



Thomas

CALCULUS AND ANALYTIC GEOMETRY

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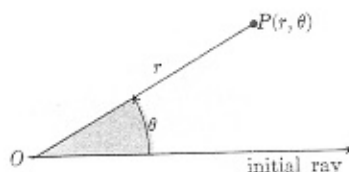
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POLAR COORDINATES

CHAPTER 11

11.1 THE POLAR COORDINATE SYSTEM

We know that a point can be located in a plane by giving its abscissa and ordinate relative to a given coordinate system. Such x - and y -coordinates are called *Cartesian* coordinates, in honor of the French mathematician-philosopher René Descartes* (1596–1650), who is credited with discovering this method of fixing the position of a point in a plane.



11.1

Another useful way to locate a point in a plane is by *polar coordinates* (see Fig. 11.1). First, we fix an *origin* O and an *initial ray*† from O . The point P has polar coordinates r, θ , with

$$r = \text{directed distance from } O \text{ to } P, \quad (1a)$$

and

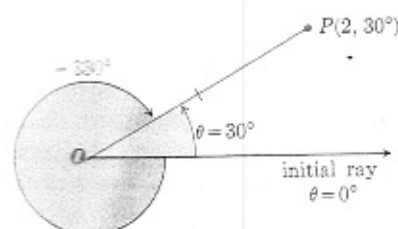
$$\theta = \text{directed angle from initial ray to } OP. \quad (1b)$$

As in trigonometry, the angle θ is *positive* when measured counterclockwise and *negative* when measured clockwise (Fig. 11.1). But the angle associated with a given point is not unique (Fig. 11.2). For instance, the point 2 units from the origin, along the ray $\theta = 30^\circ$, has polar coordinates $r = 2, \theta = 30^\circ$. It also has coordinates $r = 2, \theta = -330^\circ$, or $r = 2, \theta = 390^\circ$.

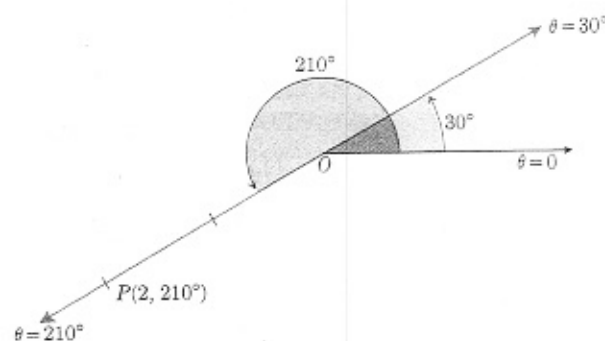
There are occasions when we wish to allow r to be negative. That's why we say "directed distance"

* For an interesting biographical account together with an excerpt from Descartes' own writings, see *World of Mathematics*, Vol. 1, pp. 235–253.

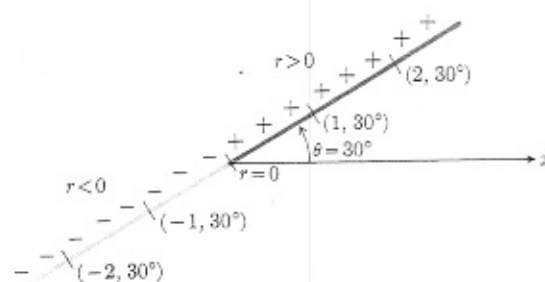
† A ray is a half-line consisting of a vertex and points of a line on one side of the vertex. For example, the origin and positive x -axis is a ray. The points on the line $y = 2x + 3$ with $x \geq 1$ is another ray; its vertex is $(1, 5)$.



11.2 The ray $\theta = 30^\circ$ is the same as the ray $\theta = -330^\circ$.



11.3 The rays $\theta = 30^\circ$ and $\theta = 210^\circ$ make a line.



11.4 The terminal ray $\theta = \pi/6$ and its negative.

in Eq. (1a). The ray $\theta = 30^\circ$ and the ray $\theta = 210^\circ$ together make up a complete line through O (see Fig. 11.3). The point $P(2, 210^\circ)$ 2 units from O on the ray $\theta = 210^\circ$ has polar coordinates $r = 2$, $\theta = 210^\circ$. It can be reached by a person standing at O and facing out along the initial ray, if he first turns 210° counterclockwise, and then goes forward

2 units. He would reach the same point by turning only 30° counterclockwise from the initial ray and then going *backward* 2 units. So we say that the point also has polar coordinates $r = -2$, $\theta = 30^\circ$.

Whenever the angle between two rays is 180° , the rays actually make a straight line. We then say that either ray is the negative of the other. Points on the ray $\theta = \alpha$ have polar coordinates (r, α) with $r \geq 0$. Points on the negative ray, $\theta = \alpha + 180^\circ$, have coordinates (r, α) with $r \leq 0$. The origin is $r = 0$. (See Fig. 11.4 for the ray $\theta = 30^\circ$ and its negative. A word of caution: The "negative" of the ray $\theta = 30^\circ$ is the ray $\theta = 30^\circ + 180^\circ = 210^\circ$ and *not* the ray $\theta = -30^\circ$. "Negative" refers to the directed distance r .)

There is a great advantage in being able to use both polar and Cartesian coordinates at once. To do this, we use a common origin and take the initial ray as the positive x -axis, and take the ray $\theta = 90^\circ$ as the positive y -axis. The coordinates, shown in Fig. 11.5, are then related by the equations

$$x = r \cos \theta, \quad y = r \sin \theta. \quad (2)$$

These are the equations that define $\sin \theta$ and $\cos \theta$ when r is positive. They are also valid if r is negative, because

$$\cos(\theta + 180^\circ) = -\cos \theta,$$

$$\sin(\theta + 180^\circ) = -\sin \theta,$$

so positive r 's on the $(\theta + 180^\circ)$ -ray correspond to negative r 's associated with the θ -ray. When $r = 0$, then $x = y = 0$, and P is the origin.

If we impose the condition

$$r = a \quad (a \text{ constant}), \quad (3)$$

then the locus of P is a circle with center O and radius a , and P describes the circle once as θ varies from 0 to 360° (see Fig. 11.6). On the other hand, if we let r vary and hold θ fixed, say

$$\theta = 30^\circ, \quad (4)$$

the locus of P is the straight line shown in Fig. 11.4.

11.5 Polar and Cartesian

11.6 The circle $r = a$

We adopt the convention, $-\infty < r < \infty$, that $r = 0$, $y = 0$ or $x = 0$

$r =$

at the origin, $x = 0$

The same point

different ways in

the point $(2, 30^\circ)$

representations: $(2, 30^\circ)$

$(-2, -150^\circ)$. The

the two formula

$(2, 30^\circ + n \cdot 360^\circ)$

$(-2, 210^\circ + n \cdot 360^\circ)$

if we represent

formulas

$(2, \frac{1}{6}\pi + 2n\pi)$

$(-2, \frac{5}{6}\pi + 2n\pi)$

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