons—especially in terms of the normalized parameter ρ.

Rectangular-Pulse System—Integrate-and-Dump Filtering

Thanks to its very special pulse shape, the ideal system achieves maximum signaling rate on a bandlimited channel. But if there is no bandwidth constraint, we might just as well use nonoverlapping rectangular pulses

$$p_S(t) = \Pi\left(\frac{t}{\tau}\right) \qquad \tau \leq \frac{1}{r} \tag{11}$$

which are much simpler to generate. Recalling our study of the random binary wave, Example 3.7, Sect. 3.5, it follows that the available bandwidth must be $B \gg 1/\tau \geq r$ to prevent pulse smearing and ISI. However, this does not mean that the receiving

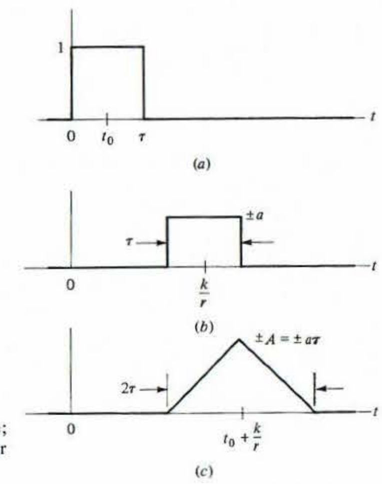


FIGURE 10.2
Matched filtering. (*a*) Impulse response; (*b*) received pulse; (*c*) matched filter output.

filter should be wide open, for that would lead to excessive output noise and high error rates. Clearly, the pivotal element of such a system is $H_R(f)$, which must be designed to minimize simultaneously the noise and the ISI due to smearing at the filter output.

As far as noise is concerned, we know that a *matched filter* has the desirable effect of maximizing the peak of an output pulse compared to the rms noise. Hence, drawing upon Eq. (12*b*), Sect. 4.4, suppose the impulse response of the receiving filter is

$$h_R(t) = \Pi\left(\frac{t_0 - t}{\tau}\right) = \Pi\left(\frac{t - t_0}{\tau}\right) \tag{12}$$

as sketched in Fig. 10.2*a* for $t_0 = \tau/2$. To analyze the performance we will assume a binary signal and ignore any transmission distortion or delay, so the kth pulse at the receiver input is of the form $\pm a\Pi[(t - k/r)/\tau]$, Fig. 10.2*b*. Convolving with $h_R(t)$ yields a *triangular* pulse of height $\pm a\tau$ and width 2τ, Fig. 10.2*c*, i.e.,

$$A_k p\left(t - \frac{k}{r}\right) = \pm A\Lambda\left(\frac{t - t_0 - k/r}{\tau}\right) \qquad A = a\tau \tag{13}$$

Clearly, the optimum sampling times are now $t_m = (m/r) + t_0$ (to which any time delay t_d must be added), and there is *no intersymbol interference* since $p(t - k/r) = 0$ for $|t - k/r| \geq 1/r$ if $\tau \leq 1/r$.

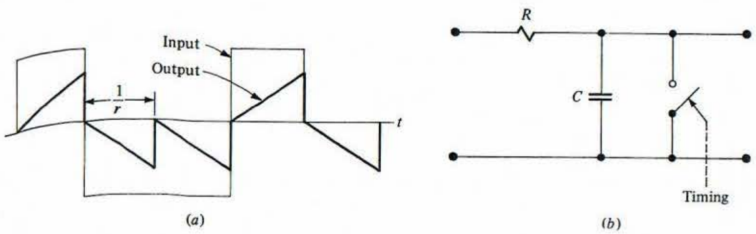


FIGURE 10.3
Integrate-and-dump filtering. (a) Waveforms; (b) circuit.

Assuming sampling at the optimum times, the error probability is $P_e = Q(A/\sigma)$ for the binary case under discussion. But from Eq. (12a), Sect. 4.4, $(A/\sigma)^2 = 2E_R/\eta$ where E_R is the energy in the received pulse. With rectangular polar binary pulses, $E_R = (\pm a)^2\tau = S_R/r$, and

$$\left(\frac{A}{\sigma}\right)^2_{\max} = \frac{2E_R}{\eta} = \frac{2S_R}{\eta r} = 2\rho \qquad (14)$$

identical to Eq. (9) with $\mu = 2$. Similar analysis for the μ-ary case shows that Eq. (9) still holds, and we conclude that the error rate of a rectangular pulse system is just as good as an ideal system providing the receiving filter is designed according to Eq. (12).

Unfortunately, that proviso turns out to be a stumbling block, for one cannot synthesize a perfectly rectangular impulse response, and all approximations have decaying tails that cause ISI. But there is usually more than one way to skin a cat, and referring to Fig. 10.2c shows that the trailing part of the triangular pulse after $t = t_0 + k/r$ is unnecessary. Therefore, suppose we simply integrate each incoming pulse and then reset or "dump" the integrator immediately after the sampling time. This *integrate-and-dump* filtering takes care of the intersymbol-interference problem, as illustrated in Fig. 10.3a, and achieves the same maximum value of A/σ. Figure 10.3b is a circuit realization; if $RC \gg 1/r$, the output rises linearly over the input pulse duration until momentary closure of the switch discharges the capacitor and resets the output to zero. Needless to say, the sampling and dumping must be carefully synchronized. Even so, the integrate-and-dump filter represents virtually the only case where matched filtering is closely approximated in practice.

EXERCISE 10.1 Consider a rectangular-pulse system with $\tau = 1/2r$ and a receiving filter that is a simple RC LPF *without* a discharging switch. Assuming RC is sufficiently small that ISI is negligible, show that

$$\left(\frac{A}{\sigma}\right)^2 = \frac{2S_R}{(\eta/4RC)} = 4RCr(2\rho)$$

Evaluate $4RCr$ such that the output pulses decay to 0.001 of their maximum value by the next sample time. *Ans.*: 0.58.

Nyquist Pulse Shaping

Between the extremes of sinc pulses ($B = r/2$) and rectangular pulses ($B \gg r$), there is a compromise made possible using Nyquist-shaped pulses—pulses bandlimited in $B \geq r/2$ having periodic zero crossings to eliminate ISI. We will consider only the case of $r/2 \leq B \leq r$, which has the greatest practical interest and for which Nyquist's *vestigial-symmetry theorem* can be presented as follows:† Let

$$P(f) = [P_\beta(f)] * \left[\left(\frac{1}{r}\right)\Pi\left(\frac{f}{r}\right)\right] \qquad (15a)$$

with

$$P_\beta(f) = 0 \qquad |f| > \beta \leq \frac{r}{2}$$
$$\int_{-\infty}^{\infty} P_\beta(f)\, df = p_\beta(0) = 1 \qquad (15b)$$

Then

$$p(t) = \mathscr{F}^{-1}[P(f)] = p_\beta(t)\,\text{sinc}\, rt \qquad (16a)$$

$$p\left(\frac{k}{r}\right) = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases} \qquad (16b)$$

and $P(f)$ is bandlimited in $B = (r/2) + \beta \leq r$, so

$$r = 2B - \beta$$

Proof of these assertions should be self-evident.

Infinitely many functions satisfy the above conditions, and they include the ideal case $p(t) = \text{sinc}\, rt$—i.e., if $P_\beta(f) = \delta(f)$, then $p_\beta(t) = 1$, $\beta = 0$, and $r = 2B$. One class of functions has the *sinusoidal roll-off* or *raised cosine* frequency characteristic plotted in Fig. 10.4a. Analytically,

$$P_\beta(f) = \frac{\pi}{4\beta}\cos\frac{\pi f}{2\beta}\,\Pi\left(\frac{f}{2\beta}\right) \qquad 0 < \beta \leq \frac{r}{2} \qquad (17a)$$

† See Lucky, Salz, and Weldon (1968, chap. 4) for the general statement and proof of the Nyquist criterion, as well as the specialized pulse shaping known as *duobinary* or *partial response* that accepts controlled amounts of ISI in exchange for faster signaling.

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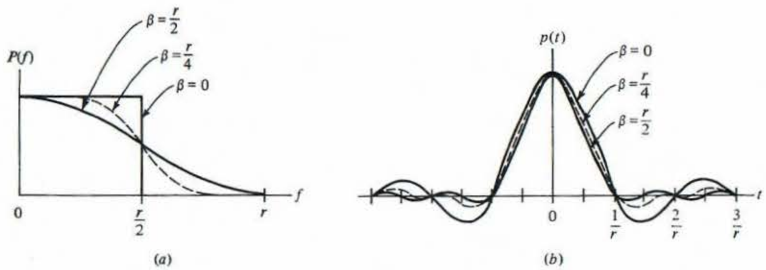


FIGURE 10.4
Nyquist pulse shaping. (*a*) Spectra; (*b*) time functions.

so

$$P(f) = \begin{cases} \dfrac{1}{r} & |f| < \dfrac{r}{2} - \beta \\ \dfrac{1}{r}\cos^2 \dfrac{\pi}{4\beta}\left(|f| - \dfrac{r}{2} + \beta\right) & \dfrac{r}{2} - \beta < |f| < \dfrac{r}{2} + \beta \\ 0 & |f| > \dfrac{r}{2} + \beta \end{cases} \qquad (17b)$$

and the required pulse-shaping filters are relatively easy to synthesize approximately. The corresponding time functions are

$$p(t) = \frac{\cos 2\pi\beta t}{1 - (4\beta t)^2}\operatorname{sinc} rt \qquad (18a)$$

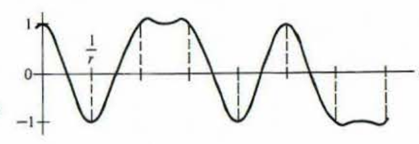
$$= \frac{\operatorname{sinc} 2rt}{1 - (2rt)^2} \qquad \beta = \frac{r}{2} \qquad (18b)$$

shown in Fig. 10.4*b* for $\beta = r/4$ and $r/2$ along with sinc rt. Note that the leading and trailing oscillations of $v(t)$ decay more rapidly than those of sinc rt; this means that synchronization will be less critical than a sinc-pulse system since modest timing errors do not cause large amounts of intersymbol interference.†

Further inspection reveals two other convenient properties of $p(t)$ when $\beta = r/2$: the half-amplitude pulse width exactly equals the pulse-to-pulse spacing $1/r$, and there are additional zero crossings at $t = \pm 3/2r, \pm 5/2r, \ldots$. Consequently, a polar digital signal constructed from such pulses will have zero crossings precisely halfway between

† On the other hand, $\sum_{k \neq m} A_k p[(m-k)/r]$ may actually be *unbounded* if $p(t) = \operatorname{sinc}(r + \epsilon)t$ where ϵ is a small error in the signaling rate.

FIGURE 10.5
Baseband waveform for **10110100** using Nyquist pulses with $\beta = r/2$.



the pulse centers whenever there is a change of polarity. Figure 10.5 illustrates this for the binary message **10110100**. These zero crossings can then be used to generate a *timing signal* for synchronization purposes, bypassing the need for separately transmitted timing information. Nyquist proved that $p(t)$ per Eq. (18*b*) is the *only* bandlimited pulse shape possessing all these convenient features. But the penalty is a 50 percent reduction of signaling speed, since $r = B$ rather than $2B$.

Optimum Terminal Filters ★

Now consider the overall design of a baseband system using Nyquist-shaped pulses; i.e., the output pulse shape $p(t)$ has the form of Eq. (16). With reference to Fig. 10.6, the input signal $x(t) = \sum_k a_k p_S(t - k/r)$ is filtered by $H_T(f)$ yielding the average transmitted power S_T; at the receiving end the signal plus noise is passed through $H_R(f)$ giving the output $y(t) = \sum_k A_k p(t - t_d - k/r) + n(t)$. If $p_S(t)$, S_T, $H_C(f)$, $G(f)$, and $p(t)$ are specified, we need to find $H_T(f)$ and $H_R(f)$ such that $\overline{n^2}/S_T$ is minimized so $(A/\sigma)^2$ is maximized and P_e is minimized. The matched-filter strategy does not apply here because we have specified the output pulse shape. Instead, our analysis will follow the lines of Sect. 4.3, where we derived optimum terminal filters for analog transmission.

Since $p(t)$ and $p_S(t)$ are given, one design constraint is

$$P_S(f)H_T(f)H_C(f)H_R(f) = K_R e^{-j\omega t_d}P(f) \qquad (19)$$

where $P(f) = \mathscr{F}[p(t)]$, etc. The second constraint involves the transmitted power S_T which we compute by finding the average power in one sample function and then take the ensemble average, this roundabout technique being necessary because the pulses

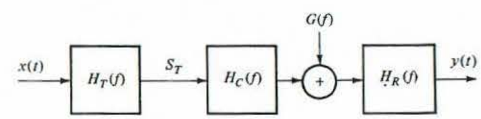


FIGURE 10.6

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are not necessarily orthogonal. Proceeding with the calculation, the signal at the output of $H_T(f)$ is $\sum_k a_k p_T(t - k/r)$ where $p_T(t) = \mathscr{F}^{-1}[H_T(f)P_S(f)]$, so

$$S_T = \mathbf{E}\left\{\lim_{M\to\infty} \frac{r}{2M}\int_{-M/r}^{M/r}\left[\sum_{k=-M}^{M} a_k p_T\left(t-\frac{k}{r}\right)\right]^2 dt\right\}$$

$$= \lim_{M\to\infty} \frac{r}{2M}\sum_{k=-M}^{M}\sum_{m=-M}^{M} \mathbf{E}[a_k a_m]\int_{-M/r}^{M/r} p_T\left(t-\frac{k}{r}\right)p_T\left(t-\frac{m}{r}\right) dt$$

If the source digits are statistically independent

$$\mathbf{E}[a_k a_m] = \begin{cases}\overline{a^2} & m = k\\ 0 & m \neq k\end{cases}$$

and

$$S_T = r\overline{a^2}\int_{-\infty}^{\infty} p_T^2(t)\,dt = r\overline{a^2}\int_{-\infty}^{\infty} |H_T(f)P_S(f)|^2\,df \tag{20a}$$

where, since $A_k = K_R a_k$,

$$\overline{a^2} = \frac{\overline{A^2}}{K_R^2} = \frac{(\mu^2-1)A^2}{3K_R^2} \tag{20b}$$

which follows from Eq. (7) even though S_R may not equal $\overline{A^2}$. As for the output noise power,

$$N = \sigma^2 = \overline{n^2} = \int_{-\infty}^{\infty} |H_R(f)|^2 G(f)\,df \tag{21}$$

Combining Eqs. (19), (20), and (21), we eliminate $H_T(f)$ to get

$$\left(\frac{A}{\sigma}\right)^2 = \frac{3S_T}{(\mu^2-1)r}\left[\int_{-\infty}^{\infty}|H_R(f)|^2G(f)\,df\int_{-\infty}^{\infty}\frac{|P(f)|^2}{|H_C(f)H_R(f)|^2}\,df\right]^{-1}$$

and the product of integrals has a form like that of Eq. (11), Sect. 4.3. Using Schwarz's inequality in the same fashion, the product is minimized when

$$|H_R(f)|_{\text{opt}}^2 = \frac{K|P(f)|}{|H_C(f)|G^{1/2}(f)} \tag{22a}$$

$$|H_T(f)|_{\text{opt}}^2 = \frac{K_R^2|P(f)|G^{1/2}(f)}{K|P_S(f)|^2|H_C(f)|} \tag{22b}$$

where K is an arbitrary constant. These may be compared with the optimum terminal filters for analog transmission, Eq. (13), Sect. 4.3. As before, the filters' phase shifts are arbitrary providing that Eq. (19) is satisfied.

With the optimum filters

$$\left(\frac{A}{\sigma}\right)_{\max}^2 = \frac{3S_T}{(\mu^2-1)r}\left[\int_{-\infty}^{\infty}\frac{|P(f)|G^{1/2}(f)}{|H_C(f)|}\,df\right]^{-2} \tag{23}$$

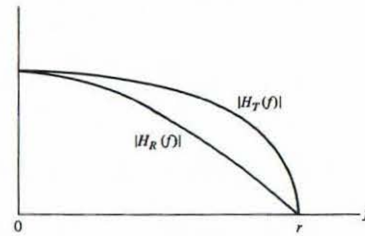

FIGURE 10.7

For the case of white noise and a distortionless channel, substituting $G(f) = \eta/2$ and $|H_C(f)| = K_R$ yields

$$\left(\frac{A}{\sigma}\right)_{\max}^2 = \frac{6K_R^2S_T}{(\mu^2-1)\eta r}\left[\int_{-\infty}^{\infty}|P(f)|\,df\right]^{-2} \tag{24}$$

But $K_R^2S_T = S_R$ when the channel is distortionless, and Nyquist-shaped pulses per Eq. (17) have $\int_{-\infty}^{\infty}|P(f)|df = 1$, so

$$\left(\frac{A}{\sigma}\right)_{\max}^2 = \frac{6S_R}{(\mu^2-1)\eta r} = \frac{6}{\mu^2-1}\,p$$

Thus system performance is again identical to an ideal baseband system.

Example 10.1

To illustrate these results, take the case of white noise and a distortionless channel with $p_S(t) = \Pi(t/\tau)$, $\tau = 1/r$, and $p(t)$ given by Eq. (18b). Then

$$P_S(f) = \frac{1}{r}\operatorname{sinc}\left(\frac{f}{r}\right) \qquad P(f) = \frac{1}{r}\cos^2\left(\frac{\pi f}{2r}\right)\Pi\left(\frac{f}{2r}\right)$$

Inserting these in Eq. (22) and dropping the porportionality constants gives

$$|H_R(f)| = \cos\left(\frac{\pi f}{2r}\right)\Pi\left(\frac{f}{2r}\right) \qquad |H_T(f)| = \frac{\cos(\pi f/2r)}{\operatorname{sinc}^{1/2}(f/r)}\Pi\left(\frac{f}{2r}\right) \tag{25}$$

as plotted in Fig. 10.7. Note the slight high-frequency rise in $|H_T(f)|$; if $\tau \ll 1/r$, this rise is negligible and $|H_T(f)| \approx |H_R(f)|$ so one design serves for both filters, a production advantage when many such systems are to be implemented. ////

Equalization

Regardless of which particular pulse shape is chosen for the baseband digital signal, some amount of residual ISI inevitably occurs owing to imperfect filter design, incomplete knowledge of the channel characteristics, etc. Hence, an adjustable

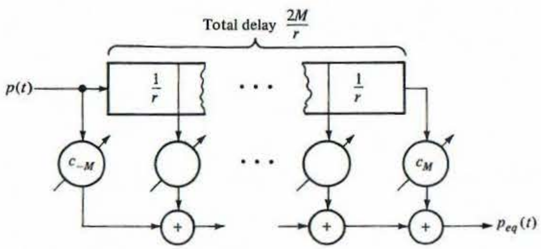

FIGURE 10.8
Transversal equalizer.

equalizing filter is often inserted between the receiving filter and the A/D converter, particularly on switched systems where the specific channel characteristics are not known in advance. Such "mop-up" equalizers usually take the form of a tapped-delay-line or transversal filter discussed before in conjunction with linear distortion, Sect. 4.2. However, the design strategy for mop-up equalization is somewhat different, and deserves consideration here.

Figure 10.8 shows a transversal equalizer with $2M + 1$ taps and total delay $2M/r$. If $p(t)$ is the input pulse shape, then the equalized output is

$$p_{eq}(t) = \sum_{m=-M}^{M} c_m p\left(t - \frac{m + M}{r}\right) \tag{26}$$

Assuming $p(t)$ has its peak at $t = 0$ and ISI on both sides, the sampling times at the output should be taken as

$$t_k = \frac{k + M}{r}$$

so

$$p_{eq}(t_k) = \sum_{m=-M}^{M} c_m p\left(\frac{k - m}{r}\right) \tag{27}$$

Equation (27) has the mathematical form known as *discrete convolution*.

Ideally, we would like to have

$$p_{eq}(t_k) = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases}$$

thereby completely eliminating ISI. But this condition cannot be realized because we have only $2M + 1$ variables at our disposal, namely, the tap gains c_m. Moreover, the optimum setting for the tap gains is not immediately obvious. One approach is to set the c_m such that

$$p_{eq}(t_k) = \begin{cases} 1 & k = 0 \\ 0 & k = \pm 1, \pm 2, \ldots, \pm M \end{cases} \tag{28}$$

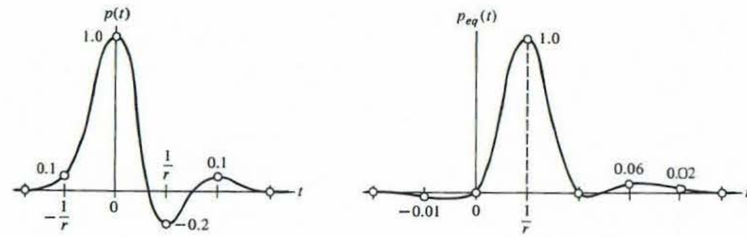

FIGURE 10.9

which together with Eq. (27) yields $2M + 1$ simultaneous linear equations that can be solved for the c_m's. Equation (28) describes a *zero-forcing equalizer* since $p_{eq}(t_k)$ has M zero values on each side. This strategy is optimum in the sense that it minimizes the peak intersymbol interference, and it has the added advantage of simplicity. Other strategies are known to be optimum in a different sense.

When the values of $p[(k - m)/r]$ are not known in advance, the tap gains must be set using an iterative process and an error criterion; i.e., starting from an initial setting, a test pulse is transmitted, the error is measured, the taps are reset, and the operation is repeated one or more times. The obvious cumbersomeness of this procedure has prompted considerable interest in automatic or *adaptive* equalizers that adjust themselves using error measures derived from the actual data signal.† Adaptive equalization has special value when the channel characteristics change with time.

Mop-up equalization (fixed or adaptive) does have one hidden catch in that the equalizer somewhat increases the noise power at the input to the A/D converter. But that effect generally is more than compensated for by the ISI reduction.

Example 10.2

A three-tap ($M = 1$) zero-forcing equalizer is to be designed for the distorted pulse $p(t)$ of Fig. 10.9*a*. The tap gains are readily calculated by expressing Eqs. (27) and (28) in the matrix form

$$\begin{bmatrix} 1.0 & 0.1 & 0.0 \\ -0.2 & 1.0 & 0.1 \\ 0.1 & -0.2 & 1.0 \end{bmatrix} \begin{bmatrix} c_{-1} \\ c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

† Lucky, Salz, and Weldon (1968, chap. 6) discusses equalizer-adjustment algorithms and implementation of adaptive equalization.

Therefore,

$$c_{-1} = -0.096 \qquad c_0 = 0.96 \qquad c_1 = 0.2$$

and the corresponding sample values of $p_{eq}(t)$ are plotted in Fig. 10.9*b* with an interpolated curve. As expected, there is one zero on each side of the peak. However, zero forcing has produced some small ISI at points further out where the unequalized pulse was zero. ////

Binary versus μ-ary Signaling

Up till now we have taken the values of r and μ as dictated by the source. But one need not stick with that limitation as long as the transmission information rate accommodates the source data. A smattering of information theory helps put this in quantitative form. Specifically, if the source parameters are r_S and μ_S, then the source information rate is

$$\mathcal{R}_S = r_S \log_2 \mu_S \qquad \text{bits/s}$$

while, ignoring errors, the system has capacity

$$\mathcal{C} = r \log_2 \mu \qquad \text{bits/s}$$

Hence, the designer may select any convenient values of r and μ providing $\mathcal{C} \geq \mathcal{R}_S$.

If power is at a premium and bandwidth is not, one should use the *smallest* value of μ—i.e., *binary* signaling. Since $\mathcal{C} = r$ when $\mu = 2$, and since $B \geq r/2$, the required bandwidth is

$$B \geq \tfrac{1}{2}\mathcal{R}_S \tag{29}$$

The extent to which the available bandwidth exceeds $\mathcal{R}_S/2$ indicates how much flexibility the designer has in selecting the pulse shape. Assuming gaussian white noise and optimum filters for the chosen pulse shape, the signal power requirement is computed from the desired error probability P_e via

$$Q\left(\sqrt{\frac{2S_R}{\eta\mathcal{R}_S}}\right) \leq P_e \tag{30}$$

which follows from Eqs. (8) to (10) with $\mu = 2$ and $r = \mathcal{R}_S$.

On the other hand, bandwidth is minimized if μ is taken as the *largest* possible value since $r = \mathcal{R}_S/\log_2 \mu$. Accordingly, one must find the largest value of μ such that

$$2\left(1 - \frac{1}{\mu}\right)Q\left(\sqrt{\frac{6 \log_2 \mu}{\mu^2 - 1}\frac{S_R}{\eta\mathcal{R}_S}}\right) \leq P_e \tag{31}$$

Then

$$B \geq \frac{\mathcal{R}_S}{2 \log_2 \mu_{max}} \tag{32}$$

where achieving the lower bound entails sinc pulses.

In any case, appropriate code translation must be provided at transmitter and receiver whenever $\mu \neq \mu_S$. This coding merely serves to create a better match between source and channel, and should not be confused with error-control coding or coding that decreases redundancy in the source messages.

EXERCISE 10.2 A certain computer produces octal digits ($\mu_S = 8$) at a rate of 10,000 per second. If the available transmission bandwidth is 20 kHz and $\eta = 5 \times 10^{-8}$ W/Hz, select appropriate system parameters such that $P_e \leq 10^{-4}$. *Ans.*: $\mu = 2$ and $r = 30{,}000$; pulse shape per Eq. (18*a*) with $\beta \leq 2B - r = 10$ kHz; $S_R \geq (3.7)^2\eta\mathcal{R}_S/2 \approx 10$ mW.

Other Design Considerations

There are numerous other factors that, quite properly, might be deemed important considerations in the design of practical baseband systems. One of these is *impulse noise*, the sporadic pulses caused by electrical storms and switching equipment which can be a particularly vexatious problem for data transmission via telephone circuits. Other factors include synchronization methods, DC removal and restoration for AC-coupled systems, and various line coding techniques.

Each, however, is a special topic in its own right and must be omitted here in deference to subjects of more general interest. The interested student will find ample material in the professional literature, to which the selected supplementary reading list is a guide and a starting point.

10.2 DIGITAL MODULATION: ASK, FSK, AND PSK

Just as there are a multitude of modulation techniques for analog signals, so is it that digital information can be impressed upon a carrier wave in many ways. This section covers the basic types and some of their variations, drawing upon the results of the previous section coupled with the modulation theory from Chaps. 5 to 7. Primary emphasis will be given to system performance in the presence of noise, i.e., error probabilities as a function of ρ. For the most part, we shall confine our attention to binary signals; the extension to μ-ary signals is not conceptually difficult but involves more arduous mathematics.

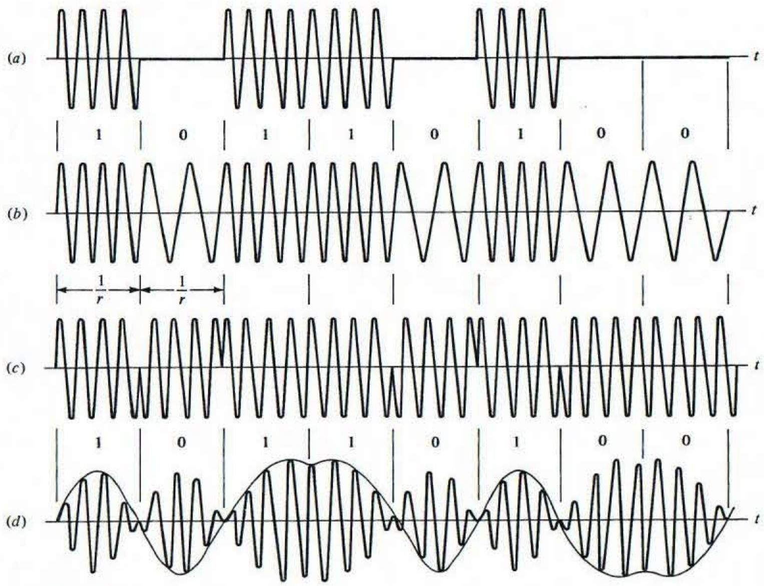

FIGURE 10.10
Digital modulation waveforms for the binary message **10110100**. (a) ASK; (b) FSK; (c) PSK; (d) DSB with baseband pulse shaping.

Given a digital message, the simplest modulation technique is *amplitude-shift keying* or ASK, wherein the carrier amplitude is switched between two or more values, usually *on* and *off* for binary signals. The resultant modulated wave then consists of RF pulses or *marks*, representing binary **1**, and *spaces*, representing binary **0**, Fig. 10.10*a*. Similarly, one could key the frequency or phase, Figs. 10.10*b* and *c*, giving *frequency-shift keying* (FSK) or *phase-shift keying* (PSK). These modulation types correspond to AM, FM, and PM, respectively, with a rectangluar-pulse modulating signal. Clearly, the price of this simplicity is excessive transmission bandwidth, i.e., $B_T \gg r$. ASK also has wasted power in the carrier, just like analog AM.

Transmission bandwidth can be reduced if the pulses are *shaped* (bandlimited) prior to modulation. For instance, Fig. 10.10*d* is the DSB-modulated version of the baseband signal in Fig. 10.5 and has $B_T = 2B = 2r$. The minimum possible bandwidth is $B_T \approx r/2$, corresponding to sinc pulses at baseband and VSB modulation. Other

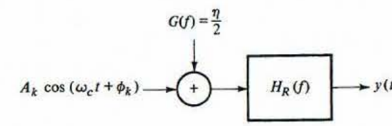

FIGURE 10.11

types of modulation include FM, PM, and VSB + C with baseband shaping.† SSB is not feasible for the reasons covered in Sect. 5.4.

Despite the number of modulation options, performance analysis depends primarily on the type of demodulation or detection—of which there are just two major classes: *coherent* (synchronous) detection and *noncoherent* (envelope) detection. Noncoherent detection is the simpler to implement and, generally speaking, is used in conjunction with ASK and FSK. Such systems usually incorporate matched (integrate-and-dump) filters since the modulated signal consists of rectangular pulses. Coherent detection is used for PSK, DSB, and VSB, with matched or otherwise optimum or near-optimum terminal filters.

We will analyze the more important noncoherent and coherent digital modulation systems. First, however, it is helpful to examine the statistical properties of a sinusoid corrupted by bandpass noise, that situation being common to all cases.

Envelope and Phase of a Sinusoid plus Bandpass Noise

Consider a digital modulation system with carrier frequency f_c. At the receiver, the signal is contaminated by white noise and then passed through a bandpass filter $H_R(f)$ centered at f_c, Fig. 10.11. The output bandpass noise is, from Sect. 7.3,

$$n(t) = n_i(t) \cos(\omega_c t + \theta) - n_q(t) \sin(\omega_c t + \theta) \qquad (1a)$$

$$= R_n(t) \cos[\omega_c t + \theta + \phi_n(t)] \qquad (1b)$$

where an arbitrary constant phase θ has been included. We recall that the in-phase and quadrature components $n_i(t)$ and $n_q(t)$ are independent zero-mean gaussian variates with

$$\overline{n_i^2} = \overline{n_q^2} = \overline{n^2} = N = \frac{\eta}{2}\int_{-\infty}^{\infty} |H_R(f)|^2\, df \qquad (2)$$

† Croisier and Pierret (1970) have devised a promising alternative they call *digital echo modulation*.

whereas the envelope $R_n = \sqrt{n_i^2 + n_q^2}$ has the *Rayleigh* PDF

$$P_{R_n}(R_n) = \frac{R_n}{N} e^{-R_n^2/2N} \qquad R_n \geq 0$$
$$\overline{R_n} = \sqrt{\frac{\pi N}{2}} \qquad \overline{R_n^2} = 2N \tag{3}$$

while the phase $\phi_n = \arctan(n_i/n_q)$ is uniformly distributed over $|\phi_n| \leq \pi$.

In the absence of noise, let the modulated signal at the output of $H_R(f)$ be of the form

$$K_R x_c(t) = A_k \cos(\omega_c t + \phi_k) \qquad t = \frac{k}{r}$$

where A_k and/or ϕ_k represent the kth message digit. Taking $\theta = \phi_k$ in Eq. (1), the signal plus noise is

$$y(t) = [A_k + n_i(t)] \cos(\omega_c t + \phi_k) - n_q(t) \sin(\omega_c t + \phi_k) \tag{4a}$$
$$= R(t) \cos[\omega_c t + \phi_k + \phi(t)] \tag{4b}$$

where

$$R^2 = (A_k + n_i)^2 + n_q^2 \qquad \phi = \arctan \frac{n_q}{A_k + n_i} \tag{5}$$

Equation (5) relates the envelope and phase of a sinusoid plus bandpass noise to the signal and noise components, and we are specifically interested in the PDFs of R and ϕ.

Before plunging into the details, let us speculate on the nature of $p_R(R)$ and $p_\phi(\phi)$ under extreme conditions. When $A_k = 0$, R and ϕ reduce to R_n and ϕ_n, the envelope and phase of the noise alone, which have Rayleigh and uniform PDFs, respectively. At the other extreme, if $A_k \gg \sqrt{N}$, $R \approx A_k + n_i$ and $\phi \approx n_q/A_k$ so both will be approximately gaussian. Since A_k^2 is proportional to the signal power S_R, this case corresponds to $S_R/N \gg 1$.

For intermediate cases we follow the procedure that led to Eq. (11), Sect. 7.3, replacing n_i by $A_k + n_i$. This yields the joint PDF

$$p(R,\phi) = \frac{R}{2\pi N} e^{-(R^2 - 2A_k R \cos\phi + A_k^2)/2N} \tag{6}$$

with $R \geq 0$ and $|\phi| \leq \pi$ by definition. Unlike the case of bandpass noise alone, the product term $R \cos\phi$ in Eq. (6) means that R and ϕ are not statistically independent.

Separation of $p_R(R)$ and $p_\phi(\phi)$ therefore requires integration, i.e.,

$$p_R(R) = \int_{-\pi}^{\pi} p(R,\phi)\, d\phi$$
$$= \frac{R}{N} e^{-(R^2+A_k^2)/2N} \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{(A_k R/N)\cos\phi}\, d\phi$$
$$= \frac{R}{N} e^{-(R^2+A_k^2)/2N} I_0\left(\frac{A_k R}{N}\right) \qquad R \geq 0 \tag{7}$$

where I_0 is the modified Bessel function of the first kind and zero order. Equation (7) is called the *Rice distribution*; its formidable appearance is perhaps discouraging, and indeed this is not a trivial function. Fortunately, most systems will satisfy the large-signal condition $A_k \gg \sqrt{N}$ for which $I_0(A_k R/N) \approx \sqrt{N/2\pi A_k R}\, e^{A_k R/N}$. Hence, if $S_R/N \gg 1$,

$$p_R(R) \approx \sqrt{\frac{R}{2\pi A_k N}} e^{-(R-A_k)^2/2N} \qquad R \geq 0 \tag{8}$$

which is essentially gaussian with $\overline{R} = A_k$ and $\sigma_R^2 = N$. That conclusion stems from the fact that $R/2\pi A_k N \approx 1/2\pi N$ in the vicinity of $R = A_k$ where $p_R(R)$ has the bulk of its area.

Turning to the phase, $p_\phi(\phi)$ can be found exactly by integrating Eq. (6) over $R = 0$ to ∞; the integration is somewhat lengthy and the result is analytically cumbersome—see Prob. 10.14. Consistent with $S_R/N \gg 1$, it simplifies to

$$p_\phi(\phi) \approx \sqrt{\frac{A_k^2}{2\pi N}} \cos\phi\, e^{-(A_k \sin\phi)^2/2N} \qquad |\phi| \leq \frac{\pi}{2} \tag{9}$$

which, for small values of ϕ, approximates a gaussian with $\overline{\phi} = 0$ and $\overline{\phi^2} = N/A_k^2$. Equation (9) is invalid for $|\phi| > \pi/2$ (why?), but the probability of that event is vanishingly small under the assumed condition.

Noncoherent ASK

Putting the above results to work, we start with noncoherent binary ASK. We assume the received waveform looks like Fig. 10.10*a* with $\tau = 1/r$ and $f_c \gg r$. If E_R is the energy in each received marking pulse (representing **1**), then the average signal power is $S_R = \frac{1}{2} r E_R$ since marks and spaces are equally likely.

The detection system consists of a bandpass filter (or integrate-and-dump filter) matched to the RF marking pulses, i.e.,

$$h_R(t) = \cos\omega_c t\, \Pi[r(t - t_0)] \tag{10}$$

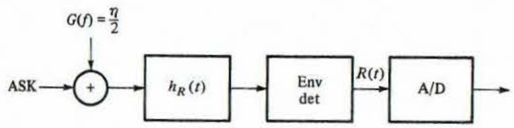

FIGURE 10.12
Noncoherent detection of ASK.

followed by an envelope detector and A/D converter, Fig. 10.12. When a marking pulse is received, the filtered output is a triangular RF pulse with peak value

$$A = \sqrt{\frac{E_R}{2r}} = \sqrt{\frac{S_R}{r^2}} \tag{11}$$

Thus, at the optimum sampling times, the envelope has $A_k = A$ or 0 (the no-pulse or space output). The filtered noise power is

$$N = \frac{\eta}{2}\int_{-\infty}^{\infty} |H_R(f)|^2\, df = \frac{\eta}{2}\int_{-\infty}^{\infty} h_R^2(t)\, dt = \frac{\eta}{4r} \tag{12}$$

so

$$\left(\frac{A^2}{N}\right)_{\max} = \frac{2E_R}{\eta} = \frac{4S_R}{\eta r} = 4\rho \tag{13}$$

where $\rho = S_R/\eta r$ as before.

Calculating the error probability involves two PDFs: when a **0** or space is sent, the resultant envelope has a Rayleigh PDF, $p(R|A_k = 0) = pR_n(R_n)$; when a **1** or mark is sent, $p(R|A_k = A)$ has a Rice PDF. Figure 10.13 shows the two density functions,

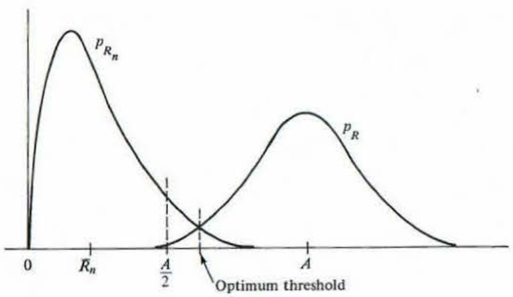

FIGURE 10.13
PDFs for noncoherent detection of ASK.

assuming $\rho \gg 1$ so Eq. (8) can be used for $p(R|A_k = A)$. Clearly, the decision threshold level of the A/D converter should be set between 0 and A, but the best value is not necessarily A/2. In point of fact, there is no "best" threshold in the sense of equalizing P_{e_1} and P_{e_0} and simultaneously minimizing P_e. Minimum P_e is achieved by taking the threshold where the two curves intersect (explain this assertion!), which turns out to be approximately $(A/2)\sqrt{1 + (2/\rho)}$.

Usually, the threshold is set at $A/2$ which is nearly optimum if $\rho \gg 1$. Then

$$P_{e_0} = \int_{A/2}^{\infty} p_{R_n}(R_n)\, dR_n = e^{-A^2/8N} = e^{-\rho/2} \tag{14a}$$

while

$$P_{e_1} = \int_0^{A/2} p_R(R)\, dR \approx Q\left(\frac{A}{2\sqrt{N}}\right) = Q(\sqrt{\rho})$$

where the approximation is the area from $-\infty$ to $A/2$ of a gaussian with $\bar{R} = A$ and $\sigma_R{}^2 = N$. Introducing the asymptotic expression for $Q(\sqrt{\rho})$, Eq. (10), Sect. 3.5, gives

$$P_{e_1} \approx \frac{1}{\sqrt{2\pi\rho}}\, e^{-\rho/2} \tag{14b}$$

thereby bringing out the fact that $P_{e_1} \ll P_{e_0}$ when $\rho \gg 1$. Finally,

$$P_e = P_0 P_{e_0} + P_1 P_{e_1}$$
$$\approx \frac{1}{2}\left(1 + \frac{1}{\sqrt{2\pi\rho}}\right) e^{-\rho/2} \tag{15}$$

To compare this result with baseband transmission under the same conditions, we have from Eq. (8), Sect. 10.1, with $\rho \gg 1$,

$$P_{e_{BB}} = Q(\sqrt{2\rho}) \approx \frac{1}{2\sqrt{\pi\rho}}\, e^{-\rho} \tag{16}$$

Therefore,

$$P_{e_{ASK}} \approx \sqrt{\pi\rho}\, e^{+\rho/2}\, P_{e_{BB}}$$

so if $\rho = 10$, the ASK error probability is about 800 times that of a well-designed baseband system—and almost all of the errors change **0**s to **1**s. Clearly, ASK does not compare very favorably. Moreover, as in most of our previous studies, Eq. (15) represents virtually the best that can be achieved by this system. Imperfect filter design, poor synchronization, etc., all work in the direction of increasing P_e. In particular, bandpass matched filters are difficult to build, so practical systems usually

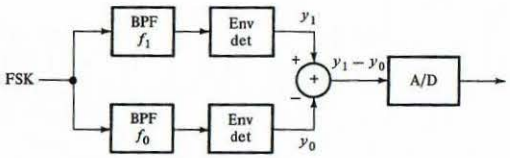

FIGURE 10.14
Noncoherent detection of binary FSK.

have an integrate-and-dump *baseband* filter after the envelope detector, further degrading the performance. But then one should not expect too much from a system having relatively unsophisticated components.

EXERCISE 10.3 Carry out the calculation of Eq. (12) and show that Eq. (11) is correct by substituting the value of N into $A^2/N = 2E_R/\eta$, which always holds for a matched filter.

Noncoherent FSK

It may seem unusual that envelope detection works for FSK as well as ASK. However, careful examination of the binary FSK signal in Fig. 10.10*b* reveals that it basically consists of two interleaved ASK signals of differeng carrier frequencies, say f_1 and f_0. Accordingly, noncoherent detection can be accomplished using a pair of matched filters and envelope detectors arranged per Fig. 10.14, one branch responding to the pulses at f_1, the other to f_0. To prevent cross talk at the sampling times, it is necessary that

$$|f_1 - f_0| = Mr$$

where M is an integer, usually taken to be unity to minimize bandwidth. (Unfortunately, this condition results in a signal that has discrete-frequency sinusoidal components, which may have an adverse effect in certain applications.)

In absence of noise and cross talk, $y_1 = A$ and $y_0 = 0$ when the received frequency is f_1, and vice versa for f_0. Since the envelope difference $y_1 - y_0$ is the input to the A/D converter, the threshold level should be set at zero, regardless of the value of A. Thus, $P_{e_1} = P(y_1 - y_0 < 0)$ and, from symmetry, $P_{e_0} = P_{e_1} = P_e$, so

$$P_e = P(y_0 > y_1) \qquad (17)$$

where, based on our study of ASK, y_0 has a Rayleigh PDF while y_1 is Rician. But because the FSK wave lacks the "spaces" of ASK, $S_R = rE_R$ and

$$\left(\frac{A^2}{N}\right)_{\max} = \frac{2E_R}{\eta} = \frac{2S_R}{\eta r} = 2\rho \qquad (18)$$

at the output of either filter, depending on the input frequency.

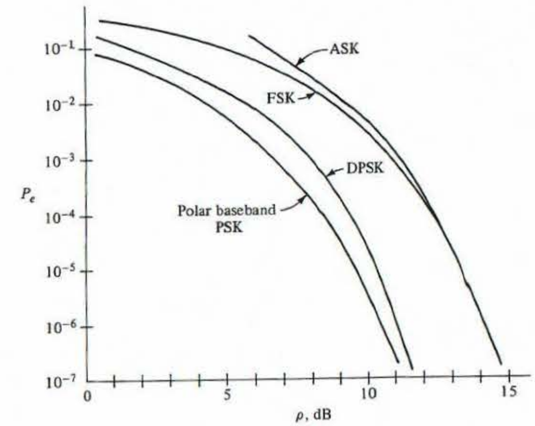

FIGURE 10.15
Error probabilities for binary digital modulation systems.

To compute the probability Eq. (17) we first note that $p_{y_0}(y_0) = p_{R_n}(y_0)$ so, for a given value of y_1,

$$P(y_0 > y_1 \mid y_1) = \int_{y_1}^{\infty} p_{R_n}(y_0)\, dy_0 = e^{-y_1^2/2N}$$

Then, accounting for all possible values of y_1,

$$P_e = \int_0^{\infty} P(y_0 > y_1 \mid y_1) p_R(y_1)\, dy_1$$

$$= \int_0^{\infty} e^{-y_1^2/2N} \frac{y_1}{N} e^{-(y_1^2 + A^2)/2N} I_0\left(\frac{Ay_1}{N}\right) dy_1$$

in which $p_{y_1}(y_1) = p_R(y_1)$ per Eq. (7) with $A_k = A$. Rather amazingly, this integral can be evaluated in closed form with no approximations. For that purpose, we let $\lambda = \sqrt{2}\, y_1$ and $a = A/\sqrt{2}$, giving

$$P_e = \frac{1}{2} e^{-A^2/4N} \int_0^{\infty} \frac{\lambda}{N} e^{-(\lambda^2 + a^2)/2N} I_0\left(\frac{a\lambda}{N}\right) d\lambda$$

and comparison with Eq. (7) shows that the integrand now has exactly the same form as a Rician PDF; consequently, the area (integral) equals unity. Finally, since $A^2/4N = \rho/2$, we have the simple result that

$$P_e = \tfrac{1}{2} e^{-\rho/2} \qquad (19)$$

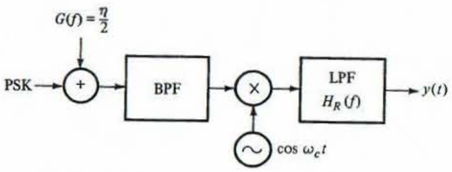

FIGURE 10.16
Synchronous detection of binary PSK.

Thus, as seen from the curves of P_e versus ρ plotted in Fig. 10.15, noncoherent FSK gives no better error rates than ASK except at relatively small values of ρ.

What then are the advantages of FSK, if any? Upon further reflection, three advantages over ASK are evident:

1 FSK has the constant-amplitude property whose merits were covered in conjunction with analog exponential modulation.
2 The per-digit error probabilities P_{e_1} and P_{e_0} are equal.
3 The optimum threshold level is independent of A and ρ, and need not be readjusted if the signal strength varies with time, a not uncommon phenomenon in radio transmission.

It is precisely for this last reason that FSK is preferred to ASK in applications where fading is expected and synchronous detection is not feasible.

One final point remains to be discussed here, namely, why binary FSK does not give the wideband noise reduction usually associated with FM. For the system here described the reason is very simple: it is basically AM rather than FM. If one uses true FM, including detection by a limiter-discriminator,† it turns out that the transmission bandwidth required to control intersymbol interference is so large that the noise-reduction effect is essentially canceled by the increased predetection noise. However, for multilevel signals ($\mu > 2$) some advantage in the form of a bandwidth-power exchange is possible; after all, if $\mu \to \infty$, digital and analog signals are equivalent.

Coherent PSK

Coherent or synchronous detection relies on precise knowledge of the phase of the received carrier wave as well as its frequency, and thus involves more sophisticated hardware. In return, it offers improved performance.

Consider, for instance, the binary PSK signal of Fig. 10.10*c*—or, for that matter, the DSB signal of Fig. 10.10*d*, since both signals can be said to have $A_k = \pm A$ or $\phi_k = 0$ or π. An appropriate synchronous detector is shown in Fig. 10.16, where the

† See Bennett and Davey (1965, chap. 9).

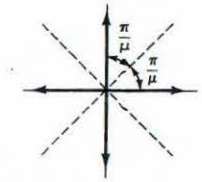

FIGURE 10.17
Phasors and thresholds of PSK with $\mu = 4$.

local oscillator is synchronized and $H_R(f)$ is a *lowpass* filter; the bandpass filter before the mixer serves only to prevent overload from excessive noise. From our study of synchronous detection for analog signals, it immediately follows that the filtered output is of the form

$$y(t) = \pm A + n_i(t) \qquad (20a)$$

and, with optimum filtering,

$$\left(\frac{A^2}{\overline{n_i^2}}\right)_{\max} = \frac{2S_R}{\eta r} = 2\rho \qquad (20b)$$

Therefore, since $n_i(t)$ is gaussian,

$$P_e = Q\left(\frac{A}{\sqrt{\overline{n_i^2}}}\right) = Q(\sqrt{2\rho}) \qquad (21)$$

the same as optimum baseband transmission and, from Fig. 10.15, 3 to 4 dB better than noncoherent modulation.

Because of its performance quality, combined with the constant-amplitude property, *multilevel* ($\mu > 2$) PSK has considerable practical value. Note, especially, that increasing μ increases the equivalent bit rate without the need for larger transmission bandwidth. It thus behooves us to spend some time on μ-ary PSK where

$$K_R x_c(t) = A \cos(\omega_c t + \phi_k) \qquad \phi_k = 0, \frac{2\pi}{\mu}, \frac{4\pi}{\mu}, \ldots, \frac{2(\mu-1)\pi}{\mu} \qquad (22)$$

Conceptually, such a signal is detected by phase discrimination with decision angles π/μ, $3\pi/\mu$, ... centered between the expected carrier-phase values. Figure 10.17 shows a phasor diagram for quaternary PSK ($\mu = 4$) and the corresponding angular thresholds.

The received signal plus noise, after bandpass filtering, has the form of Eq. (4*b*), so errors occur whenever the noise-induced phase perturbation $\phi(t)$ crosses the threshold. Therefore, by symmetry, the per-digit error probabilites are equal and

$$P_e = P\left(|\phi| > \frac{\pi}{\mu}\right) = 1 - \int_{-\pi/\mu}^{\pi/\mu} p_\phi(\phi)\, d\phi \qquad (23a)$$

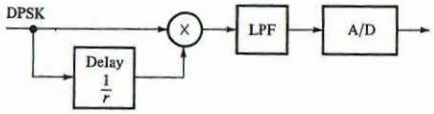

FIGURE 10.18
Phase-comparison detection of differentially coherent binary PSK.

Assuming optimum bandpass filtering, $A^2/N = 2\rho$ so if $\rho \gg 1$,

$$\int_{-\pi/\mu}^{\pi/\mu} p_\phi(\phi)\, d\phi = \frac{2}{\sqrt{2\pi}} \int_0^{\sqrt{2\rho}\sin \pi/\mu} e^{-\lambda^2/2}\, d\lambda \qquad (23b)$$

where we have used Eq. (9) and let $\lambda = \sqrt{2\rho} \sin \phi$. Consistent with the large-signal condition one finally obtains

$$P_e \approx \frac{1}{\sqrt{\pi\rho} \sin \pi/\mu} e^{-\rho \sin^2 \pi/\mu} \qquad (24)$$

which holds for $\mu \geq 4$ and $\rho \gg 1$.

Differentially Coherent PSK

A clever technique known as *phase-comparison* or *differentially coherent* PSK (DPSK) has been devised to get around the synchronization problems of coherent detection. The strategy is diagramed in Fig. 10.18 for binary DPSK; somewhat as with homodyne detection, the local oscillator is replaced by the signal itself delayed in time by exactly $1/r$, the bit spacing. If adjacent digits are of like phase, their product results in a positive output (or binary **1**); conversely, opposite phases result in a negative output (binary **0**). Thus, it is the *shift* or *no shift* between transmitted phase values that represents the message information, so appropriate coding, called *differential encoding*, is required at the transmitter.

Differential encoding starts with an arbitrary first digit and thereafter indicates the message digits by successive transition or no transition. A transition stands for message **0**, and no transition for message **1**. The coding process is illustrated below.

Input message		**1**		**0**		**1**		**1**		**0**		**1**		**0**		**0**		
Encoded message	**1**		**1**		**0**		**0**		**0**		**1**		**1**		**0**		**1**	
Transmitted phase	0		0		π		π		π		0		0		π		0	
Phase-comparison output		+		−		+		+		−		+		−		−		
Output message		1		0		1		1		0		1		0		0		

Differential encoding is most often used for PSK systems but is not restricted to such applications. In general, it is advantageous for systems having no sense of absolute polarity.

As to the noise performance of DPSK it might appear that differential detection requires twice as much power as coherent detection because the phase reference is itself contaminated by noise. However, the perturbations actually tend to cancel in the comparison process, so the degradation is not so great. We will carry out the analysis for the case where the two adjacent phases are the same, say $\phi_k = \phi_{k-1} = 0$, so an error occurs if the output is negative. Writing the received (and filtered) signal plus noise as $[A + n_i(t)] \cos \omega_c t - n_q(t) \sin \omega_c t$, the output of the delay unit will be $[A + n_i(t')] \cos \omega_c t' - n_q(t') \sin \omega_c t'$ where $t' = t - 1/r$. Thus, providing that f_c is an integer multiple of r (which can be guaranteed by heterodyning if necessary), the A/D input is proportional to

$$y(t) = [A + n_i(t)][A + n_i(t')] + n_q(t)n_q(t') \qquad (25)$$

and $P_e = P(y < 0)$.

Equation (25), involving products of gaussian variates, is called a *quadratic form*. It can be simplified through a diagonalization process by defining four new variates:

$$\alpha_i = A + \frac{n_i(t) + n_i(t')}{2} \qquad \alpha_q = \frac{n_q(t) + n_q(t')}{2}$$
$$\beta_i = \frac{n_i(t) - n_i(t')}{2} \qquad \beta_q = \frac{n_q(t) - n_q(t')}{2} \qquad (26a)$$

Then, letting

$$\alpha = |\alpha_i + j\alpha_q| \qquad \beta = |\beta_i + j\beta_q| \qquad (26b)$$

one can show that $y(t) = \alpha^2 - \beta^2$ and therefore

$$P_e = P(y < 0) = P(\alpha^2 < \beta^2) = P(\beta > \alpha) \qquad (27)$$

the last step following since α and β are nonnegative. Now we have an expression identical to that of noncoherent FSK with α and β replacing y_1 and y_0. Moreover, as the reader can check from the definitions, α is Rician and β is Rayleigh. The only difference here is that the mean-square noise terms are reduced by a factor of 2, e.g., assuming $n_q(t)$ and $n_q(t')$ are independent, $\overline{\alpha_q^2} = (\overline{n_q^2} + \overline{n_q^2})/4 = N/2$. Therefore,

$$P_e = \tfrac{1}{2} e^{-\rho} \qquad (28)$$

as obtained from Eq. (19) with ρ replaced by $\rho/2$ reflecting the noise reduction.

Referring to the curves of Fig. 10.15 we see that DPSK has a 2- to 3-dB power advantage over noncoherent detection and a penalty of less than 1 dB compared to coherent PSK at $P_e \leq 10^{-4}$. And remember that DPSK does not require separate synchronization. Additionally, with slight modification of the transmitted wave, a timing signal is easily derived for the A/D converter. The only significant disadvantage is that because of the fixed delay time $1/r$ in the detector, the system is locked in on a specific signaling speed, thereby precluding variable speed (asynchronous) transmission. A minor annoyance is the fact that errors tend to occur in groups of two (why?).

For μ-ary DPSK, the error probability has the same form as Eq. (24), save that ρ must be increased by a factor of

$$\frac{\sin^2 (\pi/\mu)}{2 \sin^2 (\pi/2\mu)} \tag{29}$$

to achieve the same P_e as coherent detection. With $\mu = 4$ the penalty is about 2 dB and increases asymptotically toward 3 dB as $\mu \to \infty$. Thus, only for large μ does differential detection fully suffer from the noisy phase reference.

Example 10.3

High-speed data transmission over voice telephone channels has been a subject of intense practical concern for many years. One of the earliest and most successful designs, used in the Bell System model 201, 205, and 207 data sets, is described here. Incorporating quaternary DPSK modulation, it achieves synchronous transmission at rates up to 2,400 bits/s on telephone lines that have been *conditioned* (equalized) for digital signals.

Figure 10.19*a* diagrams† the transmitter. Incoming binary digits are grouped into blocks of two, called *dibits*, so $\mu = \mu_s^2 = 4$ and $r = r_s/2 = 1{,}200$. The dibits differentially phase modulate the carrier and the DPSK wave is then envelope-modulated to yield

$$x_c(t) = \sum_k p\left(t - \frac{k}{r}\right) \cos (2\pi f_c t + \phi_k)$$

where

$$p(t) = \cos^2\left(\frac{\pi r t}{2}\right) \Pi(2rt)$$

This equivalent baseband pulse shape is not bandlimited but, by applying duality to Eqs. (17) and (18), one can show that $P(f)$ has negligible content for $|f| > r$. Hence, $B_T \approx 2r = 2{,}400$ Hz centered on the carrier frequency $f_c = 1{,}800$ Hz. The carrier

† Baker (1962) details the hardware realizations of transmitter and receiver functions, some of which are particularly ingenious.

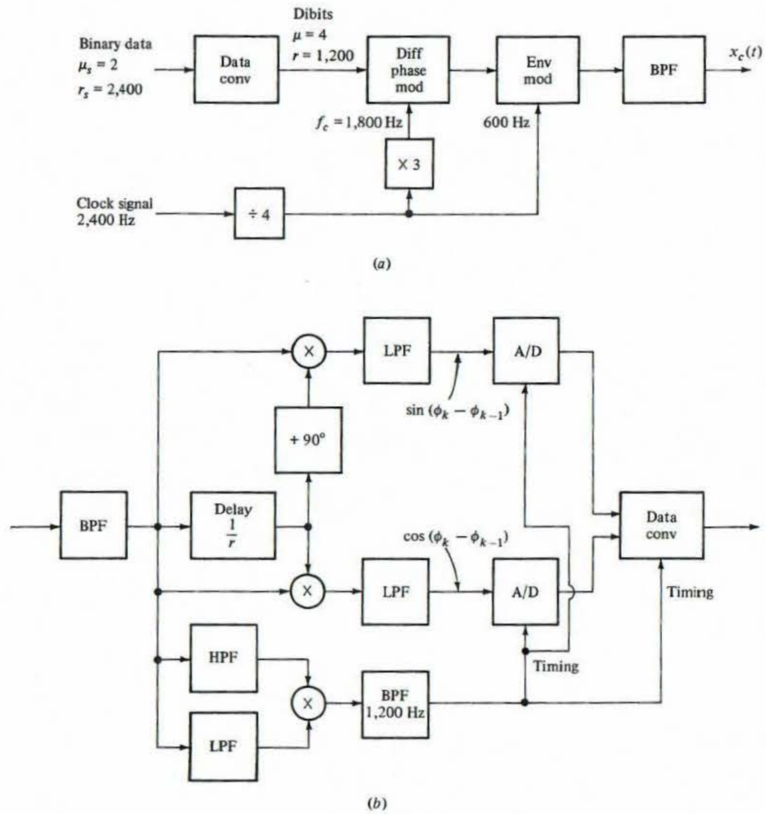

FIGURE 10.19
Quaternary PSK system. (*a*) Transmitter; (*b*) receiver.

and envelope modulation frequencies are both derived from the incoming 2,400-Hz clock frequency.

As Table 10.1 indicates, the differential phase shift has an added term of +45° so that ϕ_k can never equal ϕ_{k-1} and there is a phase shift in the modulated wave every $1/r$ seconds. This feature, combined with the envelope modulation, produces discrete frequency components at $f_c \pm 600$ that are used at the receiver to generate the timing signal.

Table 10.1

Dibit	$\phi_k - \phi_{k-1}$	$\sin(\phi_k - \phi_{k-1})$	$\cos(\phi_k - \phi_{k-1})$
00	+45°	+	+
01	+135°	+	−
11	−135°	−	−
10	−45°	−	+

Phase comparison detection is accomplished as shown in Fig. 10.19*b*, giving the outputs $\sin(\phi_k - \phi_{k-1})$ and $\cos(\phi_k - \phi_{k-1})$. Thus, from Table 10.1, there is a one-to-one correspondence between the output polarities and the binary digits in each dibit. The data converter then interleaves the two regenerated digits to yield a serial binary output. Timing is derived by mixing the outputs of the HPF and LPF, both of which cut off at 1,200 Hz, and selecting the difference frequency 1,800 − 600 = 1,200 Hz from the aforementioned sinusoidal components. This procedure is used—instead of directly filtering the 1,200-Hz component—to better compensate for delay distortion.

Tests have shown that the system error probability is less than 10^{-5} when $S/N = 15$ dB. A fully optimized DPSK system with no transmission distortion, ISI, etc., would achieve $P_e = 10^{-5}$ with about 3 dB less power. ////

Coherent Linear Modulation

As the grand finale of our study of digital modulation methods, we will look at the whole family of linear modulation (AM, DSB, VSB, etc.) with coherent detection. The student who has gotten this far will be pleased to learn that, for a change, one of the simplest cases has been saved till last.

The analog-signal portion of a coherent linear modulation system is diagramed in Fig. 10.20*a*, the primes denoting bandpass units. Assuming that the receiver oscillator is perfectly synchronized and that the baseband terminal filters do not respond above f_c, frequency-translation analysis shows that, as far as signal transmission is concerned, the entire bandpass section plus modulator and demodulator is equivalent to a *lowpass* filter

$$H_{TCR}(f) = \tfrac{1}{4}[H'_{TCR}(f - f_c) + H'_{TCR}(f + f_c)] \qquad |f| < f_c \tag{30a}$$

where

$$H'_{TCR}(f) = H'_T(f)H'_C(f)H'_R(f) \tag{30b}$$

Likewise, the lowpass equivalent noise is gaussian and has spectral density

$$G(f) = \tfrac{1}{4}[G'(f - f_c) + G'(f + f_c)] \qquad |f| < f_c \tag{31a}$$

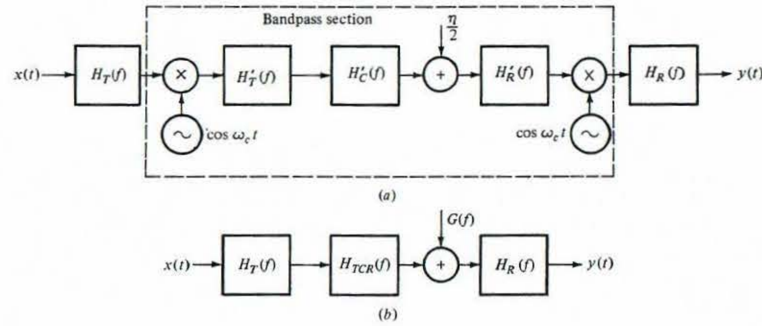

FIGURE 10.20
Coherent linear modulation. (*a*) Block diagram of analog portion; (*b*) equivalent baseband system.

where

$$G'(f) = \frac{\eta}{2}|H'_R(f)|^2 \tag{31b}$$

Therefore, for analysis or design purposes, Fig. 10.20*a* may be replaced by the *equivalent baseband system* of Fig. 10.20*b* and, in the classic tradition of mathematics, we have thereby reduced the problem to one that has already been solved, i.e., baseband data transmission.

All of the results from Sect. 10.1 apply here with the substitution of Eqs. (30) and (31). Specifically, if $G(f)$ is flat over the passband of $H_R(f)$ and if the carrier is suppressed at the transmitter (i.e., DSB or VSB), then

$$\begin{aligned} P_e &\geq 2\left(1 - \frac{1}{\mu}\right)Q\left[\sqrt{\frac{6\rho}{\mu^2 - 1}}\right] \\ &\approx \left(1 - \frac{1}{\mu}\right)\sqrt{\frac{\mu^2 - 1}{3\pi\rho}}\exp\left(-\frac{3\rho}{\mu^2 - 1}\right) \qquad \rho \gg 1 \end{aligned} \tag{32}$$

where the lower bound requires optimum terminal filters and is the same as optimum baseband transmission. Therefore, the polar baseband curve in Fig. 10.15 also holds for binary linear modulation with coherent detection and suppressed carrier. The case of *unsuppressed* carrier hardly deserves mention here since, with coherent detection, the wasted power in the carrier offers no compensating benefits.

Finally, it should be noted from the equivalent baseband system that direct *baseband equalization* is possible with coherent linear modulation and will correct linear distortion introduced by the bandpass channel. Such equalization does not work as well for incoherent systems because incoherent detection is a nonlinear process and hence linear distortion in the bandpass channel produces nonlinear distortion at baseband.

EXERCISE 10.4 Justify either Eq. (30) or (31), whichever you find more interesting. (*Hints*: For Eq. (30), consider a signal bandlimited in $B < f_c$ applied to the modulator; use the frequency-translation theorem and follow through the system with a few simple sketches of the spectra. For Eq. (31), write the bandpass noise at the detector input in quadrature-carrier form, multiply it by $\cos \omega_c t$, and use the results from Sect. 7.3.)

10.3 ERROR-CONTROL CODING

We have seen that error probability in digital transmission is a direct function of ρ or, equivalently, S/N. If, for a given system, the signal power is limited to some maximum value and errors are still unacceptably frequent, then some other means of improving reliability must be sought. Often, error-control coding provides the best solution.

In a nutshell, error-control coding is the calculated use of *redundancy*. Taking a hint from information theory, one systematically adds extra digits to the transmitted message, digits which themselves convey no information but make it possible for the receiver to detect or even correct errors in the information-bearing digits. Theoretically, near-errorless transmission is possible; more practically, there is the inevitable trade-off between transmission reliability, efficiency, and complexity of terminal equipment. Reflecting this factor, a multitude of error-detecting and error-correcting codes have been devised to suit various applications.

This section is an introduction to error-control coding. We will deal only with binary codes since relatively little has been accomplished in the realm of practical multilevel codes. (Remember the unique feature of binary digits: If one merely knows which digits are in error, the correct digits are immediately determined.) Furthermore, our treatment will be primarily qualitative because coding theory has evolved from the black art of its early days to a highly sophisticated mathematical discipline. Hence, we are barely scratching the surface of a fascinating subject; the reader whose interest is thereby aroused, and who has the essential aptitude for modern algebra, will find additional material in the supplementary reading list.

Coding Concepts and Trade-offs

By way of introduction, consider a repeated code in which each binary message digit is repeated three times—roughly analogous to repeating your words when you are trying to talk to someone on the other side of a noisy room. The allowed code words are then **000** and **111**, so any other received work such as **101** clearly indicates the presence of errors. To correct single errors, one might use a majority-rule decision and the following decoding table:

Decoded digit	0	1
Received words	**000** **001** **010** **100**	**111** **110** **101** **011**

This yields a *single-error correcting code* and decoding errors occur only when there are two or three erroneous digits in a word; e.g., two errors change **000** to **101**, etc. Therefore, assuming the *per-digit error probability* is ϵ, we find the decoding error probability from the binomial distribution, Eq. (1), Sect. 3.4, as

$$P_e = P(2 \text{ or } 3 \text{ errors in } 3 \text{ digits}) = P_3(2) + P_3(3)$$

$$= \binom{3}{2} \epsilon^2(1-\epsilon) + \binom{3}{3} \epsilon^3 = 3\epsilon^2 - 2\epsilon^3 \qquad (1)$$

Since ϵ is the error probability without coding, and since $\epsilon < \frac{1}{2}$ on any reasonable channel, coding has certainly improved the reliability.

The triple-repetition code also works for *double-error detection* if we give up single-error correction; i.e., any received code word other than **000** or **111** is treated as a detected but uncorrected error. Decoding errors, in the sense of undetected errors, occur with probability

$$P_e = P_3(3) = \epsilon^3 \qquad (2)$$

which is obviously smaller than Eq. (1). Despite the triviality of this example code, it does lead to three important and general conclusions about error-control coding.

1 Through the addition of extra digits, called *check digits*, the code words can be made "very different" from each other. Analytically, the difference between any two binary words is measured in terms of the *Hamming distance d*, defined simply as the number of places in which the words have different digits; thus, it takes d errors (in the right places) to change one word into the other. Pursuing this line of thought, a code that detects up to K errors per word or corrects up to K errors per word must consist of code words having

$$d_{\min} = \begin{cases} K+1 & \text{error detection} \\ 2K+1 & \text{error correction} \end{cases} \tag{3}$$

an assertion left for the student to ponder. The triple-repetition code clearly has $d = 3$ so, as we have seen, it can detect $K = 2$ errors or correct $K = 1$ error per word. For an arbitrary code with

$$k = \text{message digits per word}$$
$$q = \text{check digits per word}$$
$$n = k + q = \text{total digits per word}$$

there are 2^k binary code words (formed with k message digits) out of a possible $2^n = 2^q \times 2^k$ n-digit words. Accordingly, the check digits should be chosen such that the 2^k code words satisfy the distance requirement (3).

2 If the per-digit error probability ϵ is reasonably small, then the probability of $M + 1$ errors in an n-digit word will be much less than the probability of M errors, i.e., $P_n(M + 1) \ll P_n(M)$. To underscore this point, and for reference purposes, Table 10.2 lists approximate expressions for $P_n(M)$ obtained via binomial-series expansion. Therefore, if a code corrects or detects up to K errors, the decoding error probability per word is

$$P_{e,\text{word}} = \sum_{i=K+1}^{n} P_n(i) \approx P_n(K+1) \tag{4a}$$

the approximation being quite accurate if $n\epsilon \leq 0.1$. Since the majority of decoding errors are due to $K + 1$ digit errors of which the fraction k/n are erroneous message digits (the rest are check-digit errors), the net error probability per message digit or bit is

$$P_{e,\text{bit}} \approx \frac{k}{n}(K+1)P_{e,\text{word}} \tag{4b}$$

Table 10.2 SERIES APPROXIMATIONS FOR THE BINOMIAL DISTRIBUTION

M	$P_n(M) = \binom{n}{M} \epsilon^M (1-\epsilon)^{n-M}$
0	$1 - n\epsilon + \frac{1}{2}n(n-1)\epsilon^2 - \frac{1}{6}n(n-1)(n-2)\epsilon^3$
1	$n\epsilon - n(n-1)\epsilon^2 + \frac{1}{2}n(n-1)(n-2)\epsilon^3$
2	$\frac{1}{2}n(n-1)\epsilon^2 - \frac{1}{2}n(n-1)(n-2)\epsilon^3$
3	$\frac{1}{6}n(n-1)(n-2)\epsilon^3$

3 The insertion of check digits for error control reduces the effective rate at which message digits are transmitted. Quantitatively, we define the *rate efficiency factor* of a code as†

$$\mathscr{E} = \frac{k}{k+q} = \frac{k}{n} \tag{5}$$

so if the gross signaling rate is r, the message digit rate is

$$r_m = \mathscr{E} r \tag{6}$$

Generally speaking, codes that are both easily instrumented and effective in error control require a relatively large percentage of check digits. Thus, practical error control tends to go hand in hand with bit-rate reduction.

But there are other less elegant ways of decreasing errors at the expense of signaling rate, the signal power being fixed. And to properly evaluate the merits of a given code we should at least consider one other option, namely, signaling-rate reduction without coding. Reducing r increases $\rho = S_R/\eta r$ (if the terminal filters are adjusted accordingly) and thereby decreases the error probability.

Suppose, for comparison purposes, that a certain code with rate efficiency $\mathscr{E}$ has been proposed for use on a binary baseband channel having

$$\epsilon = Q(\sqrt{2\rho}) \qquad \rho = \frac{S_R}{\eta r}$$

so that $P_{e,\text{bit}}$ is given by Eq. (4*b*) with the above value of ϵ. On the other hand, one could simply reduce the signaling rate by a factor of $\mathscr{E}$, giving the same message bit rate with

$$P_{e,\text{uncoded}} = Q\left(\sqrt{\frac{2\rho}{\mathscr{E}}}\right) \tag{7}$$

If Eq. (7) is of the same order of magnitude as $P_{e,\text{bit}}$, the value of the particular code under consideration is questionable.

Example 10.4

The triple-repetition code has $k = 1$, $q = 2$, $n = 3$, and $\mathscr{E} = \frac{1}{3}$. If it is used for single-error correction on a baseband channel with $\rho = 7$ and $r = 1{,}200$, the message bit rate is $r_m = 1200/3 = 400$ and $\epsilon = Q(\sqrt{2\rho}) \approx 10^{-4}$ so, from Eq. (1),

$$P_{e,\text{bit}} \approx 3\epsilon^2 \approx 3 \times 10^{-8}$$

† Not to be confused with communication efficiency defined in Eq. (8), Sect. 9.4.

Equation (4*b*) does not apply in this case since the decoded message digit is always erroneous when there are two or three errors.

If, however, the signaling rate is reduced to $r = 300$ and no coding is used,

$$P_{e,\text{uncoded}} = Q\left(\sqrt{\frac{2\rho}{\mathscr{E}}}\right) = Q(\sqrt{42}) \approx 5 \times 10^{-9}$$

so simple signaling-rate reduction is superior to this rudimentary code. ////

Error Detection by Parity Check

For many applications, errors can be rendered harmless if they are simply *detected* with no immediate attempt at correction. This is true, for instance, in data telemetry when a large number of values are gathered for statistical analysis; erroneous values, if detected, are simply omitted from further processing, and the loss is negligible. Similarly, given a two-way communication link, the fact that an error has been detected can be sent back to the transmitter for appropriate action, i.e., retransmission. Such decision feedback is especially advantageous if the system is subject to variable transmission conditions. When conditions are good and errors infrequent, a low-redundancy code with its higher data rate is satisfactory; when conditions are unfavorable, as indicated by frequent error detection, the transmitter may switch to a code of higher redundancy or temporarily cease transmission. But with or without feedback, simple error detection suffices only if ϵ is small to begin with and the probability of undetected errors is at a suitably low level.

Most error-detecting codes are based on the notion of *parity*. The parity of a binary word is said to be even when the word includes an even number of **1**s, while odd parity means an odd number of **1**s. For error detection by parity check, we divide the message into groups of k digits and add one check digit to each group such that every $(k+1)$-digit word has the same parity, say even. Thus, the check digit is related to the message digits by

$$c = m_1 \oplus m_2 \oplus \cdots \oplus m_k \tag{8a}$$

where $\oplus$ stands for *modulo-2 addition*. Modulo-2 arithmetic, defined on the binary digits **0** and **1**, is the same as ordinary arithmetic except that $\mathbf{1} \oplus \mathbf{1} = \mathbf{0}$ and there is no difference between addition and subtraction.† Hence Eq. (8*a*) is equivalent to

$$m_1 \oplus m_2 \oplus \cdots \oplus m_k \oplus c = \mathbf{0} \tag{8b}$$

The efficiency factor is

$$\mathscr{E} = \frac{k}{k+1} \tag{9}$$

indicating reasonable efficiency if k is large.

† In hardware terms, $\oplus$ is an EXCLUSIVE-OR gate.

Of the 2^{k+1} possible binary words having $k+1$ digits, parity-check coding excludes precisely half, the half with odd parity, thereby ensuring that the code has Hamming distance $d \geq 2$ as required for single-error detection. Therefore, if the parity of a received word is odd, we know there is an error—or three errors, or, in general, an odd number of errors. Error detection can then be implemented by checking the parity of each word as it arrives. Of course error correction is not possible, since we do not know *where* the errors are located within the word. Furthermore, an even number of errors preserves valid parity and hence goes undetected.

Neglecting all but the double-error case, $P_{e,\text{word}} \approx P_{k+1}(2) \approx \frac{1}{2}(k+1)k\epsilon^2$ and hence, from Eq. (4*b*),

$$P_{e,\text{bit}} \approx \frac{k}{k+1} 2P_{e,\text{word}} \approx k^2\epsilon^2 \tag{10}$$

For example, if $k = 9$ and $\epsilon = 10^{-3}$, parity-check coding drops the error probability by more than one order of magnitude with a rate reduction of only $\frac{9}{10}$.

The probability of a detected error is also of interest, for it indicates the amount of data that must be retransmitted or discarded. Since primarily single errors are detected, we have *per word*

$$P_{de} \approx \binom{k+1}{1}\epsilon(1-\epsilon)^k \approx (k+1)\epsilon \tag{11a}$$

In a message of $N \gg 1$ total message digits, there are N/k words, of which $(N/k)(k+1)\epsilon$ have detected errors. If detected errors are discarded, the fractional number of message digits thrown away is

$$\frac{1}{N}\left[k\frac{N}{k}(k+1)\epsilon\right] = (k+1)\epsilon = P_{de} \tag{11b}$$

If $k = 9$ and $\epsilon = 10^{-3}$, $P_{de} \approx 0.01$ or 1 percent.

Two final comments with respect to practical matters are in order here. First, it is preferable to use odd-word parity and an odd number of message digits per word; this ensures that every word has at least one transition, thereby aiding synchronization and preventing apparent loss of signal if the message contains an extended string of like digits. Second, as a result of impulse noise on switched circuits or short-duration fading on radio paths, errors may tend to occur in *bursts* of several successive digits; since multiple errors wreak havoc on parity checking, the check digits should be *interlaced* such that the digits checked are widely spaced. An example of interlacing is given in Fig. 10.21, where one parity word is indicated by lines connecting the digits.

EXERCISE 10.5 Write down all the code words for a parity-check code with $k = 3$ and verify that $d \geq 2$.

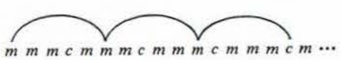

FIGURE 10.21
Interlaced parity checking; m = message digit, c = check digit.

Error-Correcting Block Codes

The idea of *error-correcting* codes is certainly far more appealing and exciting than mere error detection, suggesting as it does the one-way errorless transmission hypothesized by Shannon. And there are numerous applications, notably one-way links, for which error correction is a necessity. It also turns out that encoding at the transmitter is not very different for detection or correction; the receiving decoder is the problem in error-correcting systems. Consequently, when transmission equipment is constrained, e.g., satellites, it may prove more practical to correct errors at the receiver than to provide for retransmission facilities, even though a two-way path is available.

Parity-check coding is readily extended to error correction by observing that correction requires the detection of an error and its *location* in the word. Thus, if two checks are made on the same word but in two different patterns, errors result in characteristic symptoms of invalid parity.

Most error-correcting block codes have n-digit words in which the first k digits are message digits and the remaining $q = n - k$ are parity check digits. Such a code is called an *(n,k) systematic parity-check code*, and the ith digit of a code word is

$$x_i = \begin{cases} m_i & i = 1, 2, \ldots, k \\ c_{i-k} & i = k+1, \ldots, n \end{cases} \qquad (12)$$

For analytic convenience, a typical code word is represented by an $n \times 1$ matrix (or column vector) $\mathbf{x}$. For instance, a (7,4) code word would be

$$\mathbf{x} = [m_1 \quad m_2 \quad m_3 \quad m_4 \quad c_1 \quad c_2 \quad c_3]^T$$

which has been written as the transpose of a $1 \times n$ matrix to save space.

Given the message digits for a particular word, the check digits are chosen such that

$$\mathbf{Hx} = \mathbf{0} \qquad (13)$$

where the *parity-check matrix* $\mathbf{H}$ is a rectangular $q \times n$ matrix of the form

$$\mathbf{H} = \left[\begin{array}{cccc|cccc} \overbrace{h_{11} \quad h_{12} \quad \cdots \quad h_{1k}}^{k} & & & & \overbrace{1 \quad 0 \quad 0 \quad \cdots \quad 0}^{q} \\ h_{21} \quad h_{22} \quad \cdots \quad h_{2k} & & & & 0 \quad 1 \quad 0 \quad \cdots \quad 0 \\ \vdots \quad \vdots \quad \quad \vdots & & & & \vdots \quad \vdots \quad \vdots \quad \quad \vdots \\ h_{q1} \quad h_{q2} \quad \cdots \quad h_{qk} & & & & 0 \quad 0 \quad 0 \quad \cdots \quad 1 \end{array}\right] \Bigg\} q \qquad (14)$$

Note that the right-hand portion of $\mathbf{H}$ is a $q \times q$ unit matrix; hence, combining Eqs. (12) and (14), each row of Eq. (13) is a parity-check equation involving only one check digit, i.e.,

$$h_{j1}m_1 \oplus h_{j2}m_2 \oplus \ldots \oplus h_{jk}m_k \oplus c_j = \mathbf{0} \qquad j = 1, 2, \ldots, q$$

The components h_{11}, h_{12}, etc., of $\mathbf{H}$ are binary digits, but we have not yet specified them since they determine the error-correction properties of the code, about which we need more information.

Suppose a word $\mathbf{x}$ is transmitted and, owing to errors, results in the received word

$$\mathbf{y} = \mathbf{x} \oplus \mathbf{e} \qquad (15a)$$

which stands for the digit-by-digit sum

$$y_i = x_i \oplus e_i \qquad i = 1, 2, \ldots, n \qquad (15b)$$

with $\mathbf{e}$ being the *error pattern*, i.e.,

$$e_i = \begin{cases} 1 & y_i \neq x_i \\ 0 & y_i = x_i \end{cases}$$

As an illustration, if

$$\mathbf{x} = [0 \quad 1 \quad 0 \quad 0 \quad 1]^T$$

and

$$\mathbf{y} = [0 \quad 1 \quad 1 \quad 0 \quad 0]^T$$

then

$$\mathbf{e} = [0 \quad 0 \quad 1 \quad 0 \quad 1]^T$$

indicating errors in the third and fifth digits.

If we could determine $\mathbf{e}$ at the receiver, we could correct all errors; but finding $\mathbf{e}$ from $\mathbf{y}$ requires knowing $\mathbf{x}$, the transmitted word! We can, however, calculate a q-digit *syndrome*

$$\mathbf{s} = \mathbf{Hy} \qquad (16a)$$

which does provide some useful information. Specifically, from Eqs. (13) and (15),

$$\mathbf{s} = \mathbf{Hx} \oplus \mathbf{He} = \mathbf{He} \qquad (16b)$$

so $\mathbf{s} = \mathbf{0}$ when $\mathbf{e} = \mathbf{0}$, i.e., no errors. Furthermore, if $\mathbf{y}$ has just one error, say in the jth message digit,

$$\mathbf{s} = [h_{1j} \quad h_{2j} \quad \cdots \quad h_{qj}]^T \qquad j \le k \qquad (17)$$

and this is identical to the jth column of the parity-check matrix. Therefore, the syndrome unambiguously indicates the no-error condition or the position of a single error providing *all columns of* **H** *are different and nonzero*; and we then have a single-error-correcting code.

When used in this mode, an (n,k) code has $P_{e,\text{word}} \approx P_n(2)$ and

$$P_{e,\text{bit}} \approx \frac{k}{n} 2P_n(2) \approx k(n-1)\epsilon^2 \tag{18}$$

Multiple errors cause complications, however, since we may miss the actual errors and wrongly "correct" another message digit, making matters worse. Consequently, unless ϵ is so small that multiple errors are very rare, a more powerful code is desirable.

Unfortunately, devising an appropriate syndrome and parity-check matrix for multiple-error correction is a far more complicated task—so much so that the first double-error-correcting codes were created by inspired trial-and-error work rather than designed by a specific method. Slepian (1956) finally put coding theory on a solid mathematical foundation when he discovered its relationship to concepts of modern algebra. Soon thereafter, using the theory of Galois fields,† Bose, Chaudhuri, and Hocquenghem developed a class of multiple-error-correcting codes (now named BCH codes) that are efficient and have relatively simple hardware requirements for encoding and decoding. Needless to say, such codes are beyond the intended scope of this text.

EXERCISE 10.6 Verify that a triple-repetition code is a (3,1) systematic parity-check code with

$$\mathbf{H} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

In particular, show that $\mathbf{Hx} = \mathbf{0}$ and find $\mathbf{s}$ for each single error position.

Example 10.5 Hamming Codes

Before Slepian's discovery and even before the matrix formulation of block codes, Hamming (1950) devised a rather elegant class of block codes. In our present notation, and dealing only with single-error correction, Hamming's strategy is as follows. If there are q check digits per word, then the syndrome is a q-digit word that can be made to spell out in binary form the exact position of a single error, if any. With $q = 3$, for instance, $\mathbf{s} = \mathbf{000}$ means "no error" (as before), $\mathbf{s} = \mathbf{001}$ means "error in the first digit," and so forth.

† Rumor has it that a theoretical mathematician specializing in Galois fields abandoned the subject upon hearing of the practical application.

Since $n + 1$ error indications are required ("no error" or one error in any of the n code-word digits), and since there are 2^q different syndrome words, the numbers of check digits and message digits in a Hamming code are related by

$$2^q = k + q + 1 \qquad \text{where } k + q = n \tag{19}$$

Accordingly, the efficiency factor is

$$\mathscr{E} = \frac{k}{n} = 1 - \frac{1}{n}\log_2(n+1) \tag{20}$$

so reasonable efficiency is achieved using long code words.

The parity-check matrix is easily constructed drawing upon the above stipulation for $\mathbf{s}$ and the fact that $\mathbf{s}$ equals the jth column of $\mathbf{H}$ when there is a single error in the jth digit. Therefore, reading from left to right, the columns of $\mathbf{H}$ are simply the binary versions of the numbers 1, 2, ..., n, as illustrated below for a (7,4) Hamming code:

$$\mathbf{H} = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \tag{21a}$$

It then follows by comparison with Eq. (14) that this is not a systematic code; since the check-digit positions must correspond to the columns of $\mathbf{H}$ having just one **1**, the (7,4) code word has the form

$$\mathbf{x} = [c_1 \quad c_2 \quad m_1 \quad c_3 \quad m_2 \quad m_3 \quad m_4]^T \tag{21b}$$

and the check-digit equations are

$$\begin{aligned} c_1 &= m_1 \oplus m_2 \oplus m_4 \\ c_2 &= m_1 \oplus m_3 \oplus m_4 \\ c_3 &= m_2 \oplus m_3 \oplus m_4 \end{aligned} \tag{21c}$$

Observe that each message digit is checked by at least two check digits, which is essential for error correction. ////

EXERCISE 10.7 Taking the system parameters from Example 10.4, find $P_{e,\text{bit}}$ for a (7,4) Hamming code, and compare with $P_{e,\text{uncoded}}$ with $\mathscr{E} = 4/7$. *Ans.*: $P_{e,\text{bit}} \approx 2 \times 10^{-7}$, $P_{e,\text{uncoded}} \approx 4 \times 10^{-7}$.

Convolutional Codes

Convolutional codes, also known as sequential or recurrent codes, differ from block codes in that the check digits are continuously interleaved in the coded bit stream rather than being grouped into words. The encoding/decoding procedure therefore

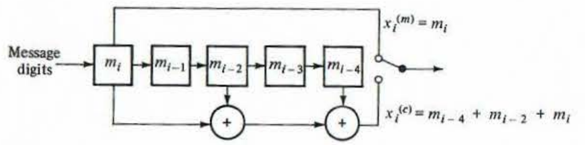

FIGURE 10.22
Convolutional encoder using shift register.

is a continuous process, eliminating the buffering or storage hardware required with block codes. The theory of convolutional codes is quite involved, but the principle can be demonstrated by a simple example.

Figure 10.22 is a convolutional encoder; it consists of a five-cell *shift register*, through which the message digits move from left to right, plus a modulo-2 adder and a switch. The switch alternately picks up the ith message digit $x_i^{(m)} = m_i$ and the ith check digit

$$\begin{aligned} x_i^{(c)} &= m_{i-4} \oplus m_{i-2} \oplus m_i \\ &= x_{i-4}^{(m)} \oplus x_{i-2}^{(m)} \oplus x_i^{(m)} \end{aligned} \qquad (22)$$

The message digits are then shifted over one cell and the process is repeated. Hence, the transmitted sequence $x_1^{(m)}x_1^{(c)}x_2^{(m)}x_2^{(c)}\ldots$ has twice the bit rate of the incoming data, and $\mathscr{E} = \frac{1}{2}$. Such high redundancy is not a necessity, but most convolutional codes do have relatively low rate efficiency factors.

At the decoder we form the ith syndrome digit from the received sequence according to the rule

$$s_i = y_{i-4}^{(m)} \oplus y_{i-2}^{(m)} \oplus y_i^{(m)} \oplus y_i^{(c)} \qquad i \geq 5$$

where, owing to errors, $y_i^{(m)} = x_i^{(m)} \oplus e_i^{(m)}$, etc. Thus, from Eq. (22) it follows that

$$s_i = e_{i-4}^{(m)} \oplus e_{i-2}^{(m)} \oplus e_i^{(m)} \oplus e_i^{(c)} \qquad i \geq 5 \qquad (23a)$$

which checks parity in the sense that $s_i = \mathbf{1}$ if there is an odd number of errors while $s_i = \mathbf{0}$ otherwise. For the start-up transient, $1 \leq i \leq 4$,

$$\begin{aligned} s_1 &= e_1^{(m)} \oplus e_1^{(c)} & s_3 &= e_1^{(m)} \oplus e_3^{(m)} \oplus e_3^{(c)} \\ s_2 &= e_2^{(m)} \oplus e_2^{(c)} & s_4 &= e_2^{(m)} \oplus e_4^{(m)} \oplus e_4^{(c)} \end{aligned} \qquad (23b)$$

Figure 10.23 displays Eq. (23) in graphical form; e.g., the ×s in the first column indicate that $e_1^{(m)}$ appears in s_1, s_3, and s_5. Studying this figure reveals that if there are two or three **1**s in $s_1 s_3 s_5$, then most likely $e_1^{(m)} = \mathbf{1}$, meaning that $y_1^{(m)}$ is erroneous and should be corrected; similarly, one should correct $y_2^{(m)}$ if there are more **1**s than

	$e_1^{(m)}$	$e_1^{(c)}$	$e_2^{(m)}$	$e_2^{(c)}$	$e_3^{(m)}$	$e_3^{(c)}$	$e_4^{(m)}$	$e_4^{(c)}$	$e_5^{(m)}$	$e_5^{(c)}$	$e_6^{(m)}$	$e_6^{(c)}$
s_1	×	×										
s_2			×	×								
s_3	×				×	×						
s_4			×				×	×				
s_5	×				×				×	×		
s_6			×				×				×	×

FIGURE 10.23

0s in $s_2 s_4 s_6$, and so on. The student can verify that this algorithm, known as *threshold decoding*, will correct up to four successive errors (check digits included) providing that the following eight digits are error-free.

Threshold decoding is particularly effective for those channels where isolated *error bursts* due to impulse noise are the major problem. Another decoding algorithm for convolutional codes is the probabilistic or *sequential* method invented by Wozencraft. Because the theory of convolutional codes is not as well developed as that of block codes, it is difficult to make an accurate assessment of their relative merits.

10.4 PROBLEMS

10.1 (Sect. 10.1) Consider a system having $p(t) = \text{sinc}\, r(1+\epsilon)t$, $0 < \epsilon \ll 1$, so the signaling rate r is slightly less than the synchronous rate $r(1+\epsilon)$. The data is a sequence of $2M+1$ alternating **1**s and **0**s, i.e., $A_k = (-1)^k A$, $k = 0, \pm 1, \pm 2, \ldots, \pm M$. Show that the ISI term in Eq. (3) at $m=0$ is $[2A\epsilon/(1+\epsilon)] \sum_{k=1}^{M} \text{sinc}\, k\epsilon \approx 2M\epsilon A$ if $M\epsilon \ll 1$.

10.2 (Sect. 10.1) A binary system suffers from ISI such that, in absence of noise, the values of $y(t_m)$ and their probabilities are as tabulated below when a **1** is sent; the table also applies when a **0** is sent, except that A is replaced by $-A$.

$y(t_m)$:	$A - \alpha$	A	$A + \alpha$
$P[y(t_m)]$:	¼	½	¼

(*a*) Assuming gaussian noise, obtain an expression for P_e in terms of A, α, and σ.
(*b*) Evaluate P_e for $A/\sigma = 4.0$ when $\alpha/A = 0.05$ and 0.25. Compare with P_e when $\alpha = 0$.

10.3 (Sect. 10.1) A polar binary signal is sent via two different ideal channels, denoted a and b, to the same destination where the signals are combined as indicated in Fig. P10.1. This arrangement is called a *diversity* system. Channel a has an amplifier with

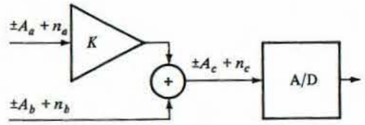

FIGURE P10.1.

adjustable voltage gain K, and n_a and n_b are independent gaussian variates with $\overline{n_a^2} = \sigma_a^2$, $\overline{n_b^2} = \sigma_b^2$.

(a) Find the value of K that maximizes A_c/σ_c and thereby minimizes P_e. *Ans.*: $K_{\text{opt}} = A_a\sigma_b^2/A_b\sigma_a^2$.

(b) Taking $K = K_{\text{opt}}$, $A_b/\sigma_b = 3.0$, and $A_a/\sigma_a = \alpha(A_b/\sigma_b)$, plot P_e for the diversity system and for channel a alone as a function of α, $0 \leq \alpha \leq 2$.

10.4 (Sect. 10.1) Let $\pm a\Pi[(t - \tau/2)/\tau]$ plus white noise $n(t)$ be the input to an ideal integrate-and-dump filter. Show that the output at $t = \tau$ has $(A/\sigma)^2 = 2E_R/\eta$, so the performance does equal that of matched filtering. (*Hint*: Use $\sigma^2 = E\{[\int_0^\tau n(t)\,dt]^2\}$ and recall that $E[n(t_1)n(t_2)] = = R_n(t_1 - t_2) = (\eta/2)\,\delta(t_1 - t_2)$.)

10.5 (Sect. 10.1) Consider a polar binary system in the form of Fig. 10.1 with $a_k = \pm a$, $H_T(f) = 1$, $G(f) = \eta/2$, and $H_R(f) = P_S^*(f)H_C^*(f)$, where $P_S(f) = \mathscr{F}[p_S(t)]$, so the receiving filter is *matched* to the received pulse shape. Because this arrangement does not shape the output pulse $p(t)$ for ISI considerations, it is used only when $p_S(t)$ is timelimited (but not necessarily rectangular) with duration much less than $1/r$.

(a) By considering the transmission of one isolated pulse, show that $P(f)$ is real so $A = ap(0) = a\int_{-\infty}^{\infty} P(f)\,df$, and hence

$$\left(\frac{A}{\sigma}\right)^2 = \left[\frac{\int_{-\infty}^{\infty} |P_S(f)H_C(f)|^2 df}{\int_{-\infty}^{\infty} |P_S(f)|^2 df}\right]\left(\frac{2S_T}{\eta r}\right)$$

where $S_T = rE_T$ and E_T is the transmitted-pulse energy.

(b) Prove that this is equivalent to Eq. (14).

10.6 (Sect. 10.1) Use Eqs. (15*a*) and (16*a*) to find and sketch $P(f)$ and $p(t)$ for:

(a) $P_\beta(f) = (1/2\beta)\Pi(f/2\beta)$, $\beta = r/4$

(b) $P_\beta(f) = (1/\beta)\Lambda(f/\beta)$, $\beta = r/2$

10.7 (Sect. 10.1) Carry out all the details leading to Eqs. (17*b*) and (18*a*) starting from Eq. (17*a*).

10.8★ (Sect. 10.1) Consider a polar binary system with $r = 2 \times 10^4$, $p_S(t) = \Pi(2rt)$, $|H_C(f)| = 10^{-2}$, $G(f) = 10^{-10}(1 + 3 \times 10^{-4}|f|)^2$, and $p(t)$ per Eq. (18*b*) so $P(f) = (1/r)\cos^2(\pi f/2r)\,\Pi(f/2r)$.

(a) Find and sketch the amplitude ratio for the optimum terminal filters.

(b) Calculate the value of S_T needed so that $P_e = 10^{-6}$. *Ans.*: $S_T = 3.49$.

10.9★ (Sect. 10.1) Repeat Prob. 10.8 for a quaternary ($\mu = 4$) system with $r = 100$, $p_S(t) = \Pi(10rt)$, $|H_C(f)| = 10^{-3}/(1 + 32 \times 10^{-4}f^2)^{1/2}$, $G(f) = 10^{-10}$, and $p(t) = \text{sinc}\, rt$.

10.10★ (Sect. 10.1) The terminal filters for a system have been optimized assuming white noise and a distortionless channel. However, it turns out that the channel does introduce some linear distortion so an equalizer with $H_{eq}(f) = K/H_C(f)$ is added after $H_R(f)$.

(a) Obtain an expression for the resulting $(A/\sigma)^2$ in terms of $H_C(f)$ and $P(f)$ at the output of the equalizer.

(b) Taking $P(f)$ as in Prob. 10.8, by what factor must S_T be increased to get the same error probability as if $H_C(f) = K_R$ when $H_C(f) = K_R/[1 + j(2f/r)]$ and $H_{eq}(f)$ is used?

10.11 (Sect. 10.1) Find the tap gains c_m for a 3-tap zero-forcing equalizer when $p(-1/r) = 0.4$, $p(0) = 1.0$, $p(1/r) = 0.2$, and $p(k/r) = 0$ for $|k| > 1$. Also compute $p_{eq}(t_k)$, $-4 \leq k \leq 4$.

10.12 (Sect. 10.1) Repeat Exercise 10.2 with an available bandwidth of 10 kHz, choosing parameters to minimize the power requirement.

10.13 (Sect. 10.2) Consider the *bandpass* pulse shape $p(t) = \text{sinc}\, Bt \cos 2\pi f_c t$, which has bandwidth B.

(a) Show that $p(k/2B) = 0$ for $k \neq 0$ if $f_c = MB/2$ with M being an odd integer. Sketch $p(t) - p(t - 1/2B)$ taking $M = 3$.

(b) Discuss the implications of this for digital modulation, including the advantages and disadvantages.

10.14 (Sect. 10.2) Integrate Eq. (6) over $0 \leq R \leq \infty$ to get

$$p_\phi(\phi) = \frac{A_k \cos\phi}{\sqrt{2\pi N}}\, e^{-(Ak2\sin 2\phi)/2N}\left[1 - Q\left(\frac{A_k \cos\phi}{\sqrt{N}}\right)\right] + \frac{1}{2\pi}\, e^{-Ak2/2N}$$

and obtain Eq. (9) by taking $A_k^2/N \gg 1$. (*Hint*: Start with the change-of-variable $\lambda = (R - A_k\cos\phi)/\sqrt{N}$ and use the fact that $Q(-\kappa) = 1 - Q(\kappa)$.)

10.15 (Sect. 10.2) Consider a *trinary* ASK system with $A_k = 0$, A, and $2A$. Find P_e in terms of ρ assuming $\rho \gg 1$. (*Hint*: Justify the approximation $P_e \approx P_{e0}/3$ and note that $S_R = (5r/3)E$ where E is the energy in the middle-level pulse.)

10.16 (Sect. 10.2) Suppose a binary ASK signal is *coherently* detected, per Fig. 10.16, with the threshold at $A/2$. Find P_e and compare with Eqs. (15) and (21).

10.17 (Sect. 10.2) A binary FSK signal is transmitted over a radio channel having *Rayleigh fading* such that A is a slowly changing Rayleigh-distributed variate with $\overline{A^2} = A_0^2$, so $p(A) = (2A/A_0^2)\exp(-A^2/A_0^2)$, $A \geq 0$.

(a) Show that the average error probability is $E[P_e] = 1/[2 + (A_0^2/2N)]$.

(b) Discuss the value of a diversity arrangement as in Prob. 10.3 for this application.

10.18 (Sect. 10.2) Figure P10.2 is a coherent detection system for quaternary PSK with $\phi_k = 0°, 90°, 180°$, and $270°$. Ignoring noise, construct a table similar to Table 10.1 giving the values of v_A and v_B for each value of ϕ_k when the phase shifts are $\psi_A = 45°$ and $\psi_B = -45°$. Repeat for $\psi_A = 0°$ and $\psi_B = -90°$. Discuss these two alternatives, giving special consideration to the case where each value of ϕ_k represents two binary digits.

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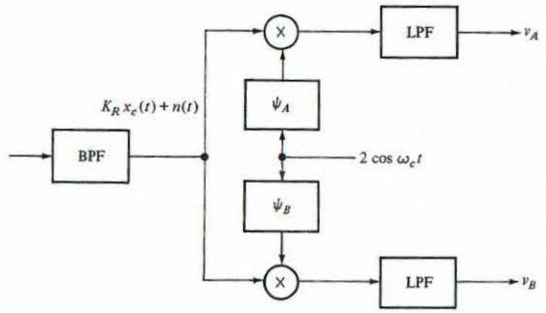

FIGURE P10.2.

10.19★(Sect. 10.2) Referring to the system in Prob. 10.18 with $\psi_A = 45°$ and $\psi_B = -45°$, write the noise $n(t)$ in the form of Eq. (1*a*) with $\theta = \psi_B$ and obtain v_A and v_B when noise is present. Show that $P(\text{no error}) = P(n_i \geq -A/\sqrt{2}$ and $n_q \geq -A/\sqrt{2})$ and from this derive $P_e = 1 - [1 - Q(\sqrt{\rho})]^2$, $\rho = A^2/2N$.

10.20★(Sect. 10.2) Differential encoding is to be used on a *baseband* polar binary system to protect against possible polarity inversions in transmission. Thus, if the *k*th message digit is a **1**, then $A_k = A_{k-1}$; if the *k*th digit is a **0**, then $A_k = -A_{k-1}$. The possible values of A_k are $\pm A$. Assuming gaussian noise, show that $P_e = 2[Q(A/\sigma) - Q^2(A/\sigma)]$ if the polarity of $A_k + n_k$ is determined using a zero-threshold A/D and then compared with the previously determined polarity of $A_{k-1} + n_{k-1}$. (*Hint*: An error occurs if there is one and only one polarity error in each pair.)

10.21★(Sect. 10.2) As an alternative to the detection scheme in Prob. 10.20, suppose the receiver has the form of Fig. P10.3. Show that the A/D must have *two* threshold levels—at $\pm A$—and hence $P_e = \frac{3}{2}Q(A/\sqrt{2}\,\sigma)$. Is this scheme better than the other?

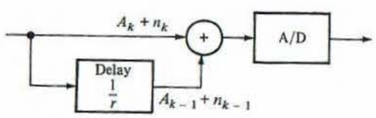

FIGURE P10.3.

10.22★(Sect. 10.2) Digital data with $\mu = 8$ modulates both the amplitude and phase of a carrier, giving $K_R x_c(t) = A_k \cos(\omega_c t + \phi_k)$ where $A_k = A$ and $2A$ and $\phi_k = 0°$, 90°, 180°, and 270°.

(*a*) Draw a block diagram of the receiver using a coherent local oscillator and integrate-and-dump filters. (*Hint*: See Prob. 10.18.)

(*b*) Analyze the performance of this system in the presence of noise.

10.23 (Sect. 10.3) A certain code with $n = 7$ and $k = 3$ is simultaneously single-error-correcting and double-error-detecting. Assuming $\epsilon \ll 1$, calculate the following probabilities: a word has a corrected error; a word has detected but uncorrected errors; $P_{e,\text{word}}$; $P_{e,\text{bit}}$.

10.24 (Sect. 10.3) An ideal binary baseband system has $r = 10{,}000$ and $\sigma = 0.2$. It is desired to send data at a usable rate of 7,000 bits/s with parity-check coding for error detection such that $P_{de} \leq 0.08$. Find appropriate values for k and A, and calculate $P_{e,\text{bit}}$. *Ans.*: $k = 3$, $A \geq 0.41$, $P_{e,\text{bit}} \leq 0.0036$.

10.25 (Sect. 10.3) Consider a system combining parity-check *error detection* and *decision feedback* so that when a word is received with a detected error, the receiver tells the transmitter to repeat that word. If $\epsilon \ll 1$, the probability of more than one repetition is negligible. Including the effects of retransmission, find $P_{e,\text{bit}}$ and the ratio of total digits transmitted per message digit in terms of k and ϵ.

10.26★(Sect. 10.3) An analog method of error detection, known as *null-zone detection*, uses a no-decision zone centered at each threshold level such that if the signal plus noise falls in this zone, it is deemed a detected but uncorrected error.

(*a*) Obtain expressions for P_{de} and P_e per bit for a polar binary baseband system with null-zone boundaries at $\pm\alpha A$.

(*b*) Compare null-zone detection with error detection by parity-check coding when both systems have the same message bit rate.

10.27 (Sect. 10.3) Figure P10.4 illustrates the *square-array parity-check code* wherein $k = k_0^2$ message digits are arranged in a square whose rows and columns are checked by $2\sqrt{k}$ check digits. A transmission error in one message digit causes a row and column parity failure with the error at the intersection point, so the code is single-error correcting and double-error detecting.

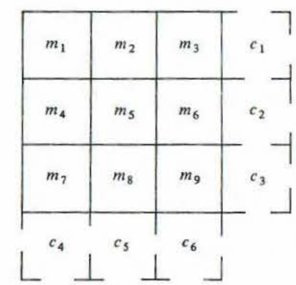

FIGURE P10.4.

(*a*) Obtain expressions for $\mathscr{E}$ and $P_{e,\text{bit}}$ in terms of k_0.

(*b*) Discuss what happens when there is one error in a check digit and when there are two or more errors.

10.28 (Sect. 10.3) Construct an **H** matrix for a single-error-correcting (6,3) systematic parity-check code. Write out the parity-check equations, the allowed code words, and the single-error syndromes. What happens when there are double errors?

10.29 (Sect. 10.3) The Hamming code in Example 10.5 provides error indications for the check digits as well as the message digits, even though correcting erroneous check digits is unnecessary. However, use Eq. (14) and the conditions on **H** to prove that any single-error-correcting (n,k) parity-check code must have $2^q \geq k + q + 1$, so the check-digit syndromes come automatically, whether or not they are used.

10.30 (Sect. 10.3) A (7,4) Hamming code is being considered for binary data transmission via FSK. Find the numerical condition on ρ such that the coding yields a lower error probability than signaling-rate reduction.

10.31 (Sect. 10.3) Figure P10.5 is the shift register for a *Hagelbarger code*, one of the first convolutional codes. The switch alternately picks up message digits and check digits, so $\mathscr{E} = 1/2$. Neglecting the start-up transient, find the parity-check linkages and verify that six successive errors can be detected if the 19 previous digits are correct.

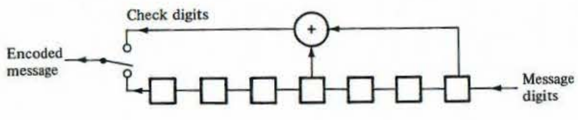

FIGURE P10.5

APPENDIX A
SIGNAL SPACE AND COMMUNICATION

By describing signals as vectors in a multidimensional space, the familiar relations and insights of geometry can be applied to numerous problems of signal analysis and communication system design. Some of the results have been drawn upon in the body of this text, particularly Schwarz's inequality and the scalar product concept. Here we derive those results and survey additional applications. The presentation is compact and informal, with a minimum of special notation.†

A.1 VECTOR SPACE THEORY

A *vector space* (or linear space) $\mathscr{S}$ is a set of elements called vectors having the property that they may always be combined linearly. Thus, for any vectors v and w in $\mathscr{S}$ and any scalars α and β there is another vector

$$z = \alpha v + \beta w \tag{1}$$

which is also in $\mathscr{S}$. In other words, the set of all elements in $\mathscr{S}$ is *closed under linear combination*. Similarly, if $\mathscr{P}$ is a subset of elements of $\mathscr{S}$ and is also closed under linear combination, then we say that $\mathscr{P}$ is a *subspace* of $\mathscr{S}$.

† See Frederick and Carlson (1971, chaps. 5 and 12) or Wozencraft and Jacobs (1965, chaps. 4 and 5) for more details.