HANDBOOK OF MATHEMATICAL FUNCTIONS

WITH FORMULAS, GRAPHS, AND MATHEMATICAL TABLES

Edited by

Milton Abramowitz and Irene A. Stegun

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The text relating to physical constants and conversion factors (page 6) has been modified to take into account the newly adopted Système International d'Unites (SI).

ERRATA NOTICE

The original printing of this Handbook (June 1964) contained errors that have been corrected in the reprinted editions. These corrections are marked with an asterisk (*) for identification. The errors occurred on the following pages: 2–3, 6–8, 10, 15, 19–20, 25, 76, 85, 91, 102, 187, 189–197, 218, 223, 225, 233, 250, 255, 260–263, 268, 271–273, 292, 302, 328, 332, 333–337, 362, 365, 415, 423, 438–440, 443, 445, 447, 449, 451, 484, 498, 505–506, 509–510, 543, 556, 558, 562, 571, 595, 599, 600, 722–723, 739, 742, 744, 746, 752, 756, 760–765, 774, 777–785, 790, 797, 801, 822–823, 832, 835, 844, 886–889, 897, 914, 915, 920, 930–931, 936, 940–941, 944–950, 953, 960, 963, 989–990, 1010, 1026.

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Reversion of Series

3.6.25 Given

$$y=ax+bx^2+cx^3+dx^4+ex^5+fx^6+gx^7+\ \dots$$

then

$$x=Ay+By^2+Cy^3+Dy^4+Ey^5+Fy^6+Gy^7+\ \dots$$

where

$aA=1$

$a^3B=-b$

$a^5C=2b^2-ac$

$a^7D=5abc-a^2d-5b^3$

$a^9E=6a^2bd+3a^2c^2+14b^4-a^3e-21ab^2c$

$a^{11}F=7a^3be+7a^3cd+84ab^3c-a^4f-28a^2bc^2-42b^5-28a^2b^2d$

$a^{13}G=8a^4bf+8a^4ce+4a^4d^2+120a^2b^3d+180a^2b^2c^2+132b^6-a^5g-36a^3b^2e-72a^3bcd-12a^3c^3-330ab^4c$

Kummer's Transformation of Series

3.6.26 Let $\sum_{k=0}^{\infty} a_k=s$ be a given convergent series and $\sum_{k=0}^{\infty} c_k=c$ be a given convergent series with known sum c such that $\lim_{k\to\infty} \frac{a_k}{c_k}=\lambda\neq 0$.

Then

$$s=\lambda c+\sum_{k=0}^{\infty}\left(1-\lambda\frac{c_k}{a_k}\right)a_k.$$

Euler's Transformation of Series

3.6.27 If $\sum_{k=0}^{\infty}(-1)^k a_k=a_0-a_1+a_2-\ \dots$ is a convergent series with sum s then

$$s=\sum_{k=0}^{\infty}\frac{(-1)^k\Delta^k a_0}{2^{k+1}},\ \Delta^k a_0=\sum_{m=0}^{k}(-1)^m\binom{k}{m}a_{k-m}$$

Euler-Maclaurin Summation Formula

3.6.28

$$\sum_{k=1}^{n-1} f_k=\int_0^n f(k)dk-\frac{1}{2}[f(0)+f(n)]+\frac{1}{12}[f'(n)-f'(0)]$$
$$-\frac{1}{720}[f'''(n)-f'''(0)]+\frac{1}{30240}[f^{(V)}(n)-f^{(V)}(0)]$$
$$-\frac{1}{1209600}[f^{(VII)}(n)-f^{(VII)}(0)]+\ \dots$$

3.7. Complex Numbers and Functions

Cartesian Form

3.7.1 $z=x+iy$

Polar Form

3.7.2 $z=re^{i\theta}=r(\cos\theta+i\sin\theta)$

3.7.3 *Modulus:* $|z|=(x^2+y^2)^{\frac{1}{2}}=r$

3.7.4 *Argument:* $\arg z=\arctan(y/x)=\theta$ (other notations for arg z are am z and ph z).

3.7.5 Real Part: $x=\mathscr{R}z=r\cos\theta$

3.7.6 Imaginary Part: $y=\mathscr{I}z=r\sin\theta$

Complex Conjugate of z

3.7.7 $\bar{z}=x-iy$

3.7.8 $|\bar{z}|=|z|$

3.7.9 $\arg\bar{z}=-\arg z$

Multiplication and Division

If $z_1=x_1+iy_1$, $z_2=x_2+iy_2$, then

3.7.10 $z_1z_2=x_1x_2-y_1y_2+i(x_1y_2+x_2y_1)$

3.7.11 $|z_1z_2|=|z_1||z_2|$

3.7.12 $\arg(z_1z_2)=\arg z_1+\arg z_2$

3.7.13 $\frac{z_1}{z_2}=\frac{z_1\bar{z}_2}{|z_2|^2}=\frac{x_1x_2+y_1y_2+i(x_2y_1-x_1y_2)}{x_2^2+y_2^2}$

3.7.14 $\left|\frac{z_1}{z_2}\right|=\frac{|z_1|}{|z_2|}$

3.7.15 $\arg\left(\frac{z_1}{z_2}\right)=\arg z_1-\arg z_2$

Powers

3.7.16 $z^n=r^ne^{in\theta}$

3.7.17 $=r^n\cos n\theta+ir^n\sin n\theta$ $(n=0,\pm1,\pm2,\dots)$

3.7.18 $z^2=x^2-y^2+i(2xy)$

3.7.19 $z^3=x^3-3xy^2+i(3x^2y-y^3)$

3.7.20 $z^4=x^4-6x^2y^2+y^4+i(4x^3y-4xy^3)$

3.7.21 $z^5=x^5-10x^3y^2+5xy^4+i(5x^4y-10x^2y^3+y^5)$

3.7.22

$$z^n=[x^n-\binom{n}{2}x^{n-2}y^2+\binom{n}{4}x^{n-4}y^4-\ \dots]$$
$$+i\,[\binom{n}{1}x^{n-1}y-\binom{n}{3}x^{n-3}y^3+\ \dots],$$
$$(n=1,2,\dots)$$