SECOND EDITION

SIGNALS & SYSTEMS

ALAN V. OPPENHEIM
ALAN S. WILLSKY
MASSACHUSETTS INSTITUTE OF TECHNOLOGY

WITH

S. HAMID NAWAB
BOSTON UNIVERSITY

PRENTICE HALL
UPPER SADDLE RIVER, NEW JERSEY 07458

Library of Congress Cataloging-in-Publication Data

Oppenheim, Alan V.
Signals and systems / Alan V. Oppenheim, Alan S. Willsky, with S. Hamid Nawab. – 2nd ed..
p. cm. – Prentice-Hall signal processing series
Includes bibliographical references and index.
ISBN 0-13-814757-4
1. System analysis. 2. Signal theory (Telecommunication)
I. Willsky, Alan S. II. Nawab, Syed Hamid. III. Title.
IV. Series.
QA402.O63 1996
621.382'23–dc20 96-19945
CIP

Acquisitions editor: **Tom Robbins**
Production service: **TKM Productions**
Editorial/production supervision: **Sharyn Vitrano**
Copy editor: **Brian Baker**
Interior and cover design: **Patrice Van Acker**
Art director: **Amy Rosen**
Managing editor: **Bayani Mendoza DeLeon**
Editor-in-Chief: **Marcia Horton**
Director of production and manufacturing: **David W. Riccardi**
Manufacturing buyer: **Donna Sullivan**
Editorial assistant: **Phyllis Morgan**

Prentice Hall
Pearson Education
Upper Saddle River, New Jersey 07458

Printed in the United States of America

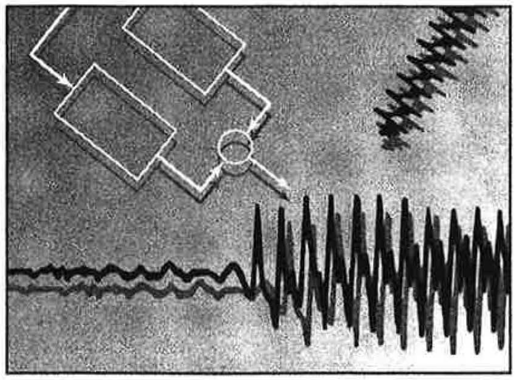
30 31 32 33 34 35 36 37 38 39 40 V092 18 17 16 15 14

ISBN 0-13-814757-4

Prentice-Hall International (UK) Limited, *London*
Prentice-Hall of Australia Pty. Limited, *Sydney*
Prentice-Hall Canada Inc., *Toronto*
Prentice-Hall Hispanoamericana, S.A., *Mexico*
Prentice-Hall of India Private Limited, *New Delhi*
Prentice-Hall of Japan, Inc., *Tokyo*
Prentice-Hall (Singapore) Asia Pte. Ltd., *Singapore*
Editora Prentice-Hall do Brasil, Ltda., *Rio de Janeiro*

1

SIGNALS AND SYSTEMS



1.0 INTRODUCTION

As described in the Foreword, the intuitive notions of signals and systems arise in a rich variety of contexts. Moreover, as we will see in this book, there is an analytical framework—that is, a language for describing signals and systems and an extremely powerful set of tools for analyzing them—that applies equally well to problems in many fields. In this chapter, we begin our development of the analytical framework for signals and systems by introducing their mathematical description and representations. In the chapters that follow, we build on this foundation in order to develop and describe additional concepts and methods that add considerably both to our understanding of signals and systems and to our ability to analyze and solve problems involving signals and systems that arise in a broad array of applications.

1.1 CONTINUOUS-TIME AND DISCRETE-TIME SIGNALS

1.1.1 Examples and Mathematical Representation

Signals may describe a wide variety of physical phenomena. Although signals can be represented in many ways, in all cases the information in a signal is contained in a pattern of variations of some form. For example, consider the simple circuit in Figure 1.1. In this case, the patterns of variation over time in the source and capacitor voltages, v_s and v_c, are examples of signals. Similarly, as depicted in Figure 1.2, the variations over time of the applied force f and the resulting automobile velocity v are signals. As another example, consider the human vocal mechanism, which produces speech by creating fluctuations in acoustic pressure. Figure 1.3 is an illustration of a recording of such a speech signal, obtained by

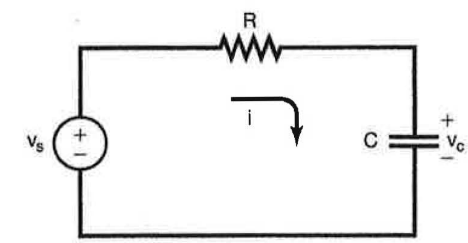


Figure 1.1 A simple *RC* circuit with source voltage v_s and capacitor voltage v_c.

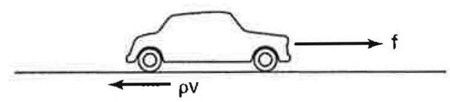


Figure 1.2 An automobile responding to an applied force f from the engine and to a retarding frictional force ρv proportional to the automobile's velocity v.

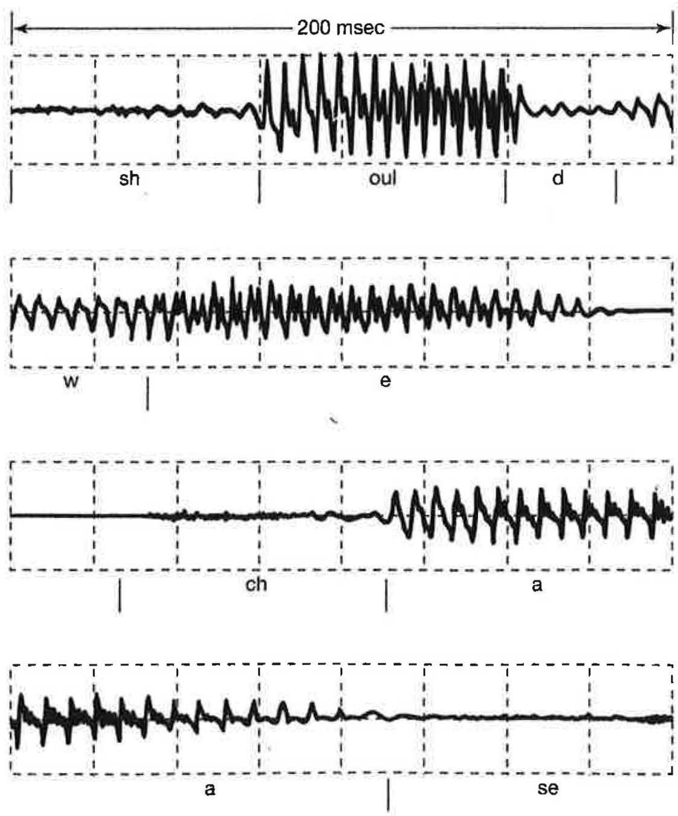


Figure 1.3 Example of a recording of speech. [Adapted from *Applications of Digital Signal Processing*, A.V. Oppenheim, ed. (Englewood Cliffs, N.J.: Prentice-Hall, Inc., 1978), p. 121.] The signal represents acoustic pressure variations as a function of time for the spoken words "should we chase." The top line of the figure corresponds to the word "should," the second line to the word "we," and the last two lines to the word "chase." (We have indicated the approximate beginnings and endings of each successive sound in each word.)

using a microphone to sense variations in acoustic pressure, which are then converted into an electrical signal. As can be seen in the figure, different sounds correspond to different patterns in the variations of acoustic pressure, and the human vocal system produces intelligible speech by generating particular sequences of these patterns. Alternatively, for the monochromatic picture, shown in Figure 1.4, it is the pattern of variations in brightness across the image that is important.

Figure 1.4 A monochromatic picture.

Signals are represented mathematically as functions of one or more independent variables. For example, a speech signal can be represented mathematically by acoustic pressure as a function of time, and a picture can be represented by brightness as a function of two spatial variables. In this book, we focus our attention on signals involving a single independent variable. For convenience, we will generally refer to the independent variable as time, although it may not in fact represent time in specific applications. For example, in geophysics, signals representing variations with depth of physical quantities such as density, porosity, and electrical resistivity are used to study the structure of the earth. Also, knowledge of the variations of air pressure, temperature, and wind speed with altitude are extremely important in meteorological investigations. Figure 1.5 depicts a typical example of annual average vertical wind profile as a function of height. The measured variations of wind speed with height are used in examining weather patterns, as well as wind conditions that may affect an aircraft during final approach and landing.

Throughout this book we will be considering two basic types of signals: continuous-time signals and discrete-time signals. In the case of continuous-time signals the independent variable is continuous, and thus these signals are defined for a continuum of values

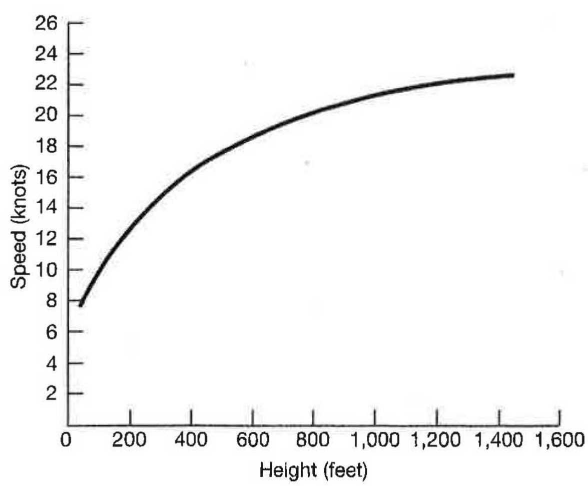


Figure 1.5 Typical annual vertical wind profile. (Adapted from Crawford and Hudson, National Severe Storms Laboratory Report, ESSA ERLTM-NSSL 48, August 1970.)

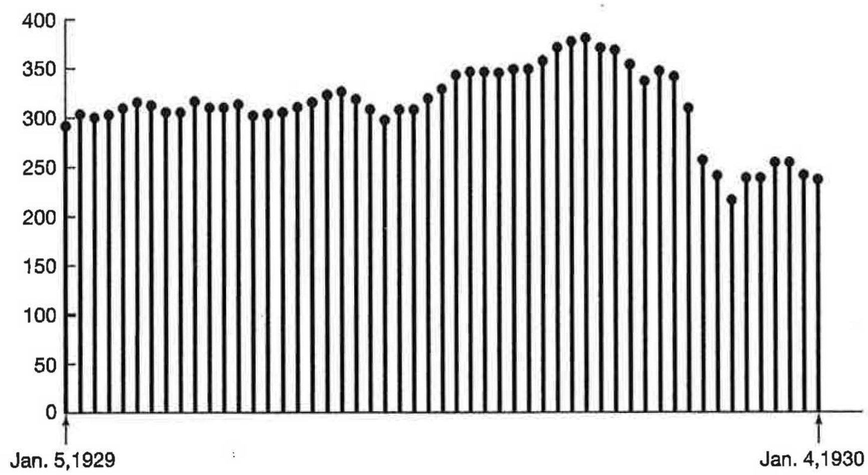


Figure 1.6 An example of a discrete-time signal: The weekly Dow-Jones stock market index from January 5, 1929, to January 4, 1930.

of the independent variable. On the other hand, discrete-time signals are defined only at discrete times, and consequently, for these signals, the independent variable takes on only a discrete set of values. A speech signal as a function of time and atmospheric pressure as a function of altitude are examples of continuous-time signals. The weekly Dow-Jones stock market index, as illustrated in Figure 1.6, is an example of a discrete-time signal. Other examples of discrete-time signals can be found in demographic studies in which various attributes, such as average budget, crime rate, or pounds of fish caught, are tabulated against such discrete variables as family size, total population, or type of fishing vessel, respectively.

To distinguish between continuous-time and discrete-time signals, we will use the symbol t to denote the continuous-time independent variable and n to denote the discrete-time independent variable. In addition, for continuous-time signals we will enclose the independent variable in parentheses $(\cdot)$, whereas for discrete-time signals we will use brackets $[\cdot]$ to enclose the independent variable. We will also have frequent occasions when it will be useful to represent signals graphically. Illustrations of a continuous-time signal $x(t)$ and a discrete-time signal $x[n]$ are shown in Figure 1.7. It is important to note that the discrete-time signal $x[n]$ is defined *only* for integer values of the independent variable. Our choice of graphical representation for $x[n]$ emphasizes this fact, and for further emphasis we will on occasion refer to $x[n]$ as a discrete-time *sequence.*

A discrete-time signal $x[n]$ may represent a phenomenon for which the independent variable is inherently discrete. Signals such as demographic data are examples of this. On the other hand, a very important class of discrete-time signals arises from the *sampling* of continuous-time signals. In this case, the discrete-time signal $x[n]$ represents successive samples of an underlying phenomenon for which the independent variable is continuous. Because of their speed, computational power, and flexibility, modern digital processors are used to implement many practical systems, ranging from digital autopilots to digital audio systems. Such systems require the use of discrete-time sequences representing sampled versions of continuous-time signals—e.g., aircraft position, velocity, and heading for an

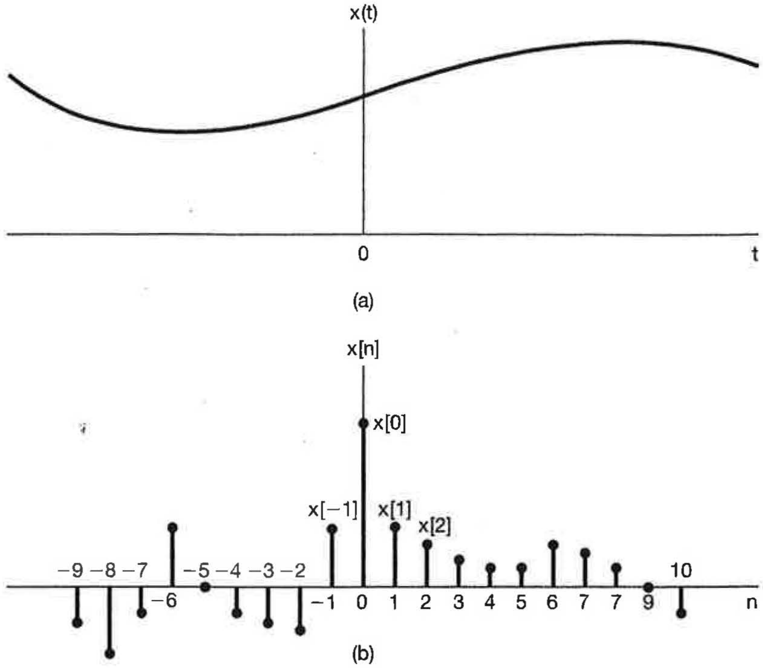


Figure 1.7 Graphical representations of (a) continuous-time and (b) discrete-time signals.

autopilot or speech and music for an audio system. Also, pictures in newspapers—or in this book, for that matter—actually consist of a very fine grid of points, and each of these points represents a sample of the brightness of the corresponding point in the original image. No matter what the source of the data, however, the signal $x[n]$ is defined only for integer values of n. It makes no more sense to refer to the $3\frac{1}{2}$th sample of a digital speech signal than it does to refer to the average budget for a family with $2\frac{1}{2}$ family members.

Throughout most of this book we will treat discrete-time signals and continuous-time signals separately but in parallel, so that we can draw on insights developed in one setting to aid our understanding of another. In Chapter 7 we will return to the question of sampling, and in that context we will bring continuous-time and discrete-time concepts together in order to examine the relationship between a continuous-time signal and a discrete-time signal obtained from it by sampling.

1.1.2 Signal Energy and Power

From the range of examples provided so far, we see that signals may represent a broad variety of phenomena. In many, but not all, applications, the signals we consider are directly related to physical quantities capturing power and energy in a physical system. For example, if $v(t)$ and $i(t)$ are, respectively, the voltage and current across a resistor with resistance R, then the instantaneous power is

$$p(t) = v(t)i(t) = \frac{1}{R}v^2(t). \tag{1.1}$$

The total *energy* expended over the time interval $t_1 \le t \le t_2$ is

$$\int_{t_1}^{t_2} p(t)\,dt = \int_{t_1}^{t_2} \frac{1}{R} v^2(t)\,dt, \tag{1.2}$$

and the *average power* over this time interval is

$$\frac{1}{t_2 - t_1}\int_{t_1}^{t_2} p(t)\,dt = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} \frac{1}{R} v^2(t)\,dt. \tag{1.3}$$

Similarly, for the automobile depicted in Figure 1.2, the instantaneous power dissipated through friction is $p(t) = bv^2(t)$, and we can then define the total energy and average power over a time interval in the same way as in eqs. (1.2) and (1.3).

With simple physical examples such as these as motivation, it is a common and worthwhile convention to use similar terminology for power and energy for *any* continuous-time signal $x(t)$ or *any* discrete-time signal $x[n]$. Moreover, as we will see shortly, we will frequently find it convenient to consider signals that take on complex values. In this case, the total energy over the time interval $t_1 \le t \le t_2$ in a continuous-time signal $x(t)$ is defined as

$$\int_{t_1}^{t_2} |x(t)|^2 dt, \tag{1.4}$$

where $|x|$ denotes the magnitude of the (possibly complex) number x. The time-averaged power is obtained by dividing eq. (1.4) by the length, $t_2 - t_1$, of the time interval. Similarly, the total energy in a discrete-time signal $x[n]$ over the time interval $n_1 \le n \le n_2$ is defined as

$$\sum_{n=n_1}^{n_2} |x[n]|^2, \tag{1.5}$$

and dividing by the number of points in the interval, $n_2 - n_1 + 1$, yields the average power over the interval. It is important to remember that the terms "power" and "energy" are used here independently of whether the quantities in eqs. (1.4) and (1.5) actually are related to physical energy.[1] Nevertheless, we will find it convenient to use these terms in a general fashion.

Furthermore, in many systems we will be interested in examining power and energy in signals over an infinite time interval, i.e., for $-\infty < t < +\infty$ or for $-\infty < n < +\infty$. In these cases, we define the total energy as limits of eqs. (1.4) and (1.5) as the time interval increases without bound. That is, in continuous time,

$$E_\infty \triangleq \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2\,dt = \int_{-\infty}^{+\infty} |x(t)|^2\,dt, \tag{1.6}$$

and in discrete time,

$$E_\infty \triangleq \lim_{N\to\infty} \sum_{n=-N}^{+N} |x[n]|^2 = \sum_{n=-\infty}^{+\infty} |x[n]|^2. \tag{1.7}$$

[1] Even if such a relationship does exist, eqs. (1.4) and (1.5) may have the wrong dimensions and scalings. For example, comparing eqs. (1.2) and (1.4), we see that if $x(t)$ represents the voltage across a resistor, then eq. (1.4) must be divided by the resistance (measured, for example, in ohms) to obtain units of physical energy.

Note that for some signals the integral in eq. (1.6) or sum in eq. (1.7) might not converge—e.g., if $x(t)$ or $x[n]$ equals a nonzero constant value for all time. Such signals have infinite energy, while signals with $E_\infty < \infty$ have finite energy.

In an analogous fashion, we can define the time-averaged power over an infinite interval as

$$P_\infty \triangleq \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |x(t)|^2\,dt \tag{1.8}$$

and

$$P_\infty \triangleq \lim_{N\to\infty} \frac{1}{2N+1}\sum_{n=-N}^{+N} |x[n]|^2 \tag{1.9}$$

in continuous time and discrete time, respectively. With these definitions, we can identify three important classes of signals. The first of these is the class of signals with finite total energy, i.e., those signals for which $E_\infty < \infty$. Such a signal must have zero average power, since in the continuous time case, for example, we see from eq. (1.8) that

$$P_\infty = \lim_{T\to\infty} \frac{E_\infty}{2T} = 0. \tag{1.10}$$

An example of a finite-energy signal is a signal that takes on the value 1 for $0 \le t \le 1$ and 0 otherwise. In this case, $E_\infty = 1$ and $P_\infty = 0$.

A second class of signals are those with finite average power P_∞. From what we have just seen, if $P_\infty > 0$, then, of necessity, $E_\infty = \infty$. This, of course, makes sense, since if there is a nonzero average energy per unit time (i.e., nonzero power), then integrating or summing this over an infinite time interval yields an infinite amount of energy. For example, the constant signal $x[n] = 4$ has infinite energy, but average power $P_\infty = 16$. There are also signals for which neither P_∞ nor E_∞ are finite. A simple example is the signal $x(t) = t$. We will encounter other examples of signals in each of these classes in the remainder of this and the following chapters.

1.2 TRANSFORMATIONS OF THE INDEPENDENT VARIABLE

A central concept in signal and system analysis is that of the transformation of a signal. For example, in an aircraft control system, signals corresponding to the actions of the pilot are transformed by electrical and mechanical systems into changes in aircraft thrust or the positions of aircraft control surfaces such as the rudder or ailerons, which in turn are transformed through the dynamics and kinematics of the vehicle into changes in aircraft velocity and heading. Also, in a high-fidelity audio system, an input signal representing music as recorded on a cassette or compact disc is modified in order to enhance desirable characteristics, to remove recording noise, or to balance the several components of the signal (e.g., treble and bass). In this section, we focus on a very limited but important class of elementary signal transformations that involve simple modification of the independent variable, i.e., the time axis. As we will see in this and subsequent sections of this chapter, these elementary transformations allow us to introduce several basic properties of signals and systems. In later chapters, we will find that they also play an important role in defining and characterizing far richer and important classes of systems.

1.2.1 Examples of Transformations of the Independent Variable

A simple and very important example of transforming the independent variable of a signal is a *time shift.* A time shift in discrete time is illustrated in Figure 1.8, in which we have two signals $x[n]$ and $x[n - n_0]$ that are identical in shape, but that are displaced or shifted relative to each other. We will also encounter time shifts in continuous time, as illustrated in Figure 1.9, in which $x(t - t_0)$ represents a delayed (if t_0 is positive) or advanced (if t_0 is negative) version of $x(t)$. Signals that are related in this fashion arise in applications such as radar, sonar, and seismic signal processing, in which several receivers at different locations observe a signal being transmitted through a medium (water, rock, air, etc.). In this case, the difference in propagation time from the point of origin of the transmitted signal to any two receivers results in a time shift between the signals at the two receivers.

A second basic transformation of the time axis is that of *time reversal.* For example, as illustrated in Figure 1.10, the signal $x[-n]$ is obtained from the signal $x[n]$ by a reflection about $n = 0$ (i.e., by reversing the signal). Similarly, as depicted in Figure 1.11, the signal $x(-t)$ is obtained from the signal $x(t)$ by a reflection about $t = 0$. Thus, if $x(t)$ represents an audio tape recording, then $x(-t)$ is the same tape recording played backward. Another transformation is that of *time scaling.* In Figure 1.12 we have illustrated three signals, $x(t)$, $x(2t)$, and $x(t/2)$, that are related by linear scale changes in the independent variable. If we again think of the example of $x(t)$ as a tape recording, then $x(2t)$ is that recording played at twice the speed, and $x(t/2)$ is the recording played at half-speed.

It is often of interest to determine the effect of transforming the independent variable of a given signal $x(t)$ to obtain a signal of the form $x(\alpha t + \beta)$, where α and β are given numbers. Such a transformation of the independent variable preserves the shape of $x(t)$, except that the resulting signal may be linearly stretched if $|\alpha| < 1$, linearly compressed if $|\alpha| > 1$, reversed in time if $\alpha < 0$, and shifted in time if β is nonzero. This is illustrated in the following set of examples.

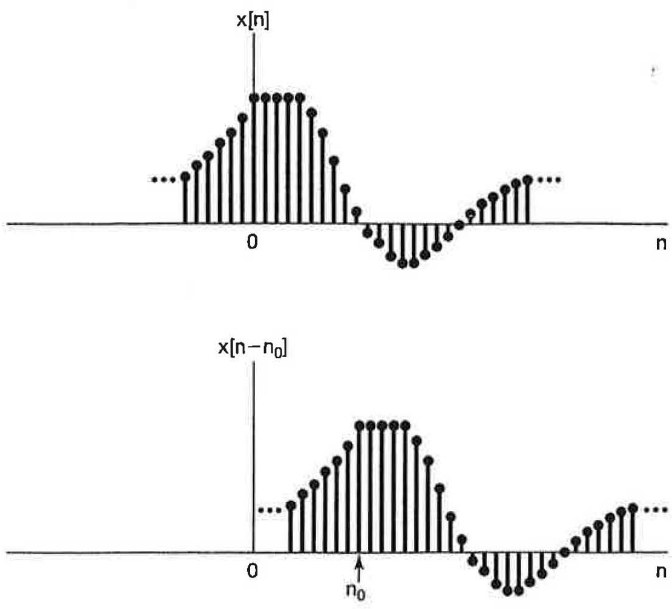


Figure 1.8 Discrete-time signals related by a time shift. In this figure $n_0 > 0$, so that $x[n - n_0]$ is a delayed verson of $x[n]$ (i.e., each point in $x[n]$ occurs later in $x[n - n_0]$).

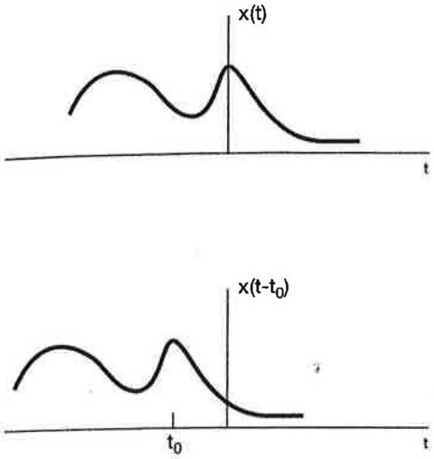


Figure 1.9 Continuous-time signals related by a time shift. In this figure $t_0 < 0$, so that $x(t - t_0)$ is an advanced version of $x(t)$ (i.e., each point in $x(t)$ occurs at an earlier time in $x(t - t_0)$).

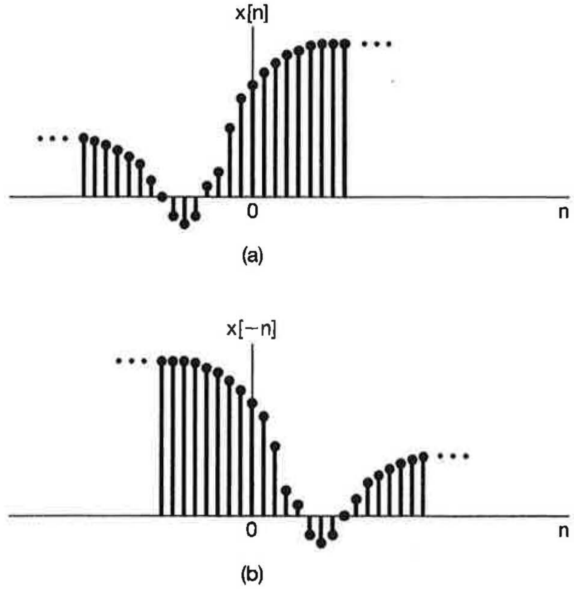


Figure 1.10 (a) A discrete-time signal $x[n]$; (b) its reflection $x[-n]$ about $n = 0$.

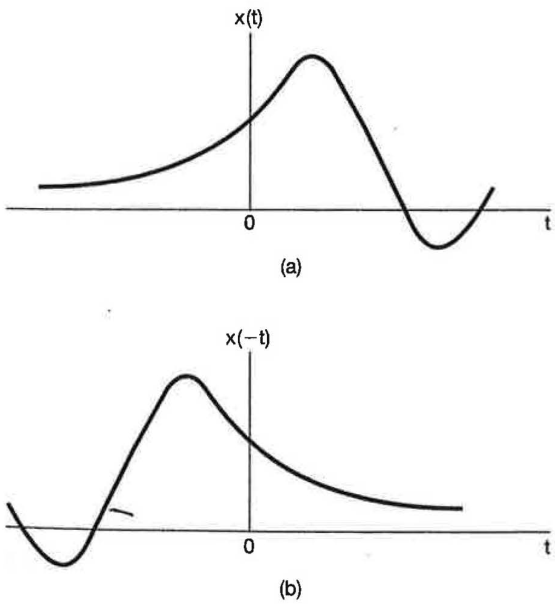


Figure 1.11 (a) A continuous-time signal $x(t)$; (b) its reflection $x(-t)$ about $t = 0$.

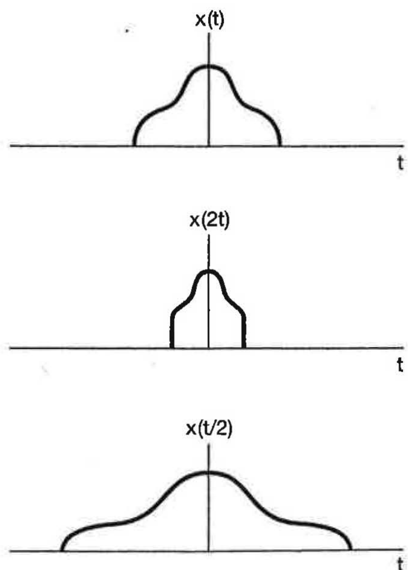


Figure 1.12 Continuous-time signals related by time scaling.

Example 1.1

Given the signal $x(t)$ shown in Figure 1.13(a), the signal $x(t+1)$ corresponds to an advance (shift to the left) by one unit along the t axis as illustrated in Figure 1.13(b). Specifically, we note that the value of $x(t)$ at $t = t_0$ occurs in $x(t+1)$ at $t = t_0 - 1$. For

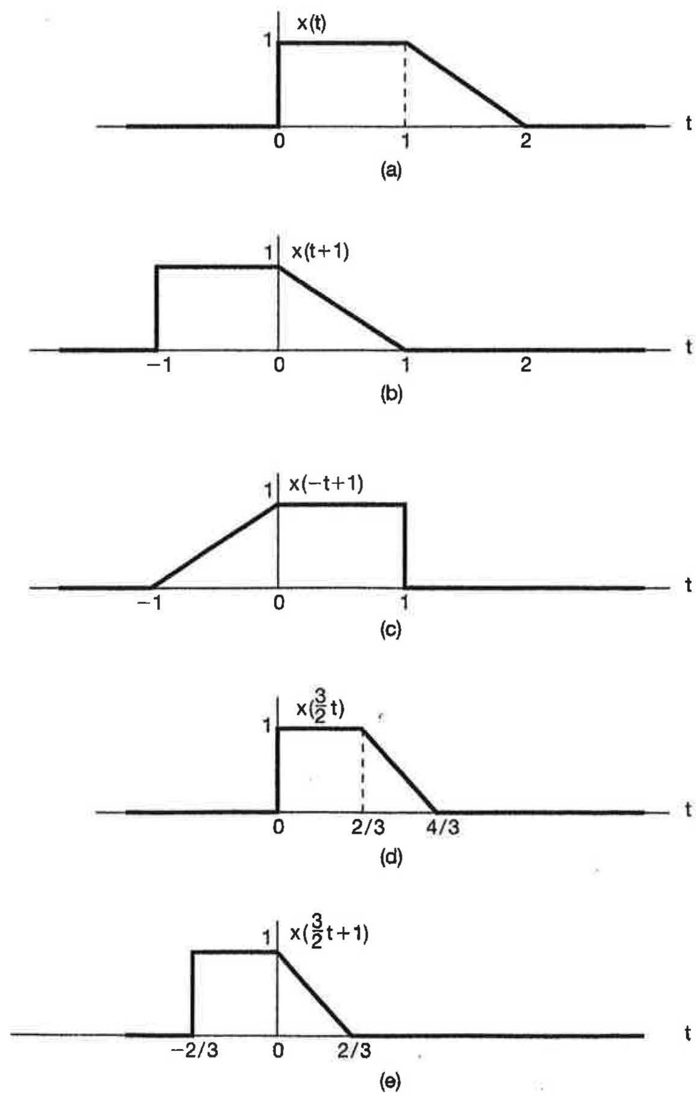


Figure 1.13 (a) The continuous-time signal $x(t)$ used in Examples 1.1–1.3 to illustrate transformations of the independent variable; (b) the time-shifted signal $x(t+1)$; (c) the signal $x(-t+1)$ obtained by a time shift and a time reversal; (d) the time-scaled signal $x(\frac{3}{2}t)$; and (e) the signal $x(\frac{3}{2}t+1)$ obtained by time-shifting and scaling.

example, the value of $x(t)$ at $t = 1$ is found in $x(t+1)$ at $t = 1 - 1 = 0$. Also, since $x(t)$ is zero for $t < 0$, we have $x(t+1)$ zero for $t < -1$. Similarly, since $x(t)$ is zero for $t > 2$, $x(t+1)$ is zero for $t > 1$.

Let us also consider the signal $x(-t+1)$, which may be obtained by replacing t with $-t$ in $x(t+1)$. That is, $x(-t+1)$ is the time reversed version of $x(t+1)$. Thus, $x(-t+1)$ may be obtained graphically by reflecting $x(t+1)$ about the t axis as shown in Figure 1.13(c).

Example 1.2

Given the signal $x(t)$, shown in Figure 1.13(a), the signal $x(\frac{3}{2}t)$ corresponds to a linear compression of $x(t)$ by a factor of $\frac{2}{3}$ as illustrated in Figure 1.13(d). Specifically we note that the value of $x(t)$ at $t = t_0$ occurs in $x(\frac{3}{2}t)$ at $t = \frac{2}{3}t_0$. For example, the value of $x(t)$ at $t = 1$ is found in $x(\frac{3}{2}t)$ at $t = \frac{2}{3}(1) = \frac{2}{3}$. Also, since $x(t)$ is zero for $t < 0$, we have $x(\frac{3}{2}t)$ zero for $t < 0$. Similarly, since $x(t)$ is zero for $t > 2$, $x(\frac{3}{2}t)$ is zero for $t > \frac{4}{3}$.

Example 1.3

Suppose that we would like to determine the effect of transforming the independent variable of a given signal, $x(t)$, to obtain a signal of the form $x(\alpha t + \beta)$, where α and β are given numbers. A systematic approach to doing this is to first delay or advance $x(t)$ in accordance with the value of β, and then to perform time scaling and/or time reversal on the resulting signal in accordance with the value of α. The delayed or advanced signal is linearly stretched if $|\alpha| < 1$, linearly compressed if $|\alpha| > 1$, and reversed in time if $\alpha < 0$.

To illustrate this approach, let us show how $x(\frac{3}{2}t + 1)$ may be determined for the signal $x(t)$ shown in Figure 1.13(a). Since $\beta = 1$, we first advance (shift to the left) $x(t)$ by 1 as shown in Figure 1.13(b). Since $|\alpha| = \frac{3}{2}$, we may linearly compress the shifted signal of Figure 1.13(b) by a factor of $\frac{2}{3}$ to obtain the signal shown in Figure 1.13(e).

In addition to their use in representing physical phenomena such as the time shift in a sonar signal and the speeding up or reversal of an audiotape, transformations of the independent variable are extremely useful in signal and system analysis. In Section 1.6 and in Chapter 2, we will use transformations of the independent variable to introduce and analyze the properties of systems. These transformations are also important in defining and examining some important properties of signals.

1.2.2 Periodic Signals

An important class of signals that we will encounter frequently throughout this book is the class of *periodic* signals. A periodic continuous-time signal $x(t)$ has the property that there is a positive value of T for which

$$x(t) = x(t+T) \tag{1.11}$$

for all values of t. In other words, a periodic signal has the property that it is unchanged by a time shift of T. In this case, we say that $x(t)$ is *periodic with period T*. Periodic continuous-time signals arise in a variety of contexts. For example, as illustrated in Problem 2.61, the natural response of systems in which energy is conserved, such as ideal *LC* circuits without resistive energy dissipation and ideal mechanical systems without frictional losses, are periodic and, in fact, are composed of some of the basic periodic signals that we will introduce in Section 1.3.

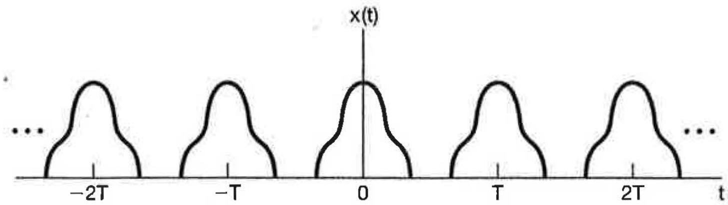


Figure 1.14 A continuous-time periodic signal.

An example of a periodic continuous-time signal is given in Figure 1.14. From the figure or from eq. (1.11), we can readily deduce that if $x(t)$ is periodic with period T, then $x(t) = x(t + mT)$ for all t and for any integer m. Thus, $x(t)$ is also periodic with period $2T, 3T, 4T, \ldots$. The *fundamental period* T_0 of $x(t)$ is the smallest positive value of T for which eq. (1.11) holds. This definition of the fundamental period works, except if $x(t)$ is a constant. In this case the fundamental period is undefined, since $x(t)$ is periodic for *any* choice of T (so there is no smallest positive value). A signal $x(t)$ that is not periodic will be referred to as an *aperiodic* signal.

Periodic signals are defined analogously in discrete time. Specifically, a discrete-time signal $x[n]$ is periodic with period N, where N is a positive integer, if it is unchanged by a time shift of N, i.e., if

$$x[n] = x[n + N] \tag{1.12}$$

for all values of n. If eq. (1.12) holds, then $x[n]$ is also periodic with period $2N, 3N, \ldots$. The *fundamental period* N_0 is the smallest positive value of N for which eq. (1.12) holds. An example of a discrete-time periodic signal with fundamental period $N_0 = 3$ is shown in Figure 1.15.

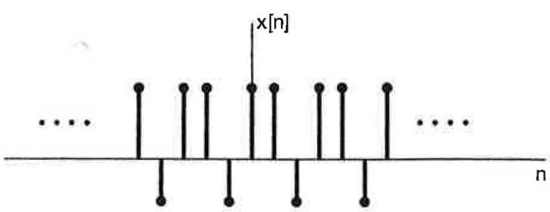


Figure 1.15 A discrete-time periodic signal with fundamental period $N_0 = 3$.

Example 1.4

Let us illustrate the type of problem solving that may be required in determining whether or not a given signal is periodic. The signal whose periodicity we wish to check is given by

$$x(t) = \begin{cases} \cos(t) & \text{if } t < 0 \\ \sin(t) & \text{if } t \geq 0 \end{cases}. \tag{1.13}$$

From trigonometry, we know that $\cos(t + 2\pi) = \cos(t)$ and $\sin(t + 2\pi) = \sin(t)$. Thus, considering $t > 0$ and $t < 0$ separately, we see that $x(t)$ does repeat itself over every interval of length 2π. However, as illustrated in Figure 1.16, $x(t)$ also has a discontinuity at the time origin that does not recur at any other time. Since every feature in the shape of a periodic signal *must* recur periodically, we conclude that the signal $x(t)$ is not periodic.

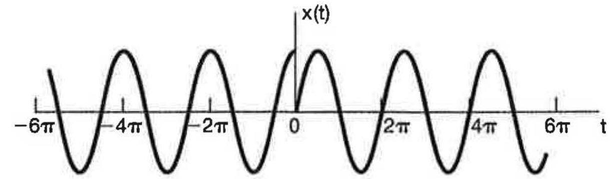


Figure 1.16 The signal $x(t)$ considered in Example 1.4.

1.2.3 Even and Odd Signals

Another set of useful properties of signals relates to their symmetry under time reversal. A signal $x(t)$ or $x[n]$ is referred to as an *even* signal if it is identical to its time-reversed counterpart, i.e., with its reflection about the origin. In continuous time a signal is even if

$$x(-t) = x(t), \tag{1.14}$$

while a discrete-time signal is even if

$$x[-n] = x[n]. \tag{1.15}$$

A signal is referred to as *odd* if

$$x(-t) = -x(t), \tag{1.16}$$

$$x[-n] = -x[n]. \tag{1.17}$$

An odd signal must necessarily be 0 at $t = 0$ or $n = 0$, since eqs. (1.16) and (1.17) require that $x(0) = -x(0)$ and $x[0] = -x[0]$. Examples of even and odd continuous-time signals are shown in Figure 1.17.

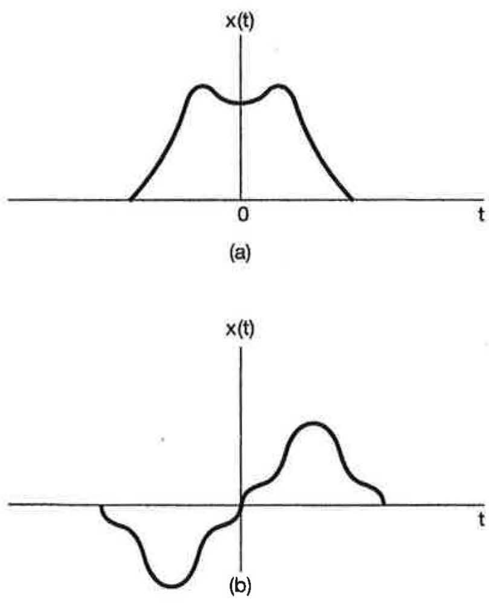


Figure 1.17 (a) An even continuous-time signal; (b) an odd continuous-time signal.

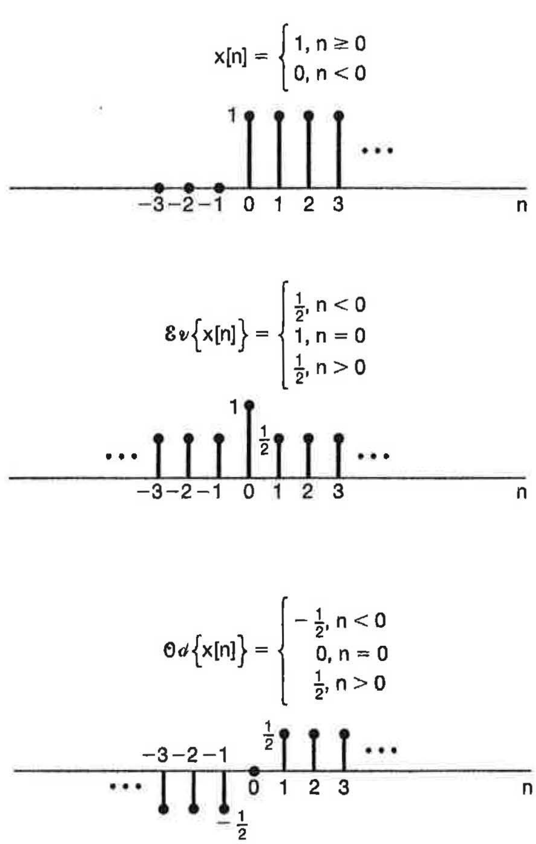


Figure 1.18 Example of the even-odd decomposition of a discrete-time signal.

An important fact is that any signal can be broken into a sum of two signals, one of which is even and one of which is odd. To see this, consider the signal

$$\mathcal{E}v\{x(t)\} = \frac{1}{2}[x(t) + x(-t)], \tag{1.18}$$

which is referred to as the *even part* of $x(t)$. Similarly, the *odd part* of $x(t)$ is given by

$$\mathcal{O}d\{x(t)\} = \frac{1}{2}[x(t) - x(-t)]. \tag{1.19}$$

It is a simple exercise to check that the even part is in fact even, that the odd part is odd, and that $x(t)$ is the sum of the two. Exactly analogous definitions hold in the discrete-time case. An example of the even-odd decomposition of a discrete-time signal is given in Figure 1.18.

1.3 EXPONENTIAL AND SINUSOIDAL SIGNALS

In this section and the next, we introduce several basic continuous-time and discrete-time signals. Not only do these signals occur frequently, but they also serve as basic building blocks from which we can construct many other signals.

1.3.1 Continuous-Time Complex Exponential and Sinusoidal Signals

The continuous-time *complex exponential signal* is of the form

$$x(t) = Ce^{at}, \tag{1.20}$$

where C and a are, in general, complex numbers. Depending upon the values of these parameters, the complex exponential can exhibit several different characteristics.

Real Exponential Signals

As illustrated in Figure 1.19, if C and a are real [in which case $x(t)$ is called a *real exponential*], there are basically two types of behavior. If a is positive, then as t increases $x(t)$ is a growing exponential, a form that is used in describing many different physical processes, including chain reactions in atomic explosions and complex chemical reactions. If a is negative, then $x(t)$ is a decaying exponential, a signal that is also used to describe a wide variety of phenomena, including the process of radioactive decay and the responses of RC circuits and damped mechanical systems. In particular, as shown in Problems 2.61 and 2.62, the natural responses of the circuit in Figure 1.1 and the automobile in Figure 1.2 are decaying exponentials. Also, we note that for $a = 0$, $x(t)$ is constant.

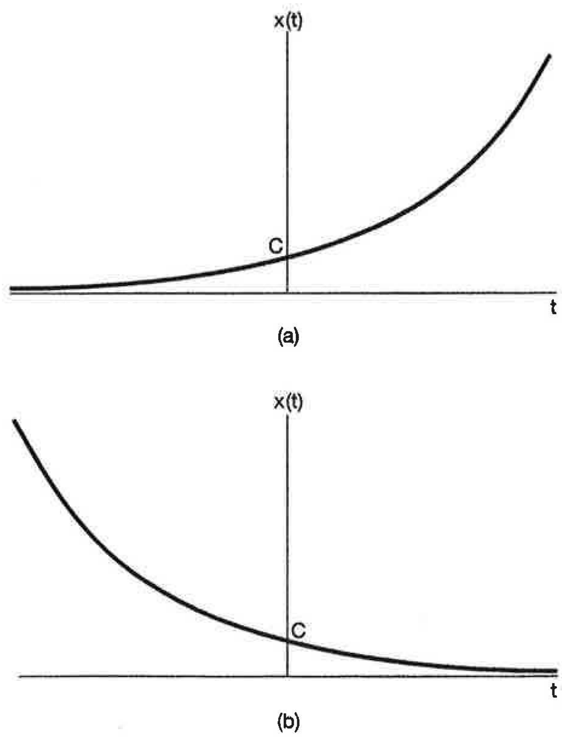


Figure 1.19 Continuous-time real exponential $x(t) = Ce^{at}$: (a) $a > 0$; (b) $a < 0$.

Periodic Complex Exponential and Sinusoidal Signals

A second important class of complex exponentials is obtained by constraining a to be purely imaginary. Specifically, consider

$$x(t) = e^{j\omega_0 t}. \quad (1.21)$$

An important property of this signal is that it is periodic. To verify this, we recall from eq. (1.11) that $x(t)$ will be periodic with period T if

$$e^{j\omega_0 t} = e^{j\omega_0 (t+T)}. \quad (1.22)$$

Or, since

$$e^{j\omega_0 (t+T)} = e^{j\omega_0 t} e^{j\omega_0 T},$$

it follows that for periodicity, we must have

$$e^{j\omega_0 T} = 1. \quad (1.23)$$

If $\omega_0 = 0$, then $x(t) = 1$, which is periodic for any value of T. If $\omega_0 \neq 0$, then the fundamental period T_0 of $x(t)$—that is, the smallest positive value of T for which eq. (1.23) holds—is

$$T_0 = \frac{2\pi}{|\omega_0|}. \quad (1.24)$$

Thus, the signals $e^{j\omega_0 t}$ and $e^{-j\omega_0 t}$ have the same fundamental period.

A signal closely related to the periodic complex exponential is the *sinusoidal signal*

$$x(t) = A\cos(\omega_0 t + \phi), \quad (1.25)$$

as illustrated in Figure 1.20. With seconds as the units of t, the units of ϕ and ω_0 are radians and radians per second, respectively. It is also common to write $\omega_0 = 2\pi f_0$, where f_0 has the units of cycles per second, or hertz (Hz). Like the complex exponential signal, the sinusoidal signal is periodic with fundamental period T_0 given by eq. (1.24). Sinusoidal and

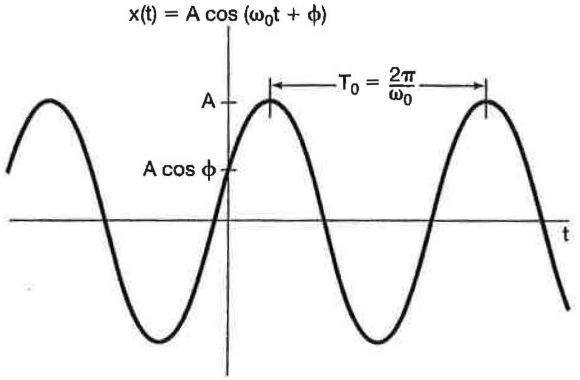


Figure 1.20 Continuous-time sinusoidal signal.

complex exponential signals are also used to describe the characteristics of many physical processes—in particular, physical systems in which energy is conserved. For example, as shown in Problem 2.61, the natural response of an *LC* circuit is sinusoidal, as is the simple harmonic motion of a mechanical system consisting of a mass connected by a spring to a stationary support. The acoustic pressure variations corresponding to a single musical tone are also sinusoidal.

By using Euler's relation,[2] the complex exponential in eq. (1.21) can be written in terms of sinusoidal signals with the same fundamental period:

$$e^{j\omega_0 t} = \cos\omega_0 t + j\sin\omega_0 t. \tag{1.26}$$

Similarly, the sinusoidal signal of eq. (1.25) can be written in terms of periodic complex exponentials, again with the same fundamental period:

$$A\cos(\omega_0 t + \phi) = \frac{A}{2}e^{j\phi}e^{j\omega_0 t} + \frac{A}{2}e^{-j\phi}e^{-j\omega_0 t}. \tag{1.27}$$

Note that the two exponentials in eq. (1.27) have complex amplitudes. Alternatively, we can express a sinusoid in terms of a complex exponential signal as

$$A\cos(\omega_0 t + \phi) = A\,\mathcal{R}e\{e^{j(\omega_0 t+\phi)}\}, \tag{1.28}$$

where, if c is a complex number, $\mathcal{R}e\{c\}$ denotes its real part. We will also use the notation $\mathcal{I}m\{c\}$ for the imaginary part of c, so that, for example,

$$A\sin(\omega_0 t + \phi) = A\,\mathcal{I}m\{e^{j(\omega_0 t+\phi)}\}. \tag{1.29}$$

From eq. (1.24), we see that the fundamental period T_0 of a continuous-time sinusoidal signal or a periodic complex exponential is inversely proportional to $|\omega_0|$, which we will refer to as the *fundamental frequency*. From Figure 1.21, we see graphically what this means. If we decrease the magnitude of ω_0, we slow down the rate of oscillation and therefore increase the period. Exactly the opposite effects occur if we increase the magnitude of ω_0. Consider now the case $\omega_0 = 0$. In this case, as we mentioned earlier, $x(t)$ is constant and therefore is periodic with period T for any positive value of T. Thus, the fundamental period of a constant signal is undefined. On the other hand, there is no ambiguity in defining the fundamental frequency of a constant signal to be zero. That is, a constant signal has a zero rate of oscillation.

Periodic signals—and in particular, the complex periodic exponential signal in eq. (1.21) and the sinusoidal signal in eq. (1.25)—provide important examples of signals with infinite total energy but finite average power. For example, consider the periodic exponential signal of eq. (1.21), and suppose that we calculate the total energy and average power in this signal over one period:

$$\begin{aligned} E_{\text{period}} &= \int_0^{T_0} |e^{j\omega_0 t}|^2\,dt \\ &= \int_0^{T_0} 1\cdot dt = T_0, \end{aligned} \tag{1.30}$$

[2]Euler's relation and other basic ideas related to the manipulation of complex numbers and exponentials are considered in the mathematical review section of the problems at the end of the chapter.

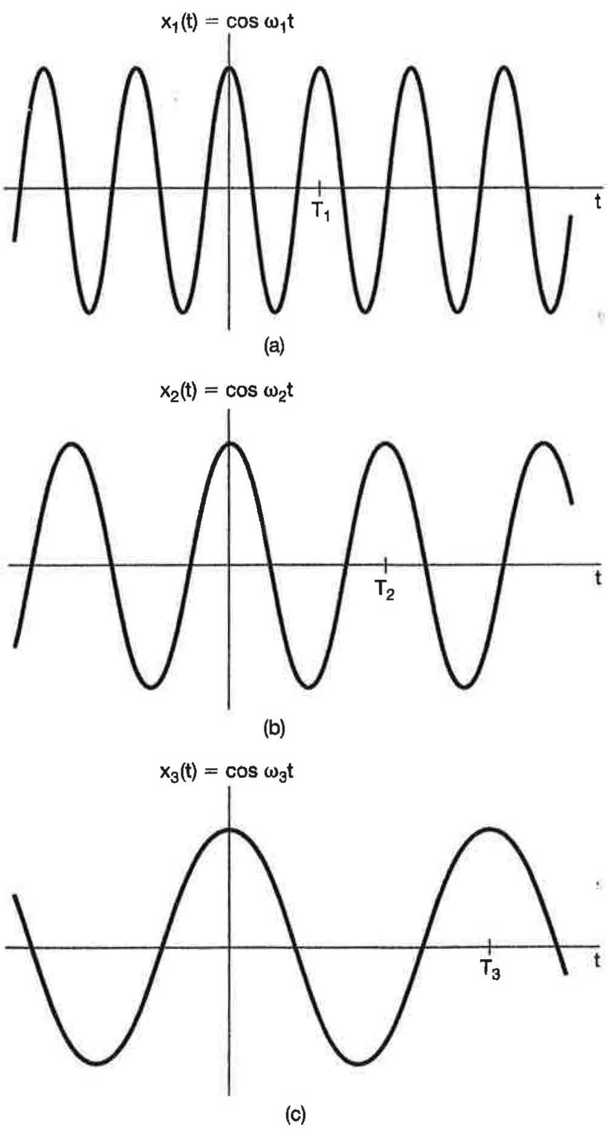


Figure 1.21 Relationship between the fundamental frequency and period for continuous-time sinusoidal signals; here, $\omega_1 > \omega_2 > \omega_3$, which implies that $T_1 < T_2 < T_3$.

$$P_{\text{period}} = \frac{1}{T_0} E_{\text{period}} = 1. \tag{1.31}$$

Since there are an infinite number of periods as t ranges from $-\infty$ to $+\infty$, the total energy integrated over all time is infinite. However, each period of the signal looks exactly the same. Since the average power of the signal equals 1 over each period, averaging over multiple periods always yields an average power of 1. That is, the complex periodic exponential signal has finite average power equal to

$$P_\infty = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |e^{j\omega_0 t}|^2\, dt = 1. \tag{1.32}$$

Problem 1.3 provides additional examples of energy and power calculations for periodic and aperiodic signals.

Periodic complex exponentials will play a central role in much of our treatment of signals and systems, in part because they serve as extremely useful building blocks for many other signals. We will often find it useful to consider sets of *harmonically related* complex exponentials—that is, sets of periodic exponentials, all of which are periodic with a common period T_0. Specifically, a necessary condition for a complex exponential $e^{j\omega t}$ to be periodic with period T_0 is that

$$e^{j\omega T_0} = 1, \tag{1.33}$$

which implies that ωT_0 is a multiple of 2π, i.e.,

$$\omega T_0 = 2\pi k, \qquad k = 0, \pm 1, \pm 2, \ldots. \tag{1.34}$$

Thus, if we define

$$\omega_0 = \frac{2\pi}{T_0}, \tag{1.35}$$

we see that, to satisfy eq. (1.34), ω must be an integer multiple of ω_0. That is, a harmonically related set of complex exponentials is a set of periodic exponentials with fundamental frequencies that are all multiples of a single positive frequency ω_0:

$$\phi_k(t) = e^{jk\omega_0 t}, \qquad k = 0, \pm 1, \pm 2, \ldots. \tag{1.36}$$

For $k = 0$, $\phi_k(t)$ is a constant, while for any other value of k, $\phi_k(t)$ is periodic with fundamental frequency $|k|\omega_0$ and fundamental period

$$\frac{2\pi}{|k|\omega_0} = \frac{T_0}{|k|}. \tag{1.37}$$

The kth harmonic $\phi_k(t)$ is still periodic with period T_0 as well, as it goes through exactly $|k|$ of its fundamental periods during any time interval of length T_0.

Our use of the term "harmonic" is consistent with its use in music, where it refers to tones resulting from variations in acoustic pressure at frequencies that are integer multiples of a fundamental frequency. For example, the pattern of vibrations of a string on an instrument such as a violin can be described as a superposition—i.e., a weighted sum—of harmonically related periodic exponentials. In Chapter 3, we will see that we can build a very rich class of periodic signals using the harmonically related signals of eq. (1.36) as the building blocks.

Example 1.5

It is sometimes desirable to express the sum of two complex exponentials as the product of a single complex exponential and a single sinusoid. For example, suppose we wish to plot the magnitude of the signal

$$x(t) = e^{j2t} + e^{j3t}. \tag{1.38}$$

To do this, we first factor out a complex exponential from the right side of eq. (1.38), where the frequency of this exponential factor is taken as the average of the frequencies of the two exponentials in the sum. Doing this, we obtain

$$x(t) = e^{j2.5t}(e^{-j0.5} + e^{j0.5t}), \tag{1.39}$$

which, because of Euler's relation, can be rewritten as

$$x(t) = 2e^{j2.5t}\cos(0.5t). \tag{1.40}$$

From this, we can directly obtain an expression for the magnitude of $x(t)$:

$$|x(t)| = 2|\cos(0.5t)|. \tag{1.41}$$

Here, we have used the fact that the magnitude of the complex exponential $e^{j2.5t}$ is always unity. Thus, $|x(t)|$ is what is commonly referred to as a full-wave rectified sinusoid, as shown in Figure 1.22.

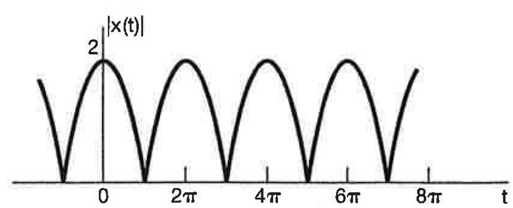


Figure 1.22 The full-wave rectified sinusoid of Example 1.5.

General Complex Exponential Signals

The most general case of a complex exponential can be expressed and interpreted in terms of the two cases we have examined so far: the real exponential and the periodic complex exponential. Specifically, consider a complex exponential Ce^{at}, where C is expressed in polar form and a in rectangular form. That is,

$$C = |C|e^{j\theta}$$

and

$$a = r + j\omega_0.$$

Then

$$Ce^{at} = |C|e^{j\theta}e^{(r+j\omega_0)t} = |C|e^{rt}e^{j(\omega_0 t+\theta)}. \tag{1.42}$$

Using Euler's relation, we can expand this further as

$$Ce^{at} = |C|e^{rt}\cos(\omega_0 t + \theta) + j|C|e^{rt}\sin(\omega_0 t + \theta). \tag{1.43}$$

Thus, for $r = 0$, the real and imaginary parts of a complex exponential are sinusoidal. For $r > 0$ they correspond to sinusoidal signals multiplied by a growing exponential, and for $r < 0$ they correspond to sinusoidal signals multiplied by a decaying exponential. These two cases are shown in Figure 1.23. The dashed lines in the figure correspond to the functions $\pm|C|e^{rt}$. From eq. (1.42), we see that $|C|e^{rt}$ is the magnitude of the complex exponential. Thus, the dashed curves act as an envelope for the oscillatory curve in the figure in that the peaks of the oscillations just reach these curves, and in this way the envelope provides us with a convenient way to visualize the general trend in the amplitude of the oscillations.

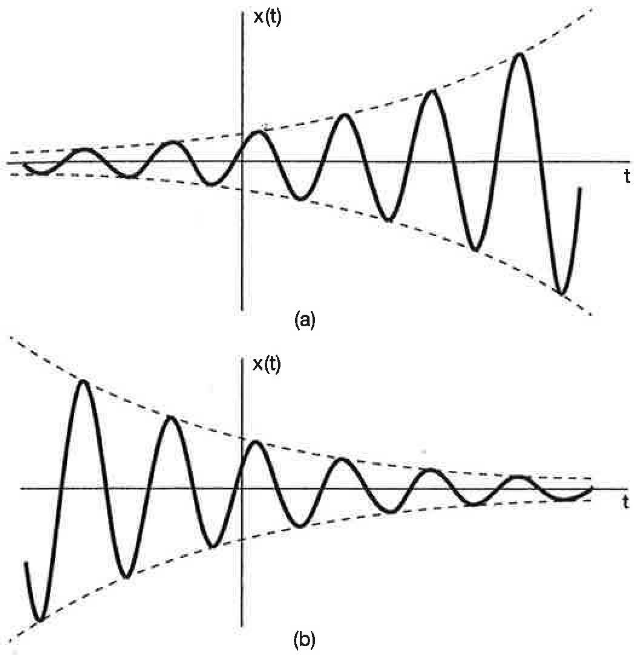


Figure 1.23 (a) Growing sinusoidal signal $x(t) = Ce^{rt}\cos(\omega_0 t + \theta)$, $r > 0$; (b) decaying sinusoid $x(t) = Ce^{rt}\cos(\omega_0 t + \theta)$, $r < 0$.

Sinusoidal signals multiplied by decaying exponentials are commonly referred to as *damped sinusoids*. Examples of damped sinusoids arise in the response of *RLC* circuits and in mechanical systems containing both damping and restoring forces, such as automotive suspension systems. These kinds of systems have mechanisms that dissipate energy (resistors, damping forces such as friction) with oscillations that decay in time. Examples illustrating such systems and their damped sinusoidal natural responses can be found in Problems 2.61 and 2.62.

1.3.2 Discrete-Time Complex Exponential and Sinusoidal Signals

As in continuous time, an important signal in discrete time is the *complex exponential signal* or *sequence*, defined by

$$x[n] = C\alpha^n, \tag{1.44}$$

where C and α are, in general, complex numbers. This could alternatively be expressed in the form

$$x[n] = Ce^{\beta n}, \tag{1.45}$$

where

$$\alpha = e^{\beta}.$$

Although the form of the discrete-time complex exponential sequence given in eq. (1.45) is more analogous to the form of the continuous-time exponential, it is often more convenient to express the discrete-time complex exponential sequence in the form of eq. (1.44).

Real Exponential Signals

If C and α are real, we can have one of several types of behavior, as illustrated in Figure 1.24. If $|\alpha| > 1$ the magnitude of the signal grows exponentially with n, while if $|\alpha| < 1$ we have a decaying exponential. Furthermore, if α is positive, all the values of $C\alpha^n$ are of the same sign, but if α is negative then the sign of $x[n]$ alternates. Note also that if $\alpha = 1$ then $x[n]$ is a constant, whereas if $\alpha = -1$, $x[n]$ alternates in value between $+C$ and $-C$. Real-valued discrete-time exponentials are often used to describe population growth as a function of generation and total return on investment as a function of day, month, or quarter.

Sinusoidal Signals

Another important complex exponential is obtained by using the form given in eq. (1.45) and by constraining β to be purely imaginary (so that $|\alpha| = 1$). Specifically, consider

$$x[n] = e^{j\omega_0 n}. \tag{1.46}$$

As in the continuous-time case, this signal is closely related to the sinusoidal signal

$$x[n] = A\cos(\omega_0 n + \phi). \tag{1.47}$$

If we take n to be dimensionless, then both ω_0 and ϕ have units of radians. Three examples of sinusoidal sequences are shown in Figure 1.25.

As before, Euler's relation allows us to relate complex exponentials and sinusoids:

$$e^{j\omega_0 n} = \cos\omega_0 n + j\sin\omega_0 n \tag{1.48}$$

and

$$A\cos(\omega_0 n + \phi) = \frac{A}{2}e^{j\phi}e^{j\omega_0 n} + \frac{A}{2}e^{-j\phi}e^{-j\omega_0 n}. \tag{1.49}$$

The signals in eqs. (1.46) and (1.47) are examples of discrete-time signals with infinite total energy but finite average power. For example, since $|e^{j\omega_0 n}|^2 = 1$, every sample of the signal in eq. (1.46) contributes 1 to the signal's energy. Thus, the total energy for $-\infty < n < \infty$ is infinite, while the average power per time point is obviously equal to 1. Other examples of energy and power calculations for discrete-time signals are given in Problem 1.3.

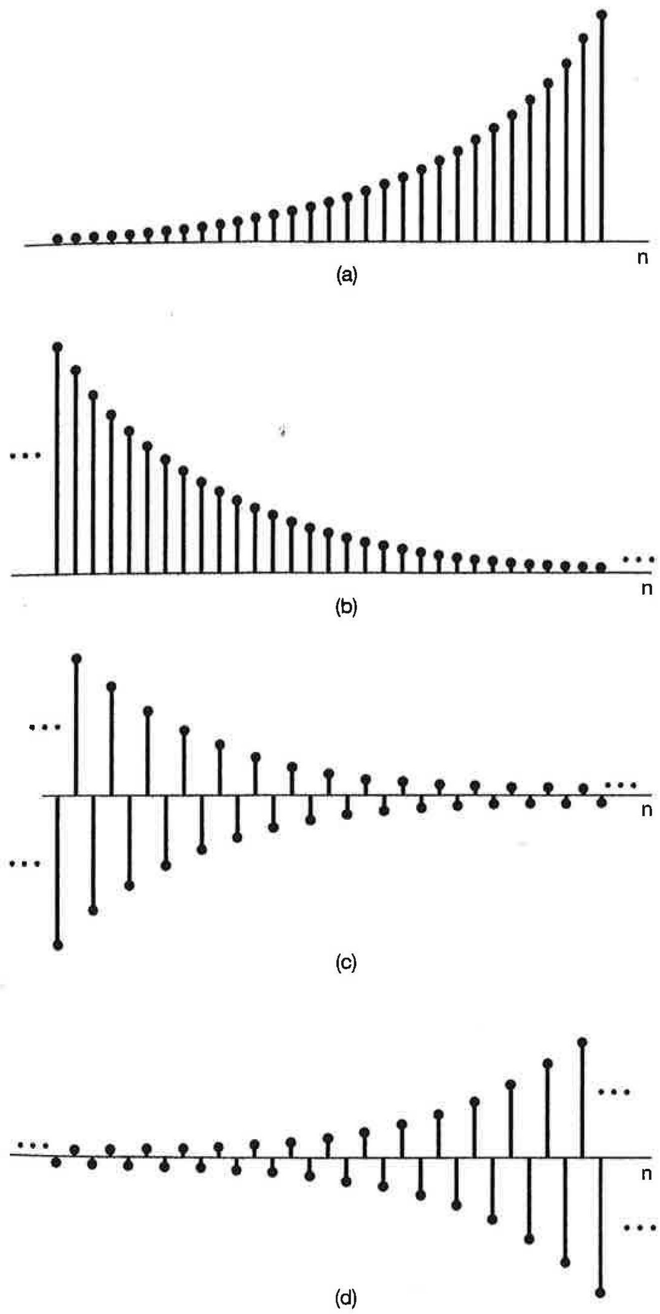


Figure 1.24 The real exponential signal $x[n] = C\alpha^n$:
(a) $\alpha > 1$; (b) $0 < \alpha < 1$;
(c) $-1 < \alpha < 0$; (d) $\alpha < -1$.

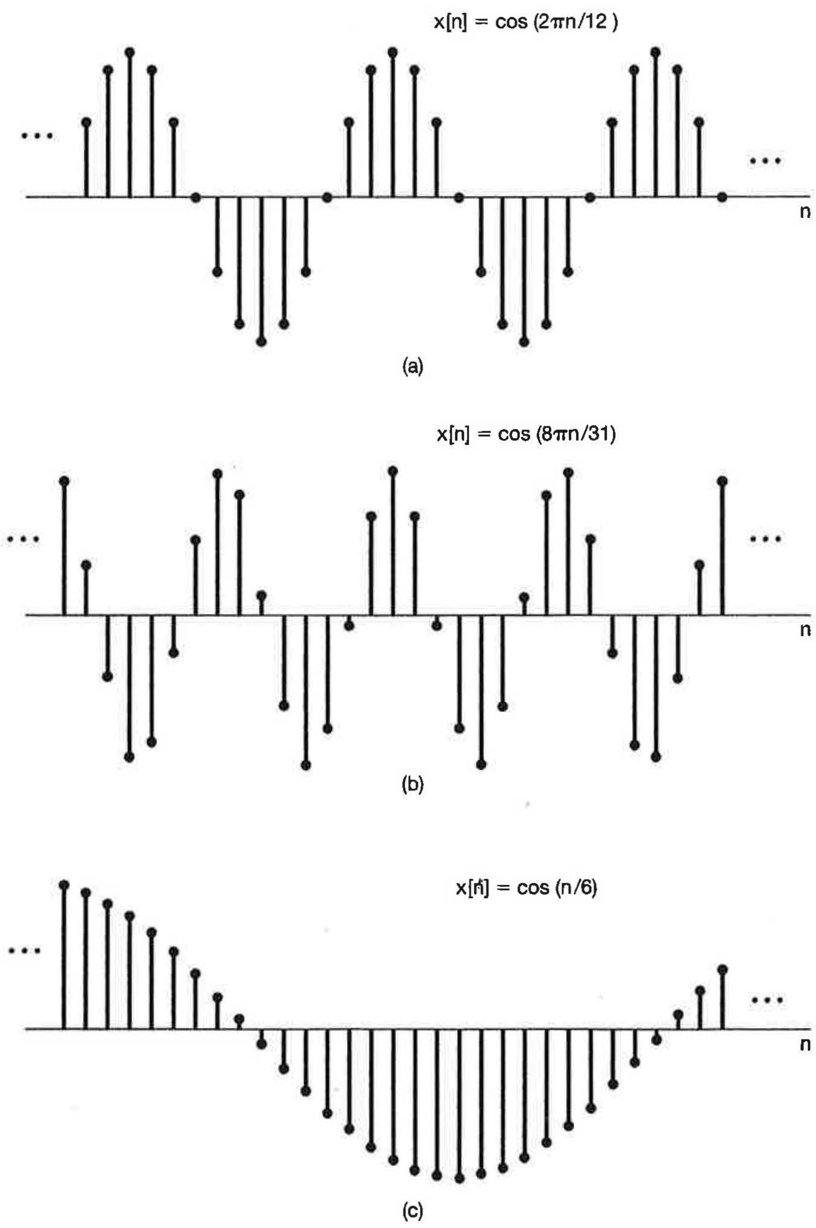


Figure 1.25 Discrete-time sinusoidal signals.

General Complex Exponential Signals

The general discrete-time complex exponential can be written and interpreted in terms of real exponentials and sinusoidal signals. Specifically, if we write C and α in polar form, viz.,

$$C = |C|e^{j\theta}$$

and

$$\alpha = |\alpha|e^{j\omega_0},$$

then

$$C\alpha^n = |C||\alpha|^n \cos(\omega_0 n + \theta) + j|C||\alpha|^n \sin(\omega_0 n + \theta). \quad (1.50)$$

Thus, for $|\alpha| = 1$, the real and imaginary parts of a complex exponential sequence are sinusoidal. For $|\alpha| < 1$ they correspond to sinusoidal sequences multiplied by a decaying exponential, while for $|\alpha| > 1$ they correspond to sinusoidal sequences multiplied by a growing exponential. Examples of these signals are depicted in Figure 1.26.

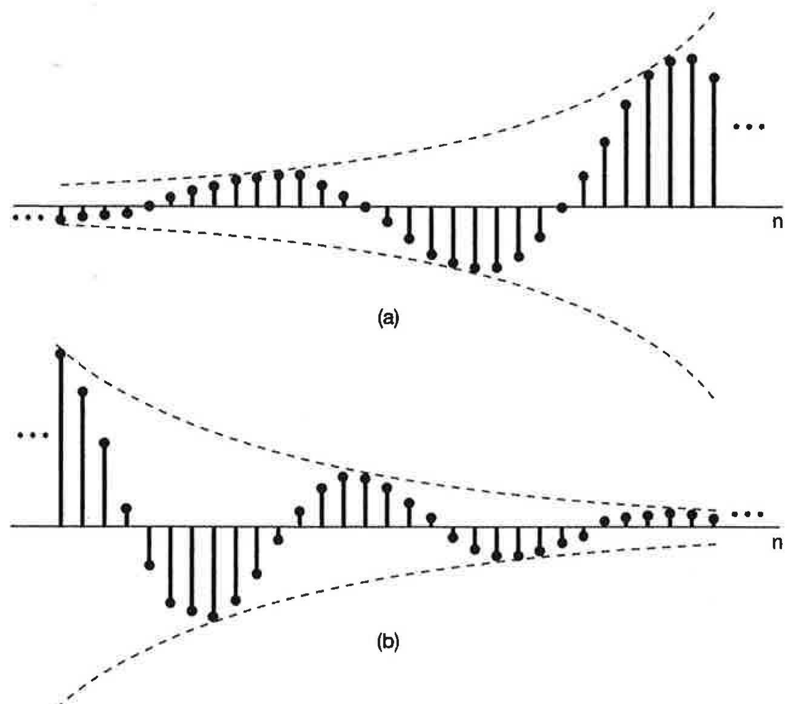


Figure 1.26 (a) Growing discrete-time sinusoidal signals; (b) decaying discrete-time sinusoid.

1.3.3 Periodicity Properties of Discrete-Time Complex Exponentials

While there are many similarities between continuous-time and discrete-time signals, there are also a number of important differences. One of these concerns the discrete-time exponential signal $e^{j\omega_0 n}$. In Section 1.3.1, we identified the following two properties of its continuous-time counterpart $e^{j\omega_0 t}$: (1) the larger the magnitude of ω_0, the higher is the rate of oscillation in the signal; and (2) $e^{j\omega_0 t}$ is periodic for any value of ω_0. In this section we describe the discrete-time versions of both of these properties, and as we will see, there are definite differences between each of these and its continuous-time counterpart.

The fact that the first of these properties is different in discrete time is a direct consequence of another extremely important distinction between discrete-time and continuous-time complex exponentials. Specifically, consider the discrete-time complex exponential with frequency $\omega_0 + 2\pi$:

$$e^{j(\omega_0+2\pi)n} = e^{j2\pi n}e^{j\omega_0 n} = e^{j\omega_0 n}. \tag{1.51}$$

From eq. (1.51), we see that the exponential at frequency $\omega_0 + 2\pi$ is the *same* as that at frequency ω_0. Thus, we have a very different situation from the continuous-time case, in which the signals $e^{j\omega_0 t}$ are all distinct for distinct values of ω_0. In discrete time, these signals are not distinct, as the signal with frequency ω_0 is identical to the signals with frequencies $\omega_0 \pm 2\pi$, $\omega_0 \pm 4\pi$, and so on. Therefore, in considering discrete-time complex exponentials, we need only consider a frequency interval of length 2π in which to choose ω_0. Although, according to eq. (1.51), any interval of length 2π will do, on most occasions we will use the interval $0 \le \omega_0 < 2\pi$ or the interval $-\pi \le \omega_0 < \pi$.

Because of the periodicity implied by eq. (1.51), the signal $e^{j\omega_0 n}$ does *not* have a continually increasing rate of oscillation as ω_0 is increased in magnitude. Rather, as illustrated in Figure 1.27, as we increase ω_0 from 0, we obtain signals that oscillate more and more rapidly until we reach $\omega_0 = \pi$. As we continue to increase ω_0, we *decrease* the rate of oscillation until we reach $\omega_0 = 2\pi$, which produces the same constant sequence as $\omega_0 = 0$. Therefore, the low-frequency (that is, slowly varying) discrete-time exponentials have values of ω_0 near 0, 2π, and any other even multiple of π, while the high frequencies (corresponding to rapid variations) are located near $\omega_0 = \pm\pi$ and other odd multiples of π. Note in particular that for $\omega_0 = \pi$ or any other odd multiple of π,

$$e^{j\pi n} = (e^{j\pi})^n = (-1)^n, \tag{1.52}$$

so that this signal oscillates rapidly, changing sign at each point in time [as illustrated in Figure 1.27(e)].

The second property we wish to consider concerns the periodicity of the discrete-time complex exponential. In order for the signal $e^{j\omega_0 n}$ to be periodic with period $N > 0$, we must have

$$e^{j\omega_0(n+N)} = e^{j\omega_0 n}, \tag{1.53}$$

or equivalently,

$$e^{j\omega_0 N} = 1. \tag{1.54}$$

For eq. (1.54) to hold, $\omega_0 N$ must be a multiple of 2π. That is, there must be an integer m such that

$$\omega_0 N = 2\pi m, \tag{1.55}$$

or equivalently,

$$\frac{\omega_0}{2\pi} = \frac{m}{N}. \tag{1.56}$$

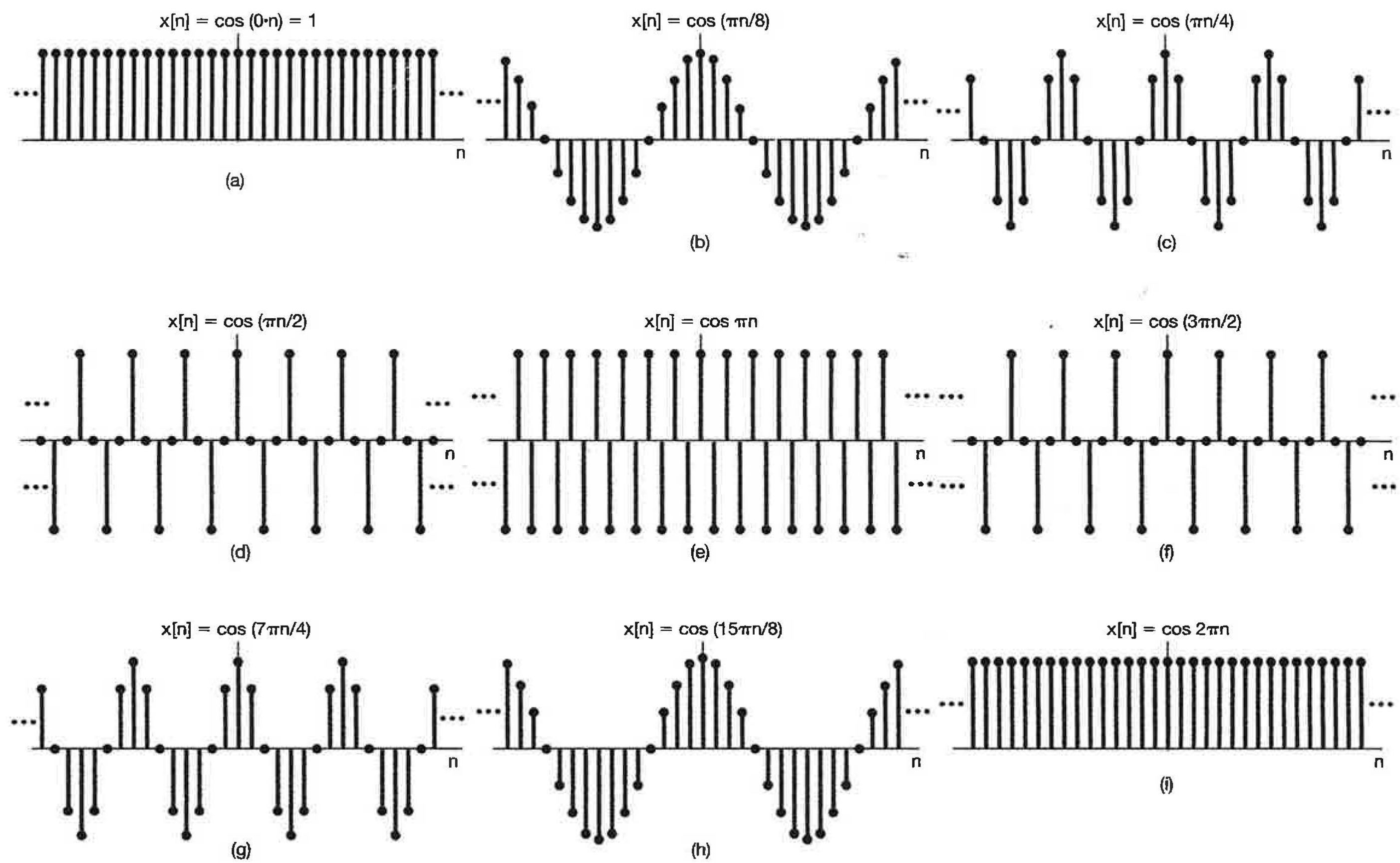


Figure 1.27 Discrete-time sinusoidal sequences for several different frequencies.

According to eq. (1.56), the signal $e^{j\omega_0 n}$ is periodic if $\omega_0/2\pi$ is a rational number and is not periodic otherwise. These same observations also hold for discrete-time sinusoids. For example, the signals depicted in Figure 1.25(a) and (b) are periodic, while the signal in Figure 1.25(c) is not.

Using the calculations that we have just made, we can also determine the fundamental period and frequency of discrete-time complex exponentials, where we define the fundamental frequency of a discrete-time periodic signal as we did in continuous time. That is, if $x[n]$ is periodic with fundamental period N, its fundamental frequency is $2\pi/N$. Consider, then, a periodic complex exponential $x[n] = e^{j\omega_0 n}$ with $\omega_0 \neq 0$. As we have just seen, ω_0 must satisfy eq. (1.56) for some pair of integers m and N, with $N > 0$. In Problem 1.35, it is shown that if $\omega_0 \neq 0$ and if N and m have no factors in common, then the fundamental period of $x[n]$ is N. Using this fact together with eq. (1.56), we find that the fundamental frequency of the periodic signal $e^{j\omega_0 n}$ is

$$\frac{2\pi}{N} = \frac{\omega_0}{m}. \tag{1.57}$$

Note that the fundamental period can also be written as

$$N = m\left(\frac{2\pi}{\omega_0}\right). \tag{1.58}$$

These last two expressions again differ from their continuous-time counterparts. In Table 1.1, we have summarized some of the differences between the continuous-time signal $e^{j\omega_0 t}$ and the discrete-time signal $e^{j\omega_0 n}$. Note that, as in the continuous-time case, the constant discrete-time signal resulting from setting $\omega_0 = 0$ has a fundamental frequency of zero, and its fundamental period is undefined.

TABLE 1.1 Comparison of the signals $e^{j\omega_0 t}$ and $e^{j\omega_0 n}$.

$e^{j\omega_0 t}$	$e^{j\omega_0 n}$
Distinct signals for distinct values of ω_0	Identical signals for values of ω_0 separated by multiples of 2π
Periodic for any choice of ω_0	Periodic only if $\omega_0 = 2\pi m/N$ for some integers $N > 0$ and m.
Fundamental frequency ω_0	Fundamental frequency* ω_0/m
Fundamental period $\omega_0 = 0$: undefined $\omega_0 \neq 0$: $\frac{2\pi}{\omega_0}$	Fundamental period* $\omega_0 = 0$: undefined $\omega_0 \neq 0$: $m\left(\frac{2\pi}{\omega_0}\right)$

*Assumes that m and N do not have any factors in common.

To gain some additional insight into these properties, let us examine again the signals depicted in Figure 1.25. First, consider the sequence $x[n] = \cos(2\pi n/12)$, depicted in Figure 1.25(a), which we can think of as the set of samples of the continuous-time sinusoid $x(t) = \cos(2\pi t/12)$ at integer time points. In this case, $x(t)$ is periodic with fundamental period 12 and $x[n]$ is also periodic with fundamental period 12. That is, the values of $x[n]$ repeat every 12 points, exactly in step with the fundamental period of $x(t)$.

In contrast, consider the signal $x[n] = \cos(8\pi n/31)$, depicted in Figure 1.25(b), which we can view as the set of samples of $x(t) = \cos(8\pi t/31)$ at integer points in time. In this case, $x(t)$ is periodic with fundamental period 31/4. On the other hand, $x[n]$ is periodic with fundamental period 31. The reason for this difference is that the discrete-time signal is defined only for integer values of the independent variable. Thus, there is no sample at time $t = 31/4$, when $x(t)$ completes one period (starting from $t = 0$). Similarly, there is no sample at $t = 2 \cdot 31/4$ or $t = 3 \cdot 31/4$, when $x(t)$ has completed two or three periods, but there is a sample at $t = 4 \cdot 31/4 = 31$, when $x(t)$ has completed *four* periods. This can be seen in Figure 1.25(b), where the pattern of $x[n]$ values does *not* repeat with each single cycle of positive and negative values. Rather, the pattern repeats after four such cycles, namely, every 31 points.

Similarly, the signal $x[n] = \cos(n/6)$ can be viewed as the set of samples of the signal $x(t) = \cos(t/6)$ at integer time points. In this case, the values of $x(t)$ at integer sample points never repeat, as these sample points never span an interval that is an exact multiple of the period, 12π, of $x(t)$. Thus, $x[n]$ is not periodic, although the eye visually interpolates between the sample points, suggesting the envelope $x(t)$, which *is* periodic. The use of the concept of sampling to gain insight into the periodicity of discrete-time sinusoidal sequences is explored further in Problem 1.36.

Example 1.6

Suppose that we wish to determine the fundamental period of the discrete-time signal

$$x[n] = e^{j(2\pi/3)n} + e^{j(3\pi/4)n}. \tag{1.59}$$

The first exponential on the right-hand side of eq. (1.59) has a fundamental period of 3. While this can be verified from eq. (1.58), there is a simpler way to obtain that answer. In particular, note that the angle $(2\pi/3)n$ of the first term must be incremented by a multiple of 2π for the values of this exponential to begin repeating. We then immediately see that if n is incremented by 3, the angle will be incremented by a single multiple of 2π. With regard to the second term, we see that incrementing the angle $(3\pi/4)n$ by 2π would require n to be incremented by 8/3, which is impossible, since n is restricted to being an integer. Similarly, incrementing the angle by 4π would require a noninteger increment of 16/3 to n. However, incrementing the angle by 6π requires an increment of 8 to n, and thus the fundamental period of the second term is 8.

Now, for the entire signal $x[n]$ to repeat, each of the terms in eq. (1.59) must go through an integer number of its own fundamental period. The smallest increment of n that accomplishes this is 24. That is, over an interval of 24 points, the first term on the right-hand side of eq. (1.59) will have gone through eight of its fundamental periods, the second term through three of its fundamental periods, and the overall signal $x[n]$ through exactly one of its fundamental periods.

As in continuous time, it is also of considerable value in discrete-time signal and system analysis to consider sets of harmonically related periodic exponentials—that is, periodic exponentials with a common period N. From eq. (1.56), we know that these are precisely the signals which are at frequencies which are multiples of $2\pi/N$. That is,

$$\phi_k[n] = e^{jk(2\pi/N)n}, \qquad k = 0, \pm 1, \ldots. \tag{1.60}$$

As the preceding examples suggest, the mathematical descriptions of systems from a wide variety of applications frequently have a great deal in common, and it is this fact that provides considerable motivation for the development of broadly applicable tools for signal and system analysis. The key to doing this successfully is identifying classes of systems that have two important characteristics: (1) The systems in this class have properties and structures that we can exploit to gain insight into their behavior and to develop effective tools for their analysis; and (2) many systems of practical importance can be accurately modeled using systems in this class. It is on the first of these characteristics that most of this book focuses, as we develop tools for a particular class of systems referred to as linear, time-invariant systems. In the next section, we will introduce the properties that characterize this class, as well as a number of other very important basic system properties.

The second characteristic mentioned in the preceding paragraph is of obvious importance for any system analysis technique to be of value in practice. It is a well-established fact that a wide range of physical systems (including those in Examples 1.8–1.10) can be well modeled within the class of systems on which we focus in this book. However, a critical point is that *any* model used in describing or analyzing a physical system represents an idealization of that system, and thus, any resulting analysis is only as good as the model itself. For example, the simple linear model of a resistor in eq. (1.80) and that of a capacitor in eq. (1.81) are idealizations. However, these idealizations are quite accurate for real resistors and capacitors in many applications, and thus, analyses employing such idealizations provide useful results and conclusions, as long as the voltages and currents remain within the operating conditions under which these simple linear models are valid. Similarly, the use of a linear retarding force to represent frictional effects in eq. (1.83) is an approximation with a range of validity. Consequently, although we will not address this issue in the book, it is important to remember that an essential component of engineering practice in using the methods we develop here consists of identifying the range of validity of the assumptions that have gone into a model and ensuring that any analysis or design based on that model does not violate those assumptions.

1.5.2 Interconnections of Systems

An important idea that we will use throughout this book is the concept of the interconnection of systems. Many real systems are built as interconnections of several subsystems. One example is an audio system, which involves the interconnection of a radio receiver, compact disc player, or tape deck with an amplifier and one or more speakers. Another is a digitally controlled aircraft, which is an interconnection of the aircraft, described by its equations of motion and the aerodynamic forces affecting it; the sensors, which measure various aircraft variables such as accelerations, rotation rates, and heading; a digital autopilot, which responds to the measured variables and to command inputs from the pilot (e.g., the desired course, altitude, and speed); and the aircraft's actuators, which respond to inputs provided by the autopilot in order to use the aircraft control surfaces (rudder, tail, ailerons) to change the aerodynamic forces on the aircraft. By viewing such a system as an interconnection of its components, we can use our understanding of the component

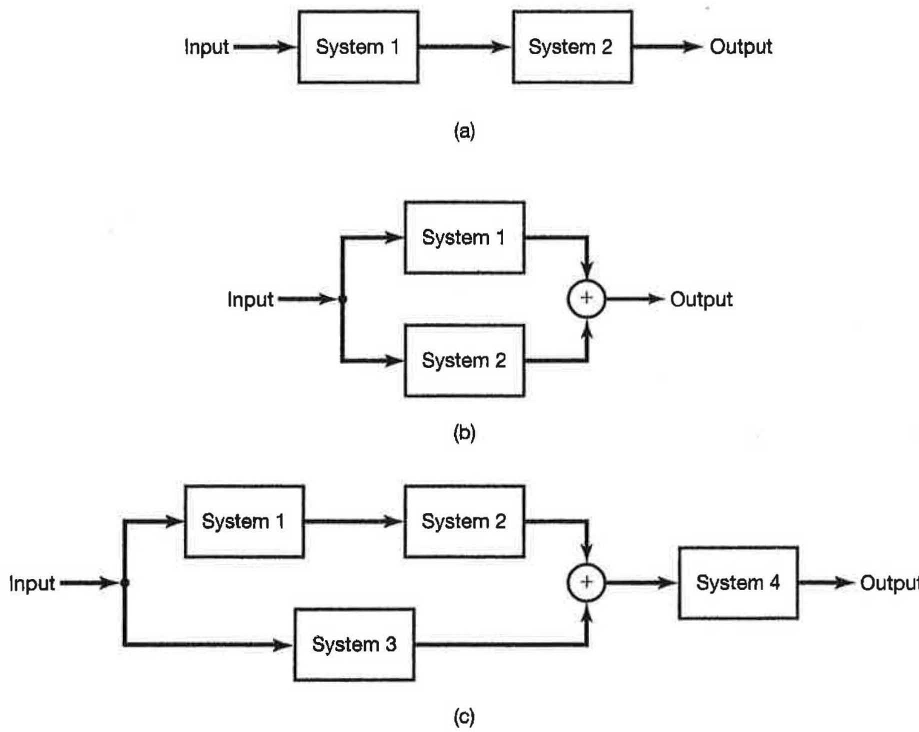


Figure 1.42 Interconnection of two systems: (a) series (cascade) interconnection; (b) parallel interconnection; (c) series-parallel interconnection.

systems and of how they are interconnected in order to analyze the operation and behavior of the overall system. In addition, by describing a system in terms of an interconnection of simpler subsystems, we may in fact be able to define useful ways in which to synthesize complex systems out of simpler, basic building blocks.

While one can construct a variety of system interconnections, there are several basic ones that are frequently encountered. A *series* or *cascade interconnection* of two systems is illustrated in Figure 1.42(a). Diagrams such as this are referred to as *block diagrams*. Here, the output of System 1 is the input to System 2, and the overall system transforms an input by processing it first by System 1 and then by System 2. An example of a series interconnection is a radio receiver followed by an amplifier. Similarly, one can define a series interconnection of three or more systems.

A *parallel interconnection* of two systems is illustrated in Figure 1.42(b). Here, the same input signal is applied to Systems 1 and 2. The symbol "$\oplus$" in the figure denotes addition, so that the output of the parallel interconnection is the sum of the outputs of Systems 1 and 2. An example of a parallel interconnection is a simple audio system with several microphones feeding into a single amplifier and speaker system. In addition to the simple parallel interconnection in Figure 1.42(b), we can define parallel interconnections of more than two systems, and we can combine both cascade and parallel interconnections

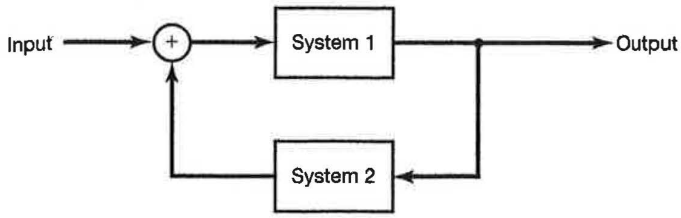
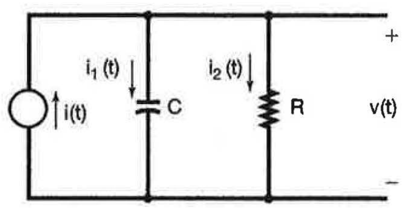


Figure 1.43 Feedback interconnection.

to obtain more complicated interconnections. An example of such an interconnection is given in Figure 1.42(c).[4]

Another important type of system interconnection is a *feedback interconnection,* an example of which is illustrated in Figure 1.43. Here, the output of System 1 is the input to System 2, while the output of System 2 is fed back and added to the external input to produce the actual input to System 1. Feedback systems arise in a wide variety of applications. For example, a cruise control system on an automobile senses the vehicle's velocity and adjusts the fuel flow in order to keep the speed at the desired level. Similarly, a digitally controlled aircraft is most naturally thought of as a feedback system in which differences between actual and desired speed, heading, or altitude are fed back through the autopilot in order to correct these discrepancies. Also, electrical circuits are often usefully viewed as containing feedback interconnections. As an example, consider the circuit depicted in Figure 1.44(a). As indicated in Figure 1.44(b), this system can be viewed as the feedback interconnection of the two circuit elements.



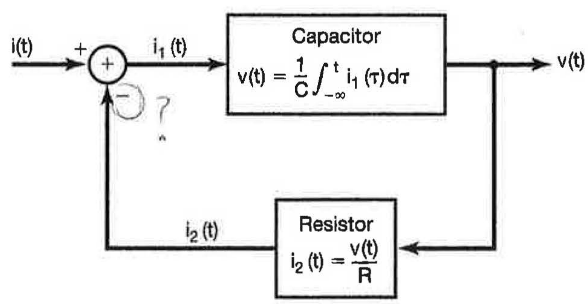


Figure 1.44 (a) Simple electrical circuit; (b) block diagram in which the circuit is depicted as the feedback interconnection of two circuit elements.

[4]On occasion, we will also use the symbol ⊗ in our pictorial representation of systems to denote the operation of multiplying two signals (see, for example, Figure 4.26).

1.6 BASIC SYSTEM PROPERTIES

In this section we introduce and discuss a number of basic properties of continuous-time and discrete-time systems. These properties have important physical interpretations and relatively simple mathematical descriptions using the signals and systems language that we have begun to develop.

1.6.1 Systems with and without Memory

A system is said to be *memoryless* if its output for each value of the independent variable at a given time is dependent on the input at only that same time. For example, the system specified by the relationship

$$y[n] = (2x[n] - x^2[n])^2 \tag{1.90}$$

is memoryless, as the value of $y[n]$ at any particular time n_0 depends only on the value of $x[n]$ at that time. Similarly, a resistor is a memoryless system; with the input $x(t)$ taken as the current and with the voltage taken as the output $y(t)$, the input-output relationship of a resistor is

$$y(t) = Rx(t), \tag{1.91}$$

where R is the resistance. One particularly simple memoryless system is the *identity system,* whose output is identical to its input. That is, the input-output relationship for the continuous-time identity system is

$$y(t) = x(t),$$

and the corresponding relationship in discrete time is

$$y[n] = x[n].$$

An example of a discrete-time system with memory is an *accumulator* or *summer*

$$y[n] = \sum_{k=-\infty}^{n} x[k], \tag{1.92}$$

and a second example is a *delay*

$$y[n] = x[n-1]. \tag{1.93}$$

A capacitor is an example of a continuous-time system with memory, since if the input is taken to be the current and the output is the voltage, then

$$y(t) = \frac{1}{C}\int_{-\infty}^{t} x(\tau)\,d\tau, \tag{1.94}$$

where C is the capacitance.

Roughly speaking, the concept of memory in a system corresponds to the presence of a mechanism in the system that retains or stores information about input values at times other than the current time. For example, the delay in eq. (1.93) must retain or store the preceding value of the input. Similarly, the accumulator in eq. (1.92) must "remember" or store information about past inputs. In particular, the accumulator computes the running sum of all inputs up to the current time, and thus, at each instant of time, the accumulator must add the current input value to the preceding value of the running sum. In other words, the relationship between the input and output of an accumulator can be described as

$$y[n] = \sum_{k=-\infty}^{n-1} x[k] + x[n], \tag{1.95}$$

or equivalently,

$$y[n] = y[n-1] + x[n]. \tag{1.96}$$

Represented in the latter way, to obtain the output at the current time n, the accumulator must remember the running sum of previous input values, which is exactly the preceding value of the accumulator output.

In many physical systems, memory is directly associated with the storage of energy. For example, the capacitor in eq. (1.94) stores energy by accumulating electrical charge, represented as the integral of the current. Thus, the simple *RC* circuit in Example 1.8 and Figure 1.1 has memory physically stored in the capacitor. Similarly, the automobile in Figure 1.2 has memory stored in its kinetic energy. In discrete-time systems implemented with computers or digital microprocessors, memory is typically directly associated with storage registers that retain values between clock pulses.

While the concept of memory in a system would typically suggest storing *past* input and output values, our formal definition also leads to our referring to a system as having memory if the current output is dependent on *future* values of the input and output. While systems having this dependence on future values might at first seem unnatural, they in fact form an important class of systems, as we discuss further in Section 1.6.3.

1.6.2 Invertibility and Inverse Systems

A system is said to be *invertible* if distinct inputs lead to distinct outputs. As illustrated in Figure 1.45(a) for the discrete-time case, if a system is invertible, then an *inverse system* exists that, when cascaded with the original system, yields an output $w[n]$ equal to the input $x[n]$ to the first system. Thus, the series interconnection in Figure 1.45(a) has an overall input-output relationship which is the same as that for the identity system.

An example of an invertible continuous-time system is

$$y(t) = 2x(t), \tag{1.97}$$

for which the inverse system is

$$w(t) = \frac{1}{2}y(t). \tag{1.98}$$

This example is illustrated in Figure 1.45(b). Another example of an invertible system is the accumulator of eq. (1.92). For this system, the difference between two successive

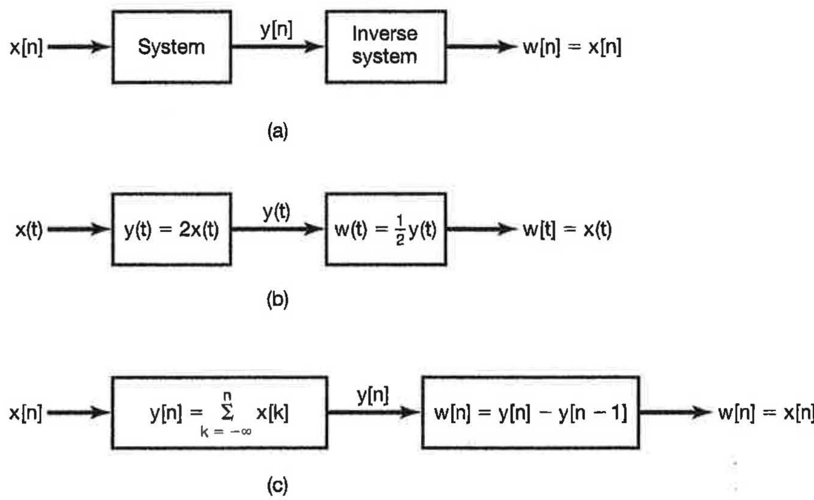


Figure 1.45 Concept of an inverse system for: (a) a general invertible system; (b) the invertible system described by eq. (1.97); (c) the invertible system defined in eq. (1.92).

values of the output is precisely the last input value. Therefore, in this case, the inverse system is

$$w[n] = y[n] - y[n-1], \tag{1.99}$$

as illustrated in Figure 1.45(c). Examples of noninvertible systems are

$$y[n] = 0, \tag{1.100}$$

that is, the system that produces the zero output sequence for any input sequence, and

$$y(t) = x^2(t), \tag{1.101}$$

in which case we cannot determine the sign of the input from knowledge of the output.

The concept of invertibility is important in many contexts. One example arises in systems for encoding used in a wide variety of communications applications. In such a system, a signal that we wish to transmit is first applied as the input to a system known as an encoder. There are many reasons for doing this, ranging from the desire to encrypt the original message for secure or private communication to the objective of providing some redundancy in the signal (for example, by adding what are known as parity bits) so that any errors that occur in transmission can be detected and, possibly, corrected. For *lossless* coding, the input to the encoder must be exactly recoverable from the output; i.e., the encoder must be invertible.

1.6.3 Causality

A system is *causal* if the output at any time depends on values of the input at only the present and past times. Such a system is often referred to as being *nonanticipative*, as the system output does not anticipate future values of the input. Consequently, if two inputs to a causal system are identical up to some point in time t_0 or n_0, the corresponding outputs must also be equal up to this same time. The *RC* circuit of Figure 1.1 is causal, since the capacitor voltage responds only to the present and past values of the source voltage. Similarly, the motion of an automobile is causal, since it does not anticipate future actions of the driver. The systems described in eqs. (1.92) – (1.94) are also causal, but the systems defined by

$$y[n] = x[n] - x[n+1] \tag{1.102}$$

and

$$y(t) = x(t+1) \tag{1.103}$$

are not. All memoryless systems are causal, since the output responds only to the current value of the input.

Although causal systems are of great importance, they do not by any means constitute the only systems that are of practical significance. For example, causality is not often an essential constraint in applications in which the independent variable is not time, such as in image processing. Furthermore, in processing data that have been recorded previously, as often happens with speech, geophysical, or meteorological signals, to name a few, we are by no means constrained to causal processing. As another example, in many applications, including historical stock market analysis and demographic studies, we may be interested in determining a slowly varying trend in data that also contain high-frequency fluctuations about that trend. In this case, a commonly used approach is to average data over an interval in order to smooth out the fluctuations and keep only the trend. An example of a noncausal averaging system is

$$y[n] = \frac{1}{2M+1} \sum_{k=-M}^{+M} x[n-k]. \tag{1.104}$$

Example 1.12

When checking the causality of a system, it is important to look carefully at the input-output relation. To illustrate some of the issues involved in doing this, we will check the causality of two particular systems.

The first system is defined by

$$y[n] = x[-n]. \tag{1.105}$$

Note that the output $y[n_0]$ at a positive time n_0 depends only on the value of the input signal $x[-n_0]$ at time $(-n_0)$, which is negative and therefore in the past of n_0. We may be tempted to conclude at this point that the given system is causal. However, we should always be careful to check the input-output relation for *all* times. In particular, for $n < 0$, e.g. $n = -4$, we see that $y[-4] = x[4]$, so that the output at this time depends on a future value of the input. Hence, the system is not causal.

It is also important to distinguish carefully the effects of the input from those of any other functions used in the definition of the system. For example, consider the system

$$y(t) = x(t)\cos(t+1). \tag{1.106}$$

If we write

$$\frac{1}{1 - \alpha e^{-j2\pi/N}} = re^{j\theta},$$

then eq. (3.135) reduces to

$$y[n] = r\cos\left(\frac{2\pi}{N}n + \theta\right). \qquad (3.136)$$

For example, if $N = 4$,

$$\frac{1}{1 - \alpha e^{-j2\pi/4}} = \frac{1}{1 + \alpha j} = \frac{1}{\sqrt{1 + \alpha^2}} e^{j(-\tan^{-1}(\alpha))},$$

and thus,

$$y[n] = \frac{1}{\sqrt{1 + \alpha^2}} \cos\left(\frac{\pi n}{2} - \tan^{-1}(\alpha)\right).$$

We note that for expressions such as eqs. (3.124) and (3.131) to make sense, the frequency responses $H(j\omega)$ and $H(e^{j\omega})$ in eqs. (3.121) and (3.122) must be well defined and finite. As we will see in Chapters 4 and 5, this will be the case if the LTI systems under consideration are stable. For example, the LTI system in Example 3.16, with impulse response $h(t) = e^{-t}u(t)$, is stable and has a well-defined frequency response given by eq. (3.125). On the other hand, an LTI system with impulse response $h(t) = e^{t}u(t)$ is unstable, and it is easy to check that the integral in eq. (3.121) for $H(j\omega)$ diverges for any value of ω. Similarly, the LTI system in Example 3.17, with impulse response $h[n] = \alpha^n u[n]$, is stable for $|\alpha| < 1$ and has frequency response given by eq. (3.134). However, if $|\alpha| > 1$, the system is unstable, and then the summation in eq. (3.133) diverges.

3.9 FILTERING

In a variety of applications, it is of interest to change the relative amplitudes of the frequency components in a signal or perhaps eliminate some frequency components entirely, a process referred to as *filtering*. Linear time-invariant systems that change the shape of the spectrum are often referred to as *frequency-shaping filters*. Systems that are designed to pass some frequencies essentially undistorted and significantly attenuate or eliminate others are referred to as *frequency-selective filters*. As indicated by eqs. (3.124) and (3.131), the Fourier series coefficients of the output of an LTI system are those of the input multiplied by the frequency response of the system. Consequently, filtering can be conveniently accomplished through the use of LTI systems with an appropriately chosen frequency response, and frequency-domain methods provide us with the ideal tools to examine this very important class of applications. In this and the following two sections, we take a first look at filtering through a few examples.

3.9.1 Frequency-Shaping Filters

One application in which frequency-shaping filters are often encountered is audio systems. For example, LTI filters are typically included in such systems to permit the listener to modify the relative amounts of low-frequency energy (bass) and high-frequency energy (treble). These filters correspond to LTI systems whose frequency responses can be changed by manipulating the tone controls. Also, in high-fidelity audio systems, a so-called equalizing filter is often included in the preamplifier to compensate for the frequency-response characteristics of the speakers. Overall, these cascaded filtering stages are frequently referred to as the equalizing or equalizer circuits for the audio system. Figure 3.22 illustrates the three stages of the equalizer circuits for one particular series of audio speakers. In this figure, the magnitude of the frequency response for each of these stages is shown on a log-log plot. Specifically, the magnitude is in units of $20\log_{10}|H(j\omega)|$, referred to as decibels or dB. The frequency axis is labeled in Hz (i.e., $\omega/2\pi$) along a logarithmic scale. As will be discussed in more detail in Section 6.2.3, a logarithmic display of the magnitude of the frequency response in this form is common and useful.

Taken together, the equalizing circuits in Figure 3.22 are designed to compensate for the frequency response of the speakers and the room in which they are located and to allow the listener to control the overall frequency response. In particular, since the three systems are connected in cascade, and since each system modifies a complex exponential input $Ke^{j\omega t}$ by multiplying it by the system frequency response at that frequency, it follows that the overall frequency response of the cascade of the three systems is the product of the three frequency responses. The first two filters, indicated in Figures 3.22(a) and (b), together make up the control stage of the system, as the frequency behavior of these filters can be adjusted by the listener. The third filter, illustrated in Figure 3.22(c), is the equalizer stage, which has the fixed frequency response indicated. The filter in Figure 3.22(a) is a low-frequency filter controlled by a two-position switch, to provide one of the two frequency responses indicated. The second filter in the control stage has two continuously adjustable slider switches to vary the frequency response within the limits indicated in Figure 3.22(b).

Another class of frequency-shaping filters often encountered is that for which the filter output is the derivative of the filter input, i.e., $y(t) = dx(t)/dt$. With $x(t)$ of the form $x(t) = e^{j\omega t}$, $y(t)$ will be $y(t) = j\omega e^{j\omega t}$, from which it follows that the frequency response is

$$H(j\omega) = j\omega. \tag{3.137}$$

The frequency response characteristics of a differentiating filter are shown in Figure 3.23. Since $H(j\omega)$ is complex in general, and in this example in particular, $H(j\omega)$ is frequently displayed (as in the figure) as separate plots of $|H(j\omega)|$ and $\sphericalangle H(j\omega)$. The shape of this frequency response implies that a complex exponential input $e^{j\omega t}$ will receive greater amplification for larger values of ω. Consequently, differentiating filters are useful in enhancing rapid variations or transitions in a signal.

One purpose for which differentiating filters are often used is to enhance edges in picture processing. A black-and-white picture can be thought of as a two-dimensional "continuous-time" signal $x(t_1, t_2)$, where t_1 and t_2 are the horizontal and vertical coordinates, respectively, and $x(t_1, t_2)$ is the brightness of the image. If the image is repeated periodically in the horizontal and vertical directions, then it can be represented by a two-dimensional Fourier series (see Problem 3.70) consisting of sums of products of complex

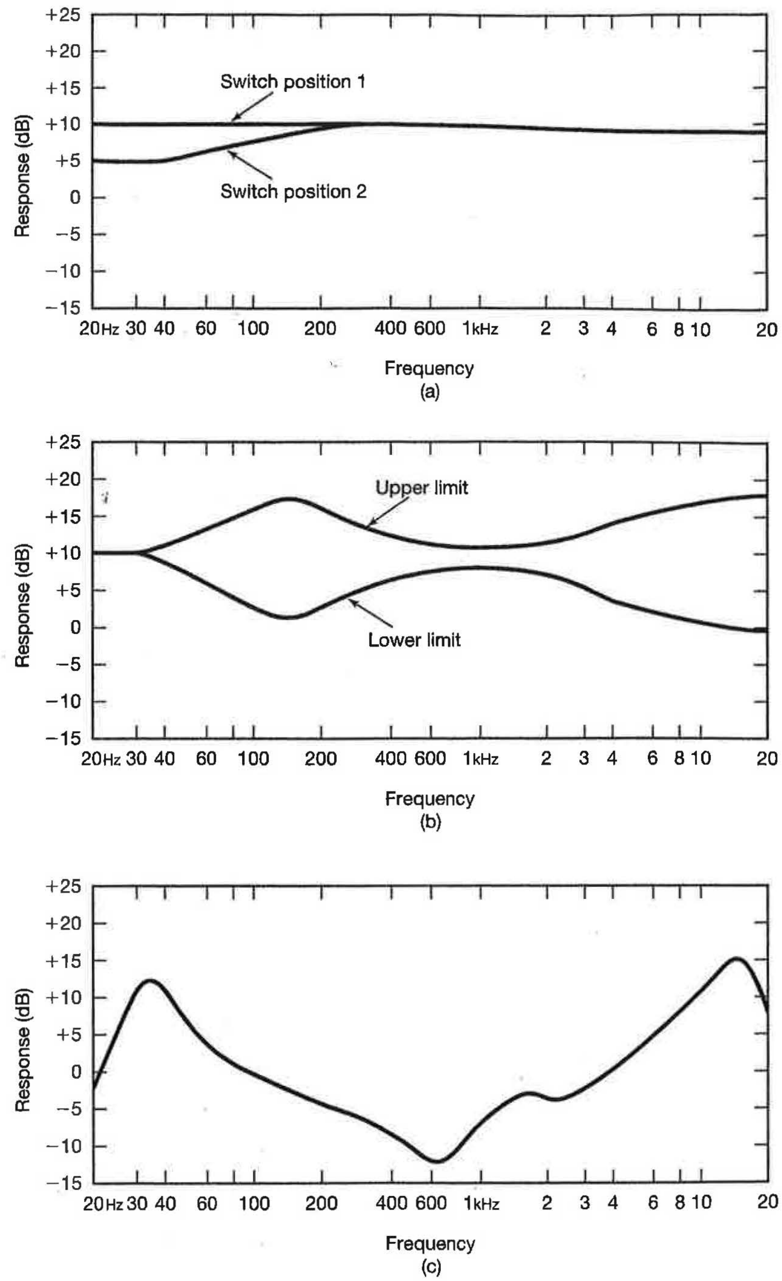


Figure 3.22 Magnitudes of the frequency responses of the equalizer circuits for one particular series of audio speakers, shown on a scale of $20\log_{10}|H(j\omega)|$, which is referred to as a decibel (or dB) scale. (a) Low-frequency filter controlled by a two-position switch; (b) upper and lower frequency limits on a continuously adjustable shaping filter; (c) fixed frequency response of the equalizer stage.

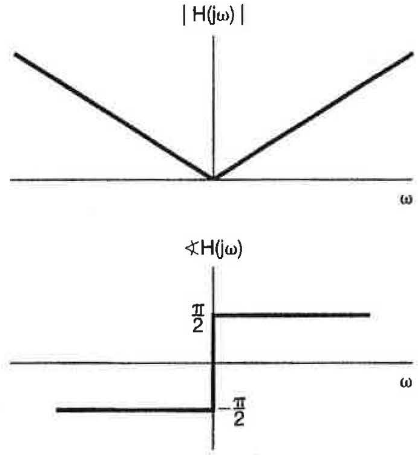


Figure 3.23 Characteristics of the frequency response of a filter for which the output is the derivative of the input.

exponentials, $e^{j\omega_1 t_1}$ and $e^{j\omega_2 t_2}$, that oscillate at possibly different frequencies in each of the two coordinate directions. Slow variations in brightness in a particular direction are represented by the lower harmonics in that direction. For example, consider an edge corresponding to a sharp transition in brightness that runs vertically in an image. Since the brightness is constant or slowly varying along the edge, the frequency content of the edge in the vertical direction is concentrated at low frequencies. In contrast, since there is an abrupt variation in brightness across the edge, the frequency content of the edge in the horizontal direction is concentrated at higher frequencies. Figure 3.24 illustrates the effect on an image of the two-dimensional equivalent of a differentiating filter.[11] Figure 3.24(a) shows two original images and Figure 3.24(b) the result of processing those images with the filter. Since the derivative at the edges of a picture is greater than in regions where the brightness varies slowly with distance, the effect of the filter is to enhance the edges.

Discrete-time LTI filters also find a broad array of applications. Many of these involve the use of discrete-time systems, implemented using general- or special-purpose digital processors, to process continuous-time signals, a topic we discuss at some length in Chapter 7. In addition, the analysis of time series information, including demographic data and economic data sequences such as the stock market average, commonly involves the use of discrete-time filters. Often the long-term variations (which correspond to low frequencies) have a different significance than the short-term variations (which correspond to high frequencies), and it is useful to analyze these components separately. Reshaping the relative weighting of the components is typically accomplished using discrete-time filters.

As one example of a simple discrete-time filter, consider an LTI system that successively takes a two-point average of the input values:

$$y[n] = \frac{1}{2}(x[n] + x[n-1]). \tag{3.138}$$

[11]Specifically each image in Figure 3.24(b) is the magnitude of the two-dimensional gradient of its counterpart image in Figure 3.24(a) where the magnitude of the gradient of $f(x, y)$ is

$$\left[\left(\frac{\partial f(x,y)}{\partial x}\right)^2 + \left(\frac{\partial f(x,y)}{\partial y}\right)^2\right]^{1/2}$$

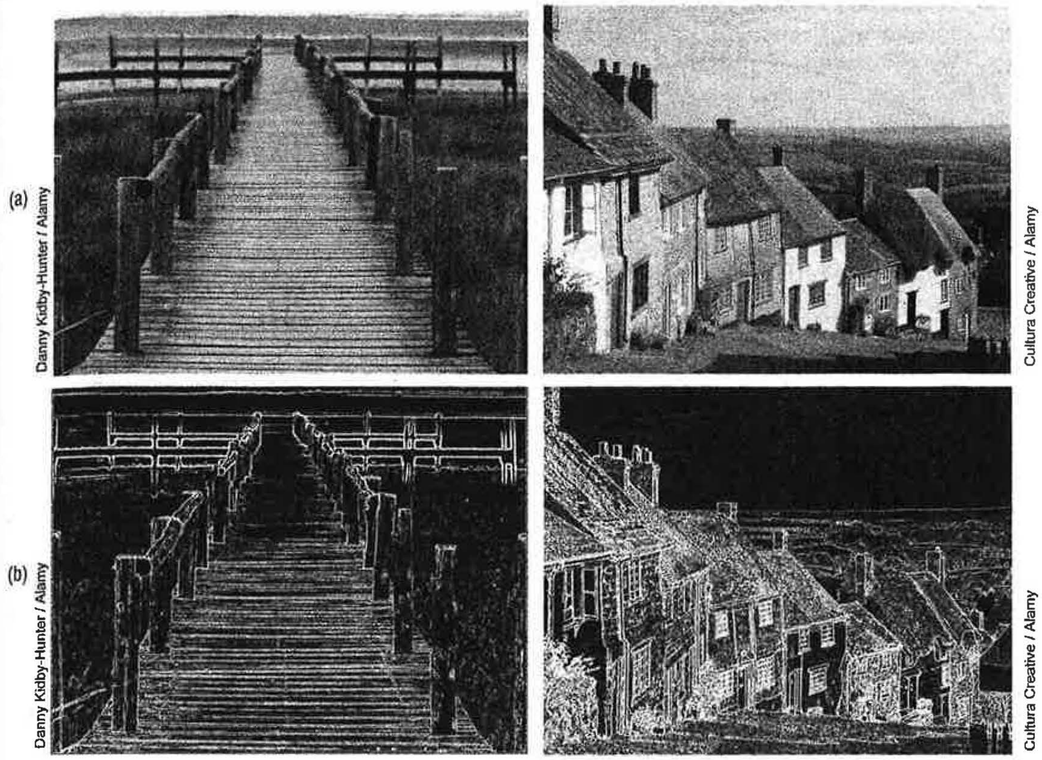


Figure 3.24 Effect of a differentiating filter on an image: (a) two original images; (b) the result of processing the original images with a differentiating filter.

In this case $h[n] = \frac{1}{2}(\delta[n] + \delta[n-1])$, and from eq. (3.122), we see that the frequency response of the system is

$$H(e^{j\omega}) = \frac{1}{2}[1 + e^{-j\omega}] = e^{-j\omega/2}\cos(\omega/2). \tag{3.139}$$

The magnitude of $H(e^{j\omega})$ is plotted in Figure 3.25(a), and $\sphericalangle H(e^{j\omega})$ is shown in Figure 3.25(b). As discussed in Section 1.3.3, low frequencies for discrete-time complex exponentials occur near $\omega = 0, \pm 2\pi, \pm 4\pi, \ldots$, and high frequencies near $\omega = \pm\pi, \pm 3\pi, \ldots$. This is a result of the fact that $e^{j(\omega+2\pi)n} = e^{j\omega n}$, so that in discrete time we need only consider a 2π interval of values of ω in order to cover a complete range of distinct discrete-time frequencies. As a consequence, any discrete-time frequency responses $H(e^{j\omega})$ must be periodic with period 2π, a fact that can also be deduced directly from eq. (3.122).

For the specific filter defined in eqs. (3.138) and (3.139), we see from Figure 3.25(a) that $|H(e^{j\omega})|$ is large for frequencies near $\omega = 0$ and decreases as we increase $|\omega|$ toward π, indicating that higher frequencies are attentuated more than lower ones. For example, if the input to this system is constant—i.e., a zero-frequency complex exponential

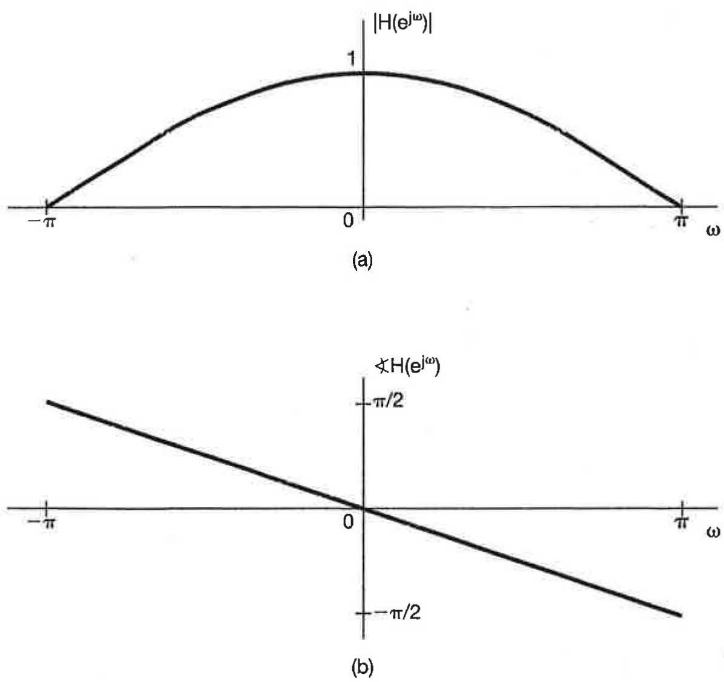


Figure 3.25 (a) Magnitude and (b) phase for the frequency response of the discrete-time LTI system $y[n] = 1/2(x[n] + x[n-1])$.

$x[n] = Ke^{j0\cdot n} = K$—then the output will be

$$y[n] = H(e^{j\cdot 0})Ke^{j\omega 0\cdot n} = K = x[n].$$

On the other hand, if the input is the high-frequency signal $x[n] = Ke^{j\pi n} = K(-1)^n$, then the output will be

$$y[n] = H(e^{j\pi})Ke^{j\pi\cdot n} = 0.$$

Thus, this system separates out the long-term constant value of a signal from its high-frequency fluctuations and, consequently, represents a first example of frequency-selective filtering, a topic we look at more carefully in the next subsection.

3.9.2 Frequency-Selective Filters

Frequency-selective filters are a class of filters specifically intended to accurately or approximately select some bands of frequencies and reject others. The use of frequency-selective filters arises in a variety of situations. For example, if noise in an audio recording is in a higher frequency band than the music or voice on the recording is, it can be removed by frequency-selective filtering. Another important application of frequency-selective filters is in communication systems. As we discuss in detail in Chapter 8, the basis for amplitude modulation (AM) systems is the transmission of information from many different sources simultaneously by putting the information from each channel into a separate frequency band and extracting the individual channels or bands at the receiver using frequency-selective filters. Frequency-selective filters for separating the individual channels and frequency-shaping filters (such as the equalizer illustrated in Figure 3.22) for adjusting the quality of the tone form a major part of any home radio and television receiver.

While frequency selectivity is not the only issue of concern in applications, its broad importance has led to a widely accepted set of terms describing the characteristics of frequency-selective filters. In particular, while the nature of the frequencies to be passed by a frequency-selective filter varies considerably from application to application, several basic types of filter are widely used and have been given names indicative of their function. For example, a *lowpass filter* is a filter that passes low frequencies—i.e., frequencies around $\omega = 0$—and attenuates or rejects higher frequencies. A *highpass filter* is a filter that passes high frequencies and attentuates or rejects low ones, and a *bandpass filter* is a filter that passes a band of frequencies and attenuates frequencies both higher and lower than those in the band that is passed. In each case, the *cutoff frequencies* are the frequencies defining the boundaries between frequencies that are passed and frequencies that are rejected—i.e., the frequencies in the *passband* and *stopband.*

Numerous questions arise in defining and assessing the quality of a frequency-selective filter. How effective is the filter at passing frequencies in the passband? How effective is it at attentuating frequencies in the stopband? How sharp is the transition near the cutoff frequency—i.e., from nearly free of distortion in the passband to highly attenuated in the stopband? Each of these questions involves a comparison of the characteristics of an actual frequency-selective filter with those of a filter with idealized behavior. Specifically, an *ideal frequency-selective filter* is a filter that exactly passes complex exponentials at one set of frequencies without any distortion and completely rejects signals at all other frequencies. For example, a continuous-time *ideal lowpass filter* with cutoff frequency ω_c is an LTI system that passes complex exponentials $e^{j\omega t}$ for values of ω in the range $-\omega_c \leq \omega \leq \omega_c$ and rejects signals at all other frequencies. That is, the frequency response of a continuous-time ideal lowpass filter is

$$H(j\omega) = \begin{cases} 1, & |\omega| \leq \omega_c \\ 0, & |\omega| > \omega_c \end{cases}, \qquad (3.140)$$

as shown in Figure 3.26.

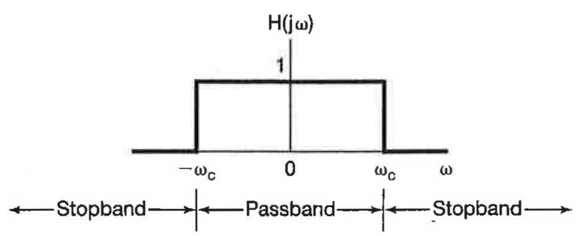


Figure 3.26 Frequency response of an ideal lowpass filter.

Figure 3.27(a) depicts the frequency response of an ideal continuous-time highpass filter with cutoff frequency ω_c, and Figure 3.27(b) illustrates an ideal continuous-time bandpass filter with lower cutoff frequency ω_{c1} and upper cutoff frequency ω_{c2}. Note that each of these filters is symmetric about $\omega = 0$, and thus, there appear to be two passbands for the highpass and bandpass filters. This is a consequence of our having adopted the

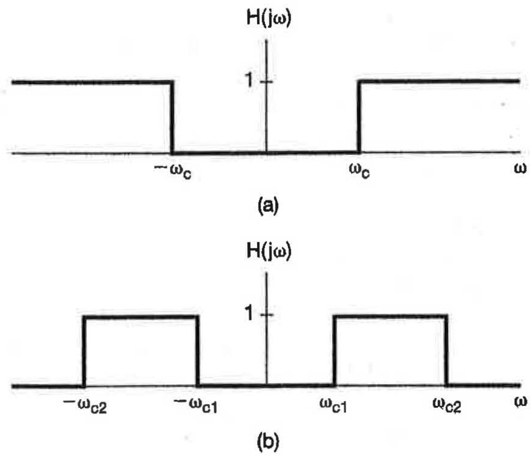


Figure 3.27 (a) Frequency response of an ideal highpass filter; (b) frequency response of an ideal bandpass filter.

use of the complex exponential signal $e^{j\omega t}$, rather than the sinusoidal signals $\sin \omega t$ and $\cos \omega t$, at frequency ω. Since $e^{j\omega t} = \cos \omega t + j \sin \omega t$ and $e^{-j\omega t} = \cos \omega t - j \sin \omega t$, both of these complex exponentials are composed of sinusoidal signals at the same frequency ω. For this reason, we usually define ideal filters so that they have the symmetric frequency response behavior seen in Figures 3.26 and 3.27.

In a similar fashion, we can define the corresponding set of ideal discrete-time frequency-selective filters, the frequency responses for which are depicted in Figure 3.28.

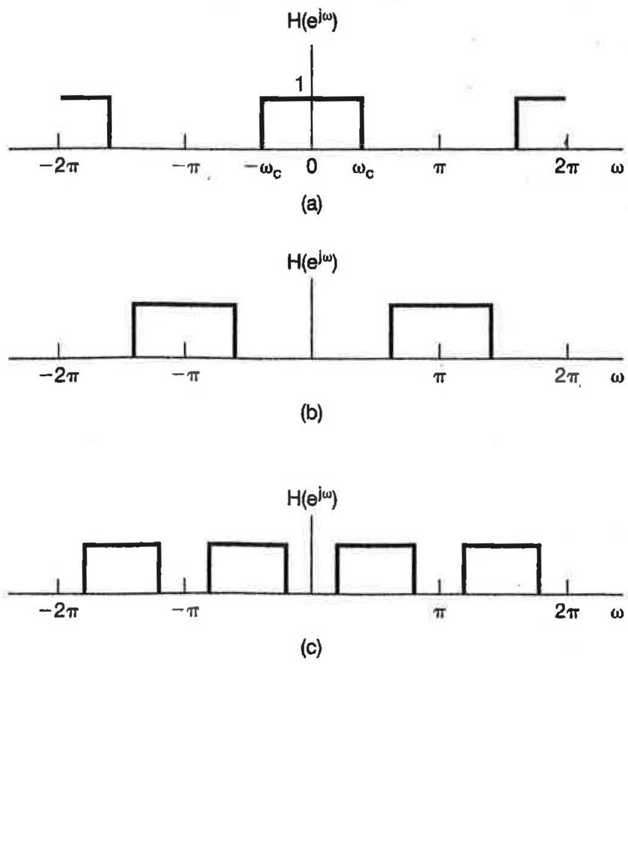


Figure 3.28 Discrete-time ideal frequency-selective filters: (a) lowpass; (b) highpass; (c) bandpass.

In particular, Figure 3.28(a) depicts an ideal discrete-time lowpass filter, Figure 3.28(b) is an ideal highpass filter, and Figure 3.28(c) is an ideal bandpass filter. Note that, as discussed in the preceding section, the characteristics of the continuous-time and discrete-time ideal filters differ by virtue of the fact that, for discrete-time filters, the frequency response $H(e^{j\omega})$ must be periodic with period 2π, with low frequencies near even multiples of π and high frequencies near odd multiples of π.

As we will see on numerous occasions, ideal filters are quite useful in describing idealized system configurations for a variety of applications. However, they are not realizable in practice and must be approximated. Furthermore, even if they could be realized, some of the characteristics of ideal filters might make them undesirable for particular applications, and a nonideal filter might in fact be preferable.

In detail, the topic of filtering encompasses many issues, including design and implementation. While we will not delve deeply into the details of filter design methodologies, in the remainder of this chapter and the following chapters we will see a number of other examples of both continuous-time and discrete-time filters and will develop the concepts and techniques that form the basis of this very important engineering discipline.

3.10 EXAMPLES OF CONTINUOUS-TIME FILTERS DESCRIBED BY DIFFERENTIAL EQUATIONS

In many applications, frequency-selective filtering is accomplished through the use of LTI systems described by linear constant-coefficient differential or difference equations. The reasons for this are numerous. For example, many physical systems that can be interpreted as performing filtering operations are characterized by differential or difference equations. A good example of this that we will examine in Chapter 6 is an automobile suspension system, which in part is designed to filter out high-frequency bumps and irregularities in road surfaces. A second reason for the use of filters described by differential or difference equations is that they are conveniently implemented using either analog or digital hardware. Furthermore, systems described by differential or difference equations offer an extremely broad and flexible range of designs, allowing one, for example, to produce filters that are close to ideal or that possess other desirable characteristics. In this and the next section, we consider several examples that illustrate the implementation of continuous-time and discrete-time frequency-selective filters through the use of differential and difference equations. In Chapters 4–6, we will see other examples of these classes of filters and will gain additional insights into the properties that make them so useful.

3.10.1 A Simple RC Lowpass Filter

Electrical circuits are widely used to implement continuous-time filtering operations. One of the simplest examples of such a circuit is the first-order *RC* circuit depicted in Figure 3.29, where the source voltage $v_s(t)$ is the system input. This circuit can be used to perform either a lowpass or highpass filtering operation, depending upon what we take as the output signal. In particular, suppose that we take the capacitor voltage $v_c(t)$ as the output. In this case, the output voltage is related to the input voltage through the linear