

Thomas

**CALCULUS  
AND  
ANALYTIC  
GEOMETRY**

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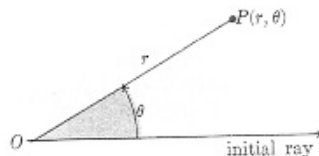
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# POLAR COORDINATES

## CHAPTER 11

### 11.1 THE POLAR COORDINATE SYSTEM

We know that a point can be located in a plane by giving its abscissa and ordinate relative to a given coordinate system. Such  $x$ - and  $y$ -coordinates are called *Cartesian* coordinates, in honor of the French mathematician-philosopher René Descartes\* (1596–1650), who is credited with discovering this method of fixing the position of a point in a plane.



11.1

Another useful way to locate a point in a plane is by *polar coordinates* (see Fig. 11.1). First, we fix an *origin*  $O$  and an *initial ray*† from  $O$ . The point  $P$  has polar coordinates  $r, \theta$ , with

$$r = \text{directed distance from } O \text{ to } P, \quad (1a)$$

and

$$\theta = \text{directed angle from initial ray to } OP. \quad (1b)$$

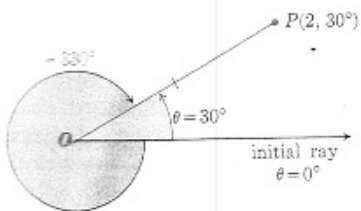
As in trigonometry, the angle  $\theta$  is *positive* when measured counterclockwise and *negative* when measured clockwise (Fig. 11.1). But the angle associated with a given point is not unique (Fig. 11.2). For instance, the point 2 units from the origin, along the ray  $\theta = 30^\circ$ , has polar coordinates  $r = 2, \theta = 30^\circ$ . It also has coordinates  $r = 2, \theta = -330^\circ$ , or  $r = 2, \theta = 390^\circ$ .

There are occasions when we wish to allow  $r$  to be negative. That's why we say "directed distance"

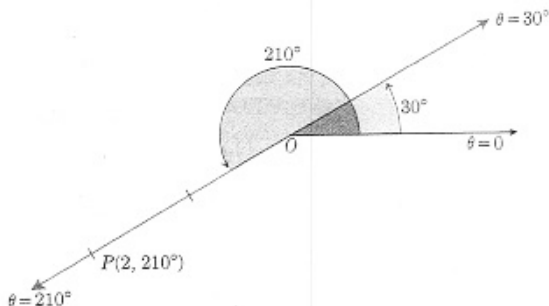
\* For an interesting biographical account together with an excerpt from Descartes' own writings, see *World of Mathematics*, Vol. 1, pp. 235–253.

† A *ray* is a half-line consisting of a vertex and points of a line on one side of the vertex. For example, the origin and positive  $x$ -axis is a ray. The points on the line  $y = 2x + 3$  with  $x \geq 1$  is another ray; its vertex is  $(1, 5)$ .

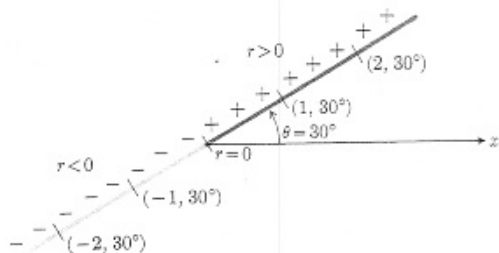




11.2 The ray  $\theta = 30^\circ$  is the same as the ray  $\theta = -330^\circ$ .



11.3 The rays  $\theta = 30^\circ$  and  $\theta = 210^\circ$  make a line.



11.4 The terminal ray  $\theta = \pi/6$  and its negative.

in Eq. (1a). The ray  $\theta = 30^\circ$  and the ray  $\theta = 210^\circ$  together make up a complete line through  $O$  (see Fig. 11.3). The point  $P(2, 210^\circ)$  2 units from  $O$  on the ray  $\theta = 210^\circ$  has polar coordinates  $r = 2$ ,  $\theta = 210^\circ$ . It can be reached by a person standing at  $O$  and facing out along the initial ray, if he first turns  $210^\circ$  counterclockwise, and then goes forward

2 units. He would reach the same point by turning only  $30^\circ$  counterclockwise from the initial ray and then going *backward* 2 units. So we say that the point also has polar coordinates  $r = -2$ ,  $\theta = 30^\circ$ .

Whenever the angle between two rays is  $180^\circ$ , the rays actually make a straight line. We then say that either ray is the negative of the other. Points on the ray  $\theta = \alpha$  have polar coordinates  $(r, \alpha)$  with  $r \geq 0$ . Points on the negative ray,  $\theta = \alpha + 180^\circ$ , have coordinates  $(r, \alpha)$  with  $r \leq 0$ . The origin is  $r = 0$ . (See Fig. 11.4 for the ray  $\theta = 30^\circ$  and its negative. A word of caution: The "negative" of the ray  $\theta = 30^\circ$  is the ray  $\theta = 30^\circ + 180^\circ = 210^\circ$  and *not* the ray  $\theta = -30^\circ$ . "Negative" refers to the directed distance  $r$ .)

There is a great advantage in being able to use both polar and Cartesian coordinates at once. To do this, we use a common origin and take the initial ray as the positive  $x$ -axis, and take the ray  $\theta = 90^\circ$  as the positive  $y$ -axis. The coordinates, shown in Fig. 11.5, are then related by the equations

$$x = r \cos \theta, \quad y = r \sin \theta. \quad (2)$$

These are the equations that define  $\sin \theta$  and  $\cos \theta$  when  $r$  is positive. They are also valid if  $r$  is negative, because

$$\cos(\theta + 180^\circ) = -\cos \theta,$$

$$\sin(\theta + 180^\circ) = -\sin \theta,$$

so positive  $r$ 's on the  $(\theta + 180^\circ)$ -ray correspond to negative  $r$ 's associated with the  $\theta$ -ray. When  $r = 0$ , then  $x = y = 0$ , and  $P$  is the origin.

If we impose the condition

$$r = a \quad (a \text{ constant}), \quad (3)$$

then the locus of  $P$  is a circle with center  $O$  and radius  $a$ , and  $P$  describes the circle once as  $\theta$  varies from 0 to  $360^\circ$  (see Fig. 11.6). On the other hand, if we let  $r$  vary and hold  $\theta$  fixed, say

$$\theta = 30^\circ, \quad (4)$$

the locus of  $P$  is the straight line shown in Fig. 11.4.

11.5 Polar and C...

11.6 The circle r =

We adopt the c  
number,  $-\infty < r$   
the  $x = 0, y = 0$  in

$r =$

the origin,  $x =$

The same point

different ways in

the point  $(2, 30^\circ)$

representations:  $($

$2, -150^\circ)$ . The

the two formula

$(2, 30^\circ + n;$

$(-2, 210^\circ + n;$

if we represent

formulas

$(2, \frac{1}{6}\pi + 2\pi$

$(-2, \frac{5}{6}\pi + 2\pi$

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