Low-Density Parity-Check Codes

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Preface

The Noisy Channel Coding Theorem discovered by C. E. Shannon in 1948 offered communication engineers the possibility of reducing error rates on noisy channels to negligible levels without sacrificing data rates. The primary obstacle to the practical use of this theorem has been the equipment complexity and the computation time required to decode the noisy received data.

This monograph presents a technique for achieving high data rates and negligible error probabilities on noisy channels with a reasonable amount of equipment. The advantages and disadvantages of this technique over other techniques for the same purpose are neither simple nor clear-cut, and depend primarily upon the channel and the type of service required. More important than the particular technique, however, is the hope that the concepts here will lead to new and better coding procedures.

The chapters of the monograph are arranged in such a way that with the exception of Chapter 5 each chapter can be read independently of the others. Chapter 1 sets the background of the study, summarizes the results, and briefly compares low-density coding with other coding schemes. Chapter 2 analyzes the distances between code words in low-density codes and Chapter 3 applies these results to the problem of bounding the probability of decoding error that can be achieved for these codes on a broad class of binary-input channels. The results of Chapter 3 can be immediately applied to any code or class of codes for which the distance properties can be bounded. Chapter 4 presents a simple decoding algorithm for these codes and analyzes the resulting error probability. Chapter 5 briefly extends all the previous results to multi-input channels, and Chapter 6 presents the results of computer simulation of the low-density decoding algorithm.

The work reported here is an expanded and revised version of my doctoral dissertation, completed in 1960 in the Department of Electrical Engineering, M.I.T. I am grateful to my thesis supervisor, Professor Peter Elias, and to my thesis readers, Professors Robert M. Fano and John M. Wozencraft, for assistance and encouragement both during the course of the thesis and later.

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Contents

Introduction

1.1 Coding for Digital Data Transmission

The need for efficient and reliable digital data communication systems has been rising rapidly in recent years. This need has been brought on by a variety of reasons, among them being the increase in automatic data processing equipment and the increased need for long range communication. Attempts to develop data systems through the use of conventional modulation and voice transmission techniques have generally resulted in systems with relatively low date rates and high error probabilities.

A more fundamental approach to the problems of efficiency and reliability in communication systems is contained in the Noisy Channel Coding theorem developed by C. E. Shannon [15, 4] in 1948. In order to understand the meaning of this theorem, consider Figure 1.1. The source produces binary digits, or binits, at some fixed time rate R_t. The encoder is a device that performs data

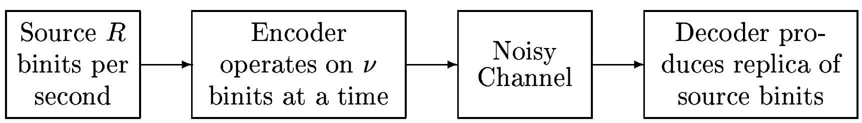

Figure 1.1: Block diagram of a communication system.

processing, modulation, and anything else that might be necessary to prepare the data for transmission over the channel. We shall assume, however, that the encoder separates the source sequence into blocks of ν binits and operates on only one block at a time. The encoder output is then transmitted over the channel and changed by some sort of random disturbance or noise. The decoder processes the channel output and produces a delayed replica of the source binits. The coding theorem states that for a large variety of channel models, encoders and decoders exist such that the probability of the decoder reproducing a source binit in error P_e is bounded by

$$e^{-\nu[E_L(R_t)+0(\nu)]} \le P_e \le e^{-\nu E(R_t)}$$

The functions $E(R_t)$ and $E_L(R_t)$ depend upon the channel but not upon ν; they are positive when $R_t = 0$, and decrease with R_t until they become 0 at some time rate C_t known as the channel capacity. The exact nature of these functions and the particular class of channels for which this theorem has been proved need not concern us here. The important result is that the coding constraint length ν is a fundamental parameter of a communication system. If a channel is to be used efficiently, that is with R_t close to C_t, then ν must be made correspondingly large to achieve a satisfactory error probability.

The obvious response of an engineer to such a theorem is: “Splendid, but how does one build encoders and decoders that behave in this way when ν is large?” It is rather sobering to observe that if an encoder stores a waveform or code word for each possible block of ν binits, then the storage requirement must be proportional to 2^{ν}, which is obviously impractical when ν is large. Fortunately, Elias [3] and Reiffen [14] have proved that for a wide variety of channel models, the results of the Noisy Channel Coding theorem can be achieved with little equipment complexity at the encoder by the use of parity-check coding. This will be described in more detail later.

Unfortunately, the problem of decoding simply but effectively when ν is large appears to be much more difficult than the problem of encoding. Enough approaches to this problem have been developed to assure one that the Coding theorem has engineering importance. On the other hand these approaches have not been carried far enough for the design of an efficient, reliable data communication system to become a matter of routine engineering.

This monograph contains a detailed study of one of the three or four most promising approaches to simple decoding for long constraint length codes. The purpose of publishing this work is primarily to show how such a coding and decoding scheme would work and where it might be useful. Also, naturally, it is hoped that this will stimulate further research on the subject. Further mathematical analysis will probably be fruitless, but there are many interesting modifications of the scheme that might be made and much experimental work that should be done.

In order the prove mathematically some results about low-density parity-check codes, we shall assume that the codes are to be used on a somewhat restricted and idealized class of channels. It is obvious that results using such channel models can be applied only to channels that closely approximate the model. However, when studying the probability of decoding error, we are interested primarily in the extremely atypical events that cause errors. It is not easy to find models that approximate both these atypical events and typical events. Consequently the analysis of codes on idealized channels can provide only limited insight about real channels, and such insight should be used with caution.

The channel models to be considered here are called symmetric binary-input channels. By this we mean a time-discrete channel for which the input is a sequence of the binary digits 0 and 1 and the output is a corresponding sequence of letters from a discrete or continuous alphabet. The channel is memoryless in the sense that given the input at a given time, the output at the corresponding time is statistically independent of all other inputs and outputs. The symmetry requirement will be defined precisely in Chapter 3, but roughly it means that the outputs can be paired in such a way that the probability of one output given an input is the same as that of the other output of the pair given the other input. The binary symmetric channel, abbreviated BSC, is a member of this class of channels in which there are only two output symbols, one corresponding to each input. The BSC can be entirely specified by the probability of a crossover from one input to the other output.

If a symmetric binary-input channel were to be used without coding, a sequence of binary digits would be transmitted through the channel and the receiver would guess the transmitted symbols one at a time from the received

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