# Some Cryptographic Techniques

# For Secure Data Communication

By

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## DECLARATION

I hereby declare that while registered as a candidate for the degree of Doctor of Philosophy with the Council for National Academic Awards I have not been a registered candidate for another award of the CNAA or other academic or professional institution.

Signed: Vijayaraghavan Varadharajan

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## Some Cryptographic Techniques For Secure Data Communication

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### Abstract

This thesis investigates conventional and public key cryptographic techniques for secure data communication.

Block and stream cipher methods to provide secure communication over an insecure channel are discussed with particular reference to the Data Encryption Standard (DES) algorithm. A microprocessor based data encryption interface unit has been designed and constructed using the DES to provide both communication and file security. Several chaining techniques using the system have also been investigated enabling a study of their error characteristics, speed of operation, level of security and their ability to overcome difficulties due to data redundancy and structure. A statistical analysis of the randomness of the output sequences in each of these techniques has been made. Furthermore , the developed system can be used on the Prestel public network allowing storage and retrieval of completely and partly encrypted frames of information on the Prestel database.

The use of such a DES based encryption system in a communication network poses problems with regard to key distribution since the keys used need to be distributed to the users over a secure, separate channel. Several methods of key distribution including the use of public key systems are discussed.

Extensions of the Rivest-Shamir-Adleman (RSA) public key scheme to matrix rings, polynomial rings and algebraic number fields have been proposed. These extensions indicate that rings other than the ring of rational integers can be used to construct public key systems with the factorization trapdoor property. The security of such systems again relies on the difficulty of factorizing a large integer.

An extension of the Diffie-Hellman public key distribution system to matrix rings is proposed. Short cycling attacks against the exponentiation system in GF($2^n$) have been analysed and are shown to be equivalent to a random search procedure. A hybrid system using exponentiation in GF($2^n$) for key distribution and the DES for data security has been implemented and the advantage of normal basis representation in the computation of the exponentiation in GF($2^n$) is examined.

The role of permutation polynomials in the design of public key systems has also been investigated. In particular, it is shown that secure public key systems can be designed using Dickson permutation polynomials and Rèdei rational functions. Further the complexity of public key systems can be increased by combining the permutation polynomials under the law of composition.

# CHAPTER 1

## INTRODUCTION

### 1.1 General

The concept of data security is becoming increasingly significant owing to the expanding role of distributed computation, distributed data bases and telecommunications applications such as electronic mail and electronic funds transfer. The computer and communications technologies have resulted in a dramatic increase in the volume and speed of information collection, distribution and storage. Greater information transfer and storage in turn imply greater risk of exposure of sensitive or confidential information to unauthorised users due to the ready availability of inexpensive miniature intercepting devices. These have resulted in an increased interest in computer data security not only in the military and political areas but also in the field of commerce, where a single transaction may involve millions of pounds. This has motivated research particularly in the art of cryptography, which forms the central technique of communication security.

Cryptography is the science and study of secret writing [1]. Cryptography can be defined as the transformation of a message or a data stream by means of an algorithm so that anyone observing the transformed data cannot deduce the hidden information. Such transformations provide solutions to two major problems of data security namely the privacy problem and the authentication problem [2]. In some environments the message can be transmitted in clear text as long as its integrity is safeguarded. A common example where the problem of authentication predominates is in telephone communication where the called party cannot determine who is calling. Other environments may require that the contents of the message be concealed during transmission from unauthorised observation and this is a privacy problem. The problems of privacy and authentication are closely related and techniques for solving one can frequently be applied to the other. Data encryption is recognised [3] as the most reliable method for not only protecting vital information from eavesdroppers but also a technique of message authentication preventing injection of false information into a communication system by illegitimate users.

This thesis is mainly concerned with the data privacy problem.

On one hand, the easy availability of enormous computer power enables the cryptographer to design complicated algorithms. But on the other hand, the computer technology also helps the code breakers to be more effective in cracking the system. So it is a never ending struggle between code makers and code breakers.

Recent developments in encryption techniques for computer communication network security including the Data Encryption Standard (DES) [11] and the evolution of public key cryptosystems provided the major thrust of the present research work. Essentially the thesis can be divided into two parts. In the first part (Chapters 2 to 9), the use of encryption and decryption techniques in communication systems is investigated. The design and operation of an encryption interface unit incorporating the DES to provide communication and file security and different key distribution schemes are discussed. The second part (Chapters 10 to 15) is mainly concerned with the design of public key cryptosystems with a particular emphasis on the extensions of the Rivest-Shamir-Adleman (RSA) [12] type factorization trapdoor systems.

## 1.2 Thesis Organization

In Chapter 2, basic concepts of symmetric and asymmetric cryptosystems and major cryptographic techniques are briefly reviewed.

An analysis of the DES is presented in Chapter 3 which includes a software implementation of the Standard, its possible weaknesses, some of its underlying design criteria and its cryptographic strength.

The design of a microprocessor based data encryption interface unit using the DES is described in Chapter 4.

The operation of the interface unit to provide a two-way secure data transfer in a two-node Apple microcomputer network is the subject of Chapter 5. Four different stream and block chaining techniques of the DES have been investigated using the developed interface unit.

In Chapter 6, a statistical analysis of the randomness characteristic of the output sequences produced by the DES under different modes has been carried out.

The use of the developed encryption interface unit has been extended in Chapter 7 to allow file security in Apple disk systems.

Chapter 8 is concerned with the incorporation of DES based encryption system in Prestel Viewdata network. This enables transfer and storage of encrypted as well as plain data between an Apple microcomputer and the Prestel database.

Different methods of key distribution for communication and file security are investigated in Chapter 9. It includes a brief review of the RSA and the Knapsack public key cryptosystems.

In Chapter 10, the prototype RSA system over rational integers has been extended to matrix and polynomial rings.

Chapter 11 discusses the notion of ideal theory and considers the RSA type factorization trapdoor systems from an ideal point of view.

The factorization trapdoor concept in some quadratic algebraic number fields and the design of public key systems in such fields are investigated in Chapter 12.

The implementation of a hybrid system using the DES and the Diffie-Hellman public key distribution [35] system is investigated in Chapter 13. An extension of the Diffie-Hellman system to matrix rings is proposed.

The role of permutation polynomials in the design of public key systems forms the subject of Chapter 14. In particular, the use of Dickson permutation polynomials and certain Rèdei functions in the construction of public key systems is discussed.

In Chapter 15, the use of chaining techniques in the matrix public key system and some precautions which must be taken when the RSA system or its extensions are used in a broadcasting type situation are described.

Chapter 16 contains the main conclusions.

# CHAPTER 2

## CRYPTOGRAPHIC CONCEPTS

### 2.1 Cryptographic Systems

Detailed treatment of cryptographic principles can be found in [2, 3, 4]. A basic cryptographic privacy system is shown in figure 2.1. The transmitter or the sender generates a plaintext message M which is to be communicated to a legitimate receiver over an insecure channel monitored by an eavesdropper. To prevent the eavesdropper from learning the contents of M, the sender encrypts M, with an invertible transformation to produce the cryptogram or ciphertext, $C = T(M)$. When the legitimate receiver obtains C, it is deciphered with the inverse transformation to obtain the plaintext message, $M = T^{-1}(C)$.

The transformation T applied at the sending and receiving ends is a key dependent mapping from a set of messages in the plaintext to a set of ciphertext messages and vice versa. The particular transformation used is chosen from a family of transformations. The parameter that selects the individual transformation to be employed is called the key. Note that there may be more than one key involved. Assuming that the same key is used in both encryption and decryption, then $C = T_k\ (M)$ and $M = T_k^{-1}(C)$.

Thus a general cryptosystem consists of the following components:

1. A plaintext message space M ;
2. A ciphertext message space C ;
3. A key space K;
4. A family of encryption transformations $E_k : M \rightarrow C$ where $k \in K$.
5. A family of decryption transformations $D_k : C \rightarrow M$ where $k \in K$.

The encryption and decryption transformations $E_k$ and $D_k$ are defined by the encrypting and decrypting algorithms E and D which may be a set of instructions, a piece of hardware or a computer program and is common to every transformation in the family. Different values of the key (s) result in totally different transformations of plaintexts and ciphertexts. This implies that the family of transformations, that is, the general cryptographic system, can be made public information without

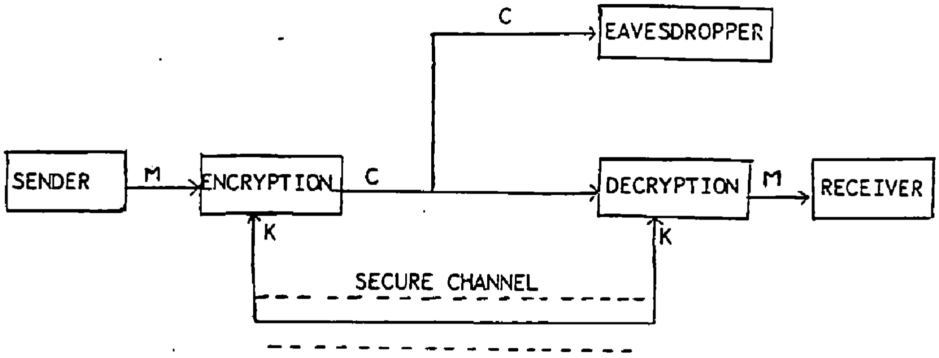


Fig. 2.1 - Basic Cryptographic Privacy System

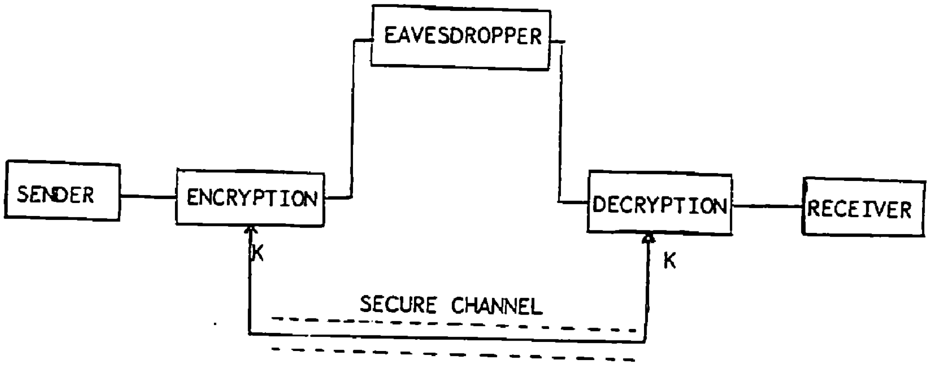


Fig. 2.2 - Basic Cryptographic Authentication System

compromising the security of the system. Only the key(s) needs to be kept secret. This satisfies one of the general requirements of a cryptosystem that the security must not depend on the secrecy of something like a cryptographic algorithm which cannot be easily changed if it is compromised. In addition, a publicly known system is necessary for standardization among commercial users. Even though the opponent knows the set of all possible keys, that is, the key space, he may still be unable to discover the correct set of keys required if the key space is large.

Simmons [5] classifies cryptosystems as symmetric (one key) and asymmetric (two key). In symmetric cryptosystems, the enciphering and deciphering keys are the same or easily determined from each other. Because the general method of encryption and decryption is known, this means that the transformations $E_k$ and $D_k$ are also easily derived from each other. Thus if both $E_k$ and $D_k$ are protected both secrecy and authenticity are achieved. However secrecy cannot be separated from authenticity because making either $E_k$ or $D_k$ available, exposes the other. Thus for secure communication, such a system requires the key to be transmitted to the receiver via some secure channel. Figure 2.2 illustrates how such a cryptographic system can be used to solve the authentication problem. In this case, the opponent not only sees all ciphertexts flowing on the channel but can alter them at will. The legitimate receiver protects himself from being deceived by an altered or injected message, by decrypting all the messages he receives and accepting only those encrypted with the correct key.

In asymmetric cryptosystems, the enciphering and deciphering keys differ in such a way that at least one key is computationally infeasible to determine from the other. Thus one of the transformations $E_k$ or $D_k$ can be revealed without endangering the other. Secrecy and authenticity are provided by protecting the separate transformations namely $D_k$ for secrecy and $E_k$ for authenticity. Such asymmetric systems are often referred to as public key systems as in addition to E and D, the encryption key is made public. Only the decrypting key is kept secret by the receiver. The use of such systems thus avoid the necessity to transmit the key used in the algorithm over a secure channel among the communicators. Moreover such systems can be used to transmit the secret key required for conventional symmetric systems.

Such asymmetric systems are also able to deal with the problem of dispute that may arise between the sender and the receiver over the actual message sent in an authentication system. The inability of the symmetric system to deal with this type of problem limits its application. This can be seen as follows.

The validity of contracts and agreements is usually guaranteed by signatures. The essence of a signature is that only one person can produce it but anybody can recognize it. For this, each user must be able to produce messages whose authenticity can be checked by anyone, but which could not have been produced by anyone else especially the intended recipient. In a symmetric system, the receiver authenticates any message that he receives from the sender by deciphering it with the key which the two hold in common. Because this key is held in common, however, the receiver has the ability to produce any ciphertext that could have been produced by the sender and hence the receiver cannot prove that it is the sender who actually sent him the disputed message. The asymmetric system provides a direct elegant solution to this signature problem. If user A wishes to send a signed message M to user B, he signs the message by producing $S = D_A (M)$. When user B receives S, he can recover M by operating on S with $E_A$, that is, $M = E_A (S)$. B keeps S as the proof that user A has sent him the particular message M as only the user A could have generated S because he is the only one who knows $D_A$. To obtain secrecy of communication as well as authentication, user A sends $E_B$ (S) instead of S to user B. As only B knows $D_B$, he is the only one who can recover S and hence M.

## 2.2 Cryptosystem Security and Complexity Theory

Any attempt by the eavesdropper either to decrypt a cryptogram C to get the plaintext M or to encrypt an inauthentic plaintext M′ to get an acceptable cryptogram C′ without prior knowledge of the key is called cryptanalysis [2]. If cryptanalysis is impossible so that a cryptanalyst cannot deduce M from C or C′ from M′ without prior knowledge of the key, the cryptographic system can be said to be secure.

In order to measure the security of a cryptosystem, Diffie and Hellman [2] have defined at least three types of attack which the system should withstand when being subjected.

(a) A'ciphertext only' attack is the weakest form of attack which the cryptographic system must withstand. In this attack, the cryptanalyst attempts to decipher the cryptogram using only the statistical properties of the message source. As an example, consider a letter written in English. Not all characters or words occur equally often; for instance, the letter 'E' occurs approximately 13% of the time [6] . Such non-uniformities in the frequency distribution of the alphabet are used to give clues about the message. There is also probably a heading which contains a date and address and a closing such as 'sincerely'. With the aid of statistical tables, the cryptanalyst uses each of these facts to determine which message was most likely sent.

(b) Under a 'known plaintext' attack, the cryptanalyst is assumed to have a substantial amount of corresponding message - cryptogram pairs and tries to determine the key used in the algorithm. This form of attack is a significant threat as frequently messages are enciphered under the same key. Hence if a system cannot withstand such an attack, all messages which have been encrypted under a common key needs to be kept secret as long as any of the messages is to be kept secret. Such an attack is quite common in practice. For instance, a typical example is when information may be transmitted in secrecy which is intended for public release at a later date.

(c) A 'chosen text' attack generally occurs less frequently than a known plaintext attack. In this case, the cryptanalyst is assumed to choose messages to be enciphered or ciphertexts to be deciphered in an attempt to determine the key.

For the purpose of certifying systems as secure, it is necessary to consider more formidable cryptanalytical threats. These not only give more realistic models of the working invironment of a cryptographic system but also make the assessment of the system's strength easier.

There are two fundamentally different ways in which cryptographic systems may be considered secure.

A cryptosystem is said to be unconditionally secure under a given form of attack if the amount of information available to the cryptanalyst is actually insufficient to determine the solution, which may be the key or the plaintext, whatever be the computing power the cryptanalyst has at his disposal [7] . As an example, consider a cryptosystem where a message-cryptogram pair uniquely determines the key. This system is not unconditionally secure under a chosen text attack or a known plaintext attack. However if the information content of the message plus the information content of the key is greater than the maximum possible amount of information in the cryptogram, then this system is unconditionally secure under a ciphertext only attack. The cryptanalyst cannot determine the complete message and key from the cryptogram alone, since he would obtain more information than that provided by the cryptogram.

Unfortunately, unconditionally secure systems require either perfect source coding or a key whose length grows linearly with respect to the sum of the lengths of all messages enciphered [7] . This requirement is not practical in most applications. Thus computationally secure systems are usually used in cryptography. A system is said to be computationally secure under a given form of attack if the amount of computation required to compute the solution exceeds the cryptanalyst's abilities or the economic value of the message to him. A measure called the work factor is often associated with a cryptosystem which gives an expression of the minimum amount of work necessary for a successful attack. In practice, there is no universally accepted fixed set of parameters used to express the work factor. Frequently, however, it is measured in one or more of the following ways; cryptanalyst hours, number of mathematical or logical operations, computing resources such as data storage and processing requirements, special hardware and calender time or more generally the cost in some money units such as dollars. This idea of computationally secure system is also related to the concept of one-way functions and complexity theory.

Algorithmic complexity theory is concerned with the computational requirements (both time and space) as a function of the size of the problem solved by a particular algorithm. Complexity theory is essentially a collection of results in computer science that attempts to quantify the statement 'Problem A is 'harder' than problem B' [ 6] . There is a class of problems called NP problems [ 8] and in particular a distinguished subclass of NP called the class of NP-complete problems which are regarded to be the 'hardest' problems. The class NP-complete is thought to be a source of problems that can be adapted to cryptographic applications and will by virtue of their computational complexity produce strong cryptographic systems. The cryptanalyst is then required to solve an NP-complete problem to break the cryptosystem which theoretically should require an exponential time algorithm.

However it is generally argued that [9] the complexity theory deals often with the worst case behaviour whereas in cryptography, the cryptanalytical task must be hard for almost all the instances. It will be a poor cryptosystem if the system allows easy decryption of all but a few cryptograms by the opponent. In addition computationally hard problems are not necessarily cryptographically hard problems since the cryptanalyst generally possesses additional side information and often tries to solve several instances of the same problem [3] . Recently a conventional and a public key cryptosystem based on NP-complete problems have been solved using polynomial time algorithms [6, 10] . This goes to show that merely starting with a computationally hard problem may not be enough to provide secure cryptosystems.

## 2.3 Cryptographic Techniques

There are two fundamental cryptographic techniques that can be used to design strong encryption-based protection schemes, namely, the block cipher technique and the stream cipher technique [3] . The suitability of either of these two techniques for use in cryptosystems depends on the nature of the application.

### 2.3.1 Block Cipher

Let the message be divided into blocks of fixed length. A block cipher then transforms these input blocks into output blocks using the same key. For instance, considering a binary system a string of input bits of fixed length is transformed into a string of output bits of fixed length using a block cipher as shown in Figure 2.3. The encryption and decryption functions are such that every bit in the output block depends on every bit in the input block as well as on every bit of the key. In the binary system, if the blocksize is n, then the size of the plaintext space and the size of the ciphertext space is $2^n$. In order that the deciphering of a ciphertext block yields an unambiguous plaintext block, the mapping must be invertible and hence injective and in this case surjective as the size of the input and output spaces are equal. Thus one can view a block cipher as defining one of the $2^n$ ! transformations on the set of n-bit blocks. In practice, it is not feasible to implement a block cipher that realizes all the possible combinations because of the size of the key required and the logical complexity of the cipher. Usually a key of n-bits is employed to select one out of a subset of $2^n$ functions.

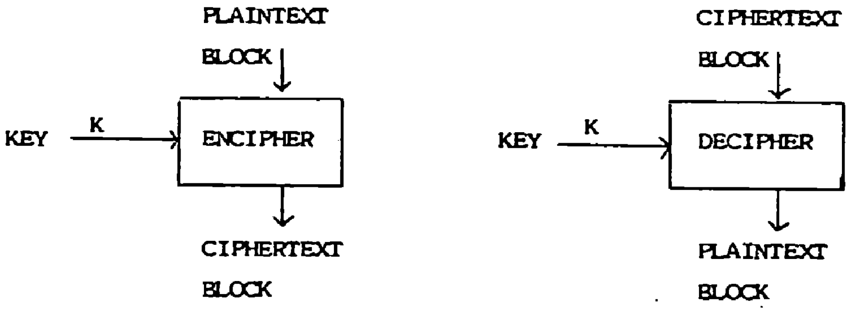


Figure 2.3 - Block Cipher.

A fundamental property of this type of cipher is that the blocksize plays an important part in determining the cryptanalytical strength of the cipher. The blocksize must be chosen large enough to foil simple message exhaustion attacks. This attack consists of encrypting all $2^n$ possible plaintexts with a given key thus building a dictionary of ciphertexts and corresponding plaintexts. A message can then be recovered by searching the dictionary and relating each intercepted ciphertext block to its corresponding plaintext block. However if the blocksize is made large enough, the dictionary can be made too large to construct or store. Other attacks must also be considered before arriving at an acceptable blocksize. An attack called block frequency analysis based on frequency of occurrence of blocks is quite common. It is similar to the analysis performed on a simple substitution cipher by taking into account letter frequencies. Analytical or deterministic attack - which consists of expressing cipher operations in mathematical form as a set of equations and solving for the unknown variables directly using analytical methods - can be thwarted by making every bit of the output block a complex mathematical function of every bit of input block and key thus giving a strong intersymbol dependence property. The block ciphers also exhibit error propagation properties which are suitable for error detection and authentication purposes.

The Data Encryption Standard [11] is an example of a symmetric block cipher of length 64 bits which well withstands the above mentioned and more sophisticated attacks. The algorithm will be discussed in Chapter 3 and is used in the design of a encryption interface unit in Chapters 4 and 5. Examples of public key block ciphers include the RSA system [12] and the trapdoor knapsack system [13]. They are discussed in Chapter 9.

### 2.3.2 Stream Cipher

A stream cipher divides the message M into successive characters or bits. It then uses a character or bit stream generator to produce a cryptographic key stream which is then combined with the plaintext message characters or bits to produce the ciphertext characters or bits. A similar procedure is carried out to recover the plaintext characters or bits by combining the key stream with the ciphertext characters or bits. The stream cipher concept is illustrated in Figures 2.4 and 2.5 where the ciphertext Y is produced from plaintext X by Exclusive-oring it with a secret binary stream R.

Let us now assume that the key to the bit stream generator is fixed and that the cryptographic key stream R produced at each iteration depends only on this key. This then implies that R does not change from one iteration to the other. Now if an opponent knows a plaintext-ciphertext pair, then he can recover the key stream R by forming $R = X \oplus Y$. Having obtained R, the opponent can decipher any intercepted ciphertext without even knowing the key to the generator which is unacceptable. Further, repetitions on the plaintext would be reflected in the ciphertext even if he did not know a ciphertext-plaintext pair. Hence to overcome this problem, the key stream must be made to change for every iteration of the ciphering algorithm. A stream cipher is said to be periodic if the key stream changes such that it repeats itself after d characters or bits for some fixed d; otherwise it is said to be non-periodic. Ideally, one would want the key stream to have a long period and to vary in a random manner. If the key stream were truly random and its length is equal to the length of the message, then this would produce an unbreakable cipher. Because the key stream is random, then it must be provided to the users in advance via some independent and secure channel which causes insurmountable logistical problems when the intended data traffic is

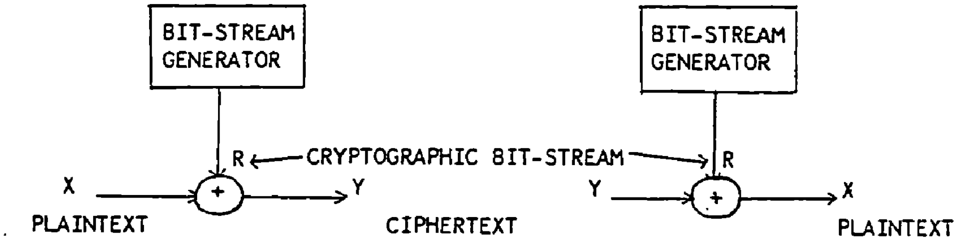


Fig. 2.4 - Stream Cipher Concept

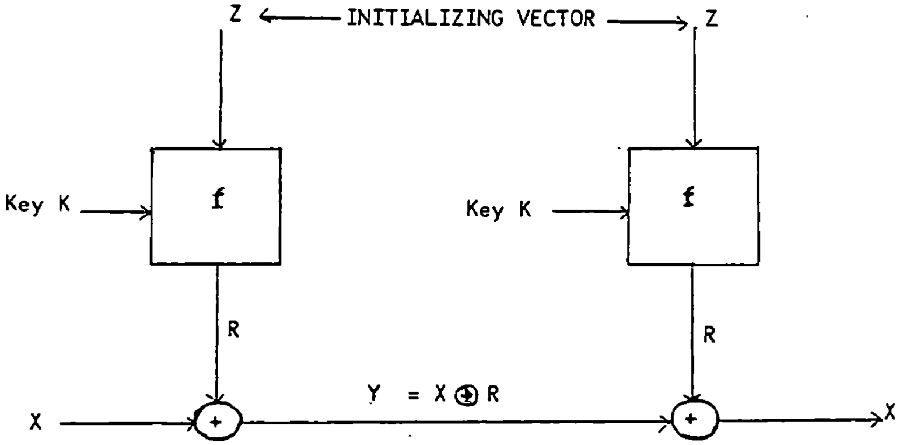


Fig. 2.5 - Stream Cipher

very large. Hence for practical reasons, the key stream must be implemented as an algorithmic procedure so that the key stream can be produced by users at both ends. Ciphers generated by rotor and Hagelin machines [1] are periodic whereas the Vernam cipher (one time pad) [16] and running key ciphers are non-periodic ciphers.

A system in which the key stream is generated independently of the message stream is said to be a synchronous stream cipher. In such a modulo 2 addition based system, each bit in the output ciphertext is dependent upon the corresponding bit in the input plaintext but not upon any other bits in the input plaintext. This is in contrast to the block cipher which exhibits a much more complex relationship between the bits in the plaintext and the bits in the ciphertext. Hence the stream ciphers can be made to have non-error propagating property. Both approaches however have comparable strength.

Various techniques may be used to generate the key stream in stream ciphers. Not only the key stream generated must have good pseudo-random properties but also the generation process must be non-linear. This limits the direct use of linear feedback shift registers for the generation of these key streams because with such generators, the cryptanalyst can derive the entire key stream given a relatively small amount of plaintext-ciphertext pair [4, 14]. It is also important that the complexity of the linear equivalent of any non-linear generating process be estimated [14]. It has been suggested in [15] that non-linear substitution-permutation functions when combined with a shift register produces cryptographically strong key streams. Since the key streams can be generated in blocks, it is also possible for a block cipher to be used to obtain a stream cipher. Because both the sender and the receiver must generate key streams that are equal and secret, it is necessary that the keys used in the algorithm must also be equal and secret. This implies that a public key block cipher algorithm can be used to obtain a stream cipher if and only if it is used as a conventional algorithm, that is, both the sender and the receiver use the same algorithm (encryption E or decryption D) and the same secret key.

But as noted earlier, a fixed key even though it is kept secret does not ensure an unpredictable cryptographic key stream. To avoid producing the same key stream at each iteration of the algorithm, another parameter called the initialization vector (IV) is introduced in the ciphering process. Different initialization vectors are generated in a pseudo-random non-repeating manner which in turn produce different cryptographic key streams. That is, R is generated using $R = f_k\ (Z)$ where Z is the initialization vector, $f_k$ defines the block cipher algorithm under key k. Encryption and decryption operations are then given by $Y = X \oplus f_k\ (Z)$ and $X = Y \oplus f_k\ (Z)$. This is shown in Figure 2.5. For the stream cipher to be cryptographically strong, the initialization vector needs to be varied in a pseudo-random manner. One way to do this is to generate independently a new initialization vector for each iteration of the ciphering algorithm. This has the disadvantage of increasing the amount of transmitted data since the initialization vectors are now added to each block of cipher-text. A more efficient approach is as follows: at the first iteration of the cipher algorithm, the initialization vector is used as before to produce a block of key stream bits which can be used to encipher the first block of plaintext (assuming the blocksize of the cipher algorithm is same as the size of the plaintext block). At all subsequent iterations of ciphering algorithm, the initialization vector is altered or determined using one of many feedback chaining techniques. Thus chaining eliminates the problem of transmitting or storing a separate initialization vector value for each ciphertext. A feedback can be obtained from several places namely the key stream itself, the plaintext, the ciphertext or some combination thereof. Each of these approaches gives rise to cryptographic systems with different characteristics with respect to recovery from errors.

Note that the initialization vector in addition to providing cryptographic strength also establishes synchoronization between communicating cryptographic devices. It assures that the same cryptographic key streams are generated at the both ends of the link. Once the initial state of the system has been set, only the current state of the system needs to be remembered to maintain synchronization.

In general feedback chaining techniques increase the overall strength of a cryptographic system. The chaining techniques when used during ciphering process make an output dependent not only on the current input and key but also on earlier input(s) and/or output(s). In effect it introduces noise into the ciphering process. This helps to eliminate the undesirable effects of redundancy and structure present within the plaintext data. Several chaining techniques will be discussed in Chapter 5 with special reference to the Data Encryption Standard.

# CHAPTER 3

## DATA ENCRYPTION ALGORITHMS

### 3.1 General

In cryptography, two main operations have been used for centuries and they still form the main elements of modern encryption algorithms. They are transpositions and substitutions and may be applied to words, symbols, letters and binary bits or groups of bits. There is also another technique, that of concealment where the symbols of the message are mixed up with many other symbols which carry no important information at all although they may appear to. This method can be used to give considerable security but it expands the message by a great amount. These operations lead to three different classes of ciphers namely the transposition ciphers, the substitution ciphers and a combination of both called the product ciphers.

#### 3.1.1 Transposition Cipher

A transposition cipher consists of rearrangement of the characters (bits) in a block of plaintext; the characters retain their identity but lose their position. If the transposition is one-to-one then the process is reversible. On the other hand, if the transposition is not one-to-one then the operation becomes irreversible. Consider, for instance, the transposition which maps an 8-bit block to a 6-bit one, say by discarding bits 3 and 6 and rearranging the others. Here the total number of zeroes and ones are no longer preserved. Transposition by itself is not a very secure type of encipherment because unless every message has a unique form of transposition, the acquisition of several plaintext-ciphertext pairs allows the cryptanalyst to discover the permutation statistically.

#### 3.1.2 Substitution Cipher

A substitution cipher consists of the replacement of characters of the plaintext with characters from another alphabet. In the case of binary operations, a look-up table characterises the substitution operation. The bits are divided into small groups which are then replaced by the contents of the look-up table addressed by each group. For example,if the message is divided into 4 bit groups, there will be 16 possible combinations for each group, so that the table will require 16 entries numbered 0 to 15. In general, the number of ones and zeroes is not preserved and a change in one bit of the input may affect several bits of the output. The Data Encryption Standard (Section 3.2) uses eight such look-up tables commonly referred to as S-boxes, each converting a six-bit input into a four bit output. An important advantage of the substitution cipher is that the contents of the look-up table can be changed frequently and implementation of such an operation can be readily done with read only memories (ROMs). A substitution may or may not be reversible depending on the form of the look-up tables. One of the oldest substitution cipher is the Caeser cipher which is a monoalphabetical substitution cipher. This cipher can be broken in ciphertext-only attack with approximately 30 alphabet characters using letter frequency analysis [14]. An important substitution cipher is the one-time pad in which the key is random, non-repeating and used only once. One-time pads are unbreakable as there is not enough information in the ciphertext to determine the key or message uniquely. The first implementation of the one-time pad cipher was the Vernam cipher [16] in which the key bits were added modulo 2 to the plaintext bits. One major problem with this cipher is that the key length grows linearly with the length of the message.

### 3.1.3 Product Cipher

A product cipher involves both the steps of substitution and transposition. Shannon [7] suggested the use of product ciphers to build a strong system out of individually weak components. He suggested that the product cipher be formed using substitution and permutation ciphers in an alternating manner. The permutation shuffles the digits providing'diffusion' and non-linear substitutions provide 'confusion'. Confusion makes the relationship between the ciphertext and the plaintext as complicated as possible, that is, it hides the key and diffusion spreads the statistics of the plaintext into the ciphertext. This formed the basis of the Lucifer system designed by the IBM [17]. The Data Encryption Standard which is considered next is based on the Lucifer system.

## 3.2 Data Encryption Standard

The cryptographic algorithm used in the design of the encryption unit is the Data Encryption Standard (DES) [11]. This algorithm is now regarded as the US Federal Standard recommended for use by non-military Government Agencies. It has also been adopted by the American National Standards Institute (ANSI) and is recommended for use by the American Bankers Association, (ABA). The adoption of DES as a Standard for encrypting data contributed to the surge in interest in this algorithm. Before considering the development of an encryption system to provide communication security using a hardware implementation of the Standard in the next chapter, in this chapter the DES algorithm is analysed to provide a deeper insight into the design of practical encryption algorithms. The algorithm has been implemented by software to study some of its characteristics. In particular, the software approach enables the study of intermediate outputs during each round, whereas the hardware (LSI) implementation only gives the final ciphertext output. The software implementation is also found to be useful when performing statistical tests on the randomness of the output obtained from the algorithm. This forms the subject of Chapter 6.

First an overview of the algorithm is given. Then the software implementation is described together with some performance figures. Some of the characteristics of the algorithm together with some of the design criteria underlying the choice of parameters in the algorithm are presented. The controversy surrounding the DES and possible weaknesses of the algorithm are then considered. Finally the complexity of the algorithm and its security are investigated.

### 3.2.1 DES Algorithm - An Overview

The Data Encryption Standard algorithm is a block product cipher system. Block because it transforms more than one character at a time. Product because it is composed of a series of transpositions, substitutions and additive encodings combined by a sequence of feedback cycles.

It is a complex non-linear algorithm which enciphers a 64-bit block of plaintext into a 64-bit block of ciphertext under the control of a 56-bit cryptographic key. DES can be regarded as a huge key-controlled substitution box (S-box) with a 64-bit input and output. With such an S-box a total of $(2^{64})!$ different transformations or functions from plaintext to ciphertext are possible. The 56-bit key thus selects only a small subset ($2^{56}$) of the total set's possible functions. As a single huge S-box is difficult to construct, DES is implemented by using several smaller S-boxes and permuting their concatenated outputs. Repetition of the substitution and permutation processes several times increases the cryptographic strength.

The complete DES algorithm is given in [11]. The three major steps in the algorithm are summarized in Figure 3.1.

1. A transposition operation, referred to as the initial permutation (IP). This fixed transposition does not utilize the 64-bit key and operates solely on the 64 data bits.

2. A complex key dependent product transformation that uses block ciphering to increase the number of substitutional and reordering patterns.

3. A final transposition operation referred to as the inverse initial permutation ($IP^{-1}$) which is actually the reversal of the transformation performed in the first step.

The second step is the most important step out of the three and it consists of 16 separate rounds of encipherment; each round using a product cipher approach or cipher function. The steps performed in each round shown in Figure 3.2 are summarized below:

(i) The 64-bit input block is divided into two parts, a left half (L) and a right half (R),each 32-bits long.

(ii) The right half of the input block becomes the left half of the output block. This is denoted in Figure 3.2 by an arrow going from $R_{i-1}$ to $L_i$.

The steps (iii) to (vii) which follow can be regarded as a complex cipher function, f, operating on both key and right half of the input block.

(iii) The 32-bits long right half (R) undergoes an expansion

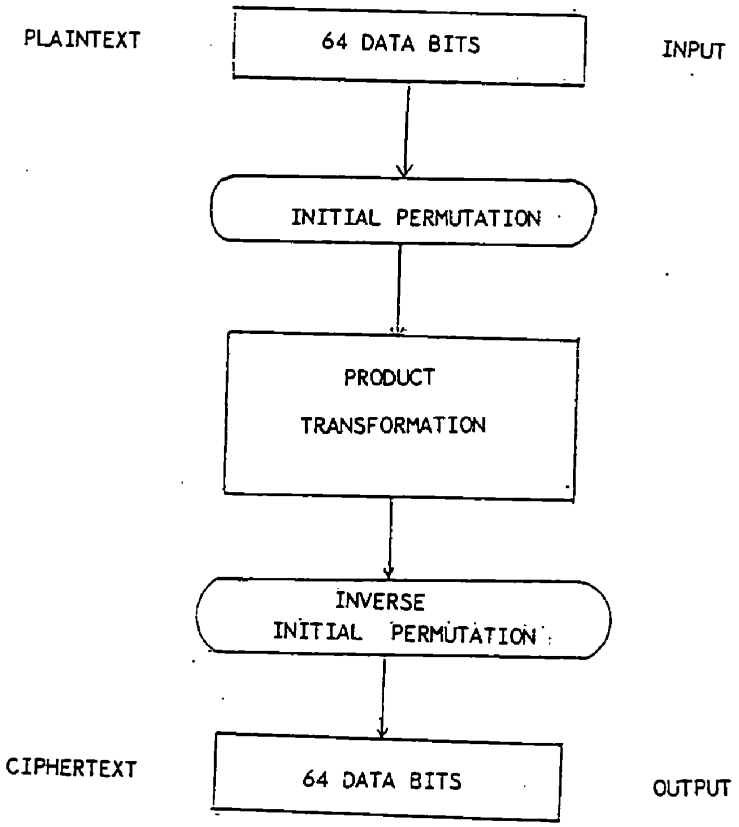


Fig.3.1 –Overview of Enciphering process for Data Encryption Standard.

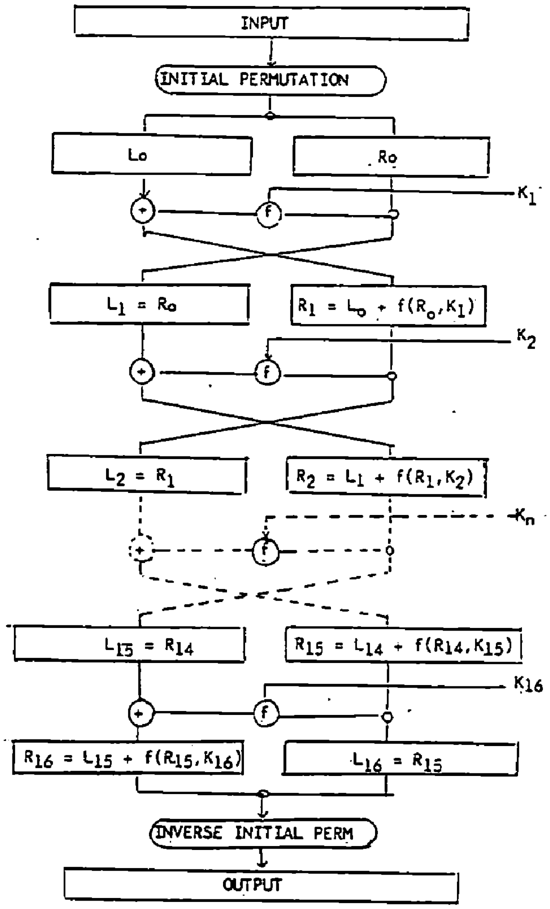


Fig.3.2(a) - Enciphering operation.

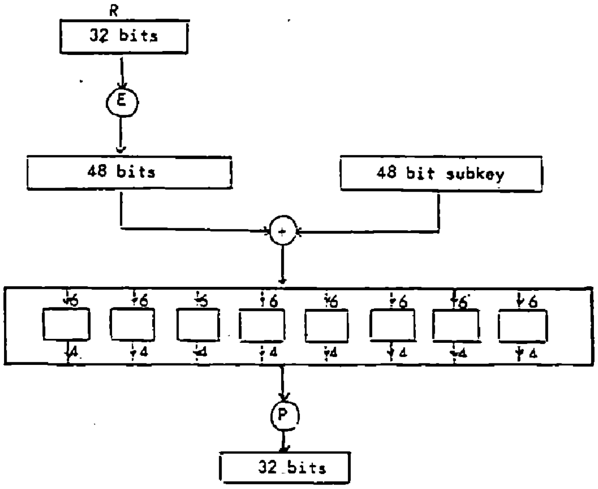


Fig.3.2(b) -Details of function f(R,E).

process (E), yielding a 48-bit data block. This is a fixed expansion permutation and not key-dependent.

(iv) The 64-bit key is used to generate a 48-bit subkey through a key scheduling procedure described in the next section.

(v) This 48-bit subkey $K_i$ is exclusive-ored with the expanded right half $E(R_i)$ yielding a 48-bit result.

(vi) This 48-bit data is divided into eight 6-bit groups each of which is subjected to a 6-to-4-bit non-reversible substitution function ($S_i$). Six groups of 4-bits are then concatenated to form a 32-bit output.

(vii) The 32-bit output is permuted to produce a 32-bit block by the fixed permutation P.

(viii) The 32-bit output of step (vii) is combined via the exclusive-or operation with the left half of the input block to form the right half of the 64-bit output block.

Details of the permutations IP, $IP^{-1}$, P and substitution boxes $S_i$ can be found in [11].

### 3.2.2 The Key Schedule Procedure

The key schedule procedure is used to enlarge the keyspace by expanding the externally supplied key into internal subkeys. The DES key schedule operation derives its sixteen 48-bit subkeys required for 16 rounds from the 56-bit key entered externally, by simple repetition. For reasons of security, all of these keys must be different. This is achieved by selecting a different subset of 48-bits from the 56-bit key. (Note that 8-bits out of 64-bits key are used as parity bits). The procedure is based on a shifting and bit selection algorithm.

Figure 3.3 illustrates the key schedule calculation used for encipherment. It begins with an initial permutation defined by Permuted Choice 1, (PC-1). PC-1 is the same for encipherment and decipherment and it selects 56 of the 64 external key bits (stripping off the 8 parity bits) and loads them into two 28-bit shift registers C and D. The parity checking of the external key is performed prior

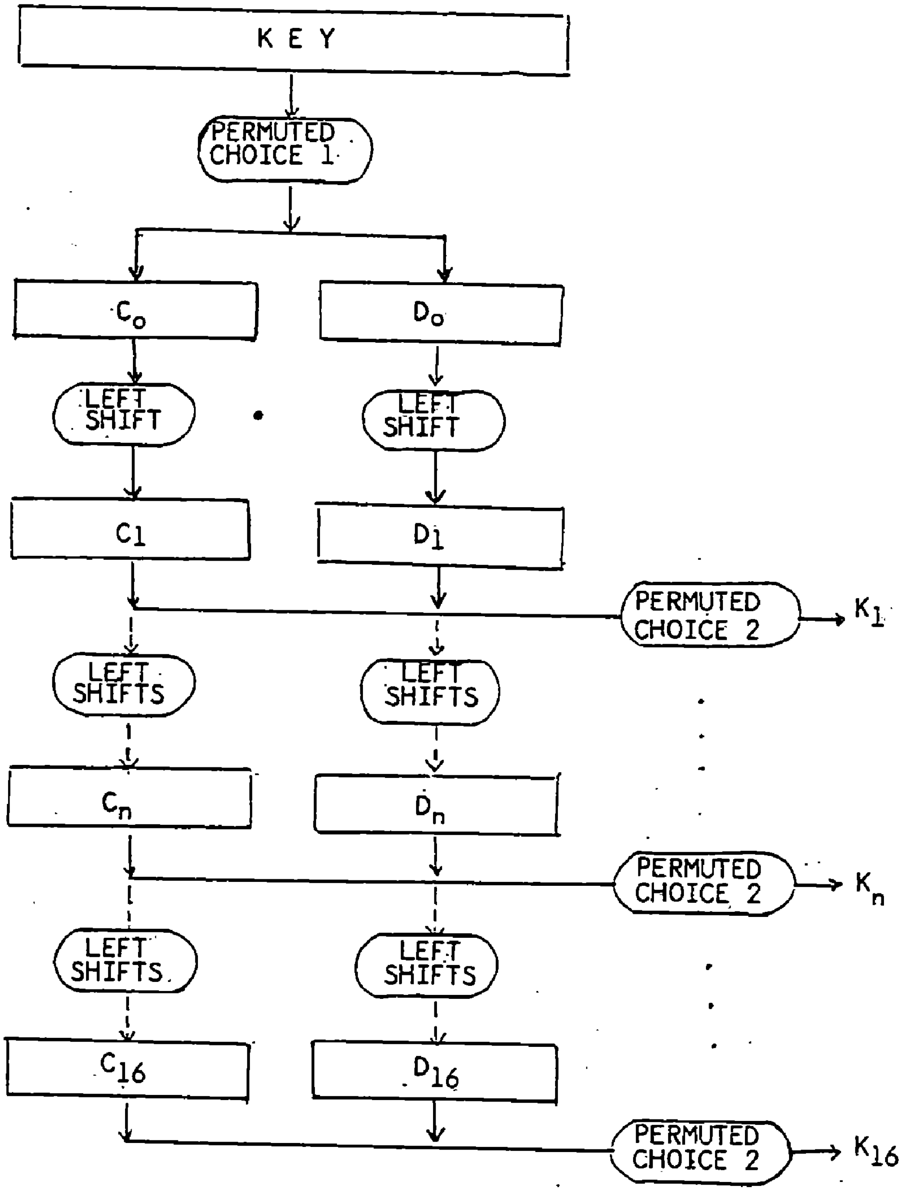


Fig.3.3 -Key Schedule Calculation.

to PC-1. During encryption, the contents of the registers $C_{i-1}$ and $D_{i-1}$ are shifted one or two positions to the left according to the schedule of circular shift operations shown in Table 3.1, Figure 3.4. The key K(i) is then derived from ($C_i$, $D_i$) through a second permutation defined by Permuted Choice, PC-2. Moreover the shift schedule is such that $C_{16} = C_o$ and $D_{16} = D_o$. It is to be noted here that PC-2 does not mix the contents of $C_i$ and $D_i$ registers. This property together with the circular shift operations allows certain values of keys K to generate identical internal subkeys. This is an inherent weakness of the algorithm and is discussed further in Section 3.6.4.

During the deciphering operation the key K(16) is used in round one and K(15) in round two and so on. But the contents of $C_o$ and $D_o$ are the same for enciphering and deciphering, since the external key is loaded in both cases via the Permuted Choice PC-1. This means that the key K(16) can be created at the first round merely by omitting the first shift operation and K(15) can be created at the second round by shifting $C_o$ and $D_o$ one bit to the right. The remainder of the internal keys are obtained in the same manner using the shift schedule in Table 3.1, Figure 3.4, in reverse order except that left shifts are changed to right shifts.

### 3.2.3 DES Encryption and Decryption

It is seen that the same algorithm can be used to perform encryption and decryption with minor changes. An outline of the proof of this property is as follows:

Let the contents of L and R registers on the $i^{th}$ round be L(i) and R(i). Let the output of the permutation P be P(i) and the output of PC-2 be K(i). Let the operation of the expansion permutation, S-boxes, permutation P be represented by f. Then during encryption,

$$\begin{aligned} L(i) &= R\,(i-1) \\ R(i) &= L\,(i-1) \oplus P(i) \\ P(i) &= f\,[R(i-1),\, K(i)] \end{aligned}$$

where ⊕ denotes Exclusive-or operation.

| Round Number | Number of Shifts |
|---|---|
| 1 | 1 |
| 2 | 1 |
| 3 | 2 |
| 4 | 2 |
| 5 | 2 |
| 6 | 2 |
| 7 | 2 |
| 8 | 2 |
| 9 | 1 |
| 10 | 2 |
| 11 | 2 |
| 12 | 2 |
| 13 | 2 |
| 14 | 2 |
| 15 | 2 |
| 16 | 1 |

Figure 3.4 - Table 3.1 : Schedule of Shift operations

Rewriting these as

$$L(i) = R(i-1) \tag{3.1}$$

$$R(i) = L(i-1) \oplus f\,[R(i-1), K(i)] \tag{3.2}$$

Hence the $i^{th}$ round is completely determined in terms of the $(i-1)^{th}$ round provided the appropriate bits of key K are available. Similarly $(i-1)^{th}$ round can be completely expressed in terms of the $i^{th}$ round. This can be done because of the reversible property of exclusive-or operation: if $A = B \oplus C$ then $B = A \oplus C$. Therefore rewriting (3.1) as

$$R(i-1) = L(i)$$

and rewriting (3.2) as

$$L(i-1) = R(i) \oplus f\,[R(i-1), K(i)]$$

$$\text{ie, } L(i-1) = R(i) \oplus f\,[L(i), K(i)] \tag{3.3}$$

it is seen that the same algorithm can be used for decryption. The involute structure of the DES can be seen by regarding the DES as a product of 33 mappings [6] as shown below

$$IP^{-1} \times \Pi_{16} \times \ldots \times \Theta \times \Pi_1 \times IP$$

where

IP is the initial permutation and $IP^{-1}$ is its inverse

$\Pi_i$, $(1 \leq i \leq 16)$ are involutions given by

$\Pi_i: (x, x') \longmapsto (x \oplus f\,(x'), x')$ where x is the left half and $x'$ is the right half

$\theta$ is the interchange involution given by

$\Theta : (x, x') \longmapsto (x', x)$

The inverse of the DES is then given by

$$(DES)^{-1} = (IP)^{-1} \times \Pi_1 \times \Theta \times \ldots \times \Theta \times \Pi_{16} \times IP$$

That is, decryption is essentially achieved by operating the DES backwards.

## 3.3 Software DES Implementation

The DES algorithm has been implemented using a 6502 based Apple microcomputer system. This was done to gain a deeper insight into the working of the algorithm and to allow a comparison between hardware and software implementation with regard to performance characteristics and the results produced. The two implementations may not be compatible and the results may not agree with each other as the FIPS publication [11]gives only the recommended values for the permutations and S-boxes.

The description of the program together with a partial listing is given in Appendix 1. The two important factors to be considered in the software design are the storage and the speed of the implementation. In this design, a mixed approach has been adopted as there was no real constraint on memory or speed. In some places storage space increases are traded to obtain speed. A typical example of such an instance would be the use of in-line code to replace loops. On the other hand, most of the transpositions and substitutions are implemented as functions instead of using tables. Consider for instance, the 48-bit key select function involving the 56-bit initial permutation and then one or two left shifts of the 56-bit string and a 48-bit permutation to produce each of the 16 operational keys required during a cycle through the algorithm. Here these subkeys are generated at the time required. But this whole process could have been implemented using sixteen 48 byte tables, listing equivalent permutations of the current keys bit positions. In the beginning of the program, all the 16 operational keys can be calculated and stored in a table. Then each round through the algorithm can obtain the operational keys directly from the storage table.

The storage space used by this program is approximately 2 kilo-bytes. The encryption times are of the order of 100 milli-seconds. This would allow a maximum throughput of about 640 bits per second. Therefore the algorithm can be implemented with reasonable performance. But it is found that 6502 instruction set is not suitable for high speed implementation of the algorithm. If bit test instruct-ions are available similar to those of Z80, encryption time can be much reduced.

A numerical example showing the output from intermediate stages within an encryption round together with different key and data values after every round is shown in Figure 3.5. It is also found that the results produced by this program and the hardware implementation (Chapter 4) agreed with each other.

Although the algorithm can be implemented in software, it suffers from two major disadvantages. Firstly as seen from above the operation is slow compared to LSI implementation of the algorithm which allows of the order of 1 M bits per second or above. Secondly, in the software approach, the key is stored within the computer system memory which is accessible when the system is in use. With the LSI hardware implementation the key becomes unavailable after it has been entered into the device and hence the security of the overall system has been greatly enhanced. In the next section a possible advantage of the software approach is discussed.

### 3.3.1 A Possible Advantage of DES Software

If the internal functions of the DES algorithm are generated using tables, then it seems that there exists a mechanism to increase the difficulty of cryptanalyzing a series of encrypted messages given an encrypted message and a copy of the corresponding plaintext. Assuming that there is no fatal flaw which is dependent on these internal tables, then it is shown that these tables (ie. permutations and substitutions) can be altered in software implementations (in contrast to the hardware approach) to produce a significantly larger effective key space. One may argue that changing the values of these tables may strengthen the algorithm, but does not reduce the throughput from the rate achieved with the specified standard. Therefore each of these tables can be used as variables, like the key, agreed between senders and receivers. (Note that this idea may not be valid from subsequent sections where it is argued that carefully chosen permutations and substitutions strengthen the DES algorithm and that randomly chosen parameters may introduce weaknesses into the algorithm).

The Initial Permutation (IP) and its inverse($IP^{-1}$) operate on 64-bit data blocks and hence there are 64! different settings possible for this permutation. Note that given IP, $IP^{-1}$ is fixed. Here any bit ordering including an identity transform is assumed to be acceptable but each bit must be used. The tables are defined by a series of 64 unique one byte addresses of bits in the Initial Permutation.

All data and key values are represented in hexadecimal form.

```
Plaintext, X:  FEDCBA9876543210
Key,       K:  FEDCBA9876543210
```

Round 1

```
Plaintext    :  L(0) = FEDCBA98
                R(0) = 76543210

Plaintext after Initial Permutation:  33FF33000F550F55
Key after Permutation choice 1     :  0E98D4BE5498C2FE
Key after 1 left shift             :  1E32AA7CAA3286FC
Key after Permutation choice 2     :  7A1E6C3032163234
Output from S-boxes, S1-S8         :  0C216D50
Output after Permutation P         :  921C209C
Ciphertext after round 1           :  0F550F55A1E3139C

          i.e.   L(1) = 0F550F55
                 R(1) = A1E3139C
```

```
Round 1 -  Data:  0F550F55A1E3139C
           Key :  7A1E6C3032163234
Round 2 -  Data:  A1E3139C7D1EC3B6
           Key :  4A4A4C4C6C522E32
Round 3 -  Data:  7D1EC3B6B66ABD06
           Key :  5C445A6A247A385A
Round 4 -  Data:  B66ABD06F229B504
           Key :  462E305A2C4E6C50
Round 5 -  Data:  F229B5048FC9FBC4
           Key :  60623E78740A487A
Round 6 -  Data:  8FC9FBC40EB8F43D
           Key :  6E1A5642606E5474
Round 7 -  Data:  0EB8F43DC6B370BA
           Key :  487E5A547A225872
Round 8 -  Data:  C6B370BACB7238B9
           Key :  540E3E0E4862527C
Round 9 -  Data:  CB7238B90C8EFF39
           Key :  1E66302C6C283E0C
Round 10-  Data:  0C8EFF39C3DD5634
           Key :  364266704E1C141A
Round 11-  Data:  C3DD5634F5C83C96
           Key :  6E5C0278784E360A
Round 12-  Data:  F5C83C96A38DA004
           Key :  465878345C541C56
Round 13-  Data:  A38DA0040B8B7529
           Key :  3646585E5A4E7806
Round 14-  Data:  0B8B75293393CBF1
           Key :  547662123E441A64
Round 15-  Data:  3393CBF145770966
           Key :  242C76747A3C4A14
Round 16-  Data:  4577096604DE0463
           Key :  1A38167822707E1A
Ciphertext = A933F6183023B310
```

Figure 3.5 - Data and Key values after each DES round

To change IP or $IP^{-1}$ requires an input of 64 bytes. There are 32! different P permutations possible and a 32 byte input vector is required to specify the ordering of 32-bits. Also there are $\binom{48}{32} \doteq 2.25 \times 10^{12}$ different possible expansion (E) permutations for mapping 32-bits into 48-bits. The 16 intermediate keys K(1) to K(16) can be obtained using a 16 x 48 table. Assuming that the 16 key select vectors must be unique, the number of unique combinations is equal to $\prod_{i=0}^{15} \binom{56}{48} - i$ which is approximately $2.77 \times 10^{146}$. Changing the key selection tables requires an input of 768 bytes. Further, there are $\binom{48}{32} = 2.25 \times 10^{12}$ possible combinations for S-boxes which reduce the 48-bit exclusive-ored product to 32-bits.

Thus the number of possible control variables using this table structure is equal to (number of $IP/IP^{-1}$ combinations) x (number of 48-bit expansion combinations) x (number of P permutation combinations) x (number of 6 bits to 4 bits reduction combination) x (number of key select combinations). This is of the order $10^{295}$. For each of these parameter settings, there are $2^{56}$ possible values of key which must be tested if the DES is to be broken by exhaustive attack. For this extended algorithm, an input of 1232 bytes is required. Changing the key merely requires changing any of these 1232 bytes.

The above argument shows how the permutations and substitutions in the DES algorithm can themselves be regarded as 'keys'. But one should be careful in determining the security of such a system. In the above analysis, it has been assumed for simplicity that any one set of permutations and substitutions can be used in the algorithm. But the security of the algorithm may be heavily dependent on the particular values of the S-boxes and the permutations [3]. Thus although the above process shows how the keysize to the DES algorithm can be extended, it must be realized that not all these 'keys' may provide the same security.

## 3.4 Some Characteristics of DES Algorithm

### 3.4.1 Avalanche Effect

If a small change in the key or plaintext were to produce a corresponding small change in the ciphertext, then this might effectively reduce the size of the plaintext or key space to be searched. Hence one of the fundamental requirements of a good cryptographic algorithm is to produce a 'large' change in the ciphertext for a 'small'change in the plaintext or key. The DES exhibits this property referred to as the 'avalanche' effect by Horst Feistel [6]. Actually Meyer has shown in [18] that after five rounds, each ciphertext bit depends on all plaintext bits and key bits. This intersymbol dependence property can be used for error detection and authentication purposes.

Table 3.2 in Figure 3.6 shows the effect of change of a single input bit change in the plaintext. The plaintexts X and X′

X = (11111111 11111111 11111111 11111111 11111111 11111111
11111111 11111111)

X′= (01111111 11111111 11111111 11111111 11111111 11111111
11111111 11111111)

are enciphered with a randomly chosen key and the effect of changing a single bit as a function of the number of rounds is seen. In Table 3.3 in Figure 3.7, the procedure is repeated, now fixing the plaintext and changing a single bit in the key.

A similar effect is seen to occur for changes of single bit in ciphertext and key on decryption.

### 3.4.2 Complementary Property

The DES is invariant under complementation of plaintext, key and ciphertext. The relationship called the complementary property of DES can be expressed as:

$$E_K(X) = \overline{E_{\bar{K}}(\bar{X})}$$

where $E_K$ stands for encryption under key K
X is the plaintext
and the bars represent complementation, that is, bit-inversion.

This property arises because of the way in which the internal subkeys are used in each round. This can be shown as follows:

Section 3.2.1 shows that the expanded version of right half

```
                Key      10000101 11001101 11001011 00011100 10011011 11010000 01000110 00011010

                Round
Plaintext 1     0   :    11111111 11111111 11111111 11111111 11111111 11111111 11111111 11111111
Plaintext 2     0   :    01111111 01111111 01111111 01111111 01111111 01111111 01111111 01111111

                1   :    11111111 11111111 11111111 11111111 01000100 01111101 11111111 00010000
                1   :    11111111 11111111 11111111 11111111 01000000 01110100 11111110 00000000

                2   :    01000100 01111101 11111111 00010000 00101010 11011000 10101000 01010111
                2   :    01000000 01110100 11111110 00000000 01100010 10100111 11010001 10110111

                3   :    00101010 11011000 10101000 01010111 00110101 01010000 01000110 11001000
                3   :    01100010 10100111 11010001 10110111 11010101 01001000 01100111 10001100

                4   :    00110101 01010000 01000110 11001000 10011011 10110000 01100010 01010001
                4   :    11010101 01001000 01100111 10001100 01110110 00001100 10100001 10011000

                5   :    10011011 10110000 01100010 01010001 00111001 00011110 00001000 01011001
                5   :    01110110 00001100 10100001 10011000 11011011 10000111 01010010 11111011

                6   :    00111001 00011110 00001001 01011001 11001101 00001111 11101001 00010010
                6   :    11011011 10000111 01010010 11111011 01101111 01111111 10101110 11001001

                7   :    11001101 00001111 11101001 00010010 10000011 01010000 00110110 00110011
                7   :    01101111 01111111 10101110 11001001 00111111 10110100 11110111 01000110

                8   :    10000011 01010000 00110110 00110011 10111010 10011110 11000100 00010110
                8   :    00111111 10110100 11110111 01000110 01000100 11111110 10001001 00110000
```

(continued on next page)

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 9 | : | 10111010 | 10011110 | 11000100 | 00010110 | 01000111 | 10011000 | 11001001 | 00011001 |
| 9 | : | 01000100 | 11111110 | 10001001 | 00110000 | 10100101 | 11010100 | 11001111 | 01100110 |
| 10 | : | 01000111 | 10011000 | 11001001 | 00011001 | 00101110 | 10110011 | 11000111 | 01010100 |
| 10 | : | 10100101 | 11010100 | 11001111 | 01100110 | 00010110 | 10100001 | 11101011 | 01000100 |
| 11 | : | 00101110 | 10110011 | 11000111 | 01010100 | 11000010 | 00100101 | 11110111 | 01001101 |
| 11 | : | 00010110 | 10100001 | 11101011 | 01000100 | 10111110 | 01111010 | 00000010 | 01010111 |
| 12 | : | 11000010 | 00100101 | 11110111 | 01001101 | 11011100 | 00000101 | 01101100 | 11011001 |
| 12 | : | 10111110 | 01111010 | 00000010 | 01010111 | 01000110 | 10110010 | 11000111 | 01110111 |
| 13 | : | 11011100 | 00000101 | 01101100 | 11011001 | 01001111 | 10100110 | 00000010 | 10111001 |
| 13 | : | 01000110 | 10110010 | 11000111 | 01110111 | 00101101 | 11011011 | 10010000 | 00100100 |
| 14 | : | 01001111 | 10100110 | 00100010 | 10111001 | 01101110 | 10010001 | 01100010 | 10101100 |
| 14 | : | 00101101 | 11011011 | 10010000 | 00100100 | 11000010 | 10100111 | 00100001 | 00011111 |
| 15 | : | 01101110 | 10010001 | 01100010 | 10101100 | 01001010 | 11100001 | 00101111 | 10000110 |
| 15 | : | 11000010 | 10100111 | 00100001 | 00011111 | 01110011 | 01000011 | 11000111 | 00101011 |
| 16 | : | 01001010 | 11100001 | 00101111 | 10110110 | 00100110 | 00011000 | 11110001 | 01101110 |
| 16 | : | 01110011 | 01000011 | 11000111 | 00101011 | 01101001 | 10011011 | 01001100 | 01011101 |
| Ciphertext 1 | : | 00101100 | 11001011 | 01001011 | 10011001 | 00010100 | 01101101 | 10100101 | 00100110 |
| Ciphertext 2 | : | 11111011 | 10111010 | 00001101 | 01010111 | 10010001 | 11000010 | 11101101 | 00011000 |

Figure 3.6 - Table 3.2 : Avalanche Effect in DES; change in plaintext

```
                 Key 1     10000101 11001101 11001011 00011100 10011011 11010000 01000110 00011010
                 Key 2     00000101 11001101 11001011 00011100 10011011 11010000 01000110 00011010

                 Round
Plaintext        0    :    11111111 11111111 11111111 11111111 11111111 11111111 11111111 11111111
Plaintext        0    :    11111111 11111111 11111111 11111111 11111111 11111111 11111111 11111111

                 1    :    11111111 11111111 11111111 11111111 01000100 01111101 11111111 00010000
                 1    :    11111111 11111111 11111111 11111111 01000100 00111101 11111111 01010000

                 2    :    01000100 01111101 11111111 00010000 00101010 11011000 10101000 01010111
                 2    :    01000100 00111101 11111111 01010000 00101110 11011001 11101100 01000110

                 3    :    00101010 11011000 10101000 01010111 00110101 01010000 01000110 11001000
                 3    :    00101110 11011001 11101100 01000110 11111011 11101110 01111001 00101001

                 4    :    00110101 01010000 01000110 11001000 10011011 10110000 01100010 01010001
                 4    :    11111011 11101110 01111001 00101001 01001010 11100010 01111101 11001000

                 5    :    10011011 10110000 01100010 01010001 00111001 00011110 00001001 01011001
                 5    :    01001010 11100010 01111101 11001000 00100001 11100000 00010011 00101011

                 6    :    00111001 00011110 00001001 01011001 11001101 00001111 11101001 00010010
                 6    :    00100001 11110000 00010011 00101011 10001110 00110101 01000111 01111010

                 7    :    11001101 00001111 11101001 00010010 10000011 01010000 00110110 00110011
                 7    :    10001110 00110101 01000111 01111010 10100001 00001100 00110001 01110110

                 8    :    10000011 01010000 00110110 00110011 10111010 10011110 11000100 00010110
                 8    :    10100001 00001100 00110001 01110110 00111011 01011111 10111001 00110010
```

(Continued on next page)

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 9 | : | 10111010 | 10011110 | 11000100 | 00010110 | 01000111 | 10011000 | 11001001 | 00011001 |
| 9 | : | 00111011 | 01011111 | 10111001 | 00110010 | 10100011 | 00001110 | 11000010 | 01011110 |
| 10 | : | 01000111 | 10011000 | 11001001 | 00011001 | 00101110 | 10110011 | 11000111 | 01010100 |
| 10 | : | 10100011 | 00001110 | 11000010 | 01011110 | 00000010 | 00010100 | 10000011 | 11011111 |
| 11 | : | 00101110 | 11010011 | 11000111 | 01010100 | 11000010 | 00100101 | 11110111 | 01001101 |
| 11 | : | 00000010 | 00010100 | 10000011 | 11011111 | 01011110 | 11001011 | 01001100 | 01000111 |
| 12 | : | 11000010 | 00100101 | 11110111 | 01001101 | 11011100 | 00000101 | 01101100 | 11011001 |
| 12 | : | 01011110 | 11001011 | 01001100 | 01000111 | 01000111 | 01010001 | 01111100 | 00011011 |
| 13 | : | 11011100 | 00000101 | 01101100 | 11011001 | 01001111 | 10100110 | 00000010 | 10111001 |
| 13 | : | 01000111 | 01010001 | 01111100 | 00011011 | 10111011 | 01010101 | 11010010 | 01110001 |
| 14 | : | 01001111 | 10100110 | 00000010 | 10111001 | 01101110 | 10010001 | 01100010 | 10101100 |
| 14 | : | 10111011 | 01010101 | 11010010 | 01110001 | 11110010 | 01101010 | 10101101 | 10001111 |
| 15 | : | 01101110 | 10010001 | 01100010 | 10101100 | 01001010 | 11100001 | 00101111 | 10000110 |
| 15 | : | 11110010 | 01101010 | 10101101 | 10001111 | 11011010 | 01000011 | 11111000 | 01010011 |
| 16 | : | 01001010 | 11100001 | 00101111 | 10110110 | 00100110 | 00011000 | 11110001 | 01101110 |
| 16 | : | 11011010 | 01000011 | 11111000 | 01010011 | 00111110 | 01010101 | 01001010 | 01111010 |
| Ciphertext 1 | : | 00101100 | 11001011 | 01001011 | 10011001 | 00010100 | 01101101 | 10100101 | 00100110 |
| Ciphertext 2 | : | 00110010 | 11100111 | 01010000 | 11001101 | 11011011 | 01001001 | 10111111 | 10001000 |

Figure 3.7 - Table 3.3 : Avalanche Effect in DES; Change in Key

of the plaintext undergoes a complicated process together with the key vectors K(i) at each round i. Let this process be denoted by function f. Then,

$$f\ [R(i), K(i+1)] = f\ [E(R(i)) \oplus K(i+1)]$$

where E(R(i)) denotes the expanded right half of plaintext at round i

$\oplus$ denotes Exclusive-or operation

Complementing both R(i) and K(i+1) therefore does not change the value of f and

$$f\ [R(i), K(i+1)] = f\ [\bar{R}(i), \bar{K}(i+1)]$$

Because

$$L(1) = R(o)$$
$$R(1) = L(o) \oplus f\ [R(o), K(1)]$$

complementing plaintext X means complementing L(o) and R(o). Complementing key K means complementing K(1), ..., K(16). Therefore this results in complementation of L(1) and R(1). By induction, this can be extended to L(2) R(2), ..., L(16)R(16) and hence to ciphertext C.

Because of the complementary property of DES, if a cryptanalyst could obtain $Y_1 = E_K(X)$ and $Y_2 = E_K(\bar{X})$ for an arbitrary X, he could reduce the size of the key space he must search from $2^{56}$ to $2^{55}$. The cryptanalyst enciphers X with all keys K that start with a 'o'. The resultant ciphertext is compared with $Y_1$ and $Y_2$. If $C \neq Y_1$ the key in use is not K, if $\bar{C} \neq Y_2$ the key in use is not $\bar{K}$ (which starts with a '1'). That is, effectively this symmetry reduces the search effort by 50 percent under a partially chosen plaintext attack. An example illustrating this complementary principle of the DES obtained using the DES software is shown in Figure 3.8. This complementary behaviour can be avoided by selecting the S-boxes using one or more key bits directly,instead of employing the modulo 2 operation. On the other hand, this complementary property can be used advantageously for testing purposes.

Key: 1000100010001000100010001000100010001000100010001000100010001000

Plaintext: 0011011110110100111010100110110000100011110010111010000111111110

Ciphertext 1: 10110001011100000110110110011001000100101011011111111110011110011

Complementation of key: 011101110111011101110111011101110111011101110111011101110110111

Complementation of Plaintext: 1100100001001011000101011001001111011100001101000101111000000001

Ciphertext 2: 01001110100011111001001001100110111011010100100000000001100001100

Ciphertext 2: Complement of Ciphertext 1.

Figure 3.8 : Complementation in DES

## 3.5 Design Criteria

Although the DES algorithm has been made public, the design criteria behind the specific choice of some permutations and substitutions remain still classified. For example,one design criterion for the DES was that the permutation schedule must ensure that each output (ciphertext) bit is a function of all input bits (plaintext and key) after a minimum number of rounds. It is reported that [3] the initial approach of random selection of the parameters such as the permutations and substitutions had to be abandoned because they introduced weaknesses into the algorithm. Instead the parameters were randomly generated and tested against the design criteria. It is believed that a significant portion of the random designs were rejected in this process. One of the most important parameters in the algorithm is the S-boxes and they are considered in the next section.

### 3.5.1 S-Boxes

All the operations involved in the DES algorithm except the S-box mappings are linear in binary arithmetic. Therefore the S-boxes play an important role in the security of the DES. It is therefore crucial that the S-boxes not be affine or the whole algorithm would be affine. A study by Hellman et al [19] found that none of the S-boxes are affine. But they point out that there are a number of questionable quasi-linear structures which may tend to weaken the algorithm. But currently no methods are found which could exploit these quasi-linearities in reducing the search effort. The designers of DES argue that it should not be surprising that certain parameters contain some structural properties different from those expected to result from the use of purely random selection. Further they claim that [3] their design effort showed that carefully selected S-box functions produce a much stronger algorithm than one based on random designs.

The DES S-boxes are non-invertible 6 to 4 non-linear mappings. Each S-box is found to have the property that changing any single bit of the input while keeping the others constant always changes at least two bits in the output. This property results in the error propagation property and is essential to avoid the key clustering attack [2]. This property plays an important role in causing the avalanche effect mentioned earlier. On average it is found that the number of output bits that change in the DES for a single input is above 2 (2.5 - 2.6) [19].

Another principle underlying the design of the S-boxes is that the difference between the number of zeroes and the number of ones in the output when any one of the input bits is held constant is a minimum. This design criteria is believed [20] to strengthen the algorithm.

### 3.5.2 Initial and Final Permutations

The plaintext input to the DES is first permuted using IP before transformed by the sixteen rounds. The output of the final round undergoes an inverse permutation $IP^{-1}$ where $IP^{-1}$. IP = I,the identity permutation. It is important to note that in a single encryption process, IP and $IP^{-1}$ do not cancel each other as IP operates on the plaintext and $IP^{-1}$ operates on the ciphertext produced by the 16th round. On the other hand, the permutation $IP^{-1}$ performed during the encryption process is cancelled by the permutation IP performed at the beginning of the decryption process.

It seems that the permutations IP and $IP^{-1}$ do not have any cryptographic significance. Possible reasons for their inclusion in the algorithm are that they facilitate the implementation of the algorithm for a particular IC layout or they result from the way the data is loaded into the DES chip. These permutations IP and $IP^{-1}$ can be absorbed by proper layout into the main algorithm and they do not slow down the algorithm. The algorithm designers claim that the permutations do have cryptographic value and they contribute to the security of the DES. This could be possible as follows: In many applications, ASCII characters are used for the plaintext. Because of the particular ASCII coding and the frequency distribution of the English alphabets, the distribution of '1's and 'o's will not be uniform. The IP permutation groups every $i^{th}$ bit of each input byte into $j^{th}$ byte of the permuted output. This preprocessing of the plaintext may redistribute the '1's and 'o's in a uniform way before inputting to the 16 rounds and thus may enhance the security.

### 3.5.3 P-Permutation

The function of the P-permutation in the DES as in all substitution-permutation ciphers is to provide the element of diffusion [7]. This concept was first introduced by Shannon to spread the dependencies of output bits on input bits through successive rounds to achieve total mixing.

### 3.6 Criticism and Weaknesses of DES

The controversy surrounding the DES essentially comes from two main objections given below:

(i) the small size of the key space and
(ii) the algorithm's unpublished design principles.

Other criticisms of the algorithm include the small number of rounds, the two 'redundant' permutations IP and $IP^{-1}$ and a relatively simple key schedule operation.

To investigate these criticisms, two workshops were organized by the National Bureau of Standards (NBS). The first workshop analysed the complexity of the algorithm and restated that no short cut methods were found [20]. The second workshop concluded that a key searching machine cannot be built before 1990 and will cost several tens of millions of dollars with a probability factor of being available even then of about 0.1 to 0.2 [21]. Let us now consider the major criticisms of DES in turn.

#### 3.6.1 The Key Length

By far the greatest source of controversy has been the choice of 56-bits as the key length. The length of the key determines the feasibility of key trial. Critics argued that the length of the key is too small and that the key space is amenable to exhaustive search. Diffie and Hellman [22] disputed the NBS claim that trying all possible keys is not economically feasible and estimated that a DES key can be recovered by exhaustive search for approximately $ 5000 worth of computation time on a special purpose machine. The special purpose machine would consist of a million LSI chips and could try all the $2^{56}$ keys in one day. The cost of such a machine using the semiconductor technology of the mid-seventies was estimated to be in 20 million dollar range. More recently [84], the cost of a 2-day average search time machine is estimated to be around 50 million dollars. This meant that it is out of reach of most groups with the possible exception of government security agencies. It was predicted that in 10 years, the machine will cost approximately 200,000 dollars and recovery of one key will be around 50 dollars. Such attacks assume that the DES is being used in the standard code book mode.

Alternative brute force attack on the DES algorithm is to use a table look-up approach. In this approach all the ciphertexts resulting from enciphering a chosen plaintext are stored and sorted using all the $2^{56}$ keys. The amount of memory required for this attack is of the order of 56 $(2^{56}) \doteq 4 \times 10^{18}$ bits which is enormous. This would require 4 billion magnetic tapes (about $10^9$ bits per tape), costing about 80 billion dollars ($20 per tape) [4]. In [23], Hellman proposed a trade off between time and memory. Like table look up approach, this technique requires precomputation and storage of a table. However, with this technique, the table size is only of the order $O(n^{2/3})$ instead of the previously $O(n)$ where $n = 56(2^{56})$ bits. This technique also requires searching and the search time is of the order $O(n^{2/3})$ instead of $O(n)$ where $n = 2^{56}$. Hellman predicts that with proper combination of precomputation and searching, a machine may be constructed that would recover the key with high probability in one day. This in turn led to the suggestion that a bigger key length of 128 bits is required.

Further the key space is reduced if the key is not chosen as a 56 randomly selected bits. For instance, if the key is selected as eight characters, each character being one of say 64 possibilities then the number of distinct keys which need to be tested is $(64)^8 = 2.8 \times 10^{14}$ which may be quite feasible.

The key space is also reduced by the invariant property of DES under complementation. As seen in Section 3.4.2, this symmetry can be used to reduce search effort by 50% under a partially chosen plaintext attack.

### 3.6.2 Unpublished Design Principles

The second major criticism of the DES algorithm is that the design criteria behind the choice of some of the parameters are not disclosed publicly. Those parameters are the S-boxes, fixed permutation P and the key schedule operation.

The classified design principles led some critics to argue that deliberate trapdoors might have been incorporated in the algorithm by the designer to his advantage. This argument is based on the fact that it is possible to design 'innocent-looking' S-boxes which contained trapdoors [19]. Further, regular structures were discovered in certain parts of the DES algorithm which were surprisingly similar to a type that can be used to build a trapdoor into the system [19, 24]. Like the complementation property, it may be possible to save another factor of two with carefully chosen S-boxes thus saving a total of 75% over exhaustive search. Lexar Corporation [24] examined the properties of the substitutions and permutations and reported some 'peculiar' properties of the S-boxes but to date no feasible cryptanalytic technique for DES has been found.

### 3.6.3 Number of Rounds

DES is also criticized for having a small number of rounds and it is suggested that the number of rounds should at least be increased to 32 [19]. The number of rounds controls the mixing of input bit values, that is, the amount of diffusion introduced by the co-ordinate permutation. An increase in the number of rounds will produce a greater avalanche effect. A two round DES was successfully cryptanalyzed [19] by exploiting the correlation between $L_2R_2$ and the input. Such correlation is believed to disappear after eight rounds [6].

Some statistical analysis to test the randomness of the DES output sequences and any correlation between plaintext input and ciphertext output is considered in Chapter 6.

### 3.6.4 Key Schedule Algorithm - Weak and Semiweak Keys

The cryptographic strength of the DES will be reduced if some of the internal subkeys derived from the external key are the same. In the extreme case, if all the internal keys are the same then the key space is reduced to $2^{48}$. Referring to the key schedule algorithm given in Figure 3.3, this situation occurs whenever all bits in the register C are all ones or zeroes and the bits in register D are all ones or zeroes. There are four such weak keys altogether [3]. These weak keys also have the property that there is no difference between the operations of encipherment and decipherment, that is, $E_kE_k(X) = X = D_kD_k(X)$ for weak keys. There is another set of keys defined as semiweak which have the property that only two different internal keys are produced each occurring 8 times. There are six such semiweak keys and they have the property $E_kE_{k'}(X) = X = E_{k'}E_k(X)$, where k and k′ are distinct semiweak keys. Further there are 48 other external keys which produce only 4 different internal keys.

These special keys do not pose any threat to the security of the algorithm because the number of such keys is small compared to the total set. Provided that the keys are randomly selected the probability of choosing such a key is very small. Further these keys can easily be avoided during key generation.

The key schedule algorithm can be improved by incorporating non-linearity in the internal key generation process [39]. A suitably shifted external key can be inputted to a substitution-permutation process in each round to produce the internal key as shown in Figure 3.9. The scheme allows, in addition to the enlargement of the key space, other cryptographically desirable properties such as error propagation. As the substitution permutation function is already available in the algorithm, this requires minimum additional hardware.

## 3.7 Cryptanalysis of DES

Cryptanalytical methods can be divided into two subcategories namely deterministic or analytical and statistical methods.

In a deterministic approach, the cryptanalyst attempts to express the desired unknown quantity (such as the key or the message) in terms of some other known quantity or quantities (such as given ciphertext or given plaintext and corresponding ciphertext). One method of attack is therefore to represent the 64 ciphertext output bits as functions of the 56 key bits in a known plaintext attack and try to solve the 64 non-linear equations over GF(2) for 56 unknowns. This problem is an NP-complete problem [8] and hence is difficult to solve in general. Such an analytical attack of the DES was proposed in [25]. The attack proved to be manageable with affine S-boxes but infeasible when the real DES was considered. IBM and NSA also reported to have conducted similar attacks as part of their validation process with no short cut solutions being found.

In the statistical approach, the cryptanalyst attempts to exploit statistical relationships between plaintext, ciphertext and key. To thwart statistical attacks, the algorithm's output should be pseudo-random even for highly structured inputs. In other words,for a large set of plaintext and key inputs one must not be able, on the basis of statistical analysis, to reject the hypothesis that the

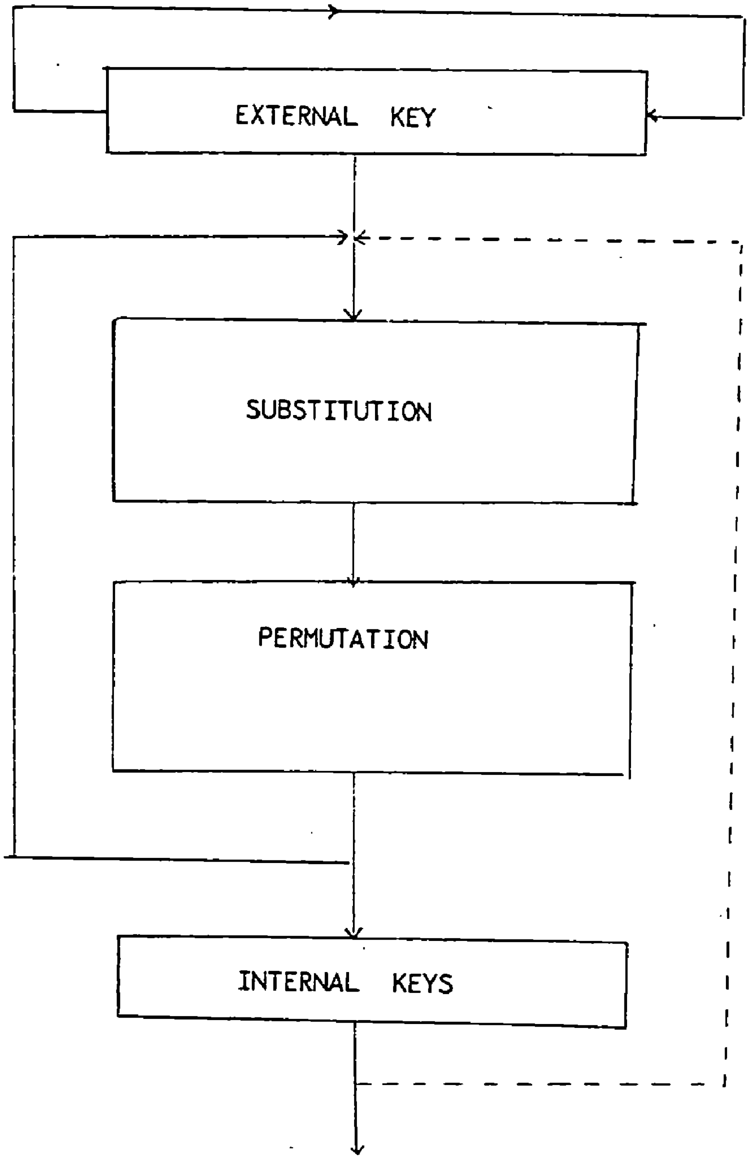


Figure 3.9 - Non-linear Key Schedule Algorithm

output bit stream is random. Some fundamental statistical tests have been performed on the DES output streams and they are discussed in Chapter 6.

Finally one should mention the brute force attack to find the unknown quantity such as the message or key by using a direct search method, trial and error or exhaustion. As mentioned earlier the key length is an important factor on this type of attack and greater the key length bigger the work factor to perform the key exhaustion.

### 3.8 Discussion

If an algorithm is defined to be cryptographically strong when the algorithm complexity is such that,

(i) using all known shortcut solutions, it is not practically possible to solve for the key or for the message

(ii) it is too costly to employ simple methods such as key exhaustion because too much time and/or hardware are required

then,despite a lot of controversy surrounding the DES, DES can be considered to be a cryptographically strong algorithm. As far as the author is aware not a single successful 'break' has been developed that can produce or determine the unknown key. Further,the life of the DES can be extended (until a shortcut solution is found) by enciphering the data two or more times with different keys. Double encryption will significantly increase the difficulty of intrusion requiring of the order of $2^{56}$ words of memory and $2^{55}$ operations [3]. However it is preferred that such a multiple encryption process is specified as part of the algorithm itself. The security can also be improved by employing DES under various chaining modes. They are considered in subsequent chapters. In account of this, the DES algorithm has been chosen to be employed in the design of an encryption interface unit to be used in a communication network. This forms the subject of the next chapter. This chapter is concluded by noting that the DES can be used as a building block for designing more sophisticated algorithms. For instance,the DES itself can be used as

**the cipher function f in Figure 3.2. This gives rise to the scheme shown in Figure 3.10 where a block of 128 bits forms the plaintext.**

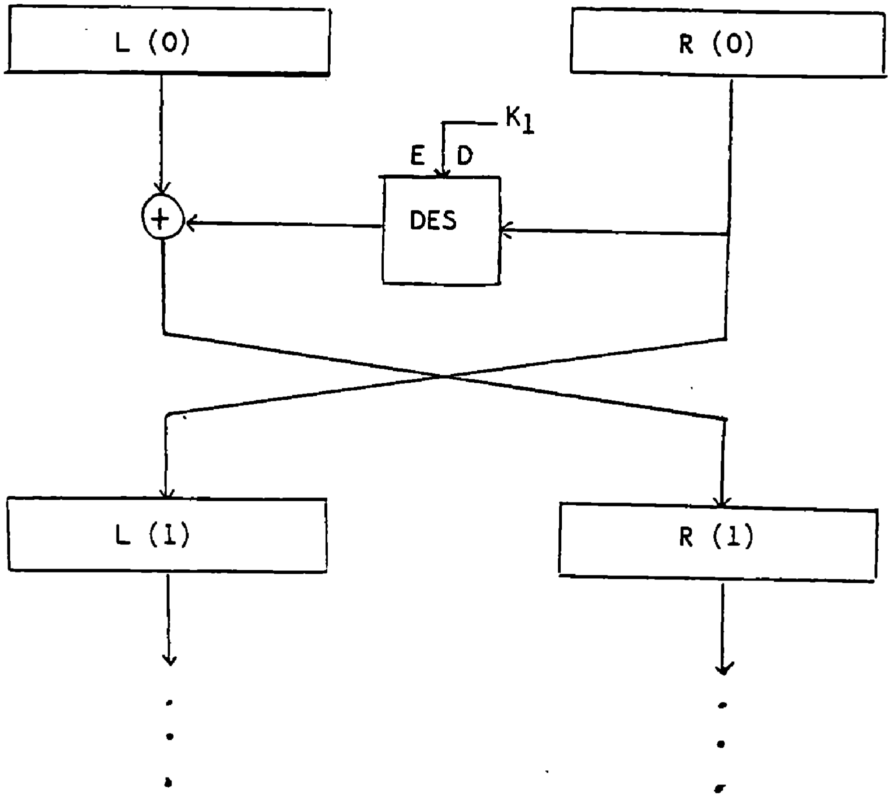


Figure 3.10 - Extended DES Algorithm

# CHAPTER 4

## DESIGN OF ENCRYPTION SYSTEM: SYSTEM HARDWARE

### 4.1 System Requirements

The requirements for the encryption system may be classified into three groups namely

(i) security requirements
(ii) Operator requirements and
(iii) technical requirements.

### 4.1.1 Security Requirements

Briefly, the security requirements are as follows:

(a) The encryption algorithm used in the system should be cryptographically strong. That is, it should withstand at least all the major cryptanalytic attacks described in Section 2.2.

(b) There should be a large number of user selectable keys to prevent the opponent employing exhaustion techniques to determine the key selected.

The DES algorithm outlined in Chapter 3 meets these conditions very well.

(c) The use of the algorithm within the system should be carried out in such a way that the system as a whole is at least as strong as the initial strength of the algorithm itself.

(d) At no time, keys used by the algorithm must be stored in plain form within the system as otherwise it would enable any unauthorized user to its recovery.

(e) Key changes should be made easy to implement.

(f) A high degree of physical security measures must be provided to protect the system against intentional or accidental threats.

### 4.1.2 Operator Requirements

The system should be made 'user friendly' thus allowing any user to operate it efficiently with little training. This aspect is very important in the case of large networks. The unit should also be easy to install.

### 4.1.3 Technical Requirements

The technical requirements for the encryption system depend to a great extent upon the use to which the system will be put. This particular encryption system is designed to be used in the following applications:

(a) transmission/reception of encrypted/plain data in a point-to-point communication security system not containing a host computer;

(b) a local encrypted/plain data storage/retrieval system using floppy disks;

(c) transmission/reception of encrypted/plain data over the communication link to a host computer and storage/retrieval of encrypted/plain data using a host computer, in particular with the Prestel Viewdata computer.

The requirements for a storage/retrieval encryption system are quite different from those necessary in a point-to-point communication system. Some of the aspects to be considered are:

1. Real time processing and transmission delays: These are particularly important in point-to-point communication where information flow is irregular and messages may not be of fixed format.

2. Transmission data rates: This aspect is important in determining whether asynchronous or synchronous transmission is needed. We are concerned here with only asynchronous transmission of data rates up to 1200 bauds. Further the

communication with Prestel computer is asymmetric full duplex in the sense that it requires 1200 bauds for transmission and 75 bauds for reception.

3. Where a mixture of plain and encrypted texts is required to be sent, switching between encrypted text mode and plaintext mode should be easy. This should be carried out within the message so that the receiving unit can follow identical mode changes.

4. In the case of storage/retrieval system with a host computer, since the encrypted information is likely to pass through the host computer control unit where certain control characters are generated and detected, there is a need to prevent the occurrence of these special characters in the encrypted information. This implies that the encryption system should be made transparent to the host computer protocols. This is particularly relevant when the system is used in Prestel network.

5. Standard editing facilities such as character delete, line delete etc. must be available to the system users.

## 4.2 General System Description

A symmetric encryption-decryption system in an end-to-end communication configuration is shown schematically in Figure 4.1. The operation of the system can be summarised as follows. Referring to Figure 4.1, the plaintext from an Apple terminal forms the input to the encryption system. The plaintext is a combination of any of the alphanumerical characters from the keyboard. The key required for the encryption process is entered from the terminal keyboard under the user control prior to the commencement of the session and is stored in an inaccessible area within the encryption device. The encryption system operates on the input sequence using the key to produce a ciphertext sequence. This cipher is transmitted through input-output communication controller to the modem. The modem converts the binary data pulses to analog frequencies using Frequency Shift Keying (FSK) scheme for transmission over the public switched telephone network.

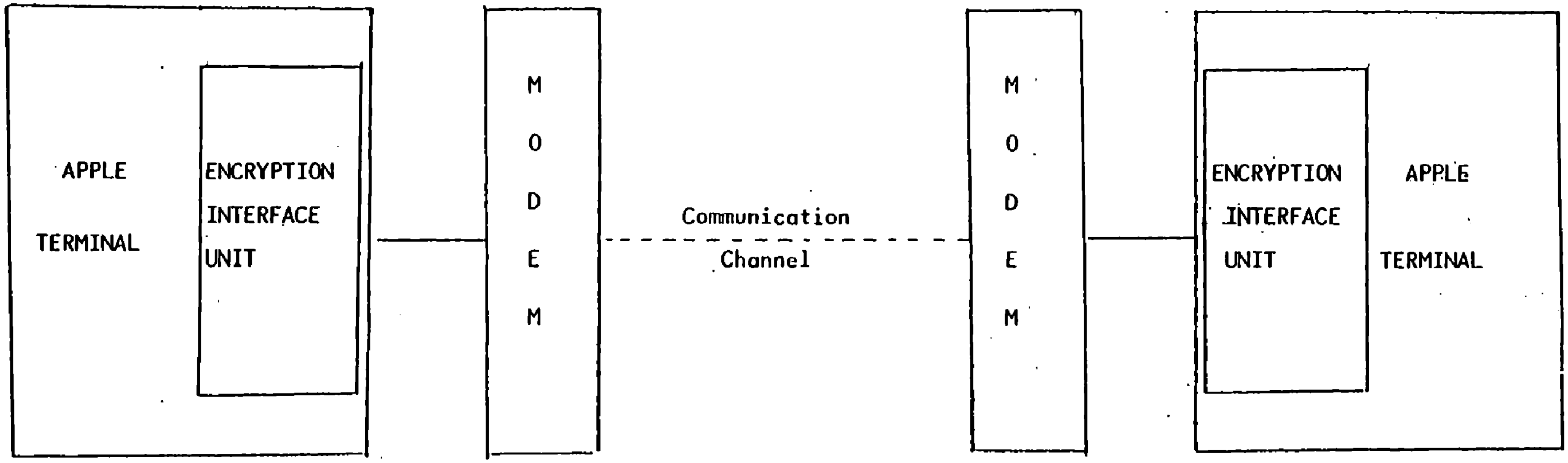


Fig. 4.1 - Point to Point System Configuration.

At the receiving end, another modem converts the FSK signals back to digital form which is then deciphered to regenerate the original plain-text. This deciphered information can then be displayed on a Visual Display Unit or printed on a line printer.

The communication link between Apple microcomputers is a half-duplex one thus allowing transfer of data in either direction but not at the same time.

From Figure 4.1 it is seen that in this implementation, the encryption system is incorporated as an in-built feature of the terminal, rather than as a separate unit. The built-in implementation is superior to stand-alone implementation in the area of access prevention as the former technique greatly reduces the chances of detection between the terminal and the encryption unit where the text is in plain form. However this technique may be difficult to implement in existing systems and can require major redesigns. This is where the stand-alone unit has the advantage in that it can simply be inserted into existing networks with little or no impact to existing equipment.

The different sections which constitute the system are:

1. Apple microcomputers - comprising 6502 microprocessor, memory, diskette interface and Language card.

2. The encryption interface unit.

3. The modulator-demodulator equipment.

4.2.1 Apple Microcomputer

In this project, Apple microcomputers have been used to form a two mode network. The Apple microcomputer is a 6502 microprocessor based system with diskette interfaces, disk driving units, Integer card and a standard keyboard. The processor can directly reference up to 64 kilobytes of memory. Along the back of the Apple's main board, is a row of eight 'slots' or peripheral connectors. In seven of these slots, peripheral interface boards designed for Apple system can be installed. In slot 0, Integer firmware card is installed. Each slot has a 2K-byte of common shared memory associated with it, with a view to holding programs or driving subroutines of the peripheral interface card. The designed encryption interface card is installed in slot number 2 providing a memory range of C800 to CFFF.

### 4.2.2 Encryption Interface Unit

The unit is designed utilizing the 6502 microprocessor of the Apple microcomputer system. A block diagram showing the various sections of the unit is illustrated in Figure 4.2. The sections are:

1. Data Security Device.

2. Serial Input-Output Controller.

3. RS-232C Interface Circuits.

4. Timing Circuits.

5. Decoding Circuits.

6. Memory.

### 4.2.2.1 Circuit Description and Operation

A complete circuit diagram is shown in Figure 4.3. Some important aspects of the circuit are now briefly considered.

### 4.2.2.2 Data Security Device

As mentioned earlier, it has been decided to use the DES algorithm in hardware in the design of the interface unit. The choice of the data security device incorporating the standard was restricted due to limitations of not only their availability but also their access. A Western Digital device has been used for this purpose because of its ready availability. The device is made in n-channel silicon gate MOS technology and is TTL compatible on all inputs and outputs. The device can be interfaced to a wide variety of processors though it is tailored to the Intel 8080A class microprocessor. The device performs 64-bit block encryption and decryption using 56-bit key. The block diagram, shown in Figure 4.4, illustrates the internal structure of the device. It contains 64-bit data register, 56-bit key register, an 8-bit command/status register plus the necessary logic to check key parity and implement the DES algorithm. The device has a single 8-bit data bus buffer with tri-state operation through which

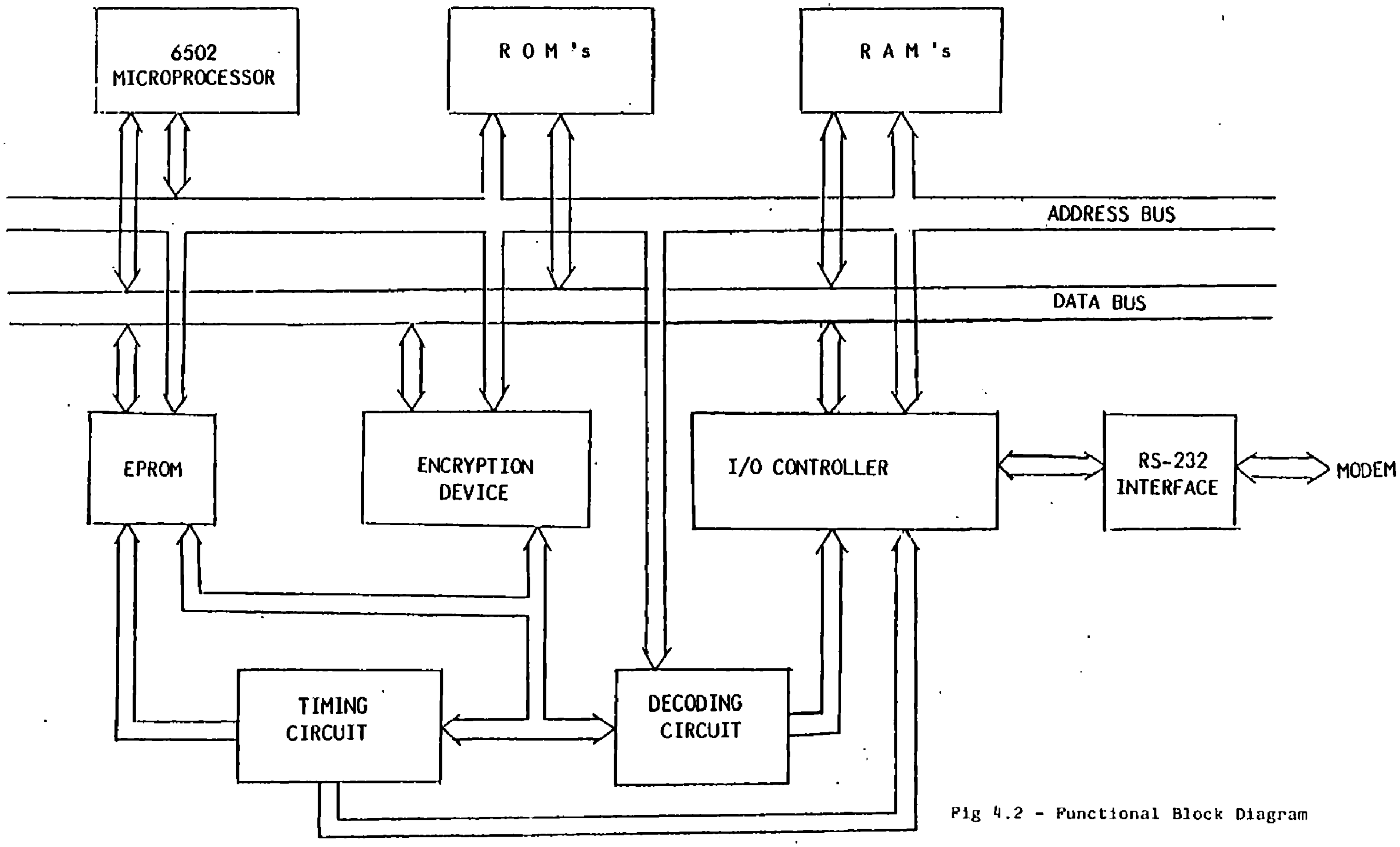


Fig 4.2 - Functional Block Diagram

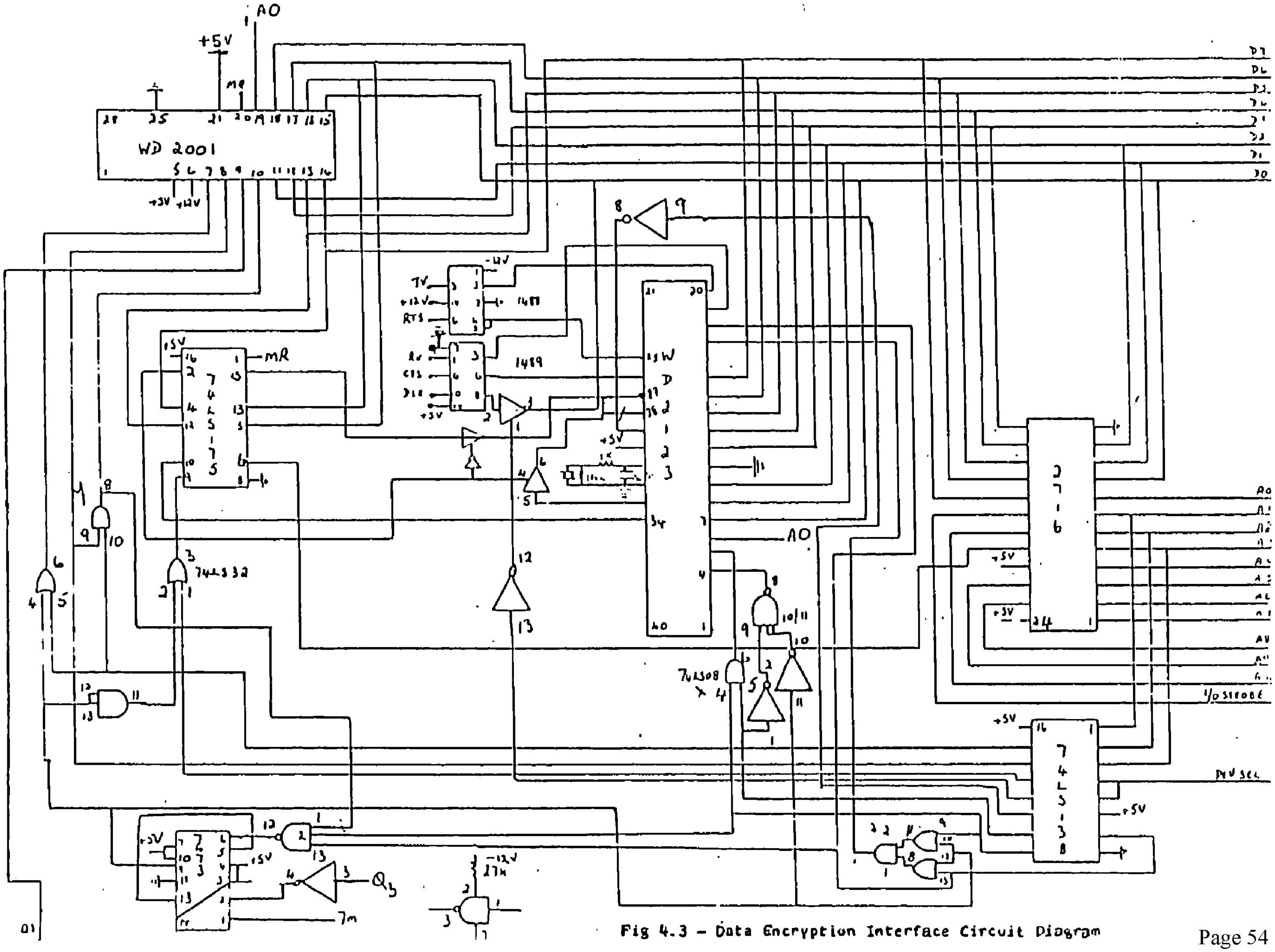


Fig 4.3 - Data Encryption Interface Circuit Diagram

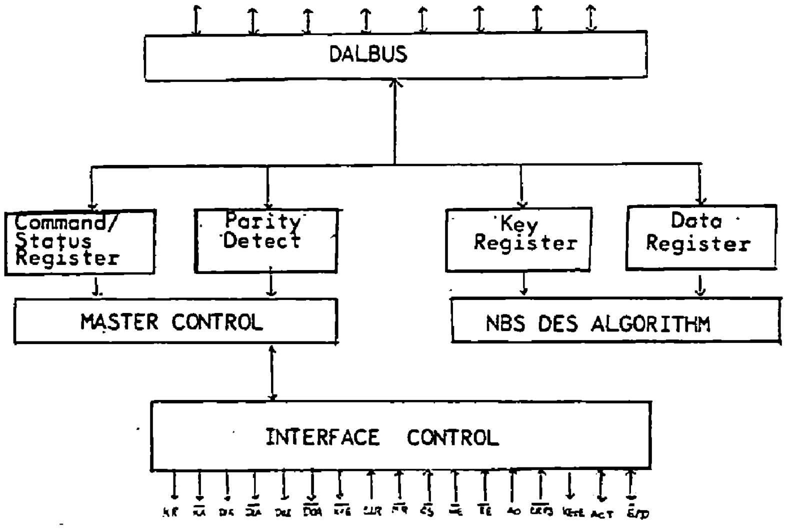


Fig. 4.4 Internal Architecture of Data Security Device

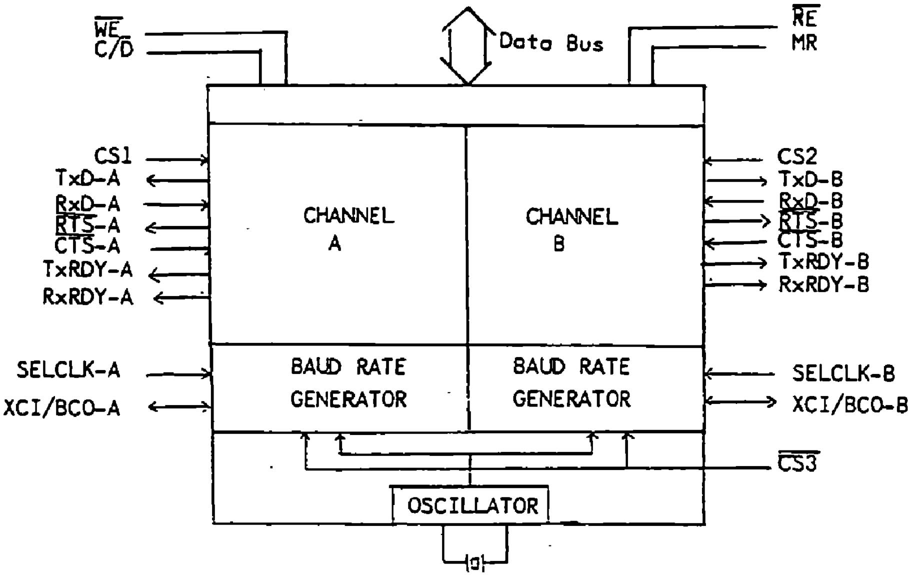


Fig. 4.5 - Internal Architecture of Serial I/O Controller

data may be entered into the key register or the data register. Output data from the status register or the data register is also switched through the data bus buffer. The device can be programmed either via the input lines or via the data bus. In this design, the second approach has been adopted. The programming of the device will be briefly discussed in system software (Chapter 5).

#### 4.2.2.3 Serial Input-Output Controller

There are a number of such devices which would be suitable for transmission and reception of plain and enciphered data. Again a Western Digital device, dual enhanced communications element (DEUCE) has been used. It is primarily designed to operate in an 8-bit microprocessor environment.

The controller contains two independent full-duplex asynchronous receiver/transmitter channels and two internal baud rate generators associated with the two channels. A block diagram of the internal architecture of the device is shown in Figure 4.5. Communications between the controlling CPU and the two receiver/transmitter channels or the two baud rate generators occurs via the 8-bit data bus through a common set of bus transceivers. The use of this complex device has enabled the development of a compact encryption system.

Each channel has associated with it a number of registers such as the Transmit Holding Register, Receive Holding Register, Command Register, Mode Register, Status Register etc,which can be programmed to transmit and receive asynchronous serial data. It performs serial to parallel conversion on data characters received from the modem and parallel to serial conversion on data characters received from the CPU. The CPU can read the status of either channel at any time. Status information includes the type and condition of transfer operations being performed by the device as well as any transmission error conditions. Internal control of each channel is achieved by means of two internal microcontrollers one for transmit and one for receive. A block diagram of one of two communication controllers is shown in Figure 4.6. The control registers,various counters and external signals provide inputs to the microcontrollers, which generate the necessary control signals to send and receive serial data according to the programmed protocol.

The device also contains Baud Rate Registers which can be

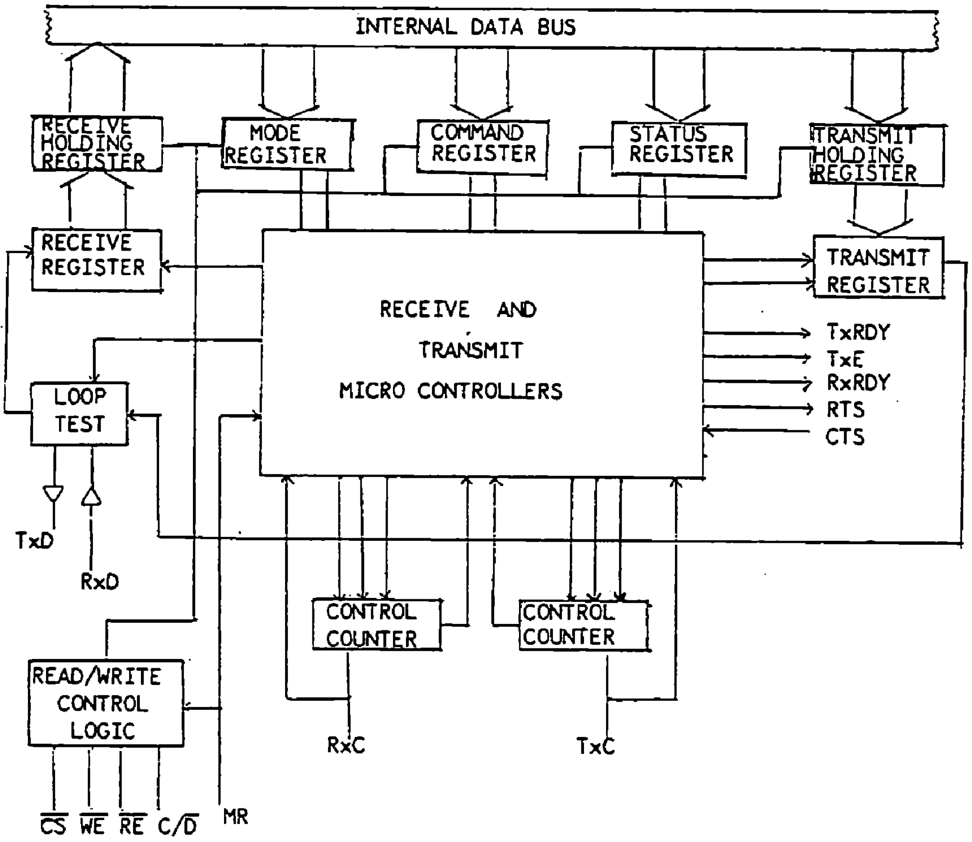


Fig. 4.6 - Block Diagram of a Receive/Transmit Communication Controller

programmed to desired data rates by loading them with the appropriate number of bauds. The contents of these registers are then decoded and then addressed to a frequency select ROM for the generation of proper frequency by the divider circuitry and the control logic during operation. A diagram of one of the two Baud Rate generators is shown in Figure 4.7. The baud clock source is selected using select clock pin (SELCLK) in conjunction with the clock select bit (CR1) in the Command Register (see Figure 4.8). When the bit CR1 is high, the external clock mode is selected. This means that the Transmit and Receive clocks are internally tied together and SELCLK input determines whether those clocks are driven from the internal baud rate generator (SELCLK high) or from the external clock input XCI/BCO (SELCLK low). If the internal baud rate generator is selected, then the external clock input becomes a baud rate generator clock output. When the bit CR1 is a logic '0' then the internal clock select mode is chosen. The transmit clock is driven by the internal baud rate generator clock and the receive clock is driven by the SELCLK input. The XCI/BCO pin then becomes the baud clock output, the transmit clock. The inputs to the SELCLK and XCI/BCO pins are derived using the simple logic circuit shown in Figure 4.8(a).

When the signal present on the line marked X is low and the signal at Y is high, this enables the tristate Z1 and sets the SELCLK to be high. With the bit CR1 at logic '1', this would result in both the transmit and receive clocks tied together and driven by internal baud rate generator. This mode shown in Figure 4.8(b) is adopted for point-to-point communication between two Apple microcomputers.

When the signal present on the line marked X is high and the bit CR1 is at logic '0' then the tristate Z2 is enabled. The XCI/BCO output is then connected to the SELCLK input which drives the receive clock. The transmit clock is driven by the internal baud rate generator. This configuration thus allows to transmit and receive at different frequencies using only one channel without using an external baud rate generator. This is one of the main reasons for using this I/O controller. This mode of operation, shown in Figure 4.8(c), is adopted when the interface is used in the Prestel network where the transmission data rate is 75 bauds and receive data rate is 1200 bauds.

The signals Clear To Send (CTS) and Request To Send (RTS) to and from the device are used in controlling the modem. These

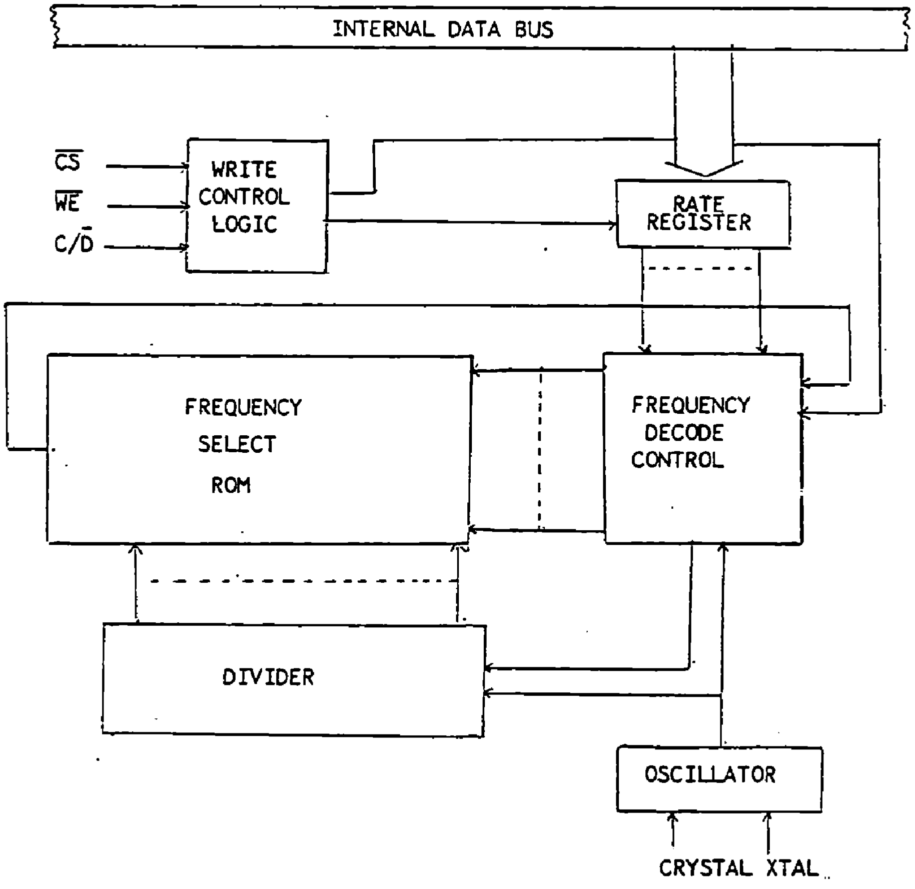


Fig. 4.7 - Baud Rate Generator Block Diagram

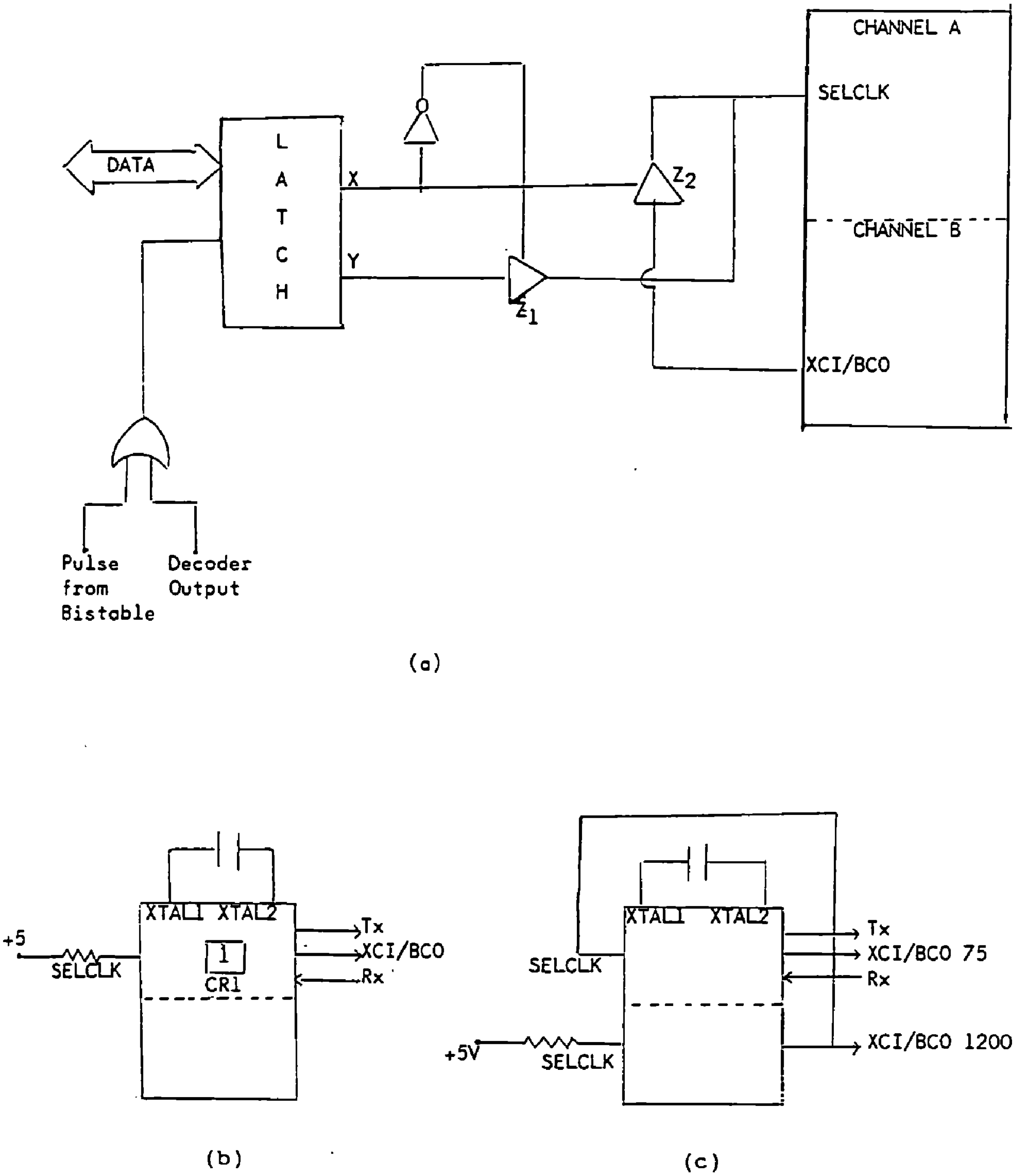

DATA
L A T C H
X
Y
Z2
Z1
CHANNEL A
SELCLK
CHANNEL B
XCI/BCO
Pulse from Bistable
Decoder Output
(a)
+5
SELCLK
XTAL1 XTAL2
1
CR1
Tx
XCI/BCO
Rx
(b)
XTAL1 XTAL2
SELCLK
+5V
SELCLK
Tx
XCI/BCO 75
Rx
XCI/BCO 1200
(c)

Fig. 4.8

signals are discussed in the next section.

### 4.2.2.4 RS-232C Interface Circuits

These circuits are used to interface between Data Terminal Equipment (eg Apple computer terminal) and Data Communications Equipment (eg Modem) in conformity with the specification of EIA standard number RS-232C. The RS-232C standard defines all the control and data signals (including signal levels) used to enable the modem and the terminal to operate together and transfer data. For this purpose, a quad line driver (MC 1488) and its companion circuit (MC 1489) quad line receiver are used to provide a complete interface system between TTL level signals from the system to RS-232C levels.

The interface control lines can be divided into two groups, one concerned with the control sequences for connecting the modem to the line and the other concerned with the sequences controlling the application (and detection) of signal to the line.

The signals Data Terminal Ready (DTR) and Data Set Ready (DSR) belong to the first group. The DTR signal from the terminal is used to tell the modem to answer the telephone (ON) and to hang it up (OFF). In this design, this signal is used to indicate whether the terminal power is ON or OFF. The DSR signal is used to indicate to the terminal that the modem is in a state in which it is capable of transmitting data.

The group of RS-232C control signals that relate to the application and detection of time signal are: Request To Send (RTS), Clear To Send (CTS) and Data Carrier Detect (DCD). The logical interaction of these signals is simple when they operate in a two-wire half-duplex mode and complex when terminals operate a four-wire full-duplex mode. In this design, the point-to-point communication between two Apple microcomputers is half-duplex whereas the communication with the Prestel computer is asymmetric full-duplex.

The Request To Send signal from the terminal is used to indicate to the modem that the terminal wishes to transmit data. The modem then responds by initiating its transmit mode. When the modem has reached a steady state carrier condition, and is ready to transmit, it sends a Clear To Send signal to the terminal. However, CTS does not imply a positive verification that communication with the other end has been established. Therefore the Data Carrier Detect signal is used to indicate that the receiving modem has detected a line signal (carrier) from the other end.

In point-to-point communication, initially when both ends are waiting for transmission to begin, the RTS signals of the serial input-output controller are at logic '1' and the modem will be holding the CTS inputs to the serial I/O device high. The terminal which wishes to transmit activates the RTS by making it low and does not start transmitting until CTS is turned ON. The RTS/CTS time delay is determined by specific hardware settings in the modem and is vitally important in determining the overall efficiency of data link use. Actually, the designed encryption system has been tested with two different modems - Modular Technology and the standard BT† ones. It is found that the former produced large amount of noise spikes when the modem switched from one mode to the other. This necessitated some complex delay routines to be introduced to reject the noise spikes. A typical interface timing diagram is shown in Figure 4.9. When connected to asymmetric full-duplex Prestel system, the RTS is kept ON continuously and the CTS ON is used to indicate that the transmission can start. This is an easy approach to ensure that the modem will operate properly in the full-duplex mode.

#### 4.2.2.5 Timing Circuits

This block in Figure 4.2 consists of logic circuitry to provide the correct timing signals such as CHIP SELECT, READ, WRITE etc for the operation of the interface unit. The inputs to this block include amongst others, the clock signals from the Apple microcomputer system for proper synchronization.

The 6502 microprocessor has two clock signals $\Phi_0$ and $\Phi_1$ of 1.023 MHz which are complementary to each other. In addition, a general purpose timing signal, twice the frequency of the system clocks but asymmetrical and an intermediate timing signal of 7.159 MHz are also available. The microprocessor uses its address and data buses only during the timing period when $\Phi_0$ is active. When $\Phi_0$ is low, the microprocessor is carrying out internal operations and does not need the address buses. These timing signals are shown in Figure 4.10.

Let us now very briefly consider the main timing signals required for the data security device and the serial input-output controller.

The timing diagram of a typical READ cycle of the encryption device is shown in Figure 4.11. The Data Output Request (DOR) bit in the Status Register of the device is used to indicate whether the

† British Telecom

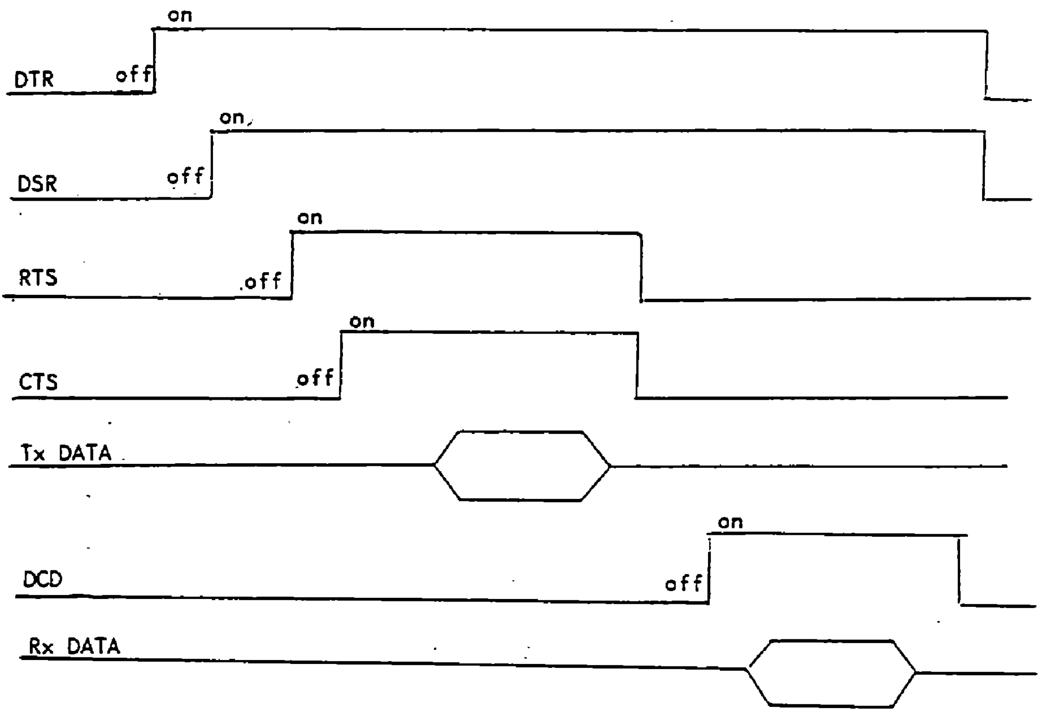


Fig. 4.9 -Interface Control Timing.

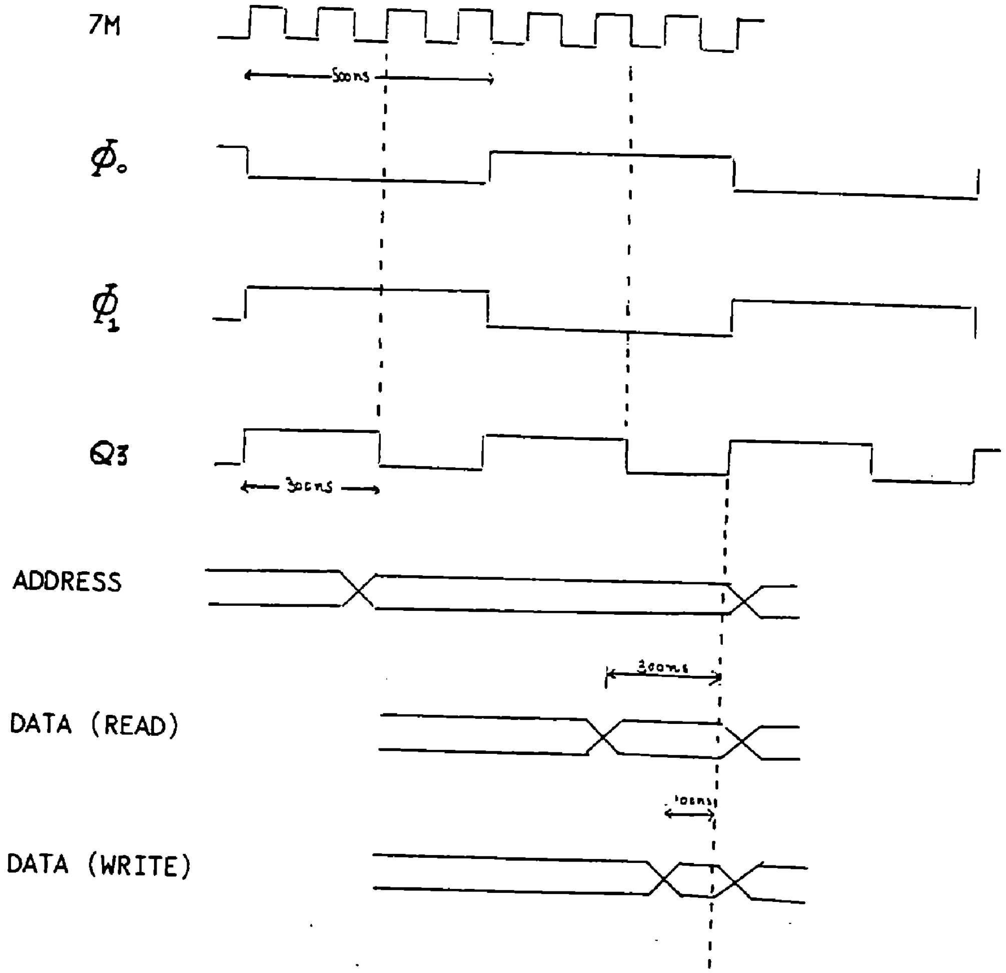


Fig. 4.10 - System Clock Signals and Timing Relationships

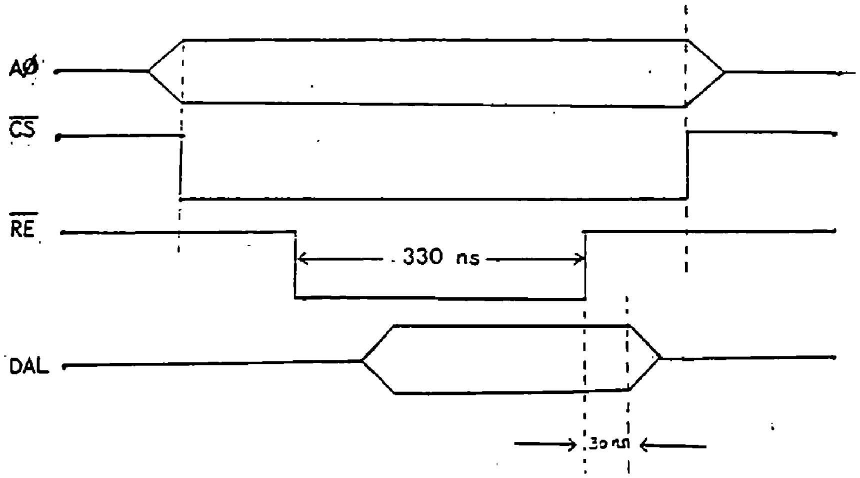


Fig.4.11 - READ Timing of Data Security Device.

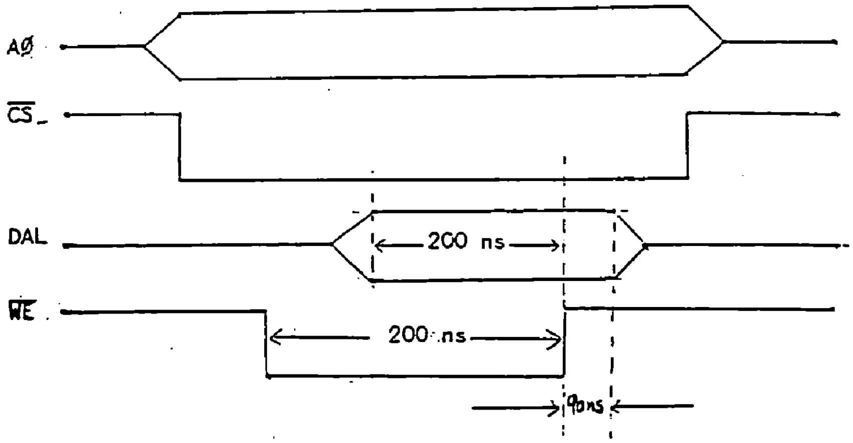


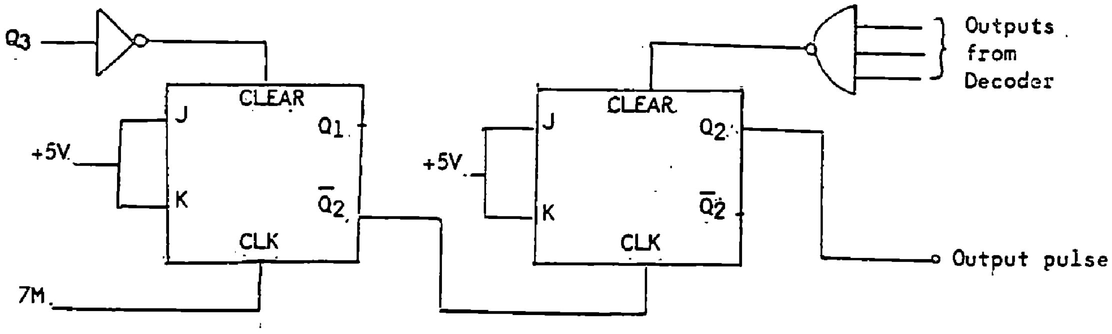
Fig. 4.12- WRITE Timing of Data Security Device.

DATA WORD present on the data bus is valid or not. When it is valid, the device is selected by making the CHIP SELECT ($\overline{CS}$) active low. This signal remains active low for at least 410 nano-seconds (ns). Then the READ signal is made active low to read the data from the bus. From the timing diagram, it is seen that the READ signal needs to be active low for 330 ns and the data must remain stable at least 30 ns after the rising edge of the READ pulse. The READ timing is not critical and it only needs to be greater than 330 ns. The READ signal is therefore derived using a standard decoder which produces an active low pulse of about 450 ns wide. This signal is also suitable for CHIP SELECT signals of both the security device and the serial I/O controller.

The WRITE timing diagram of the encryption device is shown in Figure 4.12. The device is selected by enabling the CHIP SELECT active low and then the DATA WORD or KEY WORD to be written is transferred to the data bus and a WRITE signal is produced. From the timing diagram, the data is to remain stable on the bus at least 200 ns before the rising edge of the WRITE pulse and 90 ns after the same edge. As the CHIP SELECT signal obtained from the decoder is 450 ns wide, the required pulse can be obtained by chopping off up to 200 ns from the rising edge of the CHIP SELECT signal. Actually, only a 90 ns pulse is subtracted as a 90 ns pulse is needed anyway in deriving the WRITE signal for the serial input-output controller. This pulse is then ored with the CHIP SELECT to give the WRITE signal of the security device. In Figure 4.13, this signal is denoted by Q2 + CS. The pulse is produced using a simple bistable circuit as shown in Figure 4.13 together with the timing diagram. The clock signals 7 MHz and Q3 are obtained from the Apple system. Note that the CHIP SELECT signal from the decoder has been used to control the rising edge of the WRITE signal by using it as the CLEAR input signal to the second bistable. The signal Q3 could not be used because it changes to the high state slightly before the CHIP SELECT thus producing a spike.

Now let us consider the READ, WRITE and CHIP SELECT signals required for the serial input-output controller.

The READ timing diagram of the serial input-output controller is shown in Figure 4.14. This signal is derived using a simple logic circuit which combines the pulse produced by the bistable circuit with another output line from the decoder circuit to produce the READ


Q3
+5V
J
CLEAR
Q1
K
$\overline{Q}_2$
CLK
7M
+5V
J
CLEAR
Q2
K
$\overline{Q}_2$
CLK
Outputs from Decoder
Output pulse

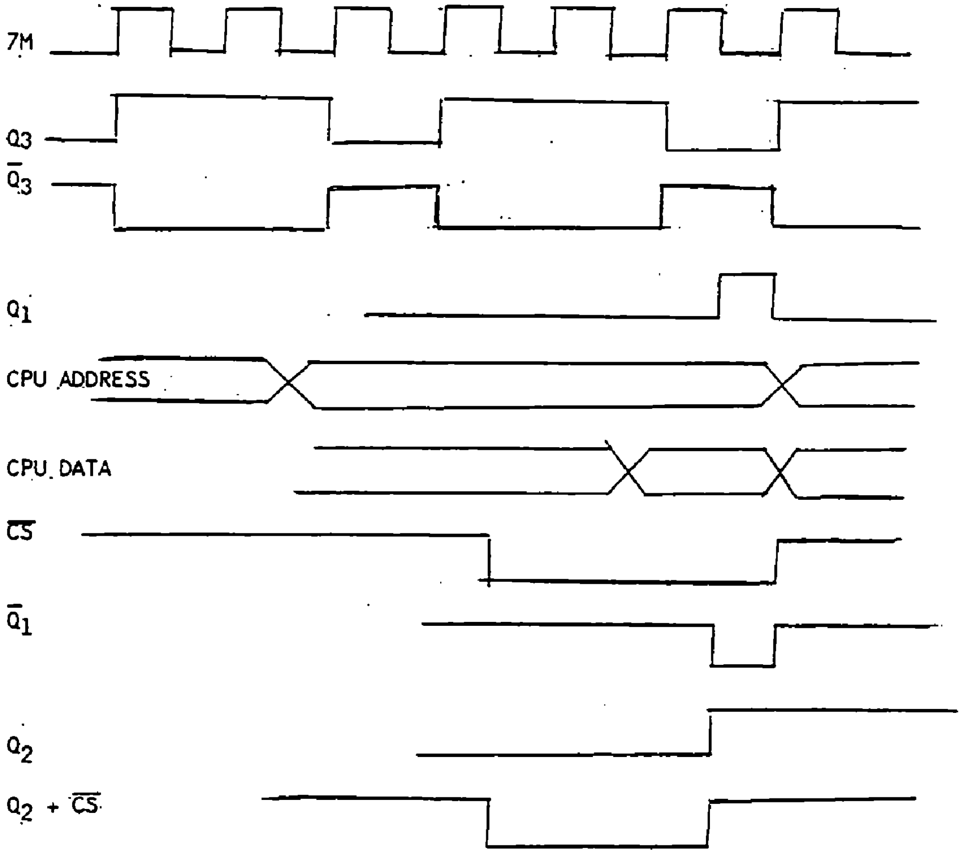

7M
Q3
$\overline{Q}_3$
Q1
CPU ADDRESS
CPU DATA
$\overline{CS}$
$\overline{Q}_1$
Q2
Q2 + $\overline{CS}$

Fig. 4.13

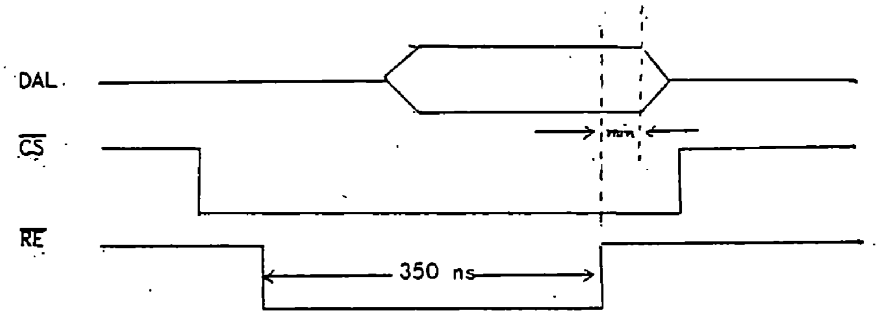


Fig. 4.14- READ Timing of I/O Controller.

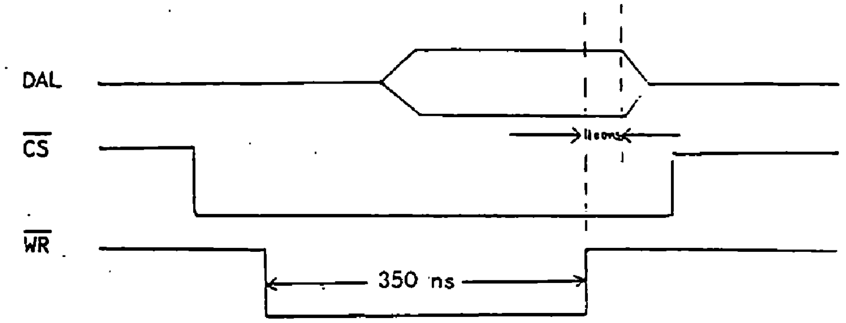


Fig.4.15 -WRITE Timing of I/O Controller.

signal. This produces a READ pulse of about 360 ns wide, with a rising edge some 90 ns from the valid data edge. These agree well with the required READ pulse in the timing diagram.

The WRITE timing diagram of the serial input-output controller is shown in Figure 4.15. In comparing this WRITE signal with that of the one required by the security device, it is seen that the controller requires the data to be stable on the bus for a longer time of at least 350 ns before the rising edge. Also the data is required to remain stable for a minimum time of 100 ns after the edge, which is 10 ns greater than that required for the encryption device. This signal is produced again by oring the 90 ns pulse with an output line from the decoder. Although this produces a WRITE pulse of width 360 ns with 90 ns of stable data after the rising edge, it is found that these timings do satisfy the requirements and does not cause any problem.

The two CHIP SELECT signals required for channel A and baud rate generator of the I/O controller are directly derived from the output of the decoder.

4.2.2.6 Decoding Circuits

This block in Figure 4.2 provides signals which together with the output signals from the timing block are used in selecting different devices when appropriate addresses are present on the address bus. In a system where there are many devices to run, it is necessary to decode the address bus down into individual addresses. This allows only one device and in particular only one mode of operation of the device to be selected. For instance, say when a control word has to be written into the serial input-output controller, three operations are required to be carried out namely

(i) only the input-output device must be selected,
(ii) only WRITE signal must be activated and
(iii) a distinction has to be made between a control word and a data word.

The inputs to the decoder include the Device Select line from the Apple microcomputer system which indicates which one of the peripheral connectors is active. Whenever the interface is selected this line will be active low. The Table 4.1 in Figure 4.16 shows the decoded address and the corresponding operations carried out by the system.

| Address | Operation Performed |
|---|---|
| COAØ | WRITE Data Word to data security device. |
| CØA1 | WRITE Control Word to data security device. |
| CØA2 | READ Data Word from data security device. |
| CØA3 | READ Status of data security device. |
| CØA4 | ACTIVATE latch. |
| CØA6 | ACTIVATE Tristate to test modem control signals. |
| CØA8 | RESET I/O Controller |
| CØAA | RECEIVE (READ) Data from I/O Controller. |
| CØAB | READ Status of I/O Controller. |
| CØAC | TRANSMIT (WRITE) Data to I/O Controller. |
| CØAD | WRITE Control Word to I/O Controller. |
| CØAE | WRITE to Baud Rate Register of I/O Controller. |

Figure 4.16 - Table 4.1: Decoding Arrangement

### 4.2.2.7 Memory

The operation of the encryption unit is completely controlled by software. Hence the proposed interface unit must have some non-volatile memory to hold the program. For this reason, a 2K-byte PROM is incorporated on the interface. The reasons for the choice are two fold. Firstly, a 2K-byte is the maximum amount of memory that can be associated with a peripheral slot in the Apple microcomputer system. Secondly, it is estimated that a 2K-byte memory should be sufficient for each of the required system tasks.

### 4.2.3 Modem

To transmit digital signals over Public Switched Telephone network, which passes frequencies in the range 300-3000 Hz,it is necessary for a data transmitter to modulate the digital stream by superimposing the '1's and '0's onto a carrier signal. The data receiver at the other end demodulates this signal. In addition to its basic function of translating between the binary digital signals of the data terminal equipment and the modulated voice frequency signals of the communication channel, as seen in section 4.2.2.4, the modem also performs a number of control functions which co-ordinate the flow of data between terminal equipments.

The digital information can be encoded by systematically changing either the amplitude, the frequency, the phase or some combination of these characteristics of the carrier signal. The type of modulation used depends on the specific application. Over the public telephone network, the maximum data rate is limited to 1200 bits per second. For speeds up to 1800 bits per second, Frequency Shift Keying (FSK) is the common choice and hence this is used. With FSK, a pair of tones $f_1$ and $f_2$ are alternatively sent over the line for the binary '1's and '0's of the asynchronous data stream. The receiving section essentially consists of two filters that sense the frequency $f_1$ or $f_2$. Whenever $f_1$ is sensed, a binary '1' is produced and whenever $f_2$ is sensed a binary '0' is output to the data terminal equipment.

Modular Technology (12 CV) mini modems and standard BT modems employing FSK scheme have been used with this system.

# CHAPTER 5

## POINT TO POINT COMMUNICATION SYSTEM : SYSTEM SOFTWARE (1)

### 5.1 General

The software has been written for the encryption interface unit to perform the following tasks

(a) Point-to-point communication between Apple microcomputers.

(b) Storage/retrieval of encrypted information with floppy disk system.

(c) Storage/retrieval of encrypted information with Prestel Viewdata system.

All the programs are written in 6502 Assembly language to minimize the execution time and memory space required. Before considering these programs, two basic I/O techniques for handling transfer of asynchronous data are briefly examined.

### 5.1.1 Polling Technique

The polling method is one of the simplest ways to handle asynchronous events. In this type of I/O, all operations are controlled by the CPU program. The processor interrogates flags associated with each possible event to determine whether any service is required. That is, it polls the peripheral device periodically to determine its readiness and hence the name polling technique. The CPU resources are tied up during the time of transfer and the time of polling and hence cannot be used for other tasks. This technique is mainly used with low speed devices.

### 5.1.2 Interrupt Driven Technique

In non-polling systems, the asynchronous event generates an interrupt request which is passed on to the processor. The processor in turn suspends the execution of the current process and starts execution of the interrupt service routine which say performs the data transfer. When the interrupt service routine is completed, the processor resumes execution of the suspended process. The response time is faster with such a system because no time is spent on interrogating the other non-active interrupts which in turn increases the system throughput.

The decision to adopt one technique or other depends on the nature of the application. In this project, the polling technique has been chosen for two reasons. Firstly in this encryption system, there are only two low speed devices which result in enough spare time for the processor to examine the status flags in a repeated fashion. Secondly the Apple 6502 system is not particularly suitable for an interrupt driven technique.

In this chapter, only the point-to-point communication program will be discussed. The other two tasks will be considered in Chapters 7 and 8 respectively.

### 5.1.3 Point-to-Point Communication

The encryption interface allows data transfer between Apple terminals (via a public switched telephone network) in either plain or encrypted or a mixture of plain and encrypted formats. Five different modes of the Data Encryption Standard have been investigated [3, 26]. They are:

1. Block Encryption (ECB)
2. Chained Block Cipher (CBC)
3. Stream Cipher Feedback (CFB)
4. Chained Block Cipher with Plaintext Feedback (CBCP)
5. Stream Cipher with Ciphertext and Vector Feedback (CFBV)

Each of these modes has been implemented using the encryption interface. They are described briefly in succeeding sections.

## 5.2 Block Encryption Mode

### 5.2.1 Principle

This is the most basic mode of operation of the Data Encryption Standard which transforms 64-bits of input to 64-bits of output as mentioned in Section 3.2. The analogy to Electronic Code Book (ECB) arises because the same plaintext block always produces the same ciphertext for a given cryptographic key. Thus it should be theoretically possible to construct a codebook of plaintext blocks and corresponding ciphertext blocks for any given key.

The ECB encryption-decryption scheme is shown in Figure 5.1. In encryption, the plaintext block (D1, D2, ..., D64) is used directly as the DES input block (I1, I2, ..., I64). The input block is processed through the DES device in the encrypt state which has been preloaded with the appropriate cryptographic key. The resultant output block (O1, O2, ..., O64) is used directly as the ciphertext (C1, C2, ..., C64). The decryption process is same as the encryption process except that the decrypt state of the DES device is used rather than the encrypt state. That is, the key schedule selection is reversed.

Mathematically, the operations of encipherment and decipherment can be described as follows:

Let the cryptographic function f define the relationship between the plaintext X and the ciphertext Y. Then,

$$Y = f_k\ (X)$$

and

$$X = f_k^{-1}\ (Y)$$

where the subscript k designates the particular key (and hence the particular function $f_k$) which is selected out of the set of all possible keys.

### 5.2.2 Implementation

Only a brief summary of the program is given here. The structure of the program can be divided mainly into five parts, namely, the initialisation routine, key input routine, data input routine, transmission routine, which includes encryption, and receive

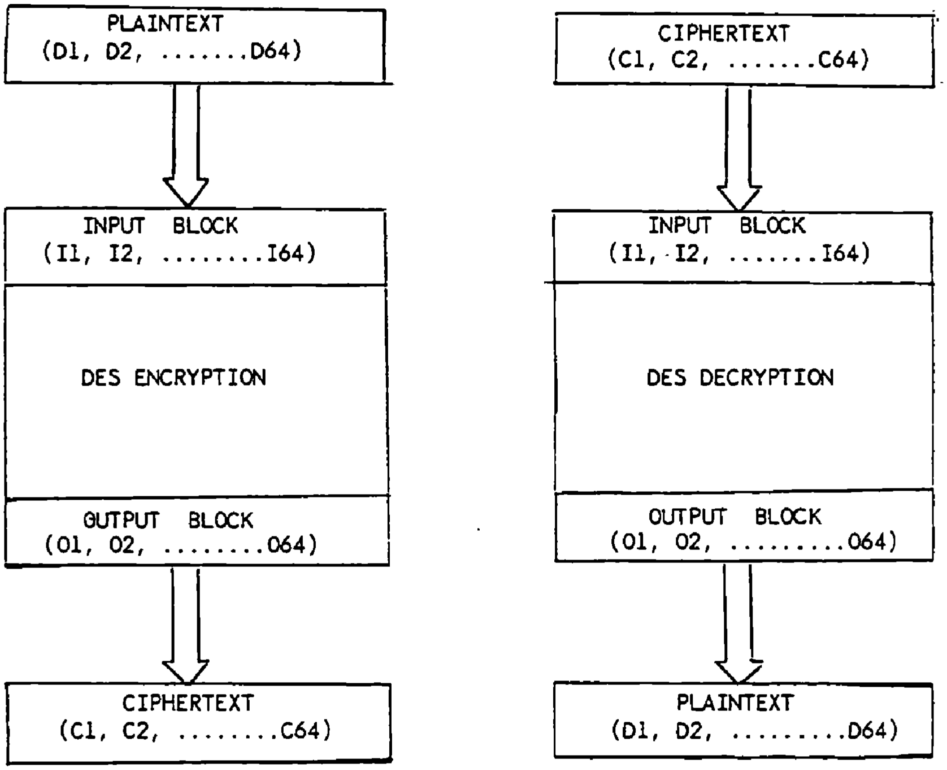


Fig. 5.1 - Block Encryption Codebook Mode (ECB)

routine, which includes decryption and display routines. The basic flowchart of the program and the complete listing can be found in Appendix 2.

5.2.2.1 Initialisation Routine

The initialisation routine itself can be divided into two subsections. The first subsection contains instructions which are executed without any user interaction whereas in the second subsection the user chooses the parameters for setting a particular mode of operation of the interface.

The first subsection consists of instructions to initialise the CPU, to set the ports of the serial I/O controller, to ensure the correct state of control signals from the modem, to set the state of the latch in the interface and to initialize some memory locations which would be later used as counters and flags in the program. The Data Set Ready (DSR) signal from the modem is tested to check whether the modem is ON and is not in the test mode. If it is not set, an error message 'NO LINK ERROR' is displayed. The format of the character to be transferred is defined to consist of 1 stop bit, 8 data bits and an even parity bit. A baud rate factor of 16 is used. The external clock mode is selected in the I/O controller. As explained in Section 4.2.2.3, this mode has been selected in point-to-point communication to drive the transmit and receive clocks with the same source. The error flags have been reset to allow error detection in subsequent transfer of data. These error flags include parity error, overflow error and framing error. The serial I/O is set to a null mode, that is, neither a transmission mode nor a receive mode.

In the second subsection, to begin with the user selects the desired data transmission rate. A choice of baud rates ranging from 50 to 1200 bits per second are available. Then the user is asked to select one of the three formats in which the interface may operate namely plain or encrypted or a mixture of plain and encrypted. If plain format is selected, no further initialization is required. If either of the other two formats is chosen, then the data security device is activated to encryption state.

5.2.2.2 Key Input Routine

If one of the two encryption formats is selected, then the secret DES key is entered from the keyboard by the user. It is assumed that both the parties concerned have the preknowledge of the key which is a prerequisite for proper communication of encrypted data. In this program, 8 characters are used to form the 64-bit key required by the DES algorithm. Any of the 88 alphanumerical characters of the keyboard can be used to form the 8 character key. As mentioned in Section 3.6.1, the total number of distinct keys possible is then equal to $(88)^8 \doteq 3.6 \times 10^{15}$ which is less than the maximum $2^{56} \doteq 7.2 \times 10^{16}$. The program can be very easily changed to accept sixteen 4-bit characters as the key thus giving the total possible key space. Here the 8 character key is chosen to enable the user to remember the key without actually recording it somewhere. It is essential that the key should be chosen randomly so that it may not be easily guessed by the opponent. Before loading every key byte into the KEY REGISTER of the device, the key parity is tested. If a parity error is detected, then an appropriate error message is displayed on the screen. The interface unit displays the entered key on the screen to enable the user to verify the correctness of the key. However the display is immediately erased to avoid detection by others during subsequent communication. The majority of users would probably use some easy to remember phrase or number combination for developing the key. In such cases, the long phrases can be converted to a form suitable for DES using a good hashing function. It should be sufficiently complicated to produce essentially unbiassed and statistically independent bits in the DES key. The program can also be modified to provide for multiple DES key encryption with different keys to achieve higher levels of security.

#### 5.2.2.3 Data Input Routine

In point-to-point communication mode, the data to be enciphered is assumed to be input from the keyboard of the terminal. File transfer is considered in Chapter 7 where file security is discussed. This routine determines whether the data to be processed is coming from the telephone line or from the keyboard of the terminal. This operation is carried out using the polling technique. The processor is allowed to poll the Data Carrier Detect (DCD) flag from the modem and the Keyboard Data input Flag (KDF). The KDF is provided by the Apple system and if set indicates that a key has been pressed. This implies that the data from the keyboard needs to be transmitted over the line in plain or encrypted or mixture format depending on the selection made earlier and hence the control is transferred to Transmission routine. On the other hand, if the DCD flag is set indicating the presence of data on the line, Receive routine is invoked to decrypt and display the received data.

### 5.2.2.4 Transmission Routine

At a given time, up to a maximum of 256 bytes of message can be input to the interface from the keyboard. A Return character is used to mark the end of the message. At any time, the key CNTRL-X can be used to cancel the entire message. All editing facilities of the Apple system such as cursor movements can be used. The program operates on the whole message(up to 256 characters) at a time. After a whole message is processed and transmitted, the program returns to fetch the next message from the keyboard. An important point to note is that with block encryption mode (ECB), the message needs to be divided into blocks of 64 bits (8 bytes) before encryption can be done.

Initially, the I/O controller is set to transmit mode. Each 64 bit block of plaintext is then input to the data security device in successive 8 bytes after testing for the correct status of the Data Input Request (DIR) flag of the device. Then the ciphertext block is sent from the device byte by byte after testing the control signals, Clear To Send (CTS) and Transmit Ready (TXRDY). Appropriate error messages are displayed if the flags are not set properly and the system is resynchronized. Special routines are developed to deal with control characters CNTRL-G (Bell), CNTRL-J (line feed) and repeated Return characters in the plaintext block. When the encrypted version of these characters are transmitted, a certain amount of delay (approximately 80 milliseconds) is included to allow time for the generation of these characters at the receiving end after decryption.

However the message need not necessarily contain an exact number of 64 bit blocks. Hence it is likely that the message will end before the last block of 64 bits is complete. Therefore some padding is required at the end of the message to fill the last block to exactly 64 bits long. This has been done by padding the last block with some random characters after the Return character . These random characters are generated as part of the key input routine. The padding results in the cryptogram expansion of up to a maximum of 7 bytes compared to the orignal plaintext message if the last block is not of full size.

5.2.2.5 Receive Routine

The Receive routine is much simpler than its counterpart Transmit routine. It is known that the incoming data is encrypted by dividing it up into blocks of 64-bits at the transmitting end. Hence this routine should simply fetch the 64-bits of ciphertext at a time from the link after testing the Receive Ready flag (RXRDY) of the I/O controller. The ciphertext is then input to the data security device which is now set to decryption mode. Then the deciphered block of data is read from the device and is displayed on the screen using Display routine. The Display routine is same as the one used by the Apple system. It displays uppercase and lowercase alphanumeric characters in either normal or inverse or flashing modes. The control characters in the decrypted block are not displayed. The block of decrypted characters is tested for the presence of the Return character. If it is not present, then it is known that another block of encrypted data must be following the current block. Hence the program loops back to test the RXRDY flag to obtain more data from the link. On the other hand if a Return character is detected within the block of deciphered data, it indicates that the current block is the last block of the message. The random characters after the Return character are treated as dummy characters and hence they are ignored. The decrypted information will be the same as the original message provided the same key has been used at both ends of the link.

The routine also performs three types of error tests on the received characters. They are parity error, framing error and overrun error. In each case, appropriate error messages are displayed. A framing error is detected when the receiver finds a logic '0' occurring at the time when stop bit, logic '1', should occur. A overrun error is detected when a new character is loaded into the Receive Holding Register of the serial I/O controller before the processor has had time to read the previous one. In all these three cases the program loops back to the start of the data input routine thus abandoning the current ciphertext block and automatically synchronises.

5.2.2.6 Plain Data Communication

The routines which allow plain data transmission and reception are very much similar to the ones described earlier except in this case there is no need to perform encryption and decryption. Hence the message is processed byte by byte rather than in blocks.

This facility of communicating in plain format is provided to allow the transfer of non-sensitive information over the link. For instance, one could envisage a situation where the users initially communicate in plain format to set various parameters needed for establishing a secure link. This immediately led to the idea of developing a program which could handle a mixture of plain and encrypted data and this is considered in the next section.

### 5.2.2.7 Mixture of Plain and Encrypted Data Communication

With this format, only parts of the message are in encrypted form while the rest of the message is transmitted in plain form. This mode is useful in many applications where it is not necessary to encrypt the whole message.

The receiver must be able to identify which parts of the received data are in encrypted form. The data transmitted is always in plain form until the change to encryption mode is initiated. This is done by typing a special character (CNTRL-A) on the keyboard that has been designated for this purpose. Characters following this character are encrypted by the sending terminal. The interface unit is automatically returned to the plain format after 8 characters. Alternatively, another special character (CNTRL-B) can be used to return to the plain format. The receiving end then checks for a CNTRL-A character and starts to decrypt when it is found. When a CNTRL-B is received, it switches back to plain mode reception. However it is necessary to ensure that the CNTRL-B character does not occur within the enciphered data to ensure unambiguous decryption at the receiving end. This can be achieved by using multiple CNTRL-B characters to indicate the end of encrypted text. The greater the number of such characters, the smaller the probability that they occur in the enciphered data and hence the smaller the ambiguity in decryption, but this increases the number of redundant characters in the transmitted data. The key required for the encryption algorithm is entered as before at the beginning of the communication.

Again with the block encryption mode, if the encrypted parts of the message are not integral multiples of 64 bits, they require padding and this results in cryptogram expansion. The difference with the mixture format is that this expansion does not occur only at the end of the message (last block), as in the complete encryption format discussed earlier, but it may occur anywhere within the message.

When implementing this mixture format in the Apple system, additional problems are encountered compared to the complete encryption format. One such problem arises from the requirement that in the multi-user network the plain information transmitted may need to be received correctly by all users whereas the encrypted information must only be deciphered correctly by the user with the right key. There may be cases where the encryption algorithm transforms a non-control character to a control character and vice versa. As a control character is not displayed by the Apple system and the screen cursor does not move, this results in a line of text, with parts of it encrypted at the transmitting end, not producing a line of text at the receiving end with the wrong key. As a certain amount of delay is required for some special characters such as CNTRL-G (Bell), CNTRL-J (Line feed), this can cause overrun error at the receiving end with the wrong key. This results in errors in the subsequent reception of data even in plain format at the receiving end. This is further complicated in this block encryption mode due to the padding with random numbers to fill the block.

A modified version of the earlier complete encryption format program is developed for this mixture format which overcomes the problems mentioned above.

### 5.2.3 Results and Discussion

An example of plaintext containing some data structure and patterns has been chosen to study the various characteristics of different encryption modes. Such an example is provided by the assembly language program shown in Figure 5.2. This whole message is enciphered using ECB encryption with the key 3131313131313131 in hexadecimal form. Note that here a non-random key has been chosen to allow the ciphertext produced to be used in some statistical tests in Chapter 6. A ciphertext character produced can be any one of the 256 possible combinations ($2^8$). To display the ciphertext, it is therefore necessary to extend the standard ASCII character set from 128 to 256. This has been done using Hershey characters [27]. The complete character set is given in Appendix 3. Note that this extended

```
61
****************************************
THIS SAMPLE PROGRAM ENABLES TRANSFER OF
DES ENCIPHERED DATA BETWEEN APPLE MICRO-
COMPUTERS.
****************************************
1000   LDA#6E      SET MODE REG.
       STA C0AD    OF UART.
       LDA#12      SET COMMAND REG.
       STA C0AD    OF UART.
       JSR 213C    INPUT SECRET KEY
       JSR 2355    CHECK PARITY
       SPACE
1011   LDA C0A6    POLL KEYBOARD AND
       AND#03      LINE CONTINUOSLY.
       BEQ(101B)
       JMP 1034
101B   LDA C000
       BPL(1011)
       SPACE
       JSR 2400    SET ENCRYPTION MODE
       JSR 3000    SUBROUTINE READ
       JSR 2500    ECD ENCRYPTION
       JSR 2600    SUBROUTINE TRANSMIT
       JSR 2700    SUBROUTINE DELAY
       JMP 1011
       SPACE
1034   JSR 2800    SET DECRYPTION MODE
       JSR 2900    SUBROUTINE RECEIVE
       JSR 2A00    ECB DECRYPTION
       JSR 3100    SUBROUTINE DISPLAY
       CMP#8D      TEST FOR END
       BEQ(1047)   OF DATA
       JMP 1011
1047   HALT
```

Fig. 5.2 - Plaintext Example

character set does not in itself introduce any secure cryptographic transformation. The ciphertext produced is shown in Figure 5.3. The program used to produce this graphical display is given in Appendix 4.

From Figure 5.3, it can be seen that a potential weakness of this block mode is that the same plaintext always produces the same ciphertext under a fixed key. This in turn implies that if the plaintext contains patterns,they will be reflected in the ciphertext as seen in Figure 5.3. Consider for instance, the asterisks in lines 1 and 5 and sequences of blanks at the beginning of each line in the assembly language program. Thus the compromise of the plaintext block underlying any ciphertext block results in the compromise of all repetitions of this same text for the remainder of the cryptographic period. Thus this block encryption mode is more susceptible to code book analysis compared to the other modes considered later. Further if the plaintext information is highly redundant then block encryption may not prevent cryptanalysis using block frequency analysis. Block frequency analysis determines the frequency of each ciphertext block from a large sample of intercepted ciphertext. By relating the observed frequencies of the ciphertext blocks to the expected frequencies of the plaintext blocks, the cryptanalyst may be able to draw certain inferences concerning the nature of the plaintext corresponding to a given ciphertext. Also if the data transmitted is highly redundant, the number of possible meaningful plaintext blocks may be small enough to construct a dictionary.

This block encryption mode is also susceptible to replay. As each block is independently enciphered with the same key, one block can be replayed for another. For instance, consider the transaction shown below

| | | | | | |
|---|---|---|---|---|---|
| CREDIT | JONES | £5000 | CREDIT | SMITH | £10 |
| C1 | C2 | C3 | C4 | C5 | C6 |

This can be changed to

| | | | | | |
|---|---|---|---|---|---|
| CREDIT | JONES | £5000 | CREDIT | SMITH | £5000 |
| C1 | C2 | C3 | C4 | C5 | C3 |

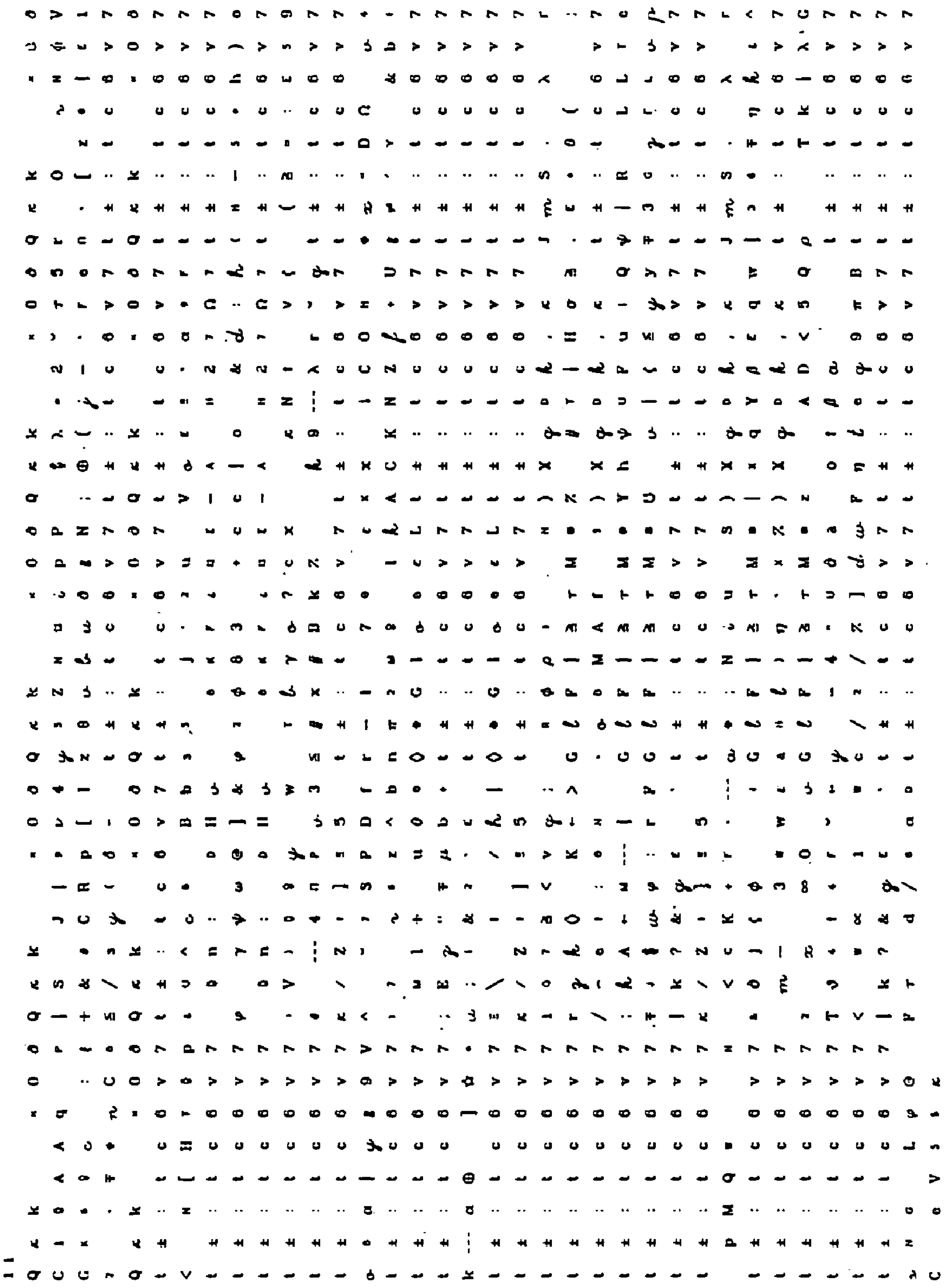
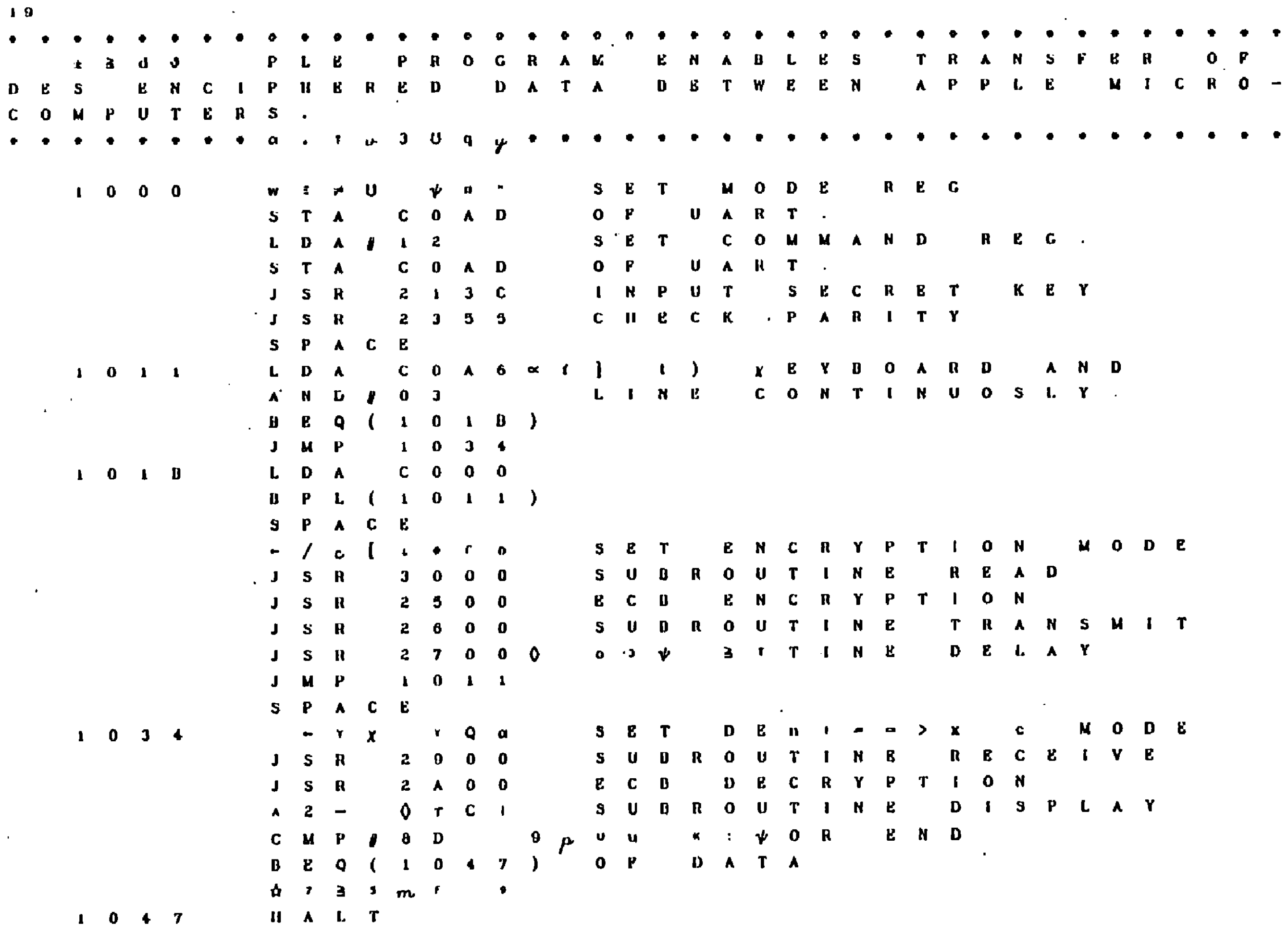


Fig. 5.3 - ECB Ciphertext Example



```
19
• • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • •
    ±ƨdɔ   PLE PROGRAM ENABLES TRANSFER OF
DES ENCIPHERED DATA BETWEEN APPLE MICRO-
COMPUTERS.
• • • • • • • • a.↑ʋ3Uqy • • • • • • • • • • • • • • • • • • • • • •

    1000    w≡≠U ψn˜     SET MODE REG
            STA C0AD     OF UART.
            LDA#12       SET COMMAND REG.
            STA C0AD     OF UART.
            JSR 213C     INPUT SECRET KEY
            JSR 2355     CHECK PARITY
            SPACE
    1011    LDA C0A6∝{]  ι) ϰEYBOARD AND
            AND#03       LINE CONTINUOSLY.
            BEQ(101B)
            JMP 1034
    101B    LDA C000
            BPL(1011)
            SPACE
            ←/c[↓•ro     SET ENCRYPTION MODE
            JSR 3000     SUBROUTINE READ
            JSR 2500     ECB ENCRYPTION
            JSR 2600     SUBROUTINE TRANSMIT
            JSR 2700◊    o·ɔψ ƐɪTINE DELAY
            JMP 1011
            SPACE
    1034     ←YX YQα     SET DEnɪ-->x c MODE
            JSR 2000     SUBROUTINE RECEIVE
            JSR 2A00     ECB DECRYPTION
            A2- ◊TCI     SUBROUTINE DISPLAY
            CMP#8D  9ρ   υu κ:ψOR END
            BEQ(1047)    OF DATA
            ✩7Ɛ1mf •
    1047    HALT
```

Fig. 5.4 - ECB Error Characteristic

by replaying the block containing the ciphertext for £5000. This type of replay is not possible with stream cipher and chained block cipher modes which are considered later. One simple solution to this problem is to append checksums to the end of messages.

This block encryption mode is also vulnerable to insertion and deletion of blocks because these changes to the message do not affect surrounding blocks. Again use of error detecting codes before encryption and checksums protects against this threat.

As mentioned in the implementation section, in this mode, the information is encrypted in integral multiples of 64 bits. This resulted in padding of last block with random characters. This causes cipher extension and therefore may be unacceptable in some applications. The padding effect can be seen in Figure 5.3 at the end producing a longer ciphertext message than the original plaintext. If the padding is done with blanks or zeroes, instead of random numbers, then this may make them vulnerable to crypanalysis.

Since each bit of the ECB output block is a complex function of every bit in the input block and the key, a single bit change in either the key or the plaintext results in a ciphertext block in which each bit is changed with approximately equal probability. Conversely, a change in 1 bit of either the key or ciphertext will produce changes in an average of fifty percent of the bits of deciphered plaintext. Although this error propagation within the block is extensive, it is strictly limited to the block in which the error occurs and the decryption of other blocks is unaffected. This can be seen from Figure 5.4 where a number of errors have been introduced in the ciphertext prior to decryption. Thus the ECB mode does not provide error extension between blocks. If block boundaries are lost between sender and receiver,then ECB cryptographic synchronization will also be lost until correct block boundaries are re-established. This may happen for instance when a bit slip occurs.

## 5.3 Cipher Block Chaining Mode

### 5.3.1 Principle

Cipher block chaining is again a block cipher in which the plaintext is exclusive-ored with a block of pseudo-random data prior to being processed through the DES device.

This scheme is shown in Figure 5.5. In order to commence the CBC encryption, the first DES input block is formed by exclusive-oring the first data block with a 64-bit initialization vector (IV). That is,

(I1, I2, ... I64) = (IV1⊕D1, IV2⊕D2, ..., IV64⊕D64).

This first CBC input block is processed through the DES device in the encrypt state, producing a 64-bit DES output block which defines the ciphertext. This first ciphertext block is then exclusive-ored with the second plaintext block to produce the second DES input block. This second input block is enciphered using the DES device to produce the second ciphertext block. This encryption process continues to chain successive ciphertext and plaintext blocks together until the last plaintext block of the message is encrypted. In CBC decryption, the first ciphertext block is processed through the DES device in the decrypt state. The first output block is then exclusive-ored with the CBC initialization vector producing the first plaintext block. The second ciphertext block is then entered into the DES device and the resultant output block is exclusive-ored with the first ciphertext block to produce the second plaintext block. The CBC decryption process continues to exclusive-or the ciphertext block at time t-1 (t>1) with the DES output block to obtain plaintext at time t until the end of the message.

Mathematically, the scheme can be expressed as follows: Let the cryptographic function $f_k$ define the relationship between the DES input block and the DES output block under the chosen key k. Let function h define how the input to the DES is altered through the introduction of the initialization vector and the feedback or intermediate initialization vectors U(1), U(2),..., U(n-1) at time t = 1 to n-1. Note that the function h may be a many-to-one function since identical inputs to this function will be available during both encipherment and decipherment. Then

$$U(1) = Z = \text{Initialization Vector}$$
$$U(i) = h[U(i-1), \text{feedback quantity}], i > 1$$

In the CBC mode, U(i) is equal to the previous ciphertext block Y(i-1), the feedback quantity. That is, U(i) = Y(i-1). Hence the encipherment and decipherment operations can be expressed as

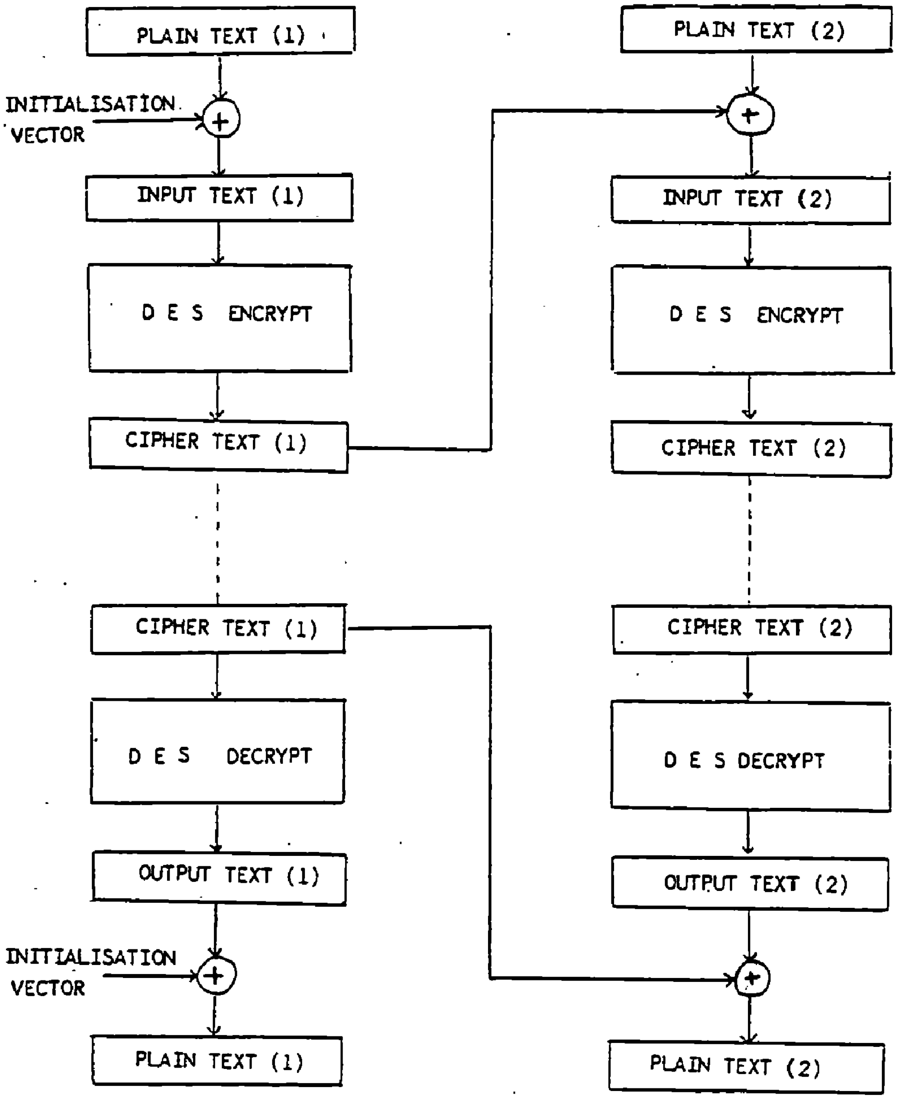


Fig. 5.5 - Cipher Block Chaining Mode. (CBC)

$$Y(i) = f_k [ X(i) \oplus Y(i-1) ] \quad i \geqslant 1$$

and

$$X(i) = f_k^{-1} [ Y(i)] \oplus Y(i-1) \quad i \geqslant 1$$

where

$$X(o) \equiv Y(o) \equiv Z$$

From the recursive nature of the above equations, there exists functions $H_1$, $H_2$..., $H_i$ such that

$$Y(i) = H_i \; [k, X(o), X(1), \ldots X(i)] \; , i \geqslant 1 \qquad (5.1)$$

Similarly there exists functions $G_1$, $G_2$, ..., $G_i$ such that

$$X(i) = G_i [ \; k, Y(i-1), Y(i)] \; , \; i \geqslant 1 \qquad (5.2)$$

From equations (5.1) and (5.2), it follows that patterns within the input are masked since the ciphertext block Y(i) depends on plaintext blocks X(1), X(2), ..., X(i). However since the received plaintext block X(i) does not depend on all ciphertext blocks Y(1), Y(2), ..., Y(i), the scheme does not represent a general block cipher (see Section 5.5).

### 5.3.2 Implementation

This program is a modified version of the ECB mode program described in Section 5.2.2. The flow chart of the program together with the listing can be found in Appendix 5. Here only the differences between this program and the ECB mode program are briefly discussed.

As in the ECB program, the user initially selects the data rate, the format of data transfer (plain, encrypted or mixture) and enters the DES secret key. Then the transmission end generates a 64 bit random block and sends it to the receiving end. The block is encrypted under the ECB mode at both the transmitting and receivng ends to form the initialization vector (IV). This vector is stored in a set of memory locations named TEMP 1 at the transmitting end and TEMP 2 at the receiving end. It is again necessary to divide the message into blocks of 64 bits. Hence padding of the last block with random characters is required to make the blocksize equal to 64. The transmission routine fetches the plaintext block which is then exclusive-ored with the memory locations TEMP 1 before inputting it to the DES device. The ciphertext block produced by the device is transmitted to the receiver as in the ECB mode. The ciphertext is also stored back into the memory locations TEMP 1 for use in the next encryption cycle. The receive routine inputs the ciphertext to the DES device and also stores it in memory locations TEMP 3. The output from the DES device is exclusive-ored with the memory locations TEMP 2 to produce the plaintext. The memory locations TEMP 3 are then transferred to TEMP 2 to form the new IV for exclusive-or operation in the decryption of the next ciphertext block.

Finally, note that in the case of mixture of plain and encrypted data communication, only the encrypted blocks are chained together.

### 5.3.3 Results and Discussion

The plaintext example shown in Figure 5.2 is enciphered under CBC encryption with the same key as in the ECB mode. The initialization vector used is 0202020202020202. The ciphertext produced is shown in Figure 5.6.

From Figure 5.6, it is seen that the CBC mode does not produce the same ciphertext even when the plaintext is the same and hence the pattern exposure problems associated with the ECB mode have been eliminated. This is because the ciphertext produced for a plaintext block with a given key is dependent on the plaintext as well as the feedback (intermediate) vector used in the process which is different at different times. So the CBC mode reproduces the same ciphertext whenever the same plaintext is encrypted under a fixed key and initialization vector. Thus with the CBC mode, ciphertext repetition occurs at the message level whereas with the ECB mode, ciphertext repetition is found to occur at block level. Thus the code book analysis problem has been reduced.

CBC is therefore less susceptible to replay than the ECB mode of encryption. The type of replay mentioned in Section 5.2.3 is not possible with CBC mode as different parts of the message are enciphered with different feedback vectors. It is also less vulnerable to insertion and deletion of blocks as these changes to the message affect the surrounding blocks.

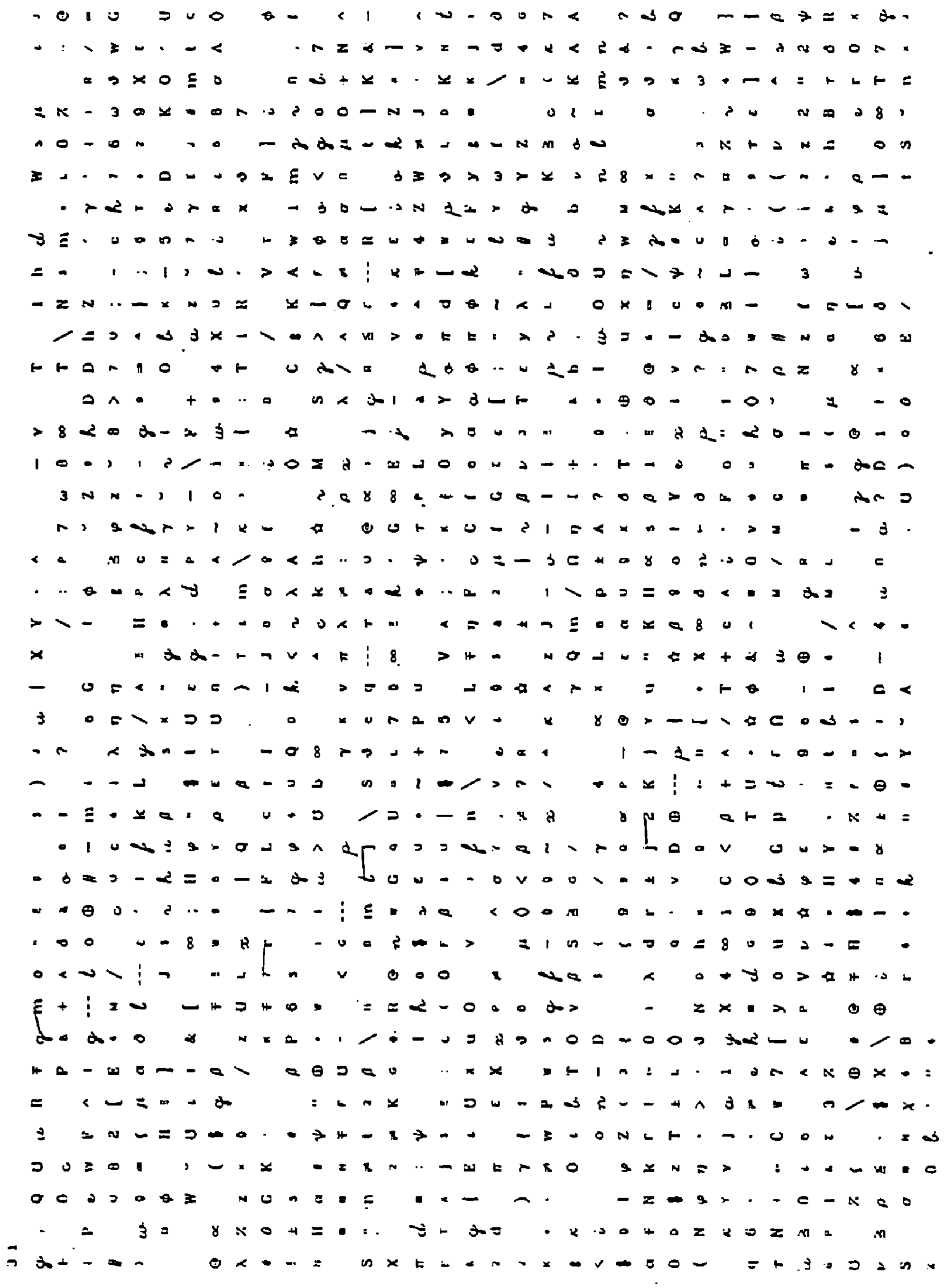


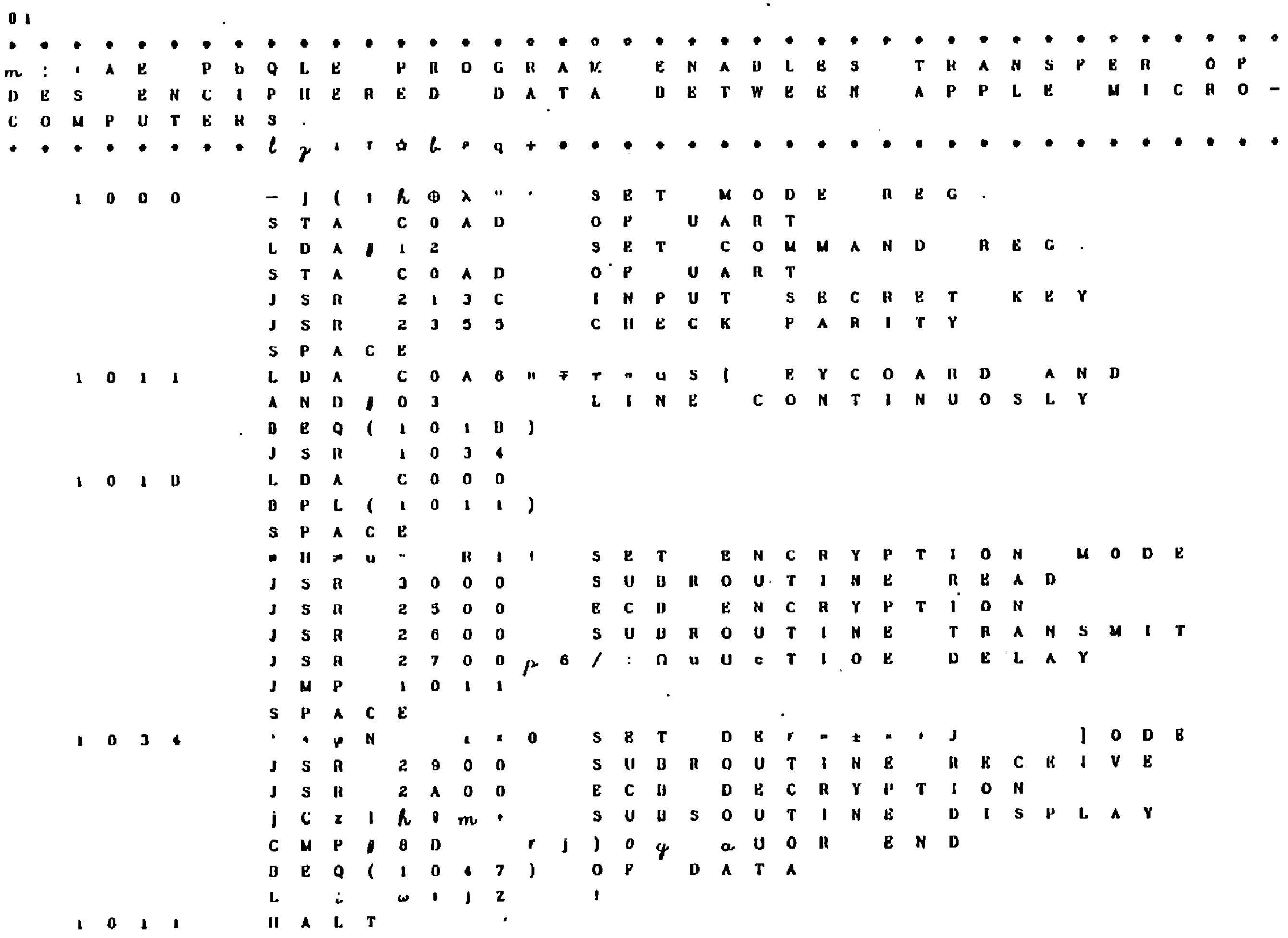
Fig. 5.6 - Ciphertext Example

CBC also protects a block cipher against the time-memory trade off attack [23] as follows.
Let Y(i) be the ciphertext corresponding to the chosen plaintext X(i). Since $f_k^{-1}$ (Y(i)) = X(i) ⊕ Y(i-1), to determine the key k, a cryptanalyst would have to generate tables of starting and ending points using X(i) ⊕ Y(i-1) rather than X(i). But this would rule out the possibility of precomputing the tables or of using the same tables to break more than one cipher.

As noted in the implementation section, the problem of padding still exists with the CBC mode. In the complete encryption format, this is confined to the last block of the message. The receiver scans the decrypted block and discards the pseudo-random bits after the Return character code in the block. One way to eliminate padding is to switch to stream cipher feedback (CFB) mode (see Section 5.4) to encipher the short block at the end of the message .

In addition, the security of the CBC mode depends among other things upon the management of the CBC initialization vectors. It is important that the initialization vector (IV) is pseudo-randomly selected. Further the vector must be protected from disclosure. In the above implementation, this is carried out by using the ECB encrypted version of the random block generated as the IV. It is also advisable to change the vectors frequently to avoid the cryptanalyst using the ciphertext search attack.

Within the ciphertext, some errors are introduced and then deciphered with the same key and initialization vector to study the error extension characteristics of the CBC mode. From Figure 5.7 one or more bit errors within a single ciphertext block are found to affect the decryption of two blocks, namely, the block in which the error occurs and the succeeding block. If the errors occur in the ith ciphertext block, then each bit of the ith plaintext block has an error rate of about fifty percent. The (i+1)-th plaintext block has only those bits in error which correspond directly to the ciphertext bits in error. Further it is seen from Figure 5.7 that after two blocks, cryptographic synchronization is automatically established and the subsequent deciphered blocks are unaffected. Compared to the ECB mode, this has extended the error propagation to two blocks. This can also be seen from equations (5.1) and (5.2)



```
01
• • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • •
m:ıAE PbQLE PROGRAM ENABLES TRANSFER OF
DES ENCIPHERED DATA BETWEEN APPLE MICRO-
COMPUTERS.
• • • • • • • • lγır☆bρq+ • • • • • • • • • • • • • • • • • • • • • • •

1000    −j(ıh⊕λ"'      SET MODE REG.
        STA C0AD        OF UART
        LDA #12         SET COMMAND REG.
        STA C0AD        OF UART
        JSR 213C        INPUT SECRET KEY
        JSR 2355        CHECK PARITY
        SPACE
1011    LDA C0A6"∓τ"uS( EYCOARD AND
        AND #03         LINE CONTINUOSLY
        BEQ (101D)
        JSR 1034
101D    LDA C000
        BPL (1011)
        SPACE
        •H≠u"  RI!      SET ENCRYPTION MODE
        JSR 3000        SUBROUTINE READ
        JSR 2500        ECB ENCRYPTION
        JSR 2600        SUBROUTINE TRANSMIT
        JSR 2700ρ6/:ΩuUcTIOE DELAY
        JMP 1011
        SPACE
1034    ·•φN   ι×0      SET DEr−±×ıJ    ]ODE
        JSR 2900        SUBROUTINE RECEIVE
        JSR 2A00        ECB DECRYPTION
        jCzIh9m+        SUBSOUTINE DISPLAY
        CMP #8D    rj)0φ aUOR END
        BEQ (1047)      OF DATA
        L ¿ ω!jZ        !
1011    HALT       '
```

Fig. 5.7 - CBC Error Characteristic

where an error in the ciphertext block Y(i-1) can affect every bit in the recovered plaintext block X(i-1) but it will affect only the corresponding bit positions in the received plaintext block X(i). Subsequent plaintext blocks X(i+1), X(i+2), ... are unaffected.

If bits are added or lost in a ciphertext block so that block boundaries are lost between sender and receiver, then synchronization is lost. However cryptographic synchronization will automatically be re-established 64 bits after block boundaries have been established.

This self synchronizing scheme may be useful when small amounts of noise are present on the data communication links.

## 5.4 Stream Cipher Feedback

### 5.4.1 Principle

The cipher feedback mode (CFB) is an additive stream cipher technique in which the DES is used to generate a pseudo-random binary stream. This stream is exclusive-ored with the binary plaintext to form the ciphertext which is fed back to form the next DES input block. The pseudo-random binary stream is sometimes referred to as the key stream and the DES the key generator.

This mode is schematically shown in Figure 5.8. One through to sixty four bit CFB operation may be used unlike the ECB mode where the message is required to be divided into blocks of a given blocksize namely 64. A 64-bit initialization vector (IV) is used as a starter input block to begin the CFB operation. (Note that if the size of IV is chosen to be less than 64 bits, then it can be padded with '0's to form 64 bits). This vector is processed through the DES device in the encrypt state to produce a pseudo-random output block. The message is divided into characters of s-bit size where $0<s<65$. The DES algorithm is operated once for each new s-bit character. Then s-bits of the DES pseudo-random output block (O1, O2, ..., Os) are used in the exclusive-or operation with the s-bits of the plaintext (D1, D2, ..., Ds) to form the ciphertext (C1, C2, ..., Cs). That is, (C1, C2, ..., Cs) = (D1 ⊕ O1, D2 ⊕ O2, ..., Ds ⊕ Os). This operation may be defined when the length of the plaintext character to be encrypted is less than

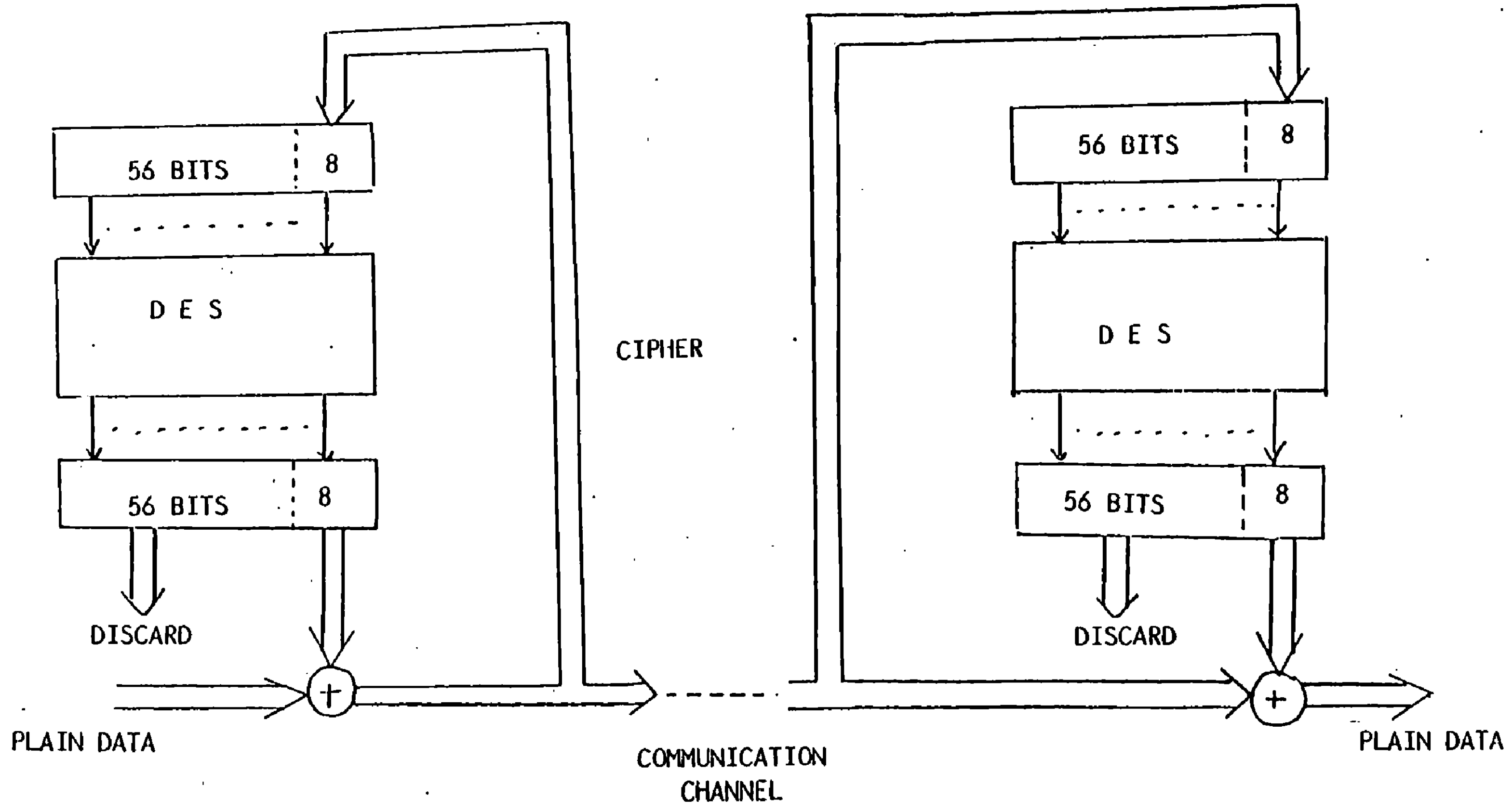


Fig. 5.8 - Cipher Feedback Mode. (CFB)

s-bits by concatenating zeroes to the left hand side or most significant bits of the plaintext character. Similarly during decryption, plaintext is produced by exclusive-oring a s-bit ciphertext character with the s-bits of DES output block. That is,

$$(D1, D2, \ldots, Ds) = (C1 \oplus O1, C2 \oplus O2, \ldots, Cs \oplus Os)$$

In both cases,the same s-bits of the DES output block are used and the unused bits are discarded. At both ends, the next input block is created by discarding the most significant s-bits of the previous input block, shifting the remaining bits s positions to the left and then inserting the s-bit ciphertext character just produced in the encryption operation or just used in the decrypt operation into the least significant bit positions as shown in Figure 5.8. That is, the input block (I1, I2, ..., I64) is given by,

$$(I1, I2, \ldots, I64) = (Is{+}1, Is{+}2, \ldots, I64, C1, C2, \ldots, Cs)$$

This input block is then processed through the DES device in the encrypt state to produce the next output block. An important difference compared with the two modes considered in Sections 5.2 and 5.3 is that even in decryption, the DES is used in its encryption state. This is because in CFB mode, the DES algorithm has been used as a pseudo-random number generator rather than as a cryptographic transformation.

Mathematically,this mode can be expressed as follows: Let X(i) be the ith plaintext input data and Y(i) be the ith ciphertext output data. Let U(i) be the intermediate initialization vector at time i. In general, the length of the intermediate initialization vector may not be equal to the length of the initialization vector Z. Let the function $h'$ define how U(1) is obtained from Z. That is, $U(1) = h'(Z)$.

Encipherment of the first s-bit plaintext is given by:

$$Y(1) = X(1) \oplus f_k [ U(1) ]$$

Let the function h define the dependency of the intermediate initialization vector at time i on the previous initialization vector U(i-1) as well as the additional feedback quantity. That is,

$$U(i) = h\ [U(i-1),\ \text{feedback quantity}]$$

where the feedback quantity = $Y(i-1)$.

In CFB mode with $s<64$,

$$U(i) = U(i-1) \,\|\, Y(i-1)\ ,\ i>1$$

where $\|$ denotes concatenation of $U(i-1)$ and $Y(i-1)$.

With $s=64$,

$$U(i) = Y(i-1)\ ,\ i>1$$

That is, effectively the ciphertext at time i-1 is fed back as input to the DES algorithm. Defining $Y(0) \equiv Z$, with $s=64$, it follows that

$$f_k\ [U(i)] = f_k\ [Y(i-1)]\ ,\ i\geqslant 1$$

Therefore equations of encipherment and decipherment can be expressed as

$$Y(i) = X(i) \oplus f_k\ [U(i)]\ ,\ i \geqslant 1 \qquad (5.3)$$

and

$$X(i) = Y(i) \oplus f_k\ [U(i)]\ ,\ i \geqslant 1 \qquad (5.4)$$

respectively where $U(1) = Z$.

### 5.4.2 Implementation

An 8-bit cipher feedback mode has been implemented on the encryption system. The flowchart of the program can be found in Appendix 6. The differences between this program and the ECB and CBC programs are now very briefly mentioned.

As in the case of the CBC program, a 64-bit initialization vector is generated at the transmitting end and sent to the receiver for proper synchronization. The major difference compared to ECB and CBC programs is that in this mode, the message is encrypted byte by byte and not in blocks. This implies that there is no need for padding at the end of the message and consequently no cryptogram extension. One additional routine SHIFT is required which shifts the intermediate initialization vector (memory locations TEMP 1 and TEMP 2) to the left and appends the 8 bits of the ciphertext to the right side of the shifted input to produce the next DES input. Finally at both the transmitting and the receiving ends, the data security device is programmed to encryption state.

### 5.4.3. Results and Discussion

The plaintext example shown in Figure 5.2 is enciphered under the CFB mode using the same key and the initialization vector as in the CBC mode. The ciphertext produced is shown in Figure 5.9.

From Figure 5.9, it is seen that as in the case of the CBC mode, the CFB mode does not produce the same ciphertext even when the input plaintext is the same. This is because the intermediate initialization vectors are different in each case. Thus chaining has again eliminated patterns occurring in the ciphertext For this mode to produce the same ciphertext when the same plaintext is encrypted, both the key and the initialization vector must be identical in the two cases. This mode is similar to CBC in its resistance to forms of attack such as ciphertext searching, replay, insertion and deletion.

While both the CBC and ECB modes discussed earlier required padding of the last block of the message (if its length is not equal to an integral multiple of 64 bits), such problems do not arise in this stream cipher mode. That is, the key stream length can be matched exactly to the length of the plaintext to be enciphered. So this mode allows easy encryption of 'non-block type' messages such as character by character or even bit by bit encryption. Hence the CFB mode can be used for instance to encipher the short last block occurring at the end of the message in CBC mode (Section 5.3.3). However, CFB is less efficient than CBC in that it requires for each plaintext character one execution of the encryption algorithm. For example, the throughput of the DES operating on 8-bit CFB is reduced by a factor of 8 or more compared to the CBC mode. This can be overcome by enciphering n plaintext characters using the pseudo-random key stream produced by one DES cycle where n is the largest

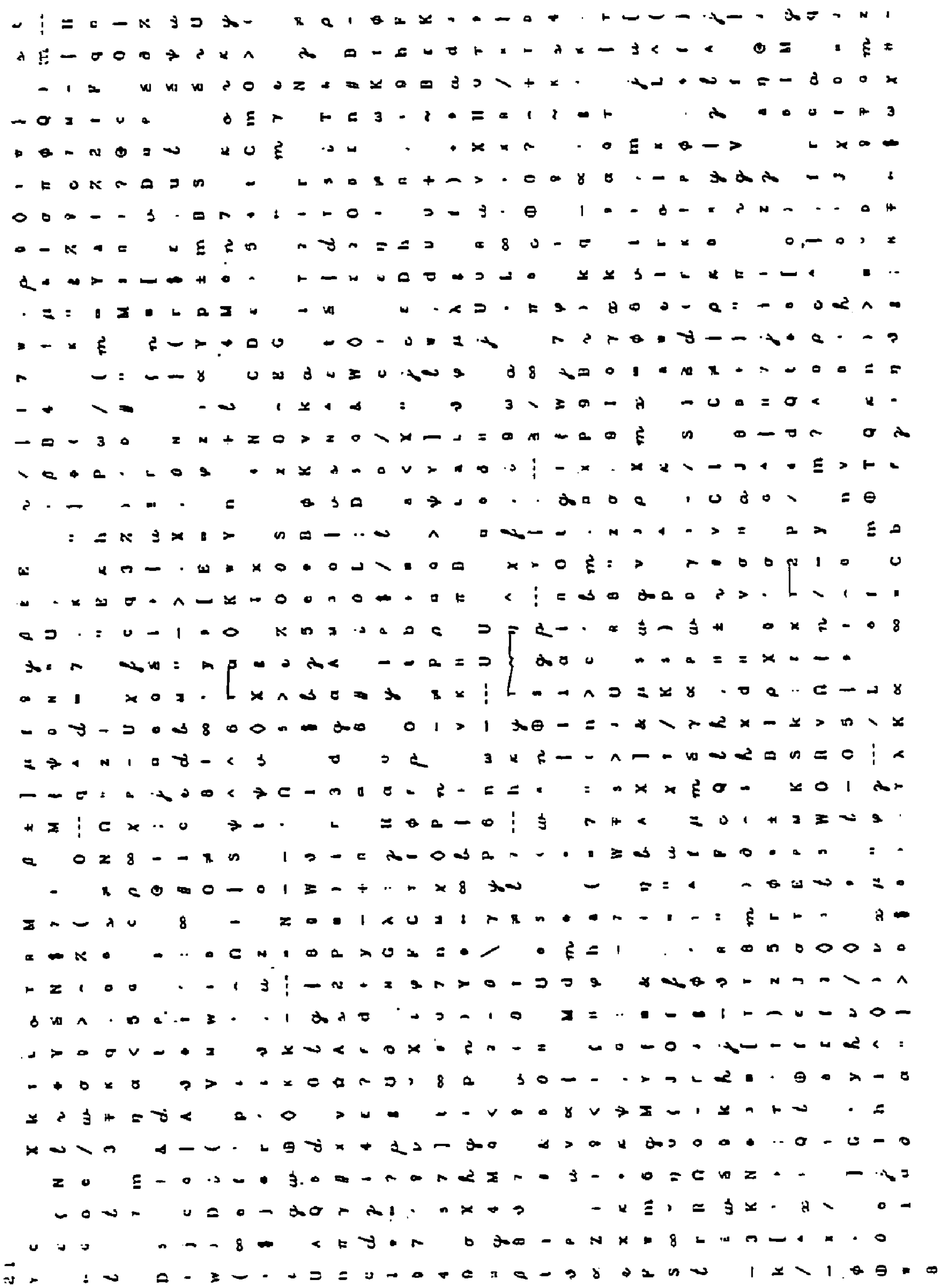


Fig. 5.9 - CFB Ciphertext Example

```
7ι
••••••••••••••••••••••••••••••••••••••
U-ƒε ,ι!ɑLE PROGRAM ENABLES TRANSFER CF
DES ENCIPHERED DATA BETWEEN APPLE MICRO-
COMPUTERS.
••••••••+rμ0<vμι •••••••••••••••••••••

1000    O=ℓ, ›-•.   SET MODE REG.
        STA C0AD    OF UART
        LDA#12      SET COMMAND REG.
        STA C0AD    OF UART
        JSR 213C    INPUT SECRET KEY
        JSR 2355    CHECK PARITY
        SPACE
1011    LDA C0A6    Qϑ•" y ?OARD AND
        AND#03      LINE CONTINUOSLY
        BEQ(101B)
        JMP 1034
101B    LDA C000
        BPL(1011)
        SPACE
        K vιu◊θrϑ   SET ENCRYPTION MODE
        JSR 3000    SUBROUTINE READ
        JSR 2500    ECB ENCRYPTION
        JSR 2600    SUBROUTINE TRANSMIT
        JSR 2700    R μ ? 16ɑE DELAY
        JMP 1011
        SPACE
1034    K›ι›8 ωMM   SET DECS(✩oωN$T"ODE
        JSR 2000    SUBROUTINE RECEIVE
        JSR 2A00    ECB DECRYPTION
        JSR 30#ᵃη;  -•›OUTINE DISPLAY
        CMP#8D      Sdγι hP¿"END
        BEQ(1047)   O·F DATA
        JMS ℓ:ρ$•∘ƒ
1047    HALT
```

Fig. 5.10 - CFB Error Characteristic

integer which is less than $\left[\frac{\text{block size of DES}}{\text{plaintext character size}}\right]$. In the case of 8-bit CFB for instance, one can encipher 8 plaintext characters (bytes) using the pseudo-random sequence produced by one DES cycle. This will produce a throughput comparable to that of the CBC without the need for padding.

Errors have been introduced within the ciphertext to study the error extension characteristics of the CFB mode. From Figure 5.10 it is seen that bit errors within one CFB ciphertext character affects not only the decryption of the garbled ciphertext character but also the decryption of the succeeding characters until the bit errors are shifted out of the CFB input block. The first affected plaintext character is garbled in exactly those places where the ciphertext character is in error. Successive plaintext characters experience an average error rate of 50% until all errors have been shifted out of the DES input block. In this 8-bit CFB case,errors in one ciphertext character is seen to affect decryption of nine characters. Further it is seen from Figure 5.10 that the CFB decryption automatically regains cryptographic synchronization. This self-synchronization property is also reflected in equations (5.3)and (5.4), Section 5.4.1. From equation (5.4), an error in ciphertext Y(i-1) can potentially affect every bit in the computed quantity $f_k$ [U(i)] and hence can cause every bit in the recovered plaintext X(i) to be in error. From equation (5.4), an error in ciphertext Y(i-1) causes only the corresponding bit position in the recovered plaintext X(i-1) to be in error. The system synchronizes when U(i) becomes equal at both ends. Thus like the CBC mode, CFB mode provides limited error extension.

The recommendations given in Section 5.3.3 regarding the management of the CBC initialization vectors and its influence on the security are also applicable to CFB.

## 5.5 Cipher Block Chaining with Plaintext Feedback

### 5.5.1 Principle

Cipher block chaining with plaintext feedback (CBCP) is a block cipher in which the plaintext is exclusive-ored not only with the previous ciphertext block but also with the previous plaintext block prior to being processed by the DES.

This scheme is illustrated in Figure 5.11. The first DES

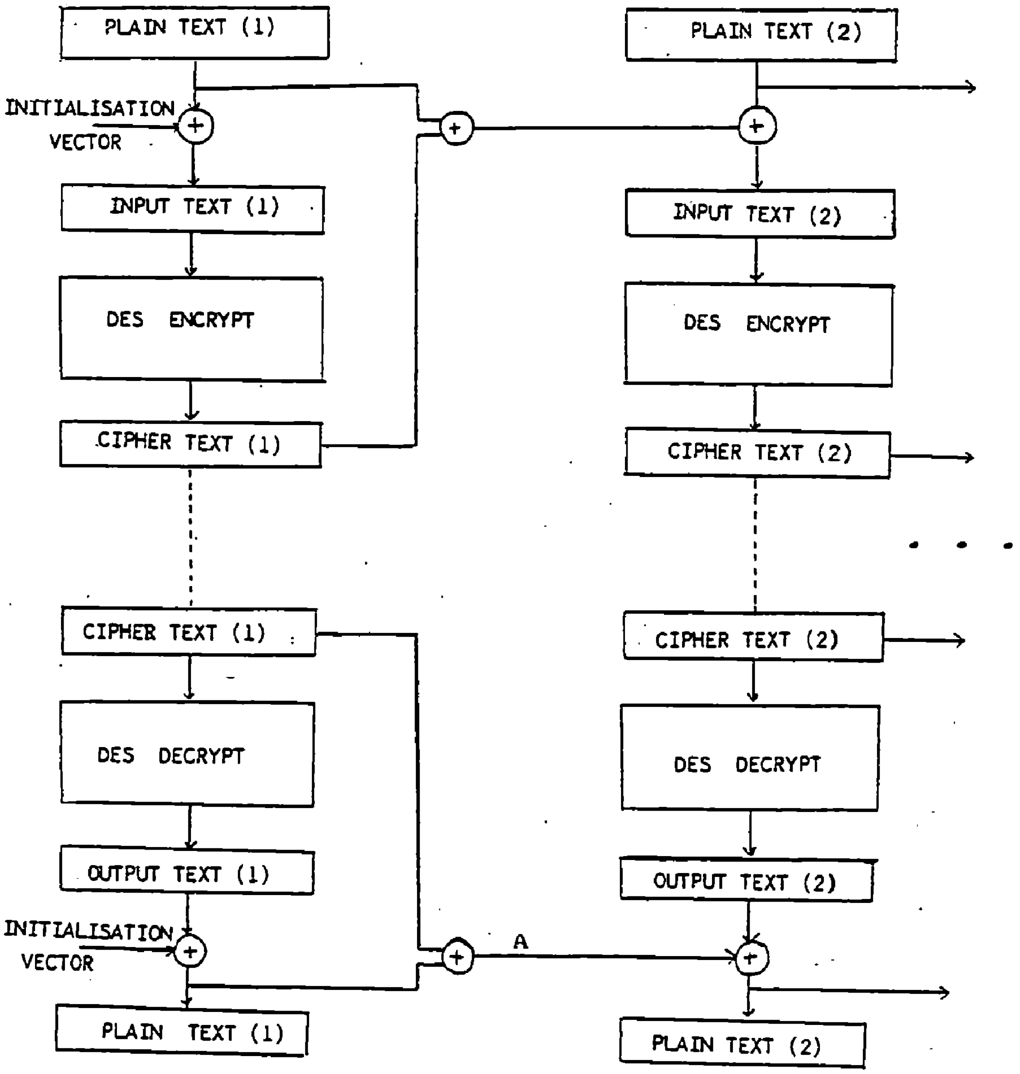


Fig 5.11 - Cipher Block Chaining with Plaintext Feedback (CBCP)

input block is formed by exclusive-oring the first plaintext block with a 64-bit initialization vector (IV). This input block is processed through the DES device producing a 64-bit DES output block which defines the first ciphertext block. Hence processing of the first plaintext block is same as in the CBC mode. This first ciphertext block is exclusive-ored with the second plaintext block as well as the first plaintext block to construct the second DES input block. The next DES operation produces the second ciphertext block. This chaining process is continued until the end of the message when the last block of ciphertext is obtained by encrypting the last input block formed by exclusive-oring last plaintext block with the (last-1) ciphertext and plaintext blocks.

In CBCP decryption, the first ciphertext block received is processed through the DES device to produce the DES output block. This first output block is exclusive-ored with the same initialization vector (IV) to produce the first plaintext block. Again the deciphering process of the first ciphertext block is same as in the CBC mode. The second ciphertext block is processed through the DES to yield the second output block which is then exclusive-ored with the first ciphertext and plaintext blocks to produce the second plaintext block. This process is continued until the end when the last block of plaintext is obtained by exclusive-oring the last DES output block with the (last-1) plaintext and ciphertext blocks.

Mathematically, the scheme can be expressed as follows:

Encipherment and decipherment procedures are given by:

$$Y(i) = f_k\,[X(i) \oplus U(i)] \quad i \geqslant 1 \qquad (5.5)$$

and

$$X(i) = f_k^{-1}\,[Y(i)] \oplus U(i) \quad i \geqslant 1 \qquad (5.6)$$

where

X(i) is the ith plaintext block
Y(i) is the corresponding ith ciphertext block

and

$$U(i) = \begin{cases} Z\text{, the initialization vector} & i = 1 \\ h[\, X(i-1),\ Y(i-1)\,] & i > 1 \end{cases}$$

Here h represents a simple exclusive-or function.

Therefore,

$$U(i) = X(i-1) \oplus Y(i-1)\ ,\ i > 1 \qquad (5.7)$$

Substituting (5.7) for U(i) into (5.5) gives

$$Y(i) = f_k\ [X(i) \oplus X(i-1) \oplus Y(i-1)]\ ,\ i > 1$$

From the recursive nature of the equations (5.5) and (5.7) it follows that there exist functions $H_1$, $H_2$, ..., $H_i$ such that

$$Y(i) = H_i\ [k,\ X(0),\ X(1),\ \ldots,\ X(i)]\ ,\ i \geq 1 \qquad (5.8)$$

where $X(0) \equiv Z$.

Similarly from (5.6) and (5.7) it follows that,

$$X(i) = \begin{cases} f_k^{-1}\ [Y(i)] \oplus Z & i = 1 \\ f_k^{-1}\ [Y(i)] \oplus Y(i-1) \oplus X(i-1) & i > 1 \end{cases}$$

Thus there exist functions $G_1$, $G_2$, ..., $G_i$ such that

$$X(i) = G_i\ [k,\ Y(0),\ Y(1),\ \ldots,\ Y(i)]\quad ,\quad i \geq 1 \qquad (5.9)$$

where $Y(0) \equiv Z$.

From equation (5.8), one can see that the enciphering process is entirely deterministic and the output ciphertext block at time i, Y(i),is dependent only on the inputs to the ciphering process from time 1 through time i, namely, the key (k), the initialization vector (Z) and all the plaintext blocks X(1) through to X(i). Furthermore, since ciphertext block Y(i) is dependent on the initial conditions established at the beginning of the ciphering process, at time 1, it is said to be origin dependent[3].

From equation (5.9), it can be seen that the recovered plaintext block at time i, X(i), depends only on the key (k), the initialization vector (Z) and all ciphertext blocks Y(1) through to Y(i). Thus X(i) is also origin dependent.

This scheme is defined to be a general block cipher[3].

### 5.5.2 Implementation

The program implementing this mode of operation is very much similar to the one used for the CBC mode. The only difference is that in this case, a plaintext block is not only exclusive-ored with the previous ciphertext block (TEMP 1) but also with the previous plaintext block prior to DES encryption. A similar difference occurs after DES decryption.

### 5.5.3 Results and Discussion

The plaintext example shown in Figure 5.2 is enciphered under this mode using the same key and initialization vector as before. The ciphertext produced is shown in Figure 5.12.

Again from Figure 5.12 it is seen that there is no repetition of ciphertext even when the plaintext is repetitive. This mode is similar to CBC and CFB in its resistance to forms of attack such as ciphertext searching, replay, insertion and deletion. The most interesting property of this mode however is that of error extension. The deciphered version of the example with some errors introduced in the ciphertext prior to decryption is shown in Figure 5.13. Two points are worth to be noted. The first one is that an error in the ciphertext affects the decryption of all subsequent blocks until the end of the message. That is, the scheme exhibits the property of error propagation. This agrees with equation (5.9) where every bit of the recovered plaintext block X(i) is a function of every bit in the ciphertext blocks Y(1) through to Y(i). The only case in which the error is not propagated occurs when the corrupted ciphertext block Y(i)* and the deciphered value of Y(i)* under key k obey the equality

$$Y(i)^* \oplus f_k^{-1}\ (Y(i)^*) = Y(i) \oplus f_k^{-1}\ (Y(i))$$

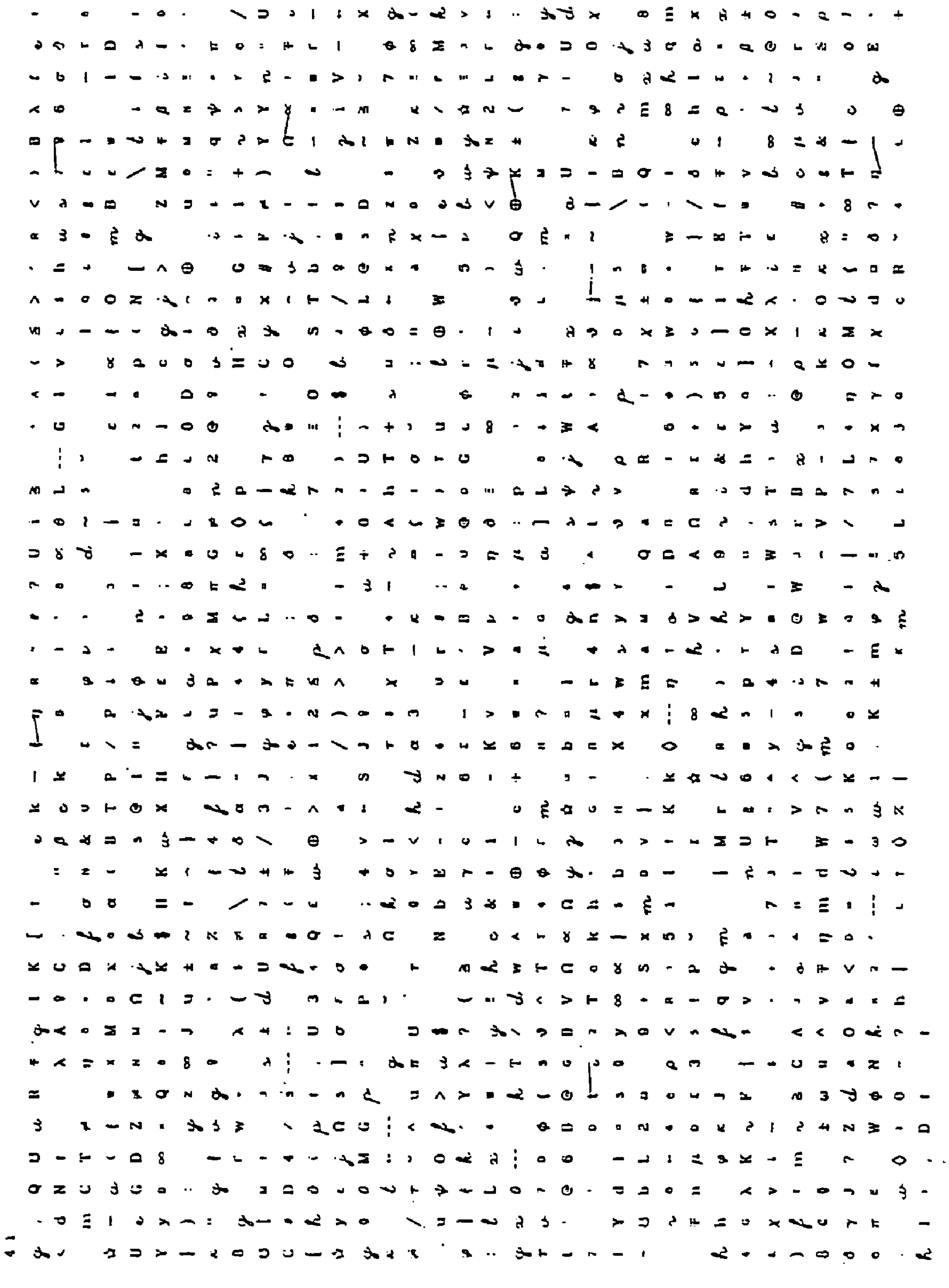


Fig. 5.12 - CBCP Ciphertext Example

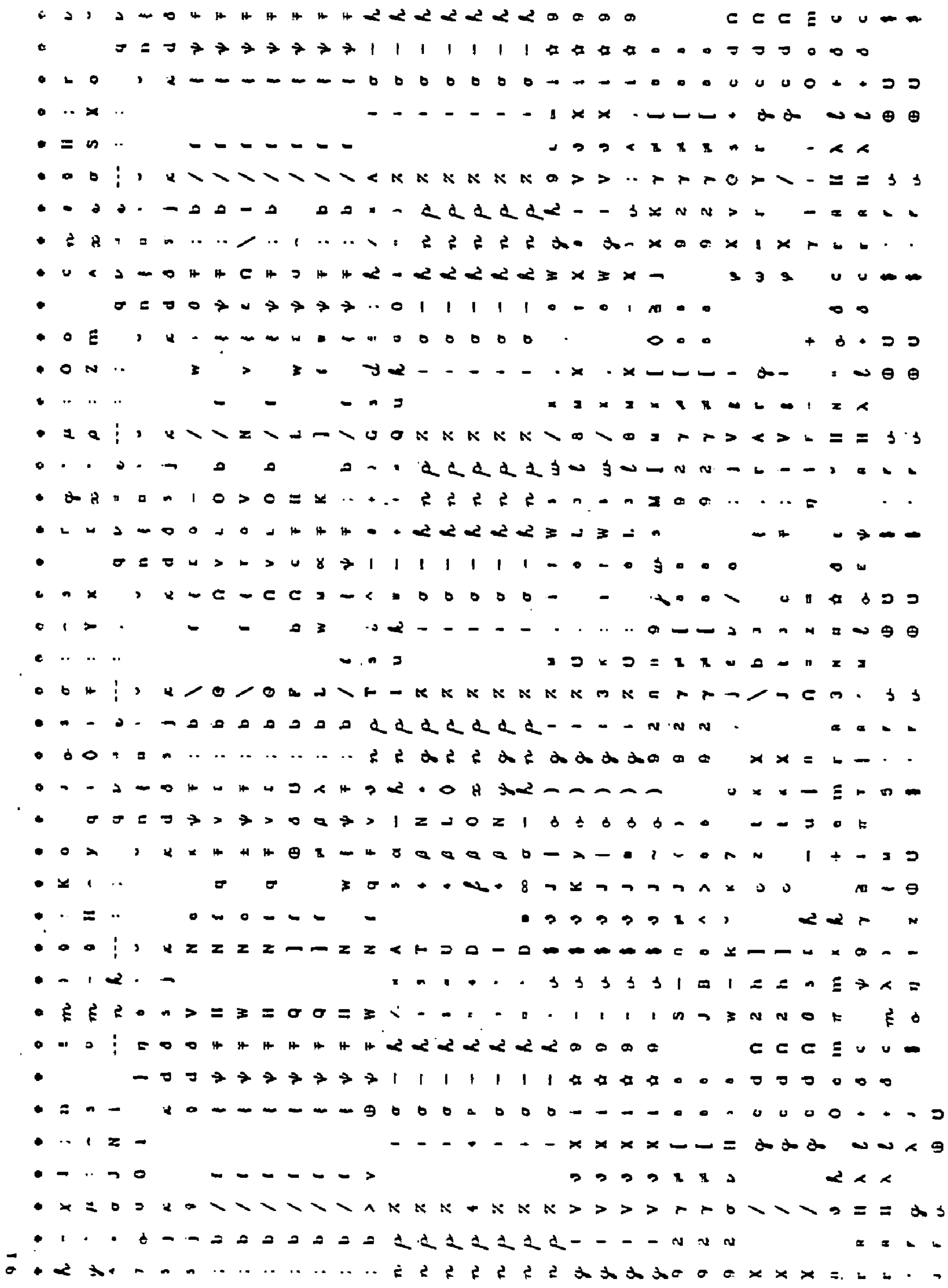


Fig. 5.13 - CBCP Error Characteristic

That is, the feedback value at point A(Figure 5.11),input to the next cycle, is unchanged. Assuming that the probability that an error in Y(i) causing each bit in $f_k^{-1}(Y(i)*)$ to differ from the corresponding bit in $f_k^{-1}(Y(i))$ is approximately equal to 0.5, the probability that an error cancellation occurs is approximately equal to $2^{-64}$.

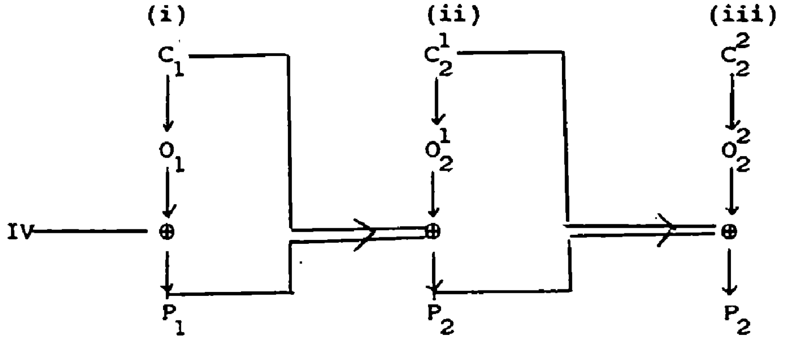
Secondly, when deciphering the ciphertext containing errors with the right key and the right initialization vector, it is seen that patterns or repetitions in the garbled deciphered text are revealed which correspond to the patterns in the plaintext. The reason for this occurrence is given as follows:

First consider the case where there are no errors in the ciphertext prior to decryption. Referring to the diagram shown below, assume that the plaintext blocks 2 and 3 are the same (but their corresponding ciphertexts $C_1$ and $C_2$ will be different due to chaining). Then, for the deciphered block 3 to be equal to block 2, $P_2$, one must have

$$O_2^2 \oplus C_2^1 \oplus P_2 = P_2$$

ie ,

$$O_2^2 \oplus C_2^1 = (0 \ldots 0) = \text{Zero block}$$



Now if the block $C_1$ has an error, $C_1^*$, then this decryption results in plaintext block, $P_1^*$. The second block $P_2^*$ is then equal to $C_1^* \oplus P_1^* \oplus O_2^1$ . The third block is given by $O_2^2 \oplus C_2^1 \oplus P_2^*$ and this is equal to $P_2^*$ because $O_2^2 \oplus C_2^1$ is still equal to the zero block as it is unaffected by error in $C_1^*$ . Thus this repetition in the deciphered version will occur as long as the two successive blocks are the same in the plaintext. When $P_2$ and $P_3$ are different, then

$P_2 \oplus C_2^1 \oplus O_2^2 = P_3$ and $C_2^1 \oplus O_2^2$ is not equal to the zero block. Therefore when an error occurs in $C_1^*$ , $P_2^* \oplus C_2^1 \oplus O_2^2 = P_3^* \neq P_2^*$.

This pattern occurrence is not of great concern as far as the security of this scheme is concerned because here we are talking about the legitimate receiver with the right key and the right initialization vector detecting patterns in the decrypted version of the garbled ciphertext. This scheme is not at all suitable for communication links prone to noise. On the other hand, this error propagation property can be used to prevent 'spoofing' attack. This scheme is very much suitable for message authentication purposes where one needs to determine with a high level of confidence whether the message has been altered.

As CBCP is a block cipher padding is again required at the end of the message like the ECB and the CBC modes.

## 5.6 Stream Cipher Feedback with Vector Feedback

### 5.6.1 Principle

This is an additive stream cipher technique similar to CFB in which the DES is used to generate a pseudo-random binary stream. This stream is exclusived-ored with the plaintext to form the ciphertext. The ciphertext together with the initialization vector is then fed back to form the next DES input block. The feedback from the initialization vector is the feature which differentiates it from the CFB mode. It is referred to as CFBV.

This scheme is illustrated in Figure 5.14. The initialization vector forms the first DES input block. This is encrypted by the DES device producing a 64-bit pseudo-random output block. The rightmost or the least significant s-bits ($1 \leqslant s \leqslant 64$) of the DES output block are exclusive-ored with the s-bits of the plaintext to form s-bits of ciphertext. These s-bits of the ciphertext are expanded to form a 64-bit block by repetition. This block is then exclusive-ored with the previous initialization vector and the result is shifted by s-bits to form the next DES input block, the new initialization vector. This is then used in the encryption of the next s-bit plaintext character. This process is repeated until the end of the message.

In CFBV decryption, the first output block, produced by encrypting the same initialization vector, is exclusive-ored with the

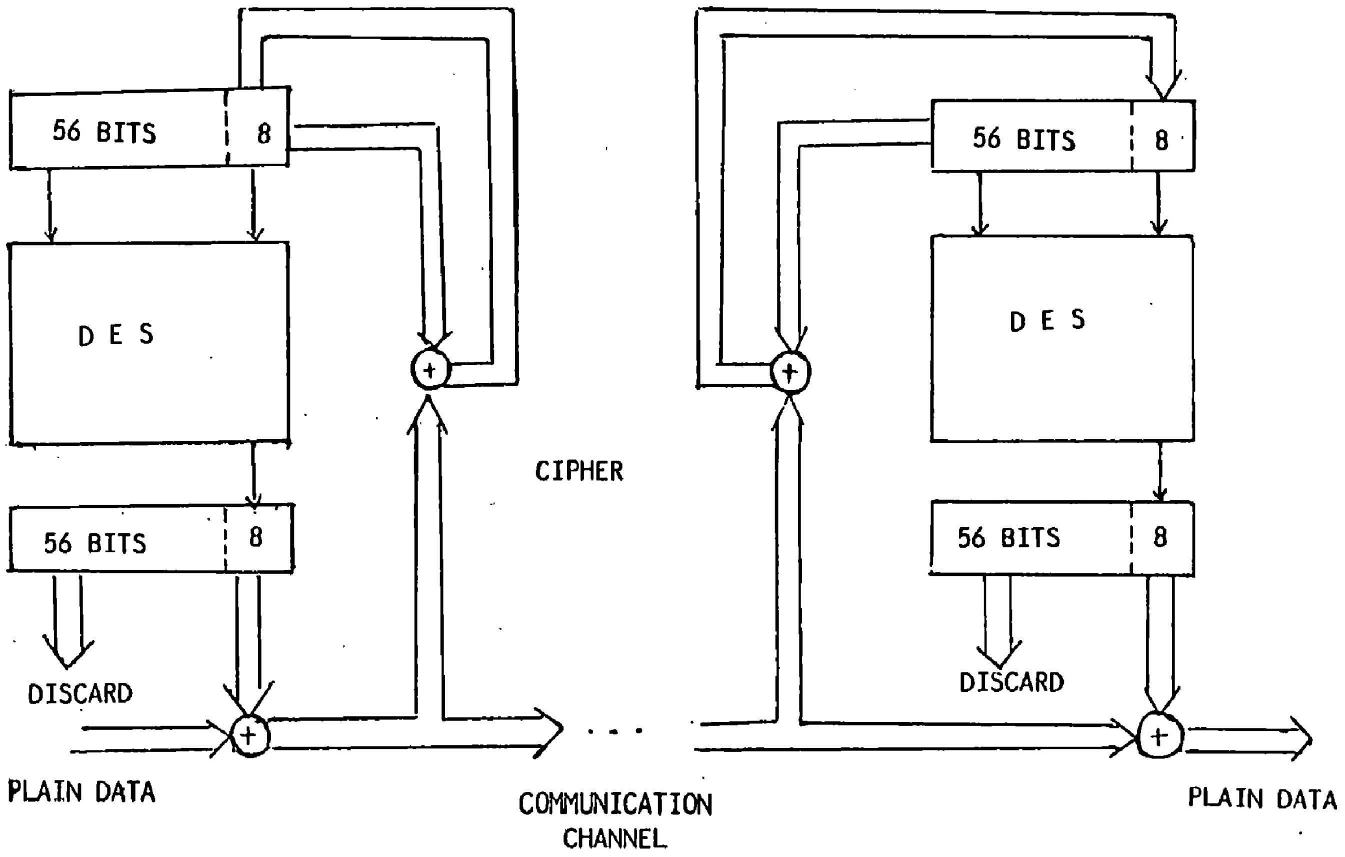


Fig. 5.14 -Stream Cipher with Ciphertext and Vector Feedback (CFBV)

first s-bit ciphertext character to produce the first s-bit plaintext character. The first s-bit ciphertext character is then expanded to form a 64-bit block which is exclusive-ored with the previous initialization vector. The result is shifted by s-bits to form the next DES input block. This process is repeated until the end of the message.

Note that the first cycle of this scheme is exactly the same as in CFB. Furthermore as in CFB, the DES device is used in its encryption state at both ends.

Mathematically, this scheme can be described as follows: The encipherment and decipherment can be expressed as

$$Y(i) = X(i) \oplus f_k [U(i)] \qquad i \geq 1 \qquad (5.10)$$

and

$$X(i) = Y(i) \oplus f_k [U(i)] \qquad i \geq 1 \qquad (5.11)$$

respectively

where

$X(i)$ is the ith plaintext character
$Y(i)$ is the corresponding ith ciphertext character

$$U(i) = \begin{cases} Z, \text{ initialization vector, } i = 1 \\ h\,[U(i-1),\ Y(i-1)]\ , \quad i > 1 \end{cases} \qquad (5.12)$$

Here the function h is given by

$$h\,[U(i-1),\ Y(i-1)] = E\,(Y(i-1)) \oplus U(i-1),\ i > 1$$

where

$E\,(Y(i-1))$ represents expansion of s-bit ciphertext characters $Y(i-1)$ repetitively to form a 64-bit block.

From the recursive nature of the equations (5.10), (5.11) and (5.12), it follows that there exist functions $H_1$, $H_2$, ..., $H_i$ and $G_1$, $G_2$, ...,

$G_i$, such that

$$Y(i) = X(i) + H_i\ [k, X(0), X(1), \ldots, X(i-1)]\ i \geq 1 \quad (5.13)$$

and

$$X(i) = Y(i) + G_i\ [k, Y(0), Y(1), \ldots, Y(i-1)]\ i \geq 1 \quad (5.14)$$

where

$$X(0) \equiv Y(0) \equiv Z$$

The equations (5.13) and (5.14) are counterparts of the equations (5.8) and (5.9) given in Section 5.5.1. Hence this scheme represents a general stream cipher [3].

From equation (5.13), it follows that the jth bit in the ciphertext character Y(i) is directly affected by only the jth bit in the plaintext character X(i) whereas it is potentially affected by every bit in plaintext characters X(1) through to X(i-1). In like manner, from equation (5.14), it follows that the jth bit in the received plaintext character X(i) is directly affected by only the jth bit in the ciphertext character Y(i) whereas it is potentially affected by every bit in ciphertext characters Y(1) through to Y(i-1).

### 5.6.2 Implementation

The program implementing this mode of operation with s=8 is very much similar to the one used for the CFB mode. The only difference compared to the CFB mode is that in this case, the new initialization vector is formed by exclusive-oring the previous initialization vector with the expanded previous 8-bit ciphertext character.

### 5.6.3 Results and Discussion

The plaintext example shown in Figure 5.2 is enciphered under this mode using the same key and initialization vector as before. The ciphertext produced is shown in Figure 5.15.

Like all the other modes except the ECB discussed earlier this mode masks the patterns in the plaintext thus reducing the code book analysis, replay, insertion and deletion attacks. It is

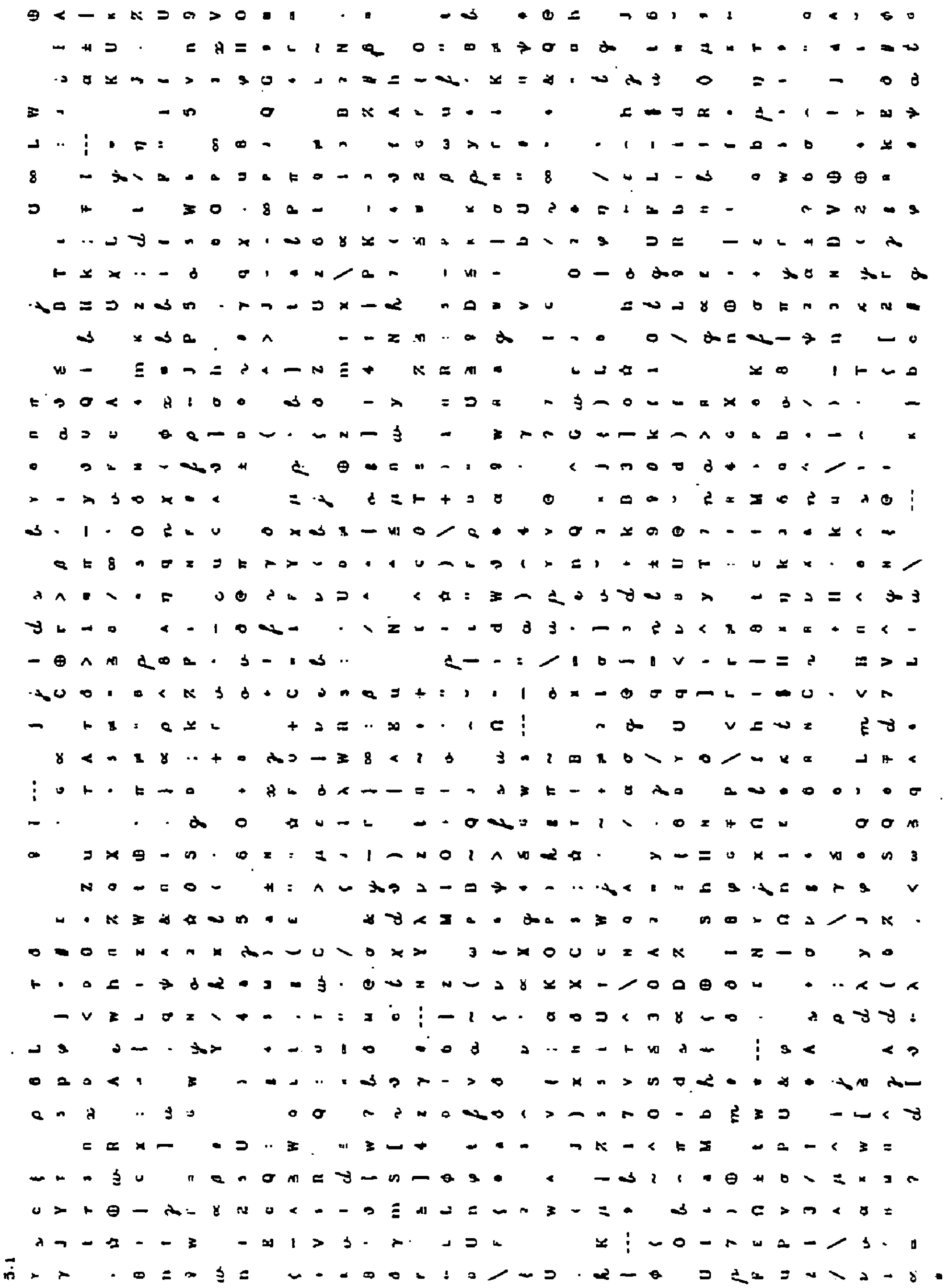


Fig. 5.15 - CFBV Ciphertext Example

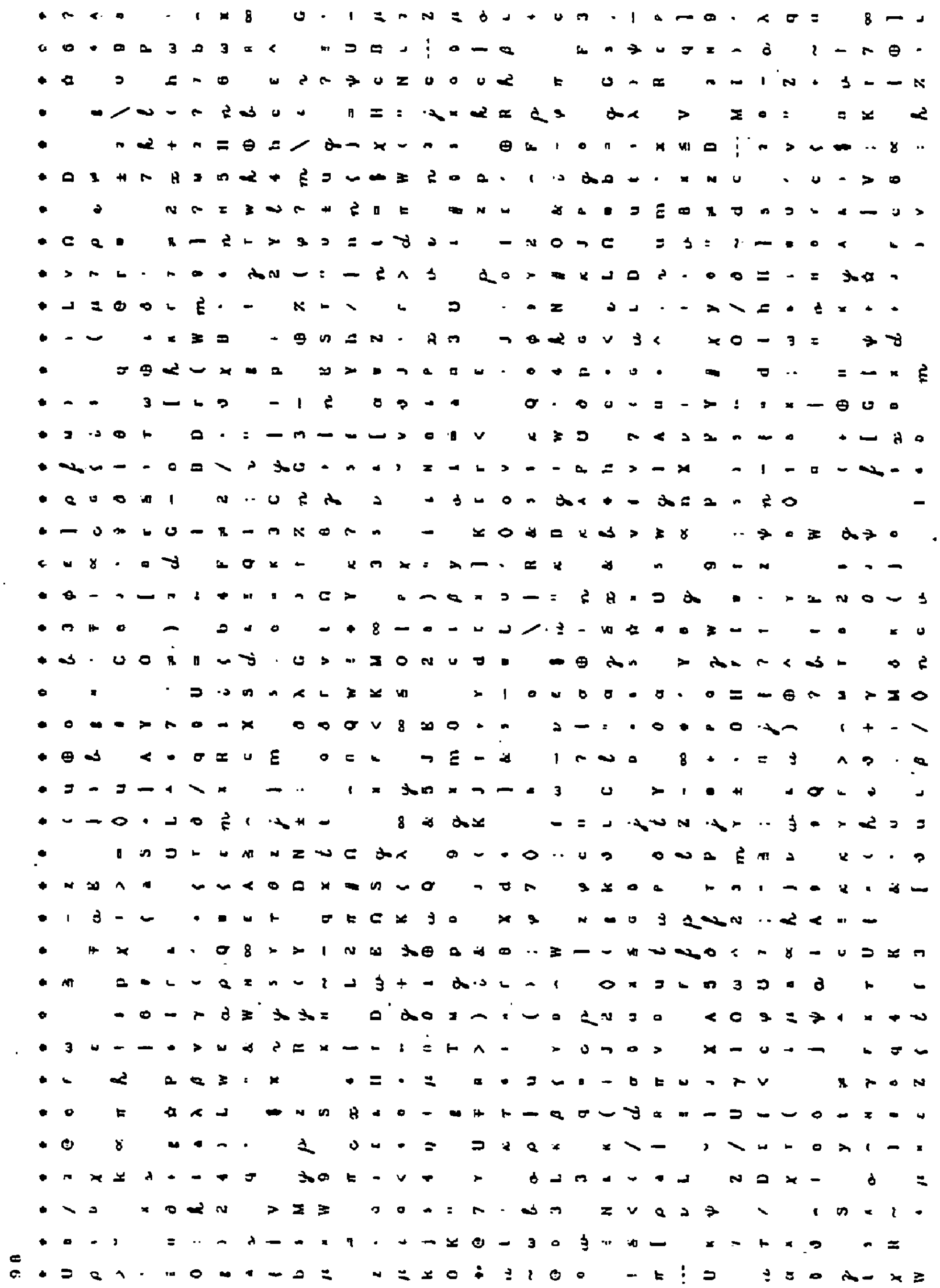


Fig. 5.16 - CFBV Error Characteristic

similar to CFB in that the messages can be processed character by character thus avoiding the padding required in ECB, CBC and CBCP modes.

The most interesting property of this stream cipher mode is that of error extension. The deciphered version of the example with some errors introduced in the ciphertext prior to decryption is shown in Figure 5.16. Two points are worth mentioning. The first one is that an error in a ciphertext character is found to affect the decryption of all subsequent ciphertext characters until the end of the message. That is, like CBCP, this stream cipher exhibits the property of error propagation. This can also be seen from equation (5.14). Since the recovered plaintext X(i) is potentially affected by every bit in the ciphertexts Y(1) through to Y(i-1), error propagation is achieved. However because the jth bit in the plaintext X(i) depends only on the jth bit in the ciphertext Y(i), the intersymbol dependence can be achieved for all but the final plaintext. On the other hand with the CBCP, there is intersymbol dependence throughout all blocks.

Secondly, when deciphering this ciphertext containing errors with correct key and initialization vector, it is seen from Figure 5.16 that there is no pattern or repetition in the garbled decrypted text. This is in contrast to the CBCP scheme considered in Section 5.5.3. So even the legitimate user with right key and right initialization vector gets a completely garbled text when the errors are introduced in the ciphertext. This means that this scheme is not suitable for links prone to noise but is very useful for message authentication purposes.

# CHAPTER 6

## STATISTICAL TESTS ON DES OUTPUT SEQUENCES

### 6.1 General

Some statistical tests are applied to the output sequences obtained using the DES algorithm under different modes (see Chapter 5) to test for their randomness properties. Some simple statistical tests are also considered with a view to detecting the dependence or correlation between the output and inputs to the DES and to determining whether plaintext-ciphertext pairs could be used to predict the bits of the key.

### 6.2 Statistical Tests for Randomness

Strictly speaking, no finite sequence is ever truly random. The best that can be done is to single out certain properties as being associated with randomness and to accept any sequence which has these properties as a random sequence. In particular, it is assumed that the opponent intercepts sections of ciphertext sequence and attempts to exploit the statistical properties of the sequence in his cryptanalytical attack. Therefore it is necessary to apply the statistical tests to sections of ciphertext sequence to check their randomness characteristics. This type of randomness is often referred to as local randomness [14].

There are several statistical tests which can be applied to a sequence. Here four fundamental tests have been considered which can be used to provide a quantitative measure of randomness [14]. They are the frequency test, the serial test, the runs test and the auto-correlation test. All these tests measure the relative frequencies of certain patterns of '0's and'1's in the sequence considered, in one way or another. The sequence under consideration is then regarded to be random if the sequence passes the test. Levels of confidence are set so as to decide if the sequence is random enough for our purpose.

Initially, it is necessary to choose the length of the section or sample to be tested. The sample size must not be too large to swamp local variations but at the same time it must not be too small preventing any reasonable conclusions. Accordingly, for the test in question, a sample size of $n = 1024$ is chosen. As the next step, the properties of the input to the DES algorithm are defined. The procedure adopted is as follows: In each case, 'non-random' inputs are applied to the DES and the output is tested for its randomness. For this reason, the inputs are chosen to be periodic sequences with different period lengths, thus allowing a variable number of cycles to be present within the selected sample size.

Six different cycle lengths are selected namely 5, 8, 10, 20, 40 and 64 digits. Within each category, five input samples are chosen. Each of these inputs is encrypted using DES under the five different modes namely, the ECB, the CBC, the CFB, the CBCP and the CFBV. This procedure is carried out using five different DES keys. Among the chosen keys are included a weak key (see Section 4.6.4) and a semi-weak key (see Section 4.6.4), a 'non-random' key and two arbitrarily selected 'random' keys. The input samples and the keys used are given in Appendix 7, Section A7.1.

The tests are performed in two parts. In the first part, the tests are applied to output samples produced by encryption under the different DES modes for a fixed key. This is done to investigate the effect of different modes of encryption on randomness of the output. In the second part, the key to the DES is varied to find the effect of key on the randomness of the output.

The four tests and the confidence levels which indicate whether a sequence is random or non-random are now briefly outlined [14].

### 6.2.1 Test 1: The Frequency Test

The frequency test checks whether there is approximately the same number of '0's and '1's in the sequence.

Let the length of the sequence be n and let it contain $n_0$ zeroes and $n_1$ ones. Defining,

$$\chi^2 = \frac{(n_0 - n_1)^2}{n}$$

$\chi^2 = 0$ when $n_0 = n_1$. Larger the value of $\chi^2$, greater the discrepancy between the observed and expected frequencies. This is a $\chi^2$- test with one degree of freedom. Thus if the value of $\chi^2$ is not greater than 3.84, then from the table of $\chi^2$ distribution given in [28], the sequence is passed at 5% significance level. (Note that if $\chi^2 = 0$, the sequence might also be rejected on the grounds of it being too good!).

### 6.2.2 Test 2: The Serial Test

The serial test checks whether the transition probabilities are reasonable, that is, the probability of consecutive entries being equal or different is about the same. This then gives some level of confidence that each bit is independent of its predecessor.

Supposing that the sequences 00 occurs $n_{00}$ times, 01 occurs $n_{01}$ times, 10 occurs $n_{10}$ times and 11 occurs $n_{11}$ times, then

$$n_{01} + n_{00} = n_0 \quad \text{or} \quad n_0-1$$
$$n_{10} + n_{11} = n_1 \quad \text{or} \quad n_1-1$$

and

$$n_{00} + n_{01} + n_{10} + n_{11} = n-1$$

(Note that n-1 occurs because in a sequence length of n bits, there are only n-1 transitions).

Ideally, we want $n_{00} = n_{01} = n_{10} = n_{11} \doteq \frac{n-1}{4}$. Good [29] has shown that

$$\frac{4}{n-1} \sum_{i=0}^{1} \sum_{j=0}^{1} (n_{ij})^2 - \frac{2}{n} \sum_{i=0}^{1} (n_i)^2 + 1 \qquad (6.1)$$

is approximately distributed as $\chi^2$ with two degrees of freedom. The value of $\chi^2$ corresponding to a 5% significance level with two degrees of freedom is 5.99. Hence the sequence is rejected if the value of (6.1) is greater than 5.99.

### 6.2.3 Test 3: The Runs Test

The runs test is based on the theory of runs where a run is a succession of identical letters (zeroes or ones) which is followed and preceded by different letters. The total number of runs is often a good indication of a possible lack of randomness.

To find the probability that $n_0$ zeroes and $n_1$ ones will form u runs when each of $\binom{n_0 + n_1}{n_1}$ possible arrangement of these letters is equally likely, first consider the case where u is even, namely, u = 2t, for some positive integer t. There are $\binom{n_0-1}{t-1}$ ways in which the $n_0$ zeroes can form t runs and $\binom{n_1-1}{t-1}$ ways in which the $n_1$ ones can form t runs. It follows then that there are altogether $2\binom{n_1-1}{t-1}\binom{n_0-1}{t-1}$ ways in which those $n_1 + n_0$ letters can form 2t runs. The factor 2 is accounted for by the fact that when the two kinds of runs are combined so that they alternate, we can begin with a run of zero or with a run of one. Thus when u = 2t, the probability of getting u runs is

$$f(u) = \frac{2\binom{n_0-1}{t-1}\binom{n_1-1}{t-1}}{\binom{n_0 + n_1}{n_1}}$$

When u = 2t + 1, similar arguments lead to

$$f(u) = \frac{\binom{n_0-1}{t}\binom{n_1-1}{t-1} + \binom{n_0-1}{t-1}\binom{n_1-1}{t}}{\binom{n_0 + n_0}{n_1}}$$

When $n_0$ and $n_1$ are both greater than 10 or more, the sampling distribution of u can be approximated with a normal distribution. Making use of this distribution, Gibbons [30] shows that, the expected value E(u) and variance Var (u) are as follows:

$$E(u) = \frac{2\, n_0 n_1}{n_0 + n_1} + 1$$

and

$$\mathrm{Var}(u) = \frac{2\, n_0 n_1\ (2n_0 n_1 - n_0 - n_1)}{(n_0 + n_1)^2\ (n_0 + n_1 - 1)}$$

Thus for sufficiently large value of $n_0$ and $n_1$, the normal test variable Z is given by

$$Z = \frac{u - E(u)}{\sqrt{\mathrm{Var}(u)}}$$

The null hypothesis that the sequence is random is rejected if

$|Z| \geqslant 1.96$ at 5% significance level.

### 6.2.4 Test 4: The Autocorrelation Test

The autocorrelation function of the sequence is an important element when testing for randomness. Random sequences possess a special kind of autocorrelation function namely peaked in the middle and tapering off rapidly at the ends. Autocorrelation also reflects the periodicity within the sequence.

If $\{x_0, x_1, \ldots, x_n\}$ is a binary sequence, then its autocorrelation function can be defined as

$$A(r) = \frac{1}{n-r} \sum_{i=1}^{n-r} x_i \cdot x_{i+r} \qquad \text{for } r = 0, 1, \ldots, m$$

Here a slightly modified version of A(r) has been used. The operation between $x_i$ and $x_{i+r}$ is defined as one of matching (comparison) instead of direct multiplication. That is,

$$A(r) = \frac{1}{n-r} \sum_{i=1}^{n-r} Z_{ir}$$

where

$$Z_{ir} = \begin{cases} 1 & \text{if } x_i = x_{i+r} \\ 0 & \text{if } x_i \neq x_{i+r} \end{cases}$$

That is, A(r) is some sort of a measure of number of times the shifted and original sequences match, both $x_i$ and $x_{i+r}$ are equal to one or zero. This operation is more sensitive than direct multiplication which just tests for matching ones.

#### 6.2.4.1 Expected Value of A(r), E (A(r))

The probability that $x_i$ and $x_{i+r}$ match is given by Prob $[Z_{ir} = 1]$ .

For $r \neq 0$

$$\text{Prob}\,[Z_{ir} = 1] = \text{Prob}\,(x_i = 1 \text{ and } x_{i+r} = 1) + \text{Prob}\,(x_i = 0 \text{ and } x_{i+r} = 0)$$

Let the probability that $x_i = 1$ be p and the probability that $x_i = 0$ be q. Then,

$$\text{Prob } [Z_{ir} = 1] = p^2 + q^2$$

$$\text{Therefore } Pr[Z_{ir} = 0] = 1 - (p^2 + q^2) = 2pq$$

$$\text{Hence } E\ (Z_{ir}) = (p^2 + q^2).1 + (2pq).0$$
$$= p^2 + q^2$$

$$\text{But } E\ (A(r)) = \frac{1}{n-r}\ E\left(\sum_{i=1}^{n-r} Z_{ir}\right)$$
$$= \frac{1}{n-r}\ (n-r)\ E\ (Z_{ir})$$

$$E\ (A(r)) = p^2 + q^2$$

For a random sequence, assuming that the probabilities p and q is a ½ this yields

$$E\ (A(r)) \longrightarrow \tfrac{1}{2}$$

This is to be expected as in a random sequence, the probability of observing a match of zeroes or ones is equal to that of not observing such a match. The mean value itself is an indication of non-randomness. From this, if the sequence is random, its autocorrelation function should vary around the 0·5 mark.

6.2.4.2 Variance of A(r), Var A(r)

$$Var(A(r)) = E\ (A^2(r)) - [E\ (A(r))]^2$$

Considering first $E\ (A^2(r))$, we have

$$E\ (A^2(r)) = \frac{1}{(n-r)^2} E\left(\sum_{i=1}^{n-r} \sum_{j=1}^{n-r} (x_i x_{i+r} x_j x_{j+r})\right)$$

$$= \frac{1}{(n-r)^2} E\left(\sum_{i=1}^{n-r} \sum_{j=1}^{n-r} (Z_{ir} Z_{jr})\right)$$

$$= \frac{1}{(n-r)^2} \sum_{i=1}^{n-r} \sum_{j=1}^{n-r} E\ (Z_{ir} Z_{jr}) \qquad (6.2)$$

Dividing the expression (6.2) into three cases namely (i) i=j, (ii) i≠j and i=j+r or j=i+r and (iii) i≠j and i≠j+r or j≠i+r, we have

$$E(A^2(r)) = \frac{1}{(n-r)^2} \sum_{i=1}^{n-r} E(Z_{ir}\, Z_{ir}) + \frac{2}{(n-r)^2} \sum_{i=r+1}^{n-r} E(Z_{ir}\, Z_{i-r,r})$$

$$+ \frac{1}{(n-r)^2} \sum_{i} \sum_{j} E(Z_{ir}\, Z_{jr})$$

$$E(Z_{ir}\, Z_{ir}) = E(Z_{ir}) = p^2 + q^2$$

Now consider the second term $E(Z_{ir}\, Z_{i-r,r})$

$$Z_{ir}\, Z_{i-r,r} = \begin{cases} 1 & \text{if } Z_{ir} = Z_{i-r,r} \\ 0 & \text{otherwise} \end{cases}$$

Note that here $Z_{ir}$ and $Z_{i-r,r}$ are not independent because

$$Z_{ir} = x_i \odot x_{i+r} \quad \text{and} \quad Z_{i-r,r} = x_{i-r} \odot x_i$$

Where ⊙ refers to matching.

Therefore,

$$\text{Prob}(Z_{ir}, Z_{i-r,r} = 1) = \text{Prob}(x_i \odot x_{i+r} \odot x_{i-r} = 1) +$$

$$\text{Prob}(x_i \odot x_{i+r} \odot x_{i-r} = 0)$$

$$= p^3 + q^3$$

Therefore,

$$\frac{2}{(n-r)^2} \sum_{i=r+1}^{n-r} E(Z_{ir}\, Z_{i-r,r}) = \frac{2(n-2r)}{(n-r)^2} (p^3 + q^3) \qquad (6.3)$$

n-2r is equal to the number of terms for which i ≠ j and i = j+r. By symmetry, the number of terms for which, i ≠ j and j = i+r is n-2r. Hence the factor 2 in the expression (6.3).

Now consider the third term $\frac{1}{(n-r)}^2 \sum_i \sum_j E(Z_{ir} Z_{jr})$ where $i \neq j$, $i \neq j+r$ and $j \neq i+r$.

$$Z_{ir}Z_{jr} = \begin{cases} 1 & Z_{ir} = Z_{jr} = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\text{Prob } (Z_{ir}\ Z_{jr} = 1) = \text{prob } (Z_{ir} = 1) \text{ and prob } (Z_{jr} = 1)$$

$$= (p^2 + q^2)\ (p^2 + q^2)$$

$$= (p^2 + q^2)^2$$

The number of terms in the double sum $\sum_i \sum_j$ is given by

$$= \text{total number of terms} - [(n-r) + 2\ (n-2r)]$$

$$= (n-r)^2 - (n-r) - 2(n-2r)$$

Therefore

$$\frac{1}{(n-r)}^2 \sum_i \sum_j E\ (Z_{ir}\ Z_{jr}) = \left(1 - \frac{2(n-2r)}{(n-r)^2} - \frac{1}{n-r}\right) (p^2 + q^2)^2$$

Hence,

$$E\ (A^2(r)) = \frac{p^2 + q^2}{(n-r)} + \frac{2(n-2r)}{(n-r)^2}\ (p^3 + q^3) + \left[1 - \frac{2(n-2r)}{(n-r)^2} - \frac{1}{n-r}\right] (p^2 + q^2)^2$$

For a random sequence, it is assumed that $p = q = \frac{1}{2}$. Substituting this in the above expression for $E(A^2(r))$ gives,

$$E\ (A^2(r)) = \frac{1}{2(n-r)} + \frac{n-2r}{2(n-r)^2} + \frac{1}{4} - \frac{(n-2r)}{2(n-r)^2} - \frac{1}{4(n-r)}$$

$$= \frac{1}{4(n-r)} + \frac{1}{4}$$

$$\text{Var } A(r) = E\ (A^2(r)) - [E\ (A(r))]^2$$

$$= \frac{1}{4(n-r)} + \frac{1}{4} - [\tfrac{1}{2}]^2 = \frac{1}{4(n-r)}$$

Thus the variance is inversely proportional to (n-r). That is, greater the sample size n, smaller the variation from the mean and as the lag r increases,the variance increases. Having calculated the expected value and the variance of A(r), one can approximate the distribution with a normal distribution for sufficiently large value of n. Thus the normal test variable N(r) is given by

$$N(r) = \left| \frac{A(r) - E(A(r))}{\sqrt{\text{Var } A(r)}} \right|$$

Thus at 5% significance level if $|N(r)| < 1\cdot 96$, the sequence is said to be random. In this test, the autocorrelation of the ciphertext sequences are computed and the number of $|N(r)|$ values which exceed 1•96 is used as a measure of non-randomness of the sequence.

## 6.3 Results and Discussion

Section A7.2 in Appendix 7 gives the results of the frequency, serial and runs tests on the ciphertext sequences produced using the ECB, CFB, CBC, CBCP and CFBV modes of DES under a fixed key namely 3131313131313131. Five input samples for each cycle length have been encrypted under the five chosen modes of DES. The notation used in Section A7.2 is : (Encryption Mode r.s), where r indicates the cycle length of the input sample and s indicates the number of the input sample within the cycle length category, r. (see Section A7.1). For instance CFB 1.2 refers to the mode CFB, input sample number 2, having a cycle length of 5 digits. The figures marked with (*) indicate that the values are in proximity to the 5% significance level and the figures marked with (**) indicate the values beyond the 5% significance level, thus showing a possible lack of randomness of the sequence under consideration.

It is seen from the results in Section A 7.2 that in general the DES under the ECB mode shows the most non-randomness characteristics out of the selected five modes. Section A7.2.1 shows, under each mode, the number of sequences which are classified as non-random by each of the three tests. In Section A7.2.2 are listed the sequences which are classified as non-random by more than one test. Again from Sections A7.2.1 and A7.2.2, it is seen that the ECB mode seems to produce the most non-random sequences out of the five modes considered.

Section A7.3 in Appendix 7 gives the results of the frequency,serial and runs tests on the ciphertext sequences produced using the five different chosen keys (see Section A7.1) under the five DES modes. In this case, one input sample from each cycle length category has been chosen for testing purposes. The input samples selected are themselves classified as non-random by more than one of the three tests. The chosen input sequences are : 1.5, 2.1, 3.3, 4.1, 5.2 and 6.5 (see Section A7.1). These six inputs are renumbered as (i) to (vi) respectively. The five keys used are labelled I to V (see Section A7.1). It is seen from the results in Section A7.3 that the variation of the key does not seem to produce any appreciable difference in the randomness characteristic of the output sequences. Section A7.3.1 shows, under each key, the number of sequences which are classified as non-random by each of the three tests. In Section A7.3.2 are listed the sequences which are classified as non-random by more than one test. Again from Sections A7.3.1 and A7.3.2, there seems to be no great significant effect on randomness due to change in the DES keys. This seems to suggest that the sequences produced are more or less random like for any key being used. If so, this may be regarded as an important positive aspect of the DES cryptographic algorithm. If there were some keys which produced significant non-random sequences, then this might be used in cryptanalytical attacks and hence may be considered as a weakness.

Two input samples having cycle lengths of 5 and 10 digits (input samples 1.5 and 3.3), encrypted using the keys I and V, have been used in the autocorrelation test. Only the autocorrelation function curves of the input sample 1.5 (denoted as sample (i))and its five ciphertext output sequences produced using the five DES modes with key V, are shown in Section A7.4. The autocorrelation function of the input reflects the periodic nature of the input sample (repetition of 5 digits). Further the minimum value of the input autocorrelation curve depends on the relative proportion of zeroes and ones present in the input sequence. Greater the proportion of '1', higher the mean value and hence higher the minimum value of the autocorrelation curve. From the ciphertext autocorrelation curves, it is seen that their mean value is around 0.5 which agrees with the expected value E(A(r)) derived earlier for a random sequence. It is also seen that the variation of the curve around the mean value seems to be the greatest in the case of ECB mode. Section A7.4.1 gives the number of points M which lie beyond the 5% significance level, calculated using normal distribution approximation given in Section 6.2.4. The value of M can be used as a measure of randomness; greater the value of M, less is the randomness of the sequence. It is seen that for the input sample (i), with ECB mode,about 19% of the points lie outside the 5% significant level thus indicating some degree of non-randomness.

To sum up, it can be said that, from the four tests - frequency, serial, runs and autocorrelation, the DES algorithm seems to be a very good pseudo-random number generator. Out of the five modes considered - ECB, CFB, CBC, CBCP and CFBV - the ECB mode seems to produce the most non-randomness characteristics. There does not seem to be any great difference between the other modes from the point of view of randomness of output sequences produced.

Having examined the randomness characteristics of the final ciphertext output from the DES algorithm, the next step is to apply these tests to the intermediate outputs namely the outputs of the 16 rounds of the DES operated in the standard ECB mode. The degree of randomness is expected to increase as the number of rounds increases. It is found that the first round shows a high degree of non-randomness and as the number of rounds increases the outputs become more and more random. Some of the results obtained are given in Section A7.5.

## 6.4 Other Statistical Tests

In this section, some other statistical tests which are carried out to detect dependence or correlation between the inputs and the output of DES are briefly mentioned.

### 6.4.1 Cross-Correlation Test

The aim of this test is to find out whether there is any significant correlation between the ciphertext sequences produced using some special inputs and whether it can be of any use in a cryptanalytical attack on DES.

A key consisting of a '1' in the most significant position and sixty three '0's is chosen to encrypt a plaintext block of all '0's. Then the bits are shifted one position to right and the key is used to encrypt the same plaintext. This procedure is repeated 64 times, by shifting each time the key bits one position to the right. Out of the 64 ciphertexts produced, only 56 ciphertext sequences are distinct, since 8 bits of the key are used as parity bits. In order to determine whether there is any correlation between these 56 sequences, it is necessary to examine the cross correlation of each sequence with the other 55 sequences. As this seemed to be impracticable, two sequences are chosen and they are cross correlated with the rest. The chosen sequences are:

(i) the first ciphertext sequence corresponding to the key 100... 0

(ii) the 29th ciphertext sequence corresponding to key $0\ldots\overset{29th}{1}\ldots0$

The results obtained showed cases with high degree of correlation; three such cases are given below:

(a) first output cross-correlated with the third output with lag $r = 1$

(b) first output cross-correlated with the fourth output with lag $r = 1$

(c) first output cross-correlated with the eleventh output with lag $r = 0$

To investigate these cases further, four other plaintext blocks are encrypted under the same key using the above procedure. The results are then examined for correlation for the three cases stated above. Further these four plaintexts are encrypted using four different keys following the above procedure and the results examined for correlation for the three cases (a), (b) and (c) above. However, these investigations showed that no conclusive evidence can be found of the existence of any systematic correlation between these 56 ciphertext sequences.

### 6.4.2 $\chi^2$-Test to Detect Dependence Between Output and Input

A $\chi^2$-test [6] has been used to detect whether there is any dependence of an output sub-block on an input sub-block for a fixed key.

Let the plaintext block $X = (x_0 \ldots x_{63})$ be enciphered by DES with key $K = (k_0, \ldots, k_{55})$ into ciphertext $Y = (y_0, \ldots, y_{63})$.

In DES, each output bit co-ordinate $y_i$ is a function of the 64 input bit co-ordinates $x_i$ and the 56 key co-ordinates $k_i$. The $\chi^2$ test enables to check whether some set of the output bit co-ordinates $CO_{out}$ are dependent on some set of the input bit co-ordinates $CO_{in}$ for a fixed key. Note that the test can also be used [6] to test for the dependency of some set of output bit co-ordinates on some set of key co-ordinates for a fixed input. Dependence might be used to estimate the key or plaintext; for example, if $k_i$ is dependent on some set of output or input bit positions, one could make this the basis of the recovery of $k_i$ from corresponding plaintext and ciphertext.

The $\chi^2$ test used is explained below [6]:

(i) **Small subsets of $CO_{in}$ and $CO_{out}$ of sizes Nin and Nout are chosen where $CO_{in} = (i_0, \ldots, i_{Nin-1})$, $CO_{out} = (j_0, \ldots, j_{Nout-1})$**

(ii) **A key $K = (k_0, \ldots k_{55})$ is chosen**

(iii) **A set of plaintexts is encrypted under the key K and the $2^{Nin}$ by $2^{Nout}$ contingency table is formed where the (s,t) entry is the number of times**

$$(x_{i,i_0}, \ldots x_{i,i_{Nin}}) = (s_0, \ldots, s_{Nin-1})$$

$$(y_{i,j_0}, \ldots, y_{i,j_{Nout}}) = (t_0, \ldots, t_{Nout-1})$$

**where $(s_0, \ldots, s_{Nin-1})$, $(t_0, \ldots, t_{Nout-1})$ are the base 2 representations of s and t.**

(iv) **Then the $\chi^2$- statistic with $2^{Nin+Nout-1}$ degrees of freedom is computed where**

$$\chi^2 = \sum_{0 \leq s < 2^{Nin}} \sum_{0 \leq t < 2^{Nout}} \frac{\left(N_{s,t}(X) - N_{samples}\, 2^{-(Nout+Nin)}\right)^2}{2^{-(Nout+Nin)}\, N_{samples}}$$

**In our case, $N_{samples}$= 250 and the key is 31313131313131. The subsets are chosen to be $CO_{in} = CO_{out} = (3,4)$. The contingency table for a two-tailed $\chi^2$-test at 1% and 99% confidence levels is given as follows:**

|  | 00 | 01 | 10 | 11 |
|---|---|---|---|---|
| 00 | 20 | 21 | 11 | 9 |
| 01 | 19 | 10 | 18 | 16 |
| 10 | 25 | 17 | 13 | 12 |
| 11 | 22 | 10 | 16 | 11 |

The computed $\chi^2$-value is 23.408. Using the two-tailed $\chi^2$-test, the 1% and 99% confidence levels are given by $\chi^2_{lower} = 5.81$ and $\chi^2_{upper} = 32$ respectively. This leads us to accept the null hypothesis that the output bit positions in $CO_{out}$ are independent of the input bit positions in $CO_{in}$ with the chosen key. In practice, the acceptance or rejection of the null hypothesis must be based upon the results of several independent $\chi^2$-tests. The evaluation of multiple $\chi^2$-tests is often made using Kolmogorov-Smirnov test [6]. Even with multiple $\chi^2$-tests for the correlation to be of any value in cryptanalyzing the DES, either the correlation is present for only a limited number of pairs (which can be predetermined) or correlation is present in a relatively large number of pairs which can be determined by random sampling. Thus it is necessary to carry out these tests on all pairs ($CO_{in}$, $CO_{out}$). The application of $\chi^2$-test is reported [6] to have been carried out by IBM and NSA as part of internal validation of DES.

# CHAPTER 7

## LOCAL FILE SECURITY: SYSTEM SOFTWARE (2)

### 7.1 General

The second application of the developed encryption interface unit is the encryption and decryption of files stored locally in the Apple disk system. This application offers the system off-line encryption facility whereas the point-to-point configuration considered in Chapter 5 provides on-line encryption facility.

### 7.2 Choice of DES Mode

Theoretically any of the DES modes previously discussed can be used for this application. However when a file is encrypted, recovery from an error must be effected with ciphertext alone. If a ciphering procedure with error propagation is used for file security, subsequent inability to read a portion of the ciphertext because of damage either to the physical medium or to the recorded bits, may prevent all the following ciphertext from being deciphered. Therefore, a self-synchronizing approach is desirable for file encryption. This constraint therefore eliminates the use of the last two of the five modes discussed in Chapter 5, leaving CBC, CFB and ECB modes. The ECB mode is to be avoided as it is the least secure of the three because of its vulnerability to the code book analysis problem. The remaining two chaining modes are the CBC, a block cipher and the CFB, a stream cipher. Any one of these two can be used. If stream cipher feedback on eight bit character is used, then the maximum speed will be one-eighth of the speed that can be achieved using the block mode. That is, if the 8-bit CFB is used,the throughput is very much reduced. Hence it is decided to adopt the CBC mode for this file security program. The limited error extension property of the CBC mode may be useful in such an application even though complete error propagation property is not suitable. Consider for instance, encryption of a database containing personnel records. Suppose a figure in the salary field of the ciphertext file is changed accidentally or deliberately, then the limited error extension property will cause two blocks of characters to be in error when decrypted. This would enable easy error detection.

On the other hand, with the CBC, the problem of padding the end of the file with pseudo-random numbers exists as it is a block cipher. Also when padding is used, an additional character called the pad count needs to be included as part of the padding characters. The pad count specifies the number of pad characters including itself which have been appended to the end of the file. This information needs to be preserved for future decipherment. Due to padding, the ciphertext will be longer than the original plaintext. This may be undesirable if the ciphertext is to replace the plaintext in some previously allocated file space. One way to avoid ciphertext expansion would be to use a stream cipher mode of operation to handle the special situation of short blocks. In this mixed mode of operation, the block cipher mode is used for ciphering standard blocks and the stream cipher mode is used for ciphering the short blocks at the end of file. Alternatively, the short blocks can be enciphered without increasing their length using the following method. To encipher the last short block of $\ell$ bytes ($\ell < 8$) the preceding full block of ciphertext is reenciphered and the first $\ell$ bytes of the result are then exclusive-ored with the plaintext short block. The preceding full block of ciphertext depends on all the preceding blocks of the file and thus is sufficiently variable. But as it is visible to the opponent, reencipherment of it provides the necessary secrecy. Thus this method provides the last short block the full strength of a standard DES encryption.

For this Apple system, as there is no stringent constraint preventing the ciphertext expansion, the padding technique has been adopted. It will be seen in the next chapter that such an approach is not possible when considering the Prestel Viewdata System and a stream cipher technique needs to be used.

## 7.3 Implementation

As far as the implementation is concerned, there is to be no change in the hardware of the encryption system. On the other hand, a different program has been developed for this purpose. The program can be divided into two sections. The first section performs encryption and decryption of files stored locally in the Apple disk system. The files can be APPLESOFT or INTEGER BASIC files. The second section invokes some of the routines developed in the point-to-point communication system to transfer the encrypted files to a remote Apple terminal where they can be automatically stored onto floppy disks. The flowchart of the first section of the program together with the listing is given in Appendix 8. The two sections of the program are now very briefly described.

The program initially asks the user to enter the name of the file to be encrypted or decrypted. It then fetches the file from the local Apple disk system automatically and stores the file in a prespecified part of the memory in the Apple microcomputer system. The APPLESOFT files are stored starting from memory locations 0800 (hex) upwards whereas the INTEGER files are stored from 9600 (hex) downwards. These addresses are referred to as 'start-of-file' addresses. Each Basic instruction stored in machine code consists of a two-byte next instruction pointer, a two-byte instruction number, a sequence of bytes representing the original source line of instructions and a byte containing the 'end-of-file' marker. The Apple system also provides an 'end-of-file' pointer. Briefly, the encryption and decryption program is described as follows. In encryption, the plain file from the start-of-file address to end-of-file address forms the input to the program. Then the file is divided into blocks of 64-bits and encrypted under cipher block chaining mode. Padding of the last block of the file is done with random characters in the usual way. The encrypted file is then stored back into the same memory locations writing on top of the plain file in the Apple system memory. Then an automatic transfer of the encrypted file from the system memory to a floppy disk is performed under the filename provided by the user. The encrypted file can be loaded back from the disk at a later time and decrypted to give the original file provided the same key and initialization vectors are used. The decryption program requires the 'end-of-file" address to be able to stop the decipherment process. This in turn implies that the 'end-of-file' address must be stored along with the cipher file during encryption. One can store this end-of-file address either at the end of the cipher file or at the head of the cipher file. If the address is stored at the end of the cipher file, the decryption program will be unable to find it as the end of file depends on the length of the cipher file which varies. The decryption program cannot distinguish between the actual ciphertext and the information containing the 'end-of-file' address; so it is stored at the head of the cipher file in plain format. In addition, the count of the number of random characters padded to the end of file, pad count, is also stored in a similar fashion. Having obtained the 'end-of-file' address and the pad count, the decryption program can find the initialization vector stored at the end of the file. Alternatively, the user may be asked to enter the initialization vector along with the key at the beginning of the program. The decryption program then deciphers the cipher file using CBC in the normal way discarding the dummy random characters at the end.

The second section uses modified versions of the Transmission routine and Receive routine (Section 5.2.2) to transfer files between two users in a point-to-point system. The user who wishes to transmit a file initially sends some plaintext to the receiving end using the terminal keyboard which contains information about his identity, the identity of the intended receiver, the type of file (APPLESOFT or INTEGER), the time at which it is sent etc. The cipher file is then sent over the communication link using the Transmission routine. The receiving end fetches the file and stores it onto a floppy disk automatically under the file name provided by the sender. The intended receiver can then decrypt the file at a later time in an off-line manner.

# CHAPTER 8

## SECURITY IN PRESTEL VIEWDATA SYSTEM: SYSTEM SOFTWARE (3)

### 8.1 General

As the need and the common use of large data bases to store sensitive information increases, the requirement to maintain secrecy becomes more and more important. The Apple microcomputer system together with the designed encryption unit is interfaced to the Prestel network, the British Telecom Viewdata System, thus allowing the Apple to act as an intelligent viewdata terminal. This enables transfer and storage of encrypted as well as plain data between Apple and Prestel computer.

### 8.2 Brief Review of Prestel Viewdata System

The Prestel system consists of a network of GEC 4082 computers linked together by high speed data links. There are two types of computer centres namely the Information Retrieval Centres (IRCs) and Update Centres (UDCs). Currently the network consists of one UDC linked to a number of IRCs.

The basic unit of information on Prestel is a frame which consists up to a maximum of 960 characters. One or more frames are linked together to form a page. These pages of information form the Prestel database. Each page is uniquely identified by a number of up to 9 digits. Frames are further identified by letters of the alphabet a to z. Frames and pages are linked together by means of pointers and they form a tree structure. Detailed information on Prestel system can be found in [31 ].

### 8.3 Encryption/Decryption in Prestel System

As the basic unit of information is a frame, a natural choice would therefore be to encipher a complete frame at a time. However, there may be instances where encipherment of sections of a frame may be required. So in our system, we should be able to encrypt parts of a frame. At the start of each frame, it is to be indicated whether encipherment has been used.

As discussed in Chapter 7,the two modes of DES which are suitable for this data base application are the cipher block chaining (CBC) and the stream cipher feedback (CFB). Since it is required to encrypt parts of a page which may be small pieces of data such as individual characters, the stream cipher feedback appears to be more suitable. Further if the CBC mode is used, when parts of a frame are encrypted, this is likely to require padding for the encryption portion. This in turn will result in cryptogram extension and pose a problem when storing the enciphered frame on the Prestel data base as each frame is limited to a maximum of 960 characters. This constraint leads us to consider the use of the CFB mode in this application. As the backward channel, that is, from the user to the Prestel computer, has a speed of only 75 bits per second the reduction in speed resulting from the use of the CFB mode does not affect the throughput of the system.

The data format of each character transferred to the Prestel computer consists of 7 data bits. For transmission down the line, these 7 data bits are sent in an asynchronous start-stop format comprising 1 start bit, 1 stop bit, 1 even parity bit and 7 data bits. If a block cipher mode such as the CBC mode is used then it is required to transmit 64-bits of ciphertext in the above 10-bit format. One way to do this is to break the block into nine seven-bit groups and a single bit group. The nine 7-bit groups can be transmitted in the normal fashion. The last bit can be grouped with the next block. This needs to be done each time a block is enciphered and this process continues until the end of the frame or page. Alternatively,the last bit can be padded with 6 other bits to form an extra character. But this results in an extra character for every block encrypted and causes problems in storage of enciphered frame as mentioned earlier. Further since all the 64-bits of a ciphertext block are required to decipher correctly, the last bit must be received before decryption can begin. In account of this, the simple approach of stream cipher feedback mode has been adopted.

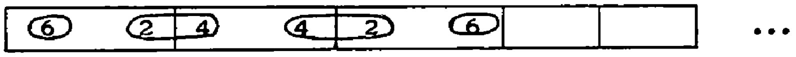
The encrypted information passes through the Prestel computer control unit which rejects any of the control characters present in the ciphertext. Referring to the coding table given in Figure 8.1,the codes belonging to the columns 1 and 0 are not accepted by Prestel computer as data. Therefore there is a need to prevent the occurrence of these control characters in the ciphertext. That is, the encryption system is to be made transparent to Prestel control

| | $b_7$ $b_6$ $b_5$ | 0 0 0 | 0 0 1 | 0 1 0 | | 0 1 1 | | | 1 0 0 | | 1 0 1 | | 1 1 0 | | 1 1 1 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| bits $b_7b_6b_5b_4b_3b_2b_1$ | Col / Row | 0 | 1 | 2 | 2a | 3 | 3a | 3b | 4 | 4b | 5 | 5b | 6 | 6a | 7 | 7a |
| 0 0 0 0 | 0 | NUL | | Sp | □ | 0 | 🬏 | | @ | | P | | – | 🬞 | p | 🬭 |
| 0 0 0 1 | 1 | | Cursor DC1 On | ! | 🬀 | 1 | 🬐 | Set Verify Mode(1) | A | Alpha$^n$ Red | Q | Graphics Red | a | 🬟 | q | 🬮 |
| 0 0 1 0 | 2 | STX | DC2 | " | 🬁 | 2 | 🬑 | Set Verify (2) | B | Alpha$^n$ Green | R | Graphics Green | b | 🬠 | r | 🬯 |
| 0 0 1 1 | 3 | ETX | DC3 | £ | 🬂 | 3 | 🬒 | Set Verify (3) | C | Alpha$^n$ Yellow | S | Graphics Yellow | c | 🬡 | s | 🬰 |
| 0 1 0 0 | 4 | | Cursor DC4 Off | $ | 🬃 | 4 | 🬓 | Set Programme Mode | D | Alpha$^n$ Blue | T | Graphics Blue | d | 🬢 | t | 🬱 |
| 0 1 0 1 | 5 | ENQ | NAK | % | 🬄 | 5 | ▌ | Tape Pause on Playback | E | Alpha$^n$ Magenta | U | Graphics Magenta | e | 🬣 | u | 🬲 |
| 0 1 1 0 | 6 | ACK | | & | 🬅 | 6 | 🬔 | Tape Start | F | Alpha$^n$ Cyan | V | Graphics Cyan | f | 🬤 | v | 🬳 |
| 0 1 1 1 | 7 | | | ' | 🬆 | 7 | 🬕 | Tape Stop | G | Alpha$^n$ White | W | Graphics White | g | 🬥 | w | 🬴 |
| 1 0 0 0 | 8 | Cursor ← BS | CAN | ( | 🬇 | 8 | 🬖 | | H | Flash | X | Conseal Display | h | 🬦 | x | 🬵 |
| 1 0 0 1 | 9 | Cursor → HT | | ) | 🬈 | 9 | 🬗 | | I | Steady | Y | Contig Graphics | i | 🬧 | y | 🬶 |
| 1 0 1 0 | 10 | Cursor ↓ LF | | * | 🬉 | : | 🬘 | | J | | Z | Separated Graphics | j | ▐ | z | 🬷 |
| 1 0 1 1 | 11 | Cursor ↑ VT | ESC | + | 🬊 | ; | 🬙 | | K | | ← | | k | 🬨 | ¼ | 🬸 |
| 1 1 0 0 | 12 | Cursor Home & Clear FF | SS2 | , | 🬋 | < | 🬚 | | L | Normal Height | ½ | Black Background | l | 🬩 | ‖ | 🬹 |
| 1 1 0 1 | 13 | Cursor ⇐ CR | SS3 | — | 🬌 | = | 🬛 | | M | Double Height | → | New Background | m | 🬪 | ¾ | 🬺 |
| 1 1 1 0 | 14 | SO | Cursor RS Home | . | 🬍 | > | 🬜 | | N | | ↑ | Hold Graphics | n | 🬫 | ÷ | 🬻 |
| 1 1 1 1 | 15 | SI | | / | 🬎 | ? | 🬝 | | O | | # | Release Graphics | o | 🬬 | ■ | █ |

Fig. 8.1 - Viewdata Transmission Codes

unit. A simple way of achieving this is to use a 6-bit cipher feedback technique. With this technique it is always possible to ensure that the ciphertext belongs to the set of accepted codes. But this allows only 64 different possible characters that can be enciphered. In this design, these 64 input codes are 0-9, A-Z, a-z, space and period. All other codes are transparent and bypass encryption. Thus the output codes are reformed into the same range as the inputs thus preserving the one to one relationship between transmission and reception. As we are mainly interested in enciphering alphanumerical characters present in the text, the above set of input codes is found to be adequate for our purpose.

This can be extended to 96 codes (32 out of the possible 128 codes being control codes) using the 'breaking-up' technique mentioned above. First consider the case where the plaintext (ciphertext) characters are 8-bits long. In this case, the encryption process can either be in block cipher mode or in stream cipher mode. The cipher is first broken into 6-bit groups and then each 6-bit group is expanded to form a 7-bit character by adding a '1' in the most significant position, on transmission. This process removes any unwanted control codes from the transmitted ciphertext character. This is shown below

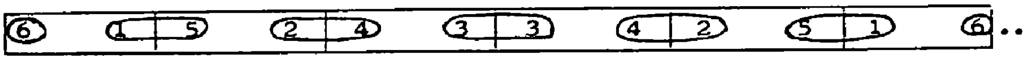


⬭ - denotes the 6-bit ciphertext character which is expanded to 7-bits by adding a '1' in the most significant position on transmission

□ - denotes the original 8-bit ciphertext character

Thus it is seen that to transmit 3 bytes of information, one needs to send 4 encrypted 7-bit characters. If the length of the plaintext to be enciphered is n-bytes long then this method will result in $\lceil \frac{8n}{6} \rceil + 1$ or $\frac{8n}{6}$ 7-bit characters depending $8n \not\equiv o$ (mod 6) or $8n \equiv o$ (mod 6) where $\lceil \frac{8n}{6} \rceil$ indicates the largest integer less than $\frac{8n}{6}$.

Now consider the case where the plaintext characters are 7-bits long. In this case, as mentioned earlier, stream cipher feedback mode seems to be more suitable than block cipher mode. To allow encryption of all 96 codes, again the breaking up technique can be used. In this case,cipher is first broken into groups of 6-bits and then each 6-bit group is expanded back to 7 bits by adding a '1' in the most significant position, on transmission. This process ensures that the transmitted character is in the normal ASCII code range and is acceptable to any host computer.



Thus if the plaintext to be enciphered consists of n 7-bit characters then this method will result in $\left\lceil \frac{7n}{6} \right\rceil$ + 1 or $\frac{7n}{6}$ characters on transmission depending on $7n \not\equiv o$ (mod 6) or $7n \equiv o$ (mod 6).

Let us now briefly consider the connection protocol involved when using the Apple encryption system with the Prestel computer. In the usual way, the system is connected to the public switched telephone network via the modem. A call to the Prestel computer is initiated using the telephone connected to the modem. The Prestel computer responds by sending a continuous tone of high frequency. At this point, the DATA switch on the telephone is pressed thus allowing the modem to get control of the line. That is, the modem aknowledges by sending a low frequency tone to the Prestel computer. Now the terminal is ready for data transfer.

The system software essentially carries out two distinctive tasks. Firstly, it emulates the Prestel terminal keyboard using Apple keyboard. That is, for the system to make use of some extra facilities provided by Prestel, the Apple keyboard is effectively extended to include some special characters. Secondly it incorporates encryption-decryption facility into the Prestel system. The software is lengthy and complicated. Hence only some important aspects are briefly considered here.

Prestel system can be used in two modes, namely, the user mode and the information provider (IP) mode.

In the user mode, two main facilities provided by the system are:

(a) Reception of plain and encrypted frames from the Prestel database.

(b) Transmission of commands such as choosing a frame etc from the Apple keyboard to the Prestel computer. Note that in this case, the only keys used are 0-9, # and *.

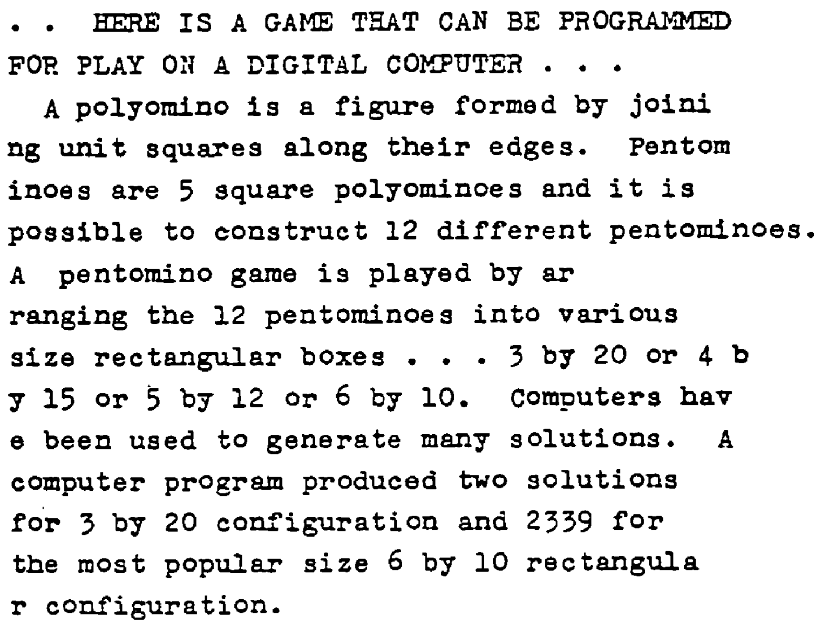
In the editing mode, in addition to (a) and (b), facilities are provided for entering, amending, copying and deleting plain or encrypted frames of Prestel. The Prestel editing terminal keyboard is given in Appendix 9. It is seen that this keyboard has additional facilities compared to the Apple keyboard which are required to provide necessary control signals. Two important ones among these are: Start Edit and End Edit. In addition,special functions for encryption and decryption are required.

Having entered the secret DES key in the normal fashion, at the beginning of the communication, in the user mode, the user must have the choice as to when to set the interface into the decryption mode. This enables him to decipher only those pages which are in enciphered form and to read the other Prestel pages in plain form. This is carried out by pressing the key CNTRL-Q. Now if any of the enciphered pages is read from the Prestel database, it is displayed on the terminal in plain form. Only the user with the right key and the correct initialization vector will be able to obtain the complete original plain frame. The interface unit is set back to normal plain mode by pressing the key CNTRL-R. This software implementation allows changes in initialization vector during communication whereas to change the DES key, the system needs to be reset and restarted again. This has been done because every user is expected to have a single secret key although he may use any number of different initialization vectors. This is particularly important when a user needs to encrypt same portions of text in different frames. Changing the initialization vector allows different ciphertext representations of the same plaintext under a fixed key. As it stands, the user needs to keep a record of the frame number together with the initialization vector he used to encrypt that frame and his single secret DES key. An improved scheme would be to generate a pseudo-random key, called the frame key, dynamically and encrypt the frame using this key. The frame key can then be enciphered under the user's secret key using ECB mode and stored at the head of the frame. The initialization vector is again generated using a pseudo-random process and can be enciphered under the frame key using ECB mode and is also stored at the top of the frame. Using this method one has effectively chained the frame key and the initialization vector used in the encryption of a frame. The decryption process can automatically recover the enciphered frame key from the top of the frame and then decipher it using the user's secret key under ECB mode to produce the frame key. The frame key is now used to decipher the next 8 bytes at the head of the frame under ECB mode to produce the initialization vector. The initialization vector and the frame key can now be used to decipher the frame under the CFB mode. This method would allow different frames and different pages to be enciphered under different keys without having to reset the system. Further, the user does not need to keep a record of each frame key and the corresponding initialization vector used in the enciphermnt of that frame. This is being done automatically. More on such key management aspects will be considered in Chapter 9.

In the editing mode, the user is able to enter and amend the encrypted as well as plain frames in Prestel. From the user point of view, it is essential that the operations that need to be done for encryption and decryption must be as simple as possible. Start (?) and stop (/) markers are used to indicate the beginning and end of enciphered data in the frame. The key CNTRL-A is pressed to set the interface unit to encryption state. All subsequent characters typed are automatically encrypted under the 6-bit CFB mode. The key CNTRL-B is used to return the interface unit to the plain mode. This allows encryption of even single bytes of data. The system initially produces upper case letters. Lower case letters are obtained by pressing the key CNTRL-V. All characters typed are now in lower case until the upper case shift, CNTRL-W is typed. Start Edit and End Edit needed to work the Prestel Editor System are obtained using the keys CNTRL-T and CNTRL-E respectively. Most of the cursor control movements such as backspace, forward, downward, upward are included in the editor facilities. The Return key behaves slightly differently compared to the normal Apple mode in that the cursor returns to the beginning of the same line. So to move to the beginning of the following line, one needs to press the Return followed by Line feed (CNTRL-J). Note that the graphic and colour keys are not included in this Prestel encryption system.

This software implementation provides on-line editing/user facilities on the Prestel Viewdata system. It is possible to merge this program and the one discussed in Chapter 7 to perform local off-line editing and encryption of a Prestel frame and then transfer the created Prestel frame to the Prestel computer.

An example of a completely encrypted frame and a partly encrypted frame together with the corresponding plain frame is shown in Figures 8.2, 8.3 and 8.4.



```
. .  HERE IS A GAME THAT CAN BE PROGRAMMED
FOR PLAY ON A DIGITAL COMPUTER . . .
  A polyomino is a figure formed by joini
ng unit squares along their edges.  Pentom
inoes are 5 square polyominoes and it is
possible to construct 12 different pentominoes.
A  pentomino game is played by ar
ranging the 12 pentominoes into various
size rectangular boxes . . . 3 by 20 or 4 b
y 15 or 5 by 12 or 6 by 10.  Computers hav
e been used to generate many solutions.  A
computer program produced two solutions
for 3 by 20 configuration and 2339 for
the most popular size 6 by 10 rectangula
r configuration.
```

Fig. 8.2 - Plain PRESTEL Page

```
Scratchpad              651314b        Op

?OnLihLcKwGbZEjxeaJgVwBCLJ BVz bbvDW Ev
 o' y w PFZQsNNaBK txvOp  hV TBB@xUmJ' K
D YnvBISJ YwQKFtrTmoBCLMv okHXebhN  GXqq
'KZdvI erePmTQax@ kyKmC Rcfi'F 'JiNaXt
@nQbwYg xkTRLsODN  IuCvSzLZvXNZb WNpYPB
X HdIEzgOnPtmjDGtOaUVIG  OdMF OgiYmdqRYA
sdTdriHkuejTd QQs  CnxkbuVgziKiokxbKtpdj
juwOC 'G  nH yMIg@MvI dhae RHX'DbQHo 'yF
xx@y Kz S UZg amqmVB'EfozyFjUIPQFcQF TR
JkzFtFA  AVsoGpciLp!Wewo wkBPJ JyiDHEz Z
uLrwEu oRPgWzOLDIWMr L  Rkqs f YRAXZXS!V
 PR  CSZCIVUet DtPCLc MhTEyQByjfALbp rqL
G Ip  psp!yee B@It! Yxo abb TxftGQpWgjLZ
yyaLmfEJvrOLzfSBwXsf ByiX wFvg h  bH Art
vbuoraZdf'xfvG!xS HRztdsp!gNuaDQ BsIyuK/
```

?  indicates start of encryption.

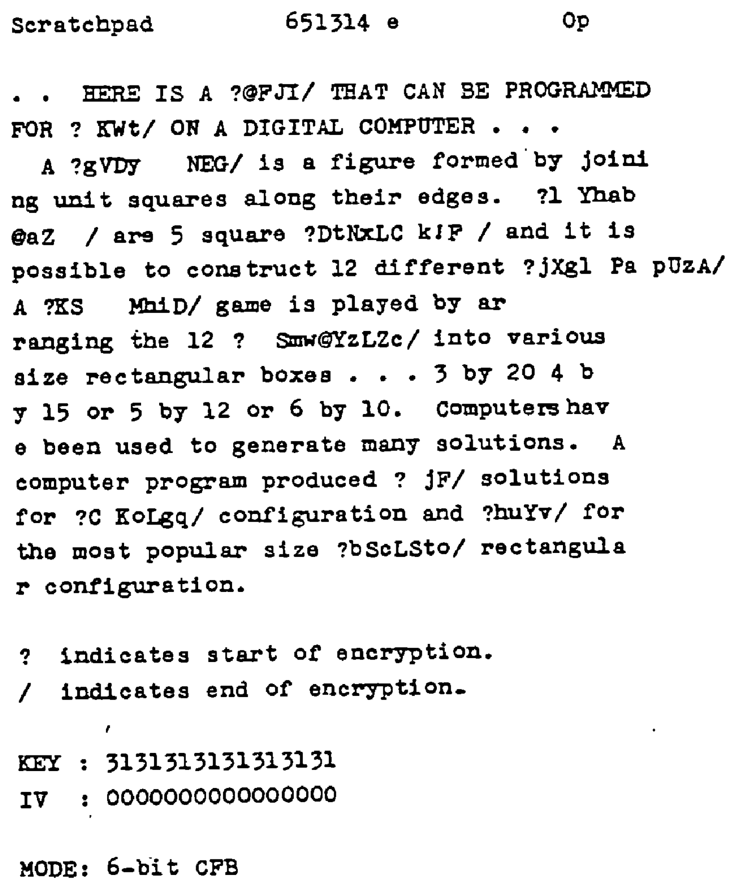
/  indicates end of encryption.

KEY  3131313131313131

IV   0000000000000000

MODE: 6-bit CFB

Fig. 8.3 - Completely Encrypted PRESTEL page



```
Scratchpad          651314 e          Op

. .  HERE IS A ?@FJI/ THAT CAN BE PROGRAMMED
FOR ? KWt/ ON A DIGITAL COMPUTER . . .
  A ?gVDy   NEG/ is a figure formed by joini
ng unit squares along their edges.  ?1 Yhab
@aZ  / are 5 square ?DtNxLC k!F / and it is
possible to construct 12 different ?jXgl Pa pUzA/
A ?KS   MhiD/ game is played by ar
ranging the 12 ?  Smw@YzLZc/ into various
size rectangular boxes . . . 3 by 20 4 b
y 15 or 5 by 12 or 6 by 10.  Computers hav
e been used to generate many solutions.  A
computer program produced ? jF/ solutions
for ?C KoLgq/ configuration and ?huYv/ for
the most popular size ?bScLSto/ rectangula
r configuration.
```

? indicates start of encryption.

/ indicates end of encryption.

KEY : 3131313131313131

IV : 0000000000000000

MODE: 6-bit CFB

Fig. 8.4 - Partly Encrypted PRESTEL page

# CHAPTER 9

## KEY DISTRIBUTION AND PUBLIC KEY CRYPTOGRAPHY

### 9.1 General

Until now, the use of DES cryptographic algorithm in protecting the data during transfer between users has been considered. However the security of the DES depends on the secrecy of its keys. Thus protecting the data depends on protecting the keys because they are the means by which the data can be decrypted. Any key controlled cryptographic algorithm thus requires a protocol for safely handling and controlling its cryptographic keys. Keys must be produced and distributed not once but constantly. In some systems they must be changed with the passage of time, or with the amount of traffic and in all systems, they must be changed when they are feared compromised. Frequent key changes limit the amount of data compromised if an opponent does learn a key. Keys must be provided to new users of the system and old keys must be retired as users withdraw. The consideration of all these aspects forms the subject of key management.

There are essentially three ways to incorporate cryptography into a communication system namely link-by-link, node-by-node and end-to-end encryption [3].

In link-by-link encryption, data is encrypted across the medium connecting two directly communicating nodes. Link-by-link encryption is independent of the system and does not necessarily imply that the cryptographic capability is integrated into the communicating nodes. It may be regarded as being implemented by a pair of cryptographic devices bracketing the line between two communicating nodes and situated between the nodes and their modems as shown in Figure 9.1.

Node-by-node encryption is similar to link-by-link encryption in that each link is protected by a unique key. However data passing through an intermediate node are not in the clear as would be the case with link encryption. Rather at an intermediate node, the enciphered data are transformed from encipherment under one key to encipherment under another key (that is,deciphered and reenciphered) within a security module which may be a peripheral device attached to the node. That is, the plaintext occurs only within the security

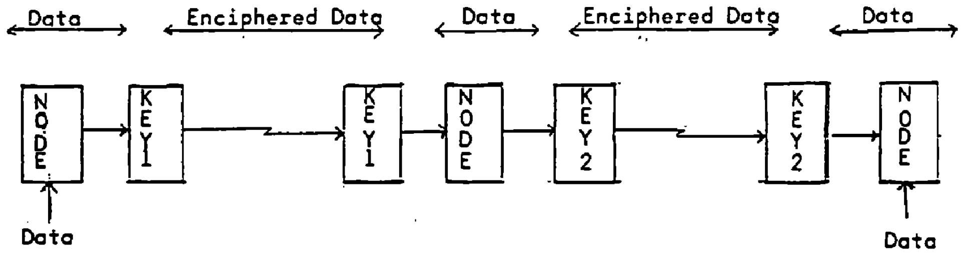


Fig. 9.1 Link Encryption

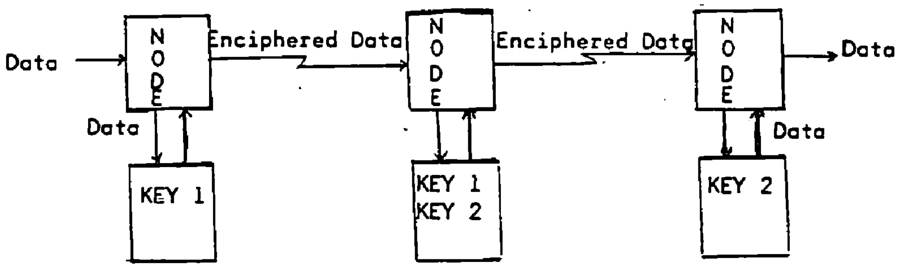


Fig. 9.2 Node Encryption

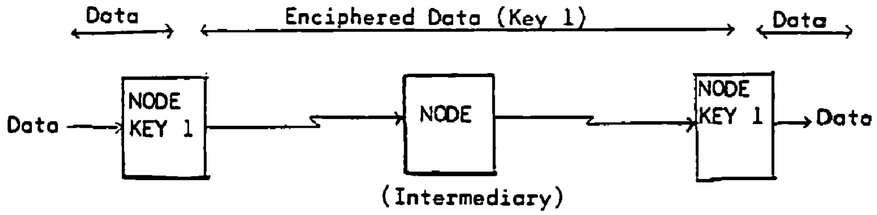


Fig. 9.3 - End Encryption

module and not within the node (Figure 9.2).

In end-to-end encryption, data encrypted at the originating node is not decrypted until it arrives at its final destination. Thus this method continuously protects data during transmission between users. Unlike link and node encryption, end-to-end encryption allows each user to have several keys, one key for each user who uses encryption (Figure 9.3)

It appears that in terms of security, cost and flexibility, end-to-end encryption seems to be the most attractive for systems requiring many protected links [3] .

The Apple encryption system discussed earlier is a simple end-to-end encryption system. More exactly, it can be referred to as a private end-to-end cryptographic system as the user needs to request for cryptography and its use is not transparent.

Some key management schemes which allow the DES interface unit to be integrated into data processing systems to provide protection for communications between individual users in an end-to-end encryption network are discussed.

## 9.2 Key Management Using Key Centre

This approach uses a Key Centre (KC) which acts as a source of session keys for encrypted calls using the DES algorithm. A detailed description of the functioning of such a centre is given in [3, 32].

Key centre can be operated manually in which the keys are sent by mail or couriers. If such an arrangement is trusted, that is, whether the risk of untoward disclosure either accidently or as a result of deliberate attempts is acceptable, it could work out very well. At least, this may be possible when the network is small and traffic volume is low. On the other hand, if the network handles large traffic volumes,the need to change the keys often demands that large amounts of keys to be distributed. In large networks, the number of possible interconnections grows as $n(n-1)/2$ where n is the number of users. This may become an expensive venture because the manual systems have to be guarded against security leaks by conventional methods and the persons involved have to be trustworthy.

In an automatic KC, data network is used to distribute and generate the keys automatically. Consider the connection protocols involved when two users wish to communicate in a secure fashion in a single Key Centre environment.

Essentially two types of keys namely data encrypting keys and key encrypting keys can be identified. The data encrypting key is active only for a duration of a single communication session and therefore is referred to as a session key (KS). The session key is protected by enciphering it under key encrypting key which varies from user to user. Therefore the Key Centre is required to store one key encrypting key for each user. These keys are themselves stored within the centre in enciphered form using the Centre's master key (KCM). Hence the problem of providing secrecy for cipher keys is reduced to providing secrecy for only one key namely the master key. This type of approach is referred to as the master key concept [3]. It is assumed that the master key is stored in some non volatile storage in an inaccessible area in the Centre referred to as the cryptographic facility so that it need be loaded into the cryptographic facility only once. Furthermore, each user is required to store only his user key (KU).

## 9.3 Communication Security

Let $KS_1$, $KS_2$, ..., $KS_n$ represent the time variant, dynamically changing data encrypting keys used for enciphering and deciphering data. It is assumed that KS is operational for the duration of a communications session. Let KCM represent the master key of the Centre and KU represent the user (or terminal) key.

To begin with, the user i requests the Centre KC for a session key (KS) to communicate with user j. The request is accompanied with a verifiable identification of the user i. The whole message is enciphered under the user i's key $KU_i$. That is,

$$i \rightarrow KC \ : \ i, (i, j, r_o)_{KU_i}$$

where $r_o$ is a random number chosen by user i. It is used to prevent an intruder impersonating the KC by replaying some previously recorded reply containing an earlier session key which the intruder would like the user i to use again. Upon receiving the request the Centre fetches the user i's key $KU_i$ which is stored in its memory under the master key KCM. Then it deciphers the request and checks against its stored information to see if the request is legitimate and if it is, it issues the session key (KS) to the user i. The session key KS is generated within the Centre using a pseudo-random procedure. The reply from the Centre to the user i is given by

$$KC \rightarrow i : (KS, r_o, (KS, i)_{KU_j})_{KU_i}$$

The random number $r_o$ is returned by the KC for the user i to verify that the reply is coming from the KC and not from an intruder. Further as the session key KS is encrypted under $KU_i$, it allows only the user i to decrypt and obtain KS and not any intruder. The session key together with the identification of user i encrypted under $KU_j$ is also sent to user i. The user i cannot decrypt this portion of the reply as he does not possess $KU_j$. The user i then sends this cipher portion to user j, that is,

$$i \rightarrow j : (KS, i)_{KU_j}$$

The user j responds by sending a random number $r_1$ to user i, encrypted under the session key KS

$$j \rightarrow i : (r_1, j)_{KS}$$

The user j does this to ensure that it is indeed user i who is requesting the call and not any intruder using parameters of a previous call. The user i then checks j's identity and modifies the random number $r_1$ in some prearranged fashion to result in $r_2$, which he returns to user j under KS

$$i \rightarrow j : (r_2)_{KS}$$

Now the users i and j can be almost certain that they are talking to each other and can communicate with each other in a secure manner using KS as the DES secret key. Most of the above steps can be made transparent to the users in the network.

A variation of the above method consists of the Key Centre KC sending the session key directly to the user j instead of sending it via user i. That is, the distinction lies in the path taken by the session key from the Centre to user j. With this approach, two possibilities may occur - either the session key has already arrived at user j when the latter receives the call request or it has not yet arrived. In the latter case, one must ensure that an old key is not used mistakenly. Further the case where user j needs to wait for the session key increases the complexity of the connection protocols. Thus the above outlined method where the session key arrives to user j via user i seems to have some advantage over the other method.

To further improve the integrity of the conversation and reduce the problem of impersonation, timestamps, T, [33] can be added to the key distribution protocol. The first three steps of the above procedure are then modified to become:

$$i \rightarrow KC \;:\; i, (i, j, r_o, T)_{KU_i}$$

$$KC \rightarrow i \;:\; (KS, r_o, (KS, i, T)_{KU_j}, T)_{KU_i}$$

$$i \rightarrow j \;:\; (KS, i, T)_{KU_j}$$

The users i and j can then verify that their messages are not replays by checking $|clock - T| < \Delta t$ where clock gives the local time, $\Delta t$ gives some time error which includes the network delay time and the time discrepancy between the sender's clock and the local clock. This requires some form of time synchronization among the users of the network.

With the above schemes, it is seen that if the session key is somehow lost within the user's system, then a fresh call is to be made by the user i to the KC to establish a new session key. It is preferable that the KC generates a new session key even when the user i did not actually use the old KS for any conversation. If on the other hand, the KC does keep a record of session keys issued to different users over a small period of time (say one day) then these keys need to be stored in enciphered form within the Centre. Rather than using the same master key for this purpose, it is advisable to use another master key KCM1 to encrypt these temporarily stored session keys.

## 9.4 File Security

Let us now consider a key management scheme for file security where one wishes to protect the stored data in the same way as the communicated data [3]. It is assumed that the encrypted files are to be stored in a database in the host processor (HP) in the network. (This could be for instance the Key Centre mentioned earlier). It is also assumed that the users in the network have distinct secret keys KU which are also stored in the host processor in enciphered form under the master key KCM. Consider the case where the user i wishes to store a file in encrypted form in the database under the name CIPHERFILE. Let the corresponding file in clear form be PLAINFILE. To begin with the procedure followed is very similar to the one outlined for communication security given earlier. A call is made to the host processor

$$i \rightarrow HP \ : \ i, (i, r_o)_{KU_i}$$

where $r_o$ is a random number chosen by user i.

The host processor responds by generating a file key (KF) using a pseudo-random process and encrypting it under the user i's key $KU_i$. This is then sent to user i

$$HP \rightarrow i \ : \ (KF, r_o)_{KU_i}$$

The user i decrypts the message to obtain KF and verifies the random number $r_o$ to ensure that the reply is coming from HP and not from an intruder. Then the user can encrypt his PLAINFILE using KF as the DES secret key to produce CIPHERFILE. This is then transmitted to the host processor to be stored under the same name. To be able to recover the PLAINFILE, it is necessary for the host processor or the user i to record the information that the file has been encrypted under KF. In a large system with a number of users and with each user having a number of files, it may not be a good idea for the Centre to keep a separate file containing the name of the data file and the corresponding file key. (If this is done, then this separate file needs to be enciphered under some master key). A better arrangement would be to store the information at the header of the file itself. The file key KF can be stored in the header in encrypted form under a master key KCM2. (KCM2 rather than KCM is chosen to achieve separation from communication security). With this method if a user r wishes to decipher the CIPHERFILE, he requests the host processor for the file key KF.

$r \rightarrow HP : r, (CIPHERFILE)$

The host processor reads the header of CIPHERFILE, decrypts it using KCM2 to recover KF. This is then reenciphered under $KU_r$ and transmitted to user r. This procedure does not allow the host processor to differentiate between users. For instance, the user i may wish that PLAINFILE not be available to user r. This can be achieved if in addition to KF, the identification of the owner of the CIPHERFILE is recorded on the file header, that is, the header contains the information $[i,(KF)_{KCM2}]$. Further,the host processor is required to maintain a record of which users are allowed by user i to obtain the PLAINFILE. Then if user r requests the host processor for the file key of CIPHERFILE, the host processor first reads the header to find the owner of the CIPHERFILE. Having found the owner, i, it checks whether the requesting user belongs to the group of users who are allowed to read PLAINFILE. If user r belongs to this group, it recovers the file key KF from the header, encrypts it under $KU_r$ and sends it to user r. If user r does not belong to this group, the above step will not be carried out by the host processor and access to the file key is prohibited.

## 9.5 Key Distribution for Groups of Users

Consider a more general case where a user in the network wishes to broadcast a message to several users [34]. Assume that a group G is a non-empty subset of n users and members of G wish to broadcast and receive messages from other members of G and to access and update files private to G. A given user may be a member of as many as $2^{n-1}$ groups and there are at most $2^n-1$ non-empty groups in the system. Again it is assumed that all aspects of key distribution for any given group is managed by a single Key Centre. One method of key distribution among the group of users is considered in this section and three other methods employing public key concept are described in Section 9.7.

### 9.5.1 Method 1

In this method, the Key Centre is assumed to keep a list of personal keys of all users. In addition, the Key Centre also keeps a record of all group keys for the groups it manages.

To establish a group G, a member i of G registers the group with the Key Centre. The Centre returns a group identifier IG to the user i who then distributes to the members of the group G using the method described earlier. The KC also generates a group key KG and creates a record identified by IG that contains KG and the users who belong to G. This record itself is stored in enciphered form under a master key.

Whenever a user j belonging to G wishes to communicate with other users or store a file to be read by other users, he obtains the group key KG from the Centre. The key distribution protocol can be described as follows:

$$j \rightarrow KC \;:\; j,\ IG$$

The user j sends to KC his identification and his group identification and requests for the group key KG from KC. KC fetches the group record identified by IG, checks whether j is a member of the group and returns KG to user j enciphered under j's personal key.

$$KC \rightarrow j \;:\; (IG,\ KG,\ T)_{KU_j}$$

where T is a timestamp used to protect against replay of previous keys. Because the group key KG is enciphered under user j's personal key it is not possible for an intruder either to intercept KG or to impersonate j and acquire a group key for a group to which he does not belong. User j can now use the group key KG to encrypt a file to be read by other users of the group or to decrypt an encrypted file created by any other user of the group.

An user i of the group can obtain the group key KG from the Centre in a similar fashion and hence the users i and j can communicate with each other in a secure manner.

The primary disadvantage of this approach is the storage requirements for the group keys. KC may need to store up to $2^n-1$ group keys. Secondly there is no identification between group members and hence no discrimination between group members.

The idea of one Key Centre in the above schemes can be extended to many such Centres and a group of m users 'belonging' to each Centre. In such a situation, each Key Centre is required to possess a shared secret key with each of the other Key Centres. (It could be two such keys,for instance,one for communication security and the other for file security). If there are n such Centres, each Centre therefore has n-1 (or 2n-2) such keys. Then for instance, if a user i belonging to Centre I wishes to communicate with a user j belonging to Centre J, then the Centre I generates the session key and sends it to user i in the usual fashion. As the Centre I does not know the user j's secret key, it sends the session key to Centre J enciphered under the shared communication secret key between Centres I and J. The Centre J recovers the session key and reenciphers it under the user j's key $KU_j$ and sends it to user j. A similar procedure can be envisaged when a user i belonging to Centre I wishes to read a file of user j belonging to Centre J.

The problem of key distribution can also be overcome using public key cryptography concept proposed by Diffie and Hellman [35]. In the next section, the underlying principles of public key cryptography are considered to see how such systems can be used to solve the key distribution problem in an elegant way.

## 9.6 Public Key Systems

Public key systems allow two users to communicate securely over an insecure channel without any prearrangement. Cryptosystems which allow this type of communication are asymmetric (see Section 2.2) in the sense that the sender and receiver have different keys at least one of which is computationally infeasible to derive from the other. These systems separate enciphering and deciphering capabilities and privacy is achieved without keeping the enciphering key secret because it is no longer used for deciphering. Hence the enciphering key is published in addition to the enciphering and deciphering algorithms without compromising the security of the system. The concept of a public key cryptosystem is shown in Figure 9.4. User i encrypts the message M using the public enciphering key of user j and sends the cipher to user j over an insecure channel. Only the user j will be able to decrypt the cipher to recover M as he is the only one who knows his secret deciphering key. The encryption (E) and decryption (D) algorithms in such a system have the following properties:

(a) Deciphering the enciphered form of a message M yields M, that is, D(E(M)) = M.

(b) Both E and D are easy to compute.

(c) By publicly revealing E, the user does not reveal an easy way to compute D. This means that only the receiver (designer) can decrypt messages encrypted with E or compute D efficiently.

(d) If a message M is first deciphered and then enciphered, then M is the result, that is E(D(M)) = M.

The property (d) is not necessary for a public key cryptosystem but if it is obeyed then it is possible to obtain the digital signature feature (see Section 2.2) [34].

The public key concept gives rise to a new class of cryptographic algorithms. One application of such algorithms is to solve the problem of key distribution in systems employing symmetric cryptosystems. The public key cryptosystems would in many instances be the ultimate solution to the key distribution problem. This can be done as follows: User i can encrypt the session key KS using the public key of user j and then send it to user j over an insecure channel. Because the deciphering key is only known to user j, he is the only one who can decrypt the cipher and obtain the session key.

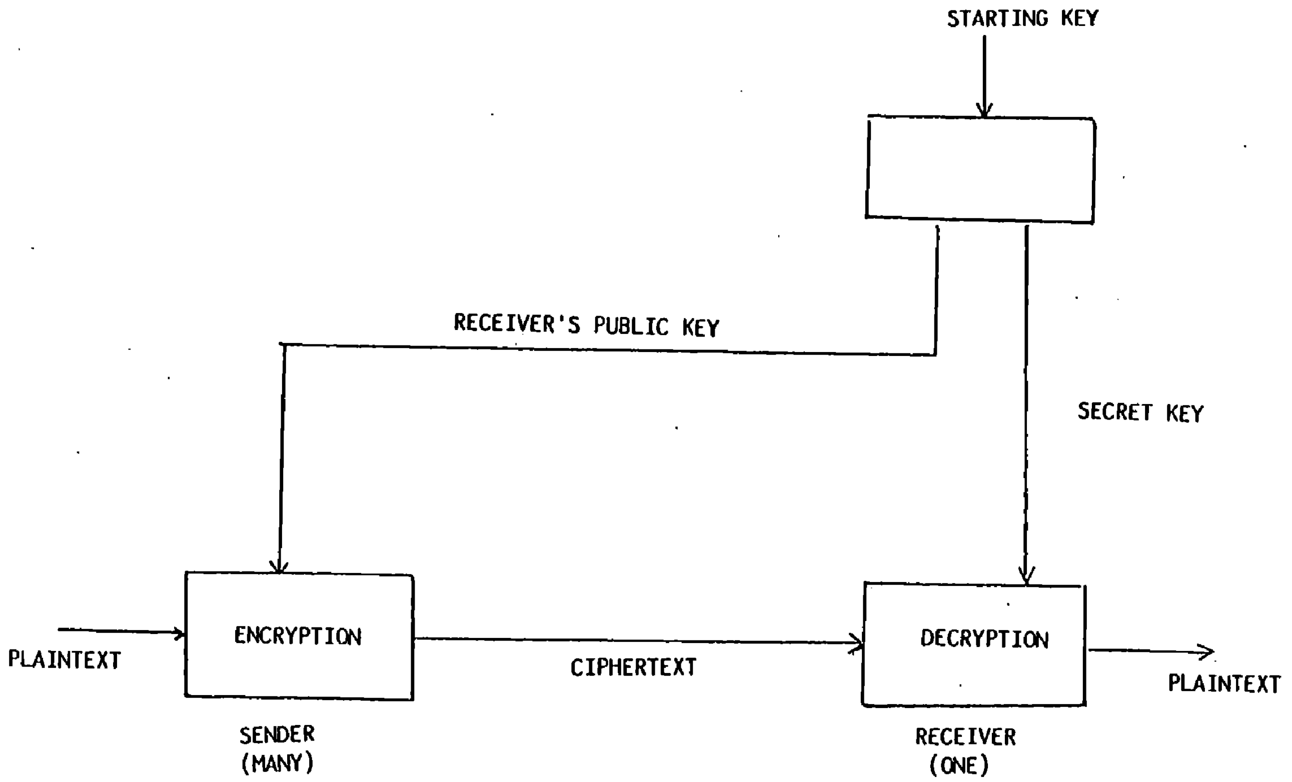


**Fig. 9.4 - Public-Key Cryptosystem.**

The users i and j can then communicate with each other using a symmetric cryptosystem such as the DES under KS. The protocols as described above pose other problems associated with the integrity of the public keys and false impersonations by an opponent. Again a trustworthy third party such as the Key Centre, KC, may be required for the maintenance of the public keys. The above set up can now be modified using a Public Key Centre (PKC) which supplies and maintains public keys of all users in the network. One can further assume that the Centre PKC has a public key ($P_k$) and a secret key ($S_k$) pair and that the key $P_k$ is known to every user in the network.

1. $i \rightarrow$ PKC : $i, (i, j, r_o)_{P_k}$

where $r_o$ is a random number chosen by user i. As in Section 9.3, this is used to prevent an opponent impersonating the PKC by replaying some previously recorded reply.

The PKC upon receiving the request from user i for user j's public key, encrypts the user j's public key $P_j$ using the public key of user i, $P_i$, and sends it to user i along with the random number $r_o$ (or some modified $r_o$ using a publicly arranged function).

2. PKC $\rightarrow i$ : $(i, P_j, r_o)_{P_i}$

Note that the cipher in step 1 can only be decrypted by the Centre and no one else and the cipher in step 2 can only be decrypted by user i and no one else. A similar procedure can be followed by user j if he wishes to obtain the public key of user i. From now on, user i can communicate with user j in a secure manner, either by generating a session key KS and transmitting it to user j enciphered under $P_j$ as mentioned above or using public key approach, that is, by encrypting the messages under the public key of the receiver j.

Note that in the set up procedure, the Centre PKC is not used for generating the session keys and it does not know the secret keys of users but is used as a distributor of public keys. Thus for the integrity to be maintained, it is crucial that the public key file be protected from unauthorized modification.

The public key concept can also be employed in a different way to the one described above to provide a solution to the key distribution problem [34]. With this method, there is no need for decryption as such at the receiving end. It is not a public key cryptosystem but it is a public key distribution system. Two users wishing to exchange a key, communicate back and forth until they arrive at a key which is common. Then this common key can be used as a session key in a symmetric cryptosystem. The opponent eavesdropping on this exchange finds it computationally infeasible to compute the key from the information overheard.

Both the public key cryptosystems and the public key distribution systems are based on one-way functions of one form or another. For instance, it is said that in public key systems, it is infeasible to determine the secret key from the knowledge of public key. (Property (c)). A one-way function has the properties that

(i) It is an easily computed function from x to y, that is, $y = f(x)$.

(ii) It has an inverse function.

(iii) It is computationally infeasible to discover the inverse function.

A precise definition of a one-way function therefore depends on a specific measure of complexity as it varies with time and technology. As mentioned in Section 2.3, the complexity measures are often defined in terms of time or storage required or as a time-memory product. If the number of operations to be done in computing the inverse is taken as a measure, then thermodynamics places a limit of approximately $10^{70}$ on the number of operations that can be performed even if the entire energy of the Sun could be harvested [36, 37]. As the legal receiver has to decrypt the cipher, the public key systems are based on 'trapdoor' one-way functions rather than one-way functions. A trapdoor one-way function is a one-way function which has the additional property that:

- it is computationally infeasible to discover what the inverse function is, unless certain specific information (trapdoor information) that is employed in the design of the function is known. A trapdoor one-way function becomes a trapdoor one-way permutation if it satisfies the property that:

- decryption algorithm followed by encryption algorithm produces the original plaintext (property (d)).

In this case, the mapping between ciphertext and plaintext becomes both injective and surjective. This is essential for implementing digital signatures.

A brief review of two well known public key cryptosystems and a public key distribution system is now presented.

### 9.6.1 Merkle-Hellman Trapdoor Knapsack Public Key Cryptosystem [13]

The knapsack problem is a combinatorial problem in which given a vector $\underset{\sim}{a}$ of n elements, it is required to select a subset of these which add up to a given sum S. The problem is to determine which $a_i$ for i = 1 to n are to be included in forming S, that is, determining whether $x_i$ = o or $x_i$ = 1 for i = 1 to n in the following equation:.

$$S = \underset{\sim}{a} \cdot \underset{\sim}{x} = \sum_{i=1}^{n} a_i \; x_i \qquad (9.1)$$

It is seen that there are $2^n$ possible ways of selecting the quantities $x_i$ and this exponential function increases very rapidly as n increases whereas it is easy to test whether a particular combination is a solution. But there are some instances in which the equation (9.1) is easy to solve. One such instance is when the elements $\{a_i'\}$ form a superincreasing sequence. That is,

$$a_i' > \sum_{j=1}^{i-1} a_j' \quad \text{for all } i > 1 \qquad (9.2)$$

When this occurs, then $x_n = 1$ and only if $S \geq a_n'$ and similarly for i = n-1, n-2, ..., 1 $x_i = 1$ if and only if

$$S - \sum_{j=i+1}^{n} x_j \cdot a_j' \geq a_i'$$

Hence the legal receiver designs the system in such a way that only he can transform the hard knapsack (9.1) into an easy knapsack using the above procedure whereas the opponents are forced to solving (9.1). He does this by choosing two numbers w and m which are relatively prime to each other and a superincreasing sequence $a_i'$ for i = 1 to n. This vector is then transformed to form the sequence $\{a_i\}$ using w and m as follows:

$$a_i = a_i' . w (\text{mod } m)$$

The vector $\underset{\sim}{a}$ is published and it forms the encrypting key of the public key system. The vector $\underset{\sim}{a}'$ and integers w and m are kept secret and form the deciphering key. The encryption procedure consists of taking a plaintext $x$ in the form of a vector $\underset{\sim}{x} = (x_1, \ldots, x_n)$ where $x_i \varepsilon \{0, 1\}$ and forming $S = \underset{\sim}{a} . \underset{\sim}{x}$. The ciphertext S is then sent over the insecure channel to the receiver. The decryption process uses w and m and $\underset{\sim}{a}'$ as follows:

$$S' = W^{-1} . S \ (\text{mod } m)$$

$$= w^{-1} . \sum_{i=1}^{n} x_i . a_i \ (\text{mod } m)$$

$$= w^{-1} . \sum_{i=1}^{n} x_i . w . a_i' \ (\text{mod } m)$$

$$S' = \sum_{i=1}^{n} x_i . a_i' \ (\text{mod } m)$$

If m is chosen such that $m > \sum_{i=1}^{n} a_i'$ then

$$S' = \sum_{i=1}^{n} x_i . a_i'$$

This knapsack is easily solved for $\underset{\sim}{x}$ which is also the solution to the apparently difficult trapdoor knapsack $S = \underset{\sim}{a} . \underset{\sim}{x}$ . It is also possible to iterate the basic transformation by generating several pairs of (w, m). For instance, rather than requiring $\underset{\sim}{a}'$ to satisfy (9.2), $\underset{\sim}{a}'$ can be transformed to a new problem $\underset{\sim}{a}''$ using

$$a_i'' = w'^{-1} . a_i' \ (\text{mod } m')$$

where $\underset{\sim}{a}''$ satisfies (9.2) and is easy to solve. With each such successive transformation, the structure in the publicly known vector

$\underset{\sim}{a}$ is made more amd more obscure.

### 9.6.2 Rivest-Shamir-Adleman(RSA) Public Key Cryptosystem [12]

The trapdoor in this asymmetric scheme is based on the difference in computational difficulty in finding large primes as opposed to factoring large numbers.

Briefly, the RSA system can be described as follows: The receiver chooses two large primes p and q so large that factoring $m = p.q$ is beyond all projected computational capabilities. The plaintext message M can be chosen from the range $1 \leq M < m$. The ciphertext C corresponding to M is derived from the permutation

$$C \equiv M^e \pmod{m}$$

The plaintext M is retrieved from C by applying the inverse transformation

$$M \equiv C^d \pmod{m}$$

The receiver chooses e and d such that

(a) $\gcd^{\dagger}(e, \phi(m)) = 1$ where $\phi(m)$ is the Euler-totient function and in this case, it is equal to $(p-1)(q-1)$

(b) $$e\,d \equiv 1 \pmod{\phi(m)} \qquad (9.3)$$

In other words, e and d are multiplcative inverses in the group formed by residue classes mod $\phi(m)$.

The reason why this encryption-decryption scheme works is based on the Euler-Fermat theorem [38] which states that for any integer M which is relatively prime to m,

$$M^{\phi(m)} \equiv 1 \pmod{m} \qquad (9.4)$$

Using (9.3) gives,

$$M^{ed} \equiv M^{K\,\phi(m)+1} \pmod{m}$$ for some integer K.

† greatest common divisor

From (9.4), for all M such that p does not divide M

$$M^{p-1} \equiv 1 \pmod p$$

and since p-1 divides $\emptyset$ (m)

$$M^{K \emptyset (m) + 1} \equiv M \pmod p \qquad (9.5)$$

When $M \equiv 0 \pmod p$, the equation (9.5) is obeyed trivially. Similarly for q

$$M^{K \emptyset (m) + 1} \equiv M \pmod q \qquad (9.6)$$

Using Chinese Remainder Theorem (Appendix 10), equations (9.5) and (9.6) imply that for all M,

$$M^{ed} \equiv M^{K \emptyset (m) + 1} \equiv M \pmod m$$

In this system, the numbers e and m are made public and they form the encrypting key. The numbers d, p and q are kept secret and form the decrypting key. If m can be easily factorized to p and q then the cryptanalyst can find $\emptyset$ (m) and d and hence can crack the system.

This cryptosystem and its possible extensions form a major part of this thesis and hence this system is considered in detail in subsequent chapters.

### 9.6.3 Diffie-Hellman Public Key Distribution System [35]

This public key distribution system makes use of the apparent difficulty of computing logarithms over a finite (Galois) field GF(q) where q is a prime.

Let 'a' be a primitive element of GF(q) and let

$$y \equiv a^x \pmod q \quad \text{for} \quad 1 \leqslant x \leqslant q - 1$$

x is referred to as the logarithm of y to the base 'a' over GF(q)

$$x = \log_a y \text{ over } GF(q)$$

Calculation of y from x is easy whereas computation of x from y is much more difficult, that is, it is an one-way function. This problem is called the logarithm problem whereas the RSA system is based on the root problem.

The key distribution occurs as follows. User i generates a random number $x_i$ chosen uniformly from the set of integers 1, 2, ..., q-1. He then computes

$$y_i \equiv a^{x_i} \pmod q$$

and publishes $y_i$ and keeps $x_i$ secret. Similarly user j publishes $y_j$ and keeps $x_j$ secret where

$$y_j \equiv a^{x_j} \pmod q$$

The private session key, $K_{ij}$, is established by forming

$$K_{ij} \equiv a^{x_i x_j} \pmod q$$

User i computes $K_{ij}$ by obtaining $y_j$ from the public file and forming

$$K_{ij} \equiv y_j^{x_i} \pmod q$$

$$\equiv a^{x_i x_j} \equiv a^{x_j x_i} \pmod q$$

User j similarly computes $K_{ij}$ using $y_i^{x_j} \pmod q$. For the opponent to form $K_{ij}$, he must compute

$$K_{ij} \equiv y_i^{(\log_a y_j)} \pmod q \equiv y_j^{(\log_a y_i)} \pmod q$$

Therefore if logarithms over GF(q) are easily computed, the system can be broken. A possible extension of this system together with a practical implementation are described in Chapter 13.

## 9.7 Key Distribution Using Public Key For Groups of Users

Let us now return to the situation where members of a group G wish to broadcast and receive messages from other members of G and to access and update files private to G and consider three other methods of key distribution for such an arrangement. Method 1 is given in Section 9.5.1.

### 9.7.1 Method 2

This approach uses a public key distribution method such as the Diffie-Hellman exponentiation method to distribute the group key.

Here each user i registers with the Key Centre KC a public key $y_i \equiv a^{x_i}$ (mod p) where $x_i$ is only known to user i. The primitive root 'a' and the prime p are known to all users in the network. The user i transmits to KC a list of members of the group G.

$$i \rightarrow \text{KC:}\ \ i,\ \ G = \{u_1, u_2, \ldots, u_n\}$$

If the user i belongs to the group, then the Key Centre generates a number $x_G$ and sends it to user i enciphered under his public key. That is,

$$\text{KC} \rightarrow i : K_{i,G} \equiv y_i^{x_G} \pmod p$$

The user i upon receiving the above message computes

$$(K_{i,G})^{x_i^{-1}} \pmod p$$
$$\equiv a^{x_i x_G x_i^{-1}} \pmod p$$
$$\equiv a^{x_G} \pmod p$$

That is, the group key KG is given by $a^{x_G}$ (mod p). With this method, storage of up to $2^n-1$ secret values of $x_G$ is necessary either by the Key Centre or by the users,where n is the total number of users and the KC does not need to know the personal keys of the users $x_i$ $(1 \leqslant i \leqslant n)$.

### 9.7.2 Method 3

If the KC is however given access to users' personal keys, a modification to the above public key distribution method is as follows:

The group key KG is now made equal to

$$\text{KG} = a^{x_1 \cdot x_2 \cdot \cdots \cdot x_n} \pmod p \qquad (9.7)$$

When the ith member of G requests KG from the Centre, KC returns KG enciphered under the personal key $x_i$. The master key is represented by the list of personal keys. Another member of G may be able to determine $a^{x_i}$ (mod p) but he cannot compute $x_i$ without computing a discrete logarithm. If p is chosen to be a large prime number,this is not feasible.

Here the key centre KC needs to store only n personal keys. Although the method uses a one-way function, it is not a public key distribution method because the KC must have access to secret personal keys of the users.

Note that in both the schemes 2 and 3, for suitably chosen primes p of the form p = 4q + 1 where q is also a prime, a = 2 is a primitive root.

### 9.7.3 Method 4

This method considers the establishment of the group key KG given by (9.7) without the use of the Key Centre. It assumes a special situation where the n users (0 to n-1) are linked together in a circular fashion, thus forming a ring. That is, user i always sends messages to user i + 1 and user n-1 sends message to user 0. The table 9.1 given in Figure 9.5 shows the messages received and transmitted by user i at various time instants.

| Time Instant | Transmitted Message | Received Message |
|---|---|---|
| 1 | $a^{x_i}$ (mod p) | $a^{x_{i-1}}$ (mod p) |
| 2 | $a^{x_{i-1}x_i}$ (mod p) | $a^{x_{i-2}x_{i-1}}$ (mod p) |
| ⋮ | ⋮ | ⋮ |
| r | $a^{x_{i-(r-1)}\cdots x_i}$(mod p) | $a^{x_{i-r}\cdots x_{i-1}}$ (mod p) |
| ⋮ | ⋮ | ⋮ |
| n-1 | $a^{x_{i-(n-2)}\cdots x_i}$(mod p) | $a^{x_{i-(n-1)}\cdots x_{i-1}}$(mod p) |

Figure 9.5 - Table 9.1: Messages received and transmitted by user i.

At time instant t = r,the user i raises the message received from user i - 1 at time instant t = r-1 to the power $x_i$ and transmits it to user i + 1. The user i forms the group key KG by raising the received message at time instant t = n - 1 to the power $x_i$. That is,

$$KG = a^{x_{i-(n-1)}\cdots x_{i-1}\, x_i} \pmod{p}$$

Every user in the network can arrive at this common group key by raising the message received at the (n-1)th time instant to the power of his secret key. Any intruder who does not belong to this ring network cannot determine the group key by monitoring the transmitted messages at all links in the network. A more generalized version of this method has recently been published in [41].

## 9.8 Key Distribution Schemes for Prestel Encryption System

This chapter is concluded by considering possible key management schemes for Prestel Viewdata system with encryption facility.

Initially consider the case where a user i (information provider) wishes to store a frame (or frames) on Prestel database which should only be read by user j. This is referred to as the 'Postbox system'. In this case, it is assumed that there is no direct network link between the two users other than via the database. If the user systems only allow DES type symmetric cryptography then the frame key (the key with which a Prestel frame has been enciphered) must be exchanged via some secure courier. This is the only method possible if it is assumed that Prestel is only used as a database and does not play an interactive role in the distribution of keys. However, if the user systems support public key cryptography, then one of the following two approaches can be adopted.

The first approach uses a public key cryptosystem such as the RSA system. Here the user i can encrypt the frame using the public key of user j (this is assuming that the public file containing enciphering keys of users is available, say, in the form of a telephone directory). User j will then be the only person who would be able to read the frame and not even user i can read the frame he had entered. Alternatively, user i can generate a random frame key which he can use to encrypt the frame using a symmetric algorithm such as the DES. User i then encrypts this frame key using the public key of user j and stores the result at the top of the frame. User j can recover the frame key using his secret RSA key and then read the Prestel frame. User i will also be able to read the frame as he can keep a copy of the frame key which he has generated (say enciphered under his own public key).

The second approach uses a public key distribution system such as the Diffie-Hellman exponentiation system. Here the user i obtains the public key information $y_j$ of user j from the public file and performs $y_j^{x_i}$,where $x_i$ is the secret key of user i,to form the common key $K_{ij}$. Then he generates a random frame key and uses it to encrypt the Prestel frame. The difference between this approach and the one above is that in this case, the frame key is enciphered under the common key and is stored at the top of the frame. User j will be able to read the frame as he can obtain the common key and hence the frame key. Here again only users i and j will be able to read the Prestel frame.

Consider now the case where a group of n users who wish to communicate with each other via the Prestel database. That is, each one of the n users should be able to read the frames entered by any one of the others in the group. Amongst the members of the group, existence of a privileged member referred to as the Manager is envisaged. [The group could represent typically a small company or an organization]. Further it is assumed that every member of the group is linked to the Manager. In such a situation, the Manager assumes the role of the Key Centre described earlier. Any user i who wishes to enter an encrypted frame on Prestel initially establishes a frame key with the Manager following the procedure outlined in Section 9.3. This frame key can then be used to encipher the frame and stored at the top of the frame enciphered under a master key of the Manager. Anuser j belonging to the group can read the enciphered frame key from the top of the frame and send it to the Manager who returns to him the frame key enciphered under the user j's personal key. Thus user j can decipher the frame entered by user i. With this approach, in addition to the master key, the Manager is required to keep a record of the personal keys of all n users and if there is more than one group, a record of members in each group. One can also use any one of the methods 1 to 3 given in Sections 9.5 and 9.7 to establish a group key among the n users using the Manager as the Key Centre. Having established the group key, the user i can follow the usual procedure of generating a random frame key and storing this at the top of the frame,this time enciphered under the group key. Thus any user belonging to the group can read the frame. Method 3 (see Section 9.7.2) is seen to be the most attractive method as the Manager needs to store a list of keys which is a polynomial function of the number of users n.

# CHAPTER 10

## EXTENSIONS OF THE RSA CRYPTOSYSTEM

### 10.1 General

As seen in the last chapter, the concept of public key cryptosystems recently proposed by Merkle and Hellman [35] not only provides a novel way of distributing the keys required for the symmetric cryptosystems but it also gives rise to a class of asymmetric public key cryptosystems with digital signature capabilities. From this stage onwards, the thesis mainly concentrates on the analysis and the design of public key systems; in particular, the RSA public key cryptosystem and the Diffie-Hellman public key distribution system. Both these systems are of immense interest among the cryptographic community at the present time.

Before considering some possible extensions of the RSA system to matrix and polynomial rings, it is useful to consider some design aspects of the RSA system over the rational integers. These aspects are more or less applicable to the extended RSA systems considered in subsequent sections.

### 10.2 Some Design Aspects of RSA System

As mentioned in Section 9.6.2, in the RSA system the encryption is performed by raising the message x ($1 \leqslant x < m$) to the eth power modulo m and the decryption is performed by raising the cipher y to the power d modulo m. That is

$$y \equiv x^e \pmod{m}$$

and

$$x \equiv y^d \pmod{m}$$

where m is equal to the product of two large distinct primes p and q. The public encryption key is the pair of integers (e, m) and the secret decryption key is (d, m). The coding exponents are chosen such that they are multiplicative inverse of each other modulo $\emptyset$ (m) and that $x^e$ is a permutation of the residue classes modulo m. That is,

$$e\ d \equiv 1 \ (\mathrm{mod}\ \phi\ (m)) \quad \text{where } \phi\ (m) = (p-1)\ (q-1)$$

To design the RSA system, the user needs to choose two large primes p and q to form the modulus m. The magnitude of the modulus m is determined by the required difficulty of breaking the designed system. The table shown in Figure 10.1, taken from [12], gives an indication of the magnitude of m. The number of operations computed for each value of m is based on the best known factoring algorithm for large intergers. (see Section 10.3.1).

| $\log_{10} m$ | Number of operations | Remarks |
|---|---|---|
| 50 | $1.4 \times 10^{10}$ | |
| 100 | $2.3 \times 10^{15}$ | At the limits of current technology |
| 200 | $1.2 \times 10^{23}$ | Beyond current technology |
| 400 | $2.7 \times 10^{34}$ | Requires significant advances in technology |
| 800 | $1.3 \times 10^{51}$ | |

Figure 10.1 Effort required to factor modulus m

Once an approximate idea of the magnitude of m is decided, then the two primes p and q need to be selected randomly. The prime number theorem states that the primes near m are spaced on the average one every (ln m) integers. Thus even for large primes several hundred digits long, only a few hundred candidates must be tested before finding a prime. Hence one needs to test whether a chosen large number is a prime. There exists elegant probabilistic algorithms [42] which decide with an arbitrarily small uncertainty if the chosen number is a prime. The probabilistic algorithm consists of making many independent tests and declaring the number to be composite when any one test fails. If a test mistakes a composite for being prime with probability $f^{-1}$then by using k such tests the probability that the algorithm will incorrectly declare a composite to be prime is $f^{-k}$. Three tests of primality are commonly used [43] namely, the Fermat test, the Solovay-Strassen test and the Rabin test. In all these cases, it is assumed that the tests are applied to the number b.

### 10.2.1 Primality Tests

#### 10.2.1.1 Fermat Test

This test is based on the Euler-Fermat theorem previously mentioned. The theorem states that if p is a prime then

$$a^{p-1} \equiv 1 \pmod p \text{ for all a, } 1 \leqslant a < p$$

Thus the test consists of choosing 'a' less than the number b and accepting b to be prime if $a^{b-1}$ (mod b) is congruent to 1 (mod b). In practice, only a few values of 'a' need to be tried and not all 'a' as indicated above. (Only a small number of composites pass this test even a few times). It is recommended that approximately one hundred tests be made to reliably conclude that the selected number is prime. It is generally believed that choosing a = 3 will identify virtually all composites. The Carmichael numbers (eg.561 = 3.11.17) are known to pass this test even though they are composite.

#### 10.2.1.2 Solovay-Strassen Test [44]

This test identifies the Carmichael numbers as being composite. It picks a random number 'a' between 1 and b-1 and tests whether

$$\gcd(a,b) = 1 \text{ and } J(a,b) = a^{(b-1)/2} \pmod b \qquad (10.1)$$

where J(a,b) is the Jacobi symbol and gcd denotes the greatest common divisor. The Jacobi symbol is defined as

$$J(a,p) = \begin{cases} 1 & \text{when } x^2 \equiv a \pmod p \text{ has a solution in } Z/pZ \\ -1 & \text{when } x^2 \equiv a \pmod p \text{ has no solution in } Z/pZ \end{cases}$$

where Z/pZ denotes the ring of integers modulo p.

If b is prime, then (10.1) is always true and if b is composite (10.1) is false with a probability of a ½. Therefore making k tests yield an answer that is wrong with a probability of $2^{-k}$ when claiming the input is prime. (When claiming the number is composite, it is always correct).

#### 10.2.1.3 Rabin Test [42]

As b is necessarily an odd integer, representing b as $b = 2^r s + 1$ where s is odd, this test then chooses a number of values of 'a' randomly in the range 1 to b-1 and accepts b to be prime if either $a^s \equiv 1 \pmod b$ or $a^{2^j s} \equiv -1 \pmod b$ for some j where $0 \leq j < r$. Otherwise b is rejected.

All the three tests can be carried out to check whether the chosen number b is a prime or not. The failure probabilities of each test are discussed in length in [43]. Note that this primality testing procedure needs to be done only once by any system designer and all these tests require computational effort of the order of $O(\log_2 b)$ operations on large integers.

To confound certain factoring algorithms [45], it is desirable to choose primes p and q such that p-1 and q-1 have a large prime factor. This can be done by generating a large prime number b and then letting p (or q) be the first prime in the sequence bi + 1 for i = 2, 4, 6 ... Additional security can be provided if b-1 also has a large prime factor. Furthermore, it is also advisable not to choose p and q too close to each other. If $p \doteq q$, then $2\sqrt{m}$ is a good approximation of p + q. Knowing p + q gives immediately $\emptyset(m)$ since $\emptyset(m) = (p-1)(q-1) = m + 1 - (p+q)$. Further, the primes should not be chosen to be any special primes such as the Fermat primes or Mersenne primes as these are well studied and the resulting product may be more likely to yield to attacks of factorization.

### 10.2.2 Choice of Coding Exponents

Having chosen the primes p and q, and hence m, the next step is to choose the coding exponents e and d. To do this, the user chooses a number d which is relatively prime to $\emptyset(m)$. This is done by selecting a number d (mod m) and computing gcd $(d, \emptyset(m))$. This is done using the Euclid's algorithm given in the Appendix 11. This not only checks whether d and $\emptyset(m)$ are relatively prime but also gives the multiplicative inverse e. It is known that the gcd function is computable in $O(\log_2 m)$ time [45]. It is necessary to choose both e and d such that they are greater than $\log_2 m$. If $e < \log_2 m$, then small messages will not be disguised by the modulo reduction process. That is, $x^e \pmod m) = x^e$ and the cipher is breakable by brute force attack. If d is smaller than $\log_2 m$,again the system can be broken by a random search by the opponent to determine its value. There is a further condition on the choice of e. As mentioned previously, e needs to be chosen relatively prime to $\phi$ (m) or more exactly to $\ell cm$[†] (p-1,q-1), for the function $x^e$ (mod m) to permute the residue classes (mod m) (see Section 14.1). But this permutation has fixed messages, that is, there are residue classes x (mod m) which satisfy the congruence

$$x^e \equiv x \pmod m \text{ where } e \geqslant 3 \text{ is odd and } m = p.q \qquad (10.2)$$

It is known [46] that any solution of the congruence

$$x^3 \equiv x \pmod m \qquad (10.3)$$

also satisfies (10.2). Note that the congruence (10.3) has exactly 9 solutions in the range $1 \leqslant x \leqslant m-1$, (m = pq). Thus the congruence (10.2) will have at least 9 fixed messages. Blakely [47] has shown that for the congruence to have only 9 fixed messages e must be chosen such that gcd $\{e-1, \ell cm\ (p-1,q-1)\} = 2$.

## 10.3 Cryptanalysis of RSA System

The main cryptanalytical attack seems to be the determination of the secret coding exponent d. There are basically three ways a cryptanalyst might try to determine d from the publicly revealed information (e, m).

### 10.3.1 Factorization of m

The factorization of m would allow the opponent to compute $\phi$ (m) and hence the secret coding exponent d using ed $\equiv$ 1 (mod $\phi$ (m)). A large number of factoring algorithms exist [45]. The fastest algorithm known at the present time can factor m in approximately $(\ln m)^{(\ln m/\ln \ln m)^{\frac{1}{2}}}$ steps and is due to R.Schroeppel (unpublished). The table given in Section 10.1 is based on Schroeppel's method and it shows that a number m of 200 digits (decimal) long would provide a margin of safety against future developments.

[†] $\ell cm$ : least common multiple

### 10.3.2 Computation of $\emptyset$ (m) Without Factorization of m

Another method of cryptanalysis would be to somehow directly determine $\emptyset$ (m) without factorizing m. It is shown in [12] that the approach of computing $\emptyset$ (m) directly is no easier than factoring m since finding $\emptyset$ (m) enables the opponent to easily factor m. This can be seen as follows:

$$\text{Let } x = p + q = m + 1 - \emptyset(m)$$
$$\text{and } y = (p-q)^2 = x^2 - 4m$$

Knowing $\emptyset$ (m), one can determine x and hence y. Using x and y, the primes p and q are given by $p = \frac{x + \sqrt{y}}{2}$ and $q = \frac{x - \sqrt{y}}{2}$.

### 10.3.3. Determining d Without Factoring m or Computing $\emptyset$ (m)

The third method of cryptanalysis consists of computing the secret exponent d without factorizing m or determining $\emptyset$ (m). Again it is argued in [12] that provided d is chosen large enough to make a direct search attack infeasible, computing d is no easier than factoring m, since once d is known, m could be factorized easily. This can be seen as follows: If d is known, then it is possible to calculate some multiple of $\emptyset$ (m) using

$$ed - 1 = k\,\emptyset(m) \text{ for some integer } k$$

Miller [40] has demonstrated that m can be factored using any multiple of $\emptyset$ (m). The opponent,on the other hand,may hope to find a d′ such that it is equivalent to the secret exponent d of the designer. If there are a lot of such d′, then one could use a brute force search method to break the system. But all the d′ differ by the least common multiple of (p-1) and (q-1) and if one is found then m can be factored. Thus finding any such d′ is as difficult as factoring m.

Now some possible extensions of the RSA system are considered.

## 10.4 Extension of RSA System to Matrix Rings

### 10.4.1 Trapdoor Rings

Assume R is a finite ring [48] with 1 which is associative but not necessarily commutative. Suppose that members of the ring R are used as messages and that $r \in R$ is enciphered as $r^e$ where e is the published encrypting exponent. The trapdoor property can now be stated as follows:- there exists some integer $n > o$ such that $r^{n+1} = r$ for all $r \in R$. These rings are to be referred to as trapdoor rings. For instance in $Z/pZ$, $r^p = r$ for all $r \in R$. More generally, if we let $R = F_q = GF(q)$, the field of q elements where q is a prime power $(p^k)$ then $r^q = r$ for all $r \in R$. Further, if R and S are any two such trapdoor rings, then the direct sum $R \oplus S$ consists of vectors $(r,s)$ with $r \in R$ and $s \in S$ is another trapdoor ring say T. The number of elements in the ring T is equal to the product of the number of elements in R and S. This above process can be applied repeatedly taking vectors of arbitrarily many components, each taken from some finite field. Considering finite fields $F_{q_i}$ for $1 \leqslant i \leqslant j$ where $q_i$'s can be the same or different, the trapdoor ring R is formed by all vectors $x = (x_1, \ldots, x_j)$ where $x_i \in F_{q_i}$ for $1 \leqslant i \leqslant j$. The ring R consists of $q_1 . q_2 . \ldots . q_j$ elements and the equality $r^{n+1} = r$ is obeyed for all $r \in R$ where n is equal to $(q_1 - 1)(q_2 - 1) \ldots (q_j - 1)$ or any multiple of it.

There are many finite rings which are not trapdoor rings. Consider for instance, $R = Z/p^2 Z$ where p is a prime. Then, $p^2 = p^3 = \ldots = 0$ in the ring R but $p \neq 0$ in ring R. So the property that $p^{n+1} = p$ is not satisfied for any $n > 0$. More generally, for a ring R to be a trapdoor ring, it is necessary that R have no nilpotent elements except zero. (An element, x, is said to be nilpotent if $x^a = 0$ and $x^{a-1} \neq 0$ for some $a > 0$). However, if we take an integer m to be a square free positive integer say $m = p_1 . \ldots . p_j$, where all the $p_i$'s are distinct primes, then the ring $R = Z/mZ$ is a trapdoor ring. This ring can in fact be regarded as a direct sum of finite fields $F_{p_1} \oplus F_{p_2} \oplus \ldots \oplus F_{p_j}$ as described above. If $j=2$, then this becomes the standard trapdoor ring used by the RSA cryptosystem.

It has been suggested by Dr R Odoni that every finite trapdoor ring is isomorphic to a direct sum of finite fields. The argument relies on the use of Wedderburn's structure theory [49] for semisimple rings. The main steps involved are as follows:

1. The ring R is trapdoor implies that R has no nilpotent elements except 0.

2. A finite ring without nonzero nilpotent elements must have a 1 [50].

3. A finite ring with 1 and lacking nilpotents ($\neq 0$) is a direct sum of matrix rings with entries in a division algebra (skew field) (Wedderburn's theorem).

4. If any of these matrices is not 1 x 1 then there will be non-zero nilpotent elements in R.

5. Hence R is a direct sum of finite skew fields.

6. A finite skew field is necessarily commutative.

Thus a trapdoor ring R is a commutative ring with 1 which is isomorphic to a direct sum of finite fields. (Note that two rings R and S are isomorphic if there exists a function $f : R \rightarrow S$ which is one-to-one and onto and satisfies $f(r_1 \pm r_2) = f(r_1) \pm f(r_2)$, $f(r_1 r_2) = f(r_1) f(r_2)$ for all $r_1, r_2 \in R$)

The original RSA scheme derived its message space from Z/mZ, the ring of integers modulo m where m is the product of two large distinct primes p and q. Here other finite systems that might serve as a basis for an extended RSA cryptosystem are investigated.

### 10.4.2 Non-Singular Matrices Over Z/mZ

If the ring of all n x n matrices over the ring R = Z/mZ is considered, it is seen that the ring contains nilpotent elements when $n > 1$. Consider for instance $M = \begin{pmatrix} 0 & 3 \\ 0 & 0 \end{pmatrix}$ over Z/6Z ; $M^2 \equiv 0 \pmod 6$ but $M \not\equiv 0 \pmod 6$. To overcome this problem, initially only the group $M_n$ formed by the non-singular matrices of order n, is selected to form the message space of this extended system.

Let us first consider the finite group formed by matrices of order n whose determinants are relatively prime to p and whose elements are in Z/pZ (p prime). That is, the non-singular matrices over Z/pZ are considered.

To begin with, the non-singular matrices over Z/pZ form a finite group because (i) the product of any two members of the set is congruent (mod p) with some member of the set, (ii) every member has its reciprocal, that is, for any two members A and B, there exist members C and D such that $AB \equiv C \pmod p$ and $AD \equiv I \pmod p$ and (iii) the unit matrix is the 'identical' member of the group.

The order of the group formed by these elements can be shown to be equal to $N_p$ where

$$N_p = (p^n-1)\cdot(p^n-p) \dots (p^n - p^{n-1}) \qquad (10.4)$$

The argument goes as follows: The elements of the first row vector $\underset{\sim}{U}_1$ of the matrix may consist of any of the p numbers except that they cannot all be zero since this would make the matrix singular. There are therefore $p^n-1$ ways of selecting this vector. When $\underset{\sim}{U}_1$ has been selected, then $\underset{\sim}{U}_2$ may be any of the $p^n$ possible vectors except those which are congruent with $k_1\ \underset{\sim}{U}_1$ (mod p) for $k_1 = \{0, 1, \dots, p-1\}$. This will be the case if and only if $\underset{\sim}{U}_1$ and $\underset{\sim}{U}_2$ are chosen to be linearly independent. Therefore there are $p^n-p$ ways of selecting $\underset{\sim}{U}_2$. Continuing this procedure, the expression (10.4) for the order $N_p$ is obtained.

If non-singular matrices with elements over Z/pZ are used as messages, then one can form a conventional cryptographic system where the secret key contains the modulus p itself. The encrypting (e) and decrypting (d) exponents can then be determined using

$$ed \equiv 1 \pmod{N_p} \qquad (10.5)$$

The encrypting key is therefore (e, p, n) and the decrypting key is (d, p, n). None of these keys can be made public and the encryption and decryption procedures are as in the RSA system.

$$\left.\begin{array}{l} M^e \equiv C \pmod{p} \\ \text{and} \\ C^d \equiv M \pmod{p} \end{array}\right\} \qquad (10.6)$$

where M, C $\varepsilon$ $M_n$ (Z/pZ), non-singular matrices over Z/pZ

The above system can be modified to include the public key property as follows: Suppose the modulus is a composite number m whose factorization is

$$m = \prod_{j=1}^{s} p_j^{r_j} \qquad (10.7)$$

Then the order $N_m$ of the multiplicative group formed by non-singular matrices of order n over Z/mZ is given by

$$N_m = \prod_{j=1}^{s} N_{p_j^{r_j}} \qquad (10.8)$$

This ia a consequence of the Chinese Remainder Theorem (Appendix 10).

Let us first determine the order $N_{p^r}$ of the group of non-singular n x n matrices over $Z/p^rZ$ when p is prime and $r > 1$.

Let $\theta$ be the homomorphism [48], mapping an n x n matrix A over $Z/p^{r+1}Z$ to $A'$, a matrix over $Z/p^rZ$, via $a_{ij} \pmod{p^{r+1}} \longmapsto a_{ij} \pmod{p^r}$.

$$\theta : \quad A \longmapsto A'$$

$$\theta : \; M_n(Z/p^{r+1}Z) \longmapsto M_n\,(Z/p^rZ)$$

This induces a surjective homomorphism between the linear groups formed by these matrices, that is ,

$$\theta' : \; GL\,(n,\, p^{r+1}) \twoheadrightarrow GL\,(n,\, p^r)$$

Therefore using group theory [48]

$$\frac{GL\,(n,\, p^{r+1})}{\text{Kernel } (\theta')} \cong GL\,(n,\, p^r)$$

(where $\cong$ denotes isomorphic to)

The kernel consists of the set of matrices which are mapped to the identity matrix I (mod $p^r$), ie,

$$a_{ii} \equiv 1 \pmod{p^r} \quad \text{for } 1 \leq i \leq n \qquad (10.9)$$

$$a_{ij} \equiv 0 \pmod{p^r} \quad \text{for } i \neq j \qquad (10.10)$$

There are p possibilities for each of the equations (10.9) and (10.10) giving rise to $p^{n^2}$ possibilities. Therefore using group theory, and denoting order by the symbol #,

$$\#\, GL\,(n,\, p^{r+1}) = p^{n^2} \;\#\, GL\,(n,\, p^r) = \ldots =$$

$$= p^{rn^2} \;\#\, GL\,(n,\, p)$$

But $\# GL(n, p) = N_p$

$= (p^n-1)(p^n-p) \ldots (p^n-p^{n-1})$ from (10.4)

Therefore

$$\# GL(n, p^r) = p^{(r-1)n^2} (p^n-1) \ldots (p^n-p^{n-1})$$

$$= N_{p^r}$$

$$\therefore \quad N_{p_j^{r_j}} = p_j^{(r_j-1)n^2} (p_j^n-1)(p_j^n-p_j) \ldots (p_j^n-p_j^{n-1}) \qquad (10.11)$$

Substituting (10.11) into (10.8) gives the order $N_m$. Now as in the RSA cryptosystem if m is made to be the product of two distinct primes p and q this simplifies to

$$N_m = N_p \cdot N_q$$

$$= (p^n-1)(p^n-p) \ldots (p^n-p^{n-1})(q^n-1)(q^n-q) \ldots (q^n-q^{n-1})$$

Therefore for a message M

$$M^{N_m} \equiv I \pmod m$$

and the coding exponents e and d are determined using

$$ed \equiv 1 \pmod{N_m} \qquad (10.12)$$

The expression of the order $N_m$ depends on the structure of m, that is, on its prime factors. This therefore can be used to form a public key cryptosystem by choosing m to be a large integer (say 200 decimal digits) whose security is the same as that of the RSA system.

Although the order $N_m$ can be used in finding e and d as in (10.12), it is usually a large number. For instance, even for small primes such as p = 13 and q = 23, the order is approximately $1.6 \times 10^{22}$ for 3 x 3 non-singular matrices. Therefore it is desirable to find the exponent, EXP, of the group, that is, the least integer greater than zero such that

$$M^{EXP} \equiv I \pmod m \text{ for all } M.$$

EXP is a divisor of the order $N_m$ of the group.

Let us first consider the exponent of the group formed by the non-singular matrices over $Z/pZ$, $M_n(Z/pZ)$.
Let the exponent be $\ell$ such that

$$A^{\ell} \equiv I \text{ for all } A \in M_n \ (Z/pZ)$$

Assume that $p > n$. $A^{\ell} \equiv I \pmod p$ implies that $x^{\ell}-1$ is divisible by the minimum polynomial of A. As A ranges over the non-singular matrices of order n over $Z/pZ$, $x^{\ell}-1$ must be divisible by every monic irreducible polynomial $P(x)$ $(\neq x)$ of degree $\leqslant n$ in $Z/pZ$. Every irreducible polynomial $P(x)_u (\neq x)$ of degree u divides $x^{p^u-1}-1$. Thus $x^{\ell}-1$ must be divisible by $x^{p^u-1}-1$.

But

$$x^b-1 \equiv 0 \pmod{x^a-1}$$

implies that $a|b$

Hence,

$$\ell \equiv 0 \pmod{p^u-1} \text{ for } 1 \leqslant u \leqslant n$$

Therefore,

$$\ell \equiv 0 \ (\text{mod } \ell cm \ \{p-1, \ldots, p^n-1\}) \text{ certainly} \qquad (10.14)$$

Furthermore, the matrix A given by

$$A = I + \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & \bigcirc & \vdots \\ \vdots & \bigcirc & & \ddots & 0 \\ 0 & \cdots & & & 0 \end{bmatrix}$$

satisfies $A^p = I \neq A$ $(p > n)$
That is, A has order p and hence $p \mid \ell$
Hence the exponent of $GL(n,p)$, $p > n$, is given by

$$\ell = p \ \ \ell cm \ \ \{p-1, p^2-1, \ldots, p^n-1\} \qquad (10.15)$$

Now for any $A \in M_n \ (Z/pZ)$, using Jordan's Canonical form, there exists a non-singular matrix E such that

$$E^{-1}AE = \begin{bmatrix} B_1 & & & \\ & B_2 & & \\ & & \ddots & \\ & & & B_r \end{bmatrix}$$

$$= D$$

Each block $B_i$ is of the form

$$B_i = \begin{bmatrix} \lambda_i & 1 & & O \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ O & & & \lambda_i \end{bmatrix} .$$

ie, $B_i = \lambda_i I_i + N_i$ for some upper triangular nilpotent matrix $N_i$

where $\lambda_i$'s are non-zero in $F_{p^{r_i}}$ for some $r_i \leq n$.

If the order of D is k, that is, $D^k \equiv I \pmod p$, then as $D = E^{-1}AE$, this gives

$$\begin{aligned} (E^{-1}AE)^k &= (E^{-1}AE)\,(E^{-1}AE) \ldots^{k \text{ times}} \\ &= E^{-1}A^kE \end{aligned}$$

$$\text{Thus } A^k \equiv I \pmod p$$

Order of A = order of D = k = $\ell$cm of orders of $B_i$. If $N_i = 0$, then order of $B_i$ is a divisor of $p^{r_i}-1$. Hence the order of A divides $\ell\text{cm}\,\{p-1, \ldots, p^n-1\}$. Otherwise, $B_i^p = \lambda_i^p I_i$ will have such an order and hence the order of A divides p $\ell\text{cm}\,\{p-1, \ldots, p^n-1\}$.

A multiple of p in the expression (10.15) for the exponent $\ell$ is expected as the order given by (10.4) contains multiples of p.

Similarly,

$$A^s \equiv I \ (\text{mod } q) \quad \text{for all } A \text{ in } M_n \ (Z/qZ)$$

where

$$s \ = \ q \ \ \ell cm \ \ \{q-1, \ q^2-1, \ \ldots, \ q^n-1\}$$

Therefore exponent EXP of the group GL(n,m) where m = pq is given by

$$\ell cm \ (p \ \ell cm \ \ \{p-1, \ p^2-1, \ \ldots \ p^n-1\} \ , \ q \ \ \ell cm \ \ \{q-1, q^2-1, \ \ldots \ q^n-1\} \ )$$

Let us now extend this argument to non-square free modulus m. First consider a matrix A in $M_n(Z/p^2Z)$. Let θ be the natural homomorphism from $Z/p^2Z$ onto $Z/pZ$ (p prime). From the above argument

$$\begin{aligned} \theta(A)^{EXP} &\equiv I \ \text{ in } \ M_n(Z/pZ) \\ &\equiv \theta \ (I) \end{aligned}$$

Therefore,

$$\theta(A^t-I) \ \equiv \ 0 \ (\text{mod } p) \text{ where } t = EXP$$

This means that every entry in $A^t-I$ is some multiple of prime p and hence

$$A^t-I \ = \ p \ B \quad \text{for some matrix B.}$$

That is,

$$\begin{aligned} A^t &= I + p \ B \\ A^{tp} &= (I + p \ B)^p \end{aligned}$$

Using the binomial theorem,

$$\begin{aligned} A^{tp} &= I + \binom{p}{1} \ p \ B + \binom{p}{2} \ p^2 \ B^2 + \ldots \\ &\equiv I \ (\text{mod } p) \end{aligned}$$

Therefore considering in general a matrix in $M_n(Z/p^kZ)$

$$\theta: \quad \begin{aligned} M_n(Z/p^kZ) &\rightarrow M_n(Z/pZ) \\ A &\rightarrow \theta(A) \end{aligned}$$

If θ (A) has order t then A has order t or pt or $p^2t\ldots$ or $p^{k-1}t$.

If

$$m = \prod_{i=1}^{s} p_i^{r_i}$$

then

$$\left.\begin{aligned} EXP &= \ell cm \;\{v_1, v_2, \ldots, v_s\} \\ \text{where} \quad v_i &= p_i^{r_i - 1} \; (w_i) \\ \text{and} \quad w_i &= p_i \; \ell cm \; (p_i - 1, p_i^2 - 1, \ldots p_i^n - 1) \end{aligned}\right\} \qquad (10.16)$$

(assuming $p_i$ is greater than n for all i)

Again from (10.16), it is clearly seen that the exponent EXP depends upon the prime factor decomposition of m.

The message space has so far been restricted to only non-singular elements , $M_n(Z/mZ)$. Theoretically, the message space can also include some singular non-nilpotent elements. Consider for instance, $M_2(Z/3Z)$. There are 48 non-singular matrices using (10.4). But there are in addition, some non-nilpotent singular matrices which could also be used as messages. Consider such a matrix X

$$X = \begin{pmatrix} a & b \\ \lambda a & \lambda b \end{pmatrix} \qquad \text{where } \lambda, a, b \text{ are in } Z/3Z$$

From the Cayley-Hamilton theorem [51], this matrix satisfies a quadratic characteristic equation of the form

$$X^2 - (\text{trace } X)\, X + (\det X)\, I = 0$$

In the singular case, det X = 0. If trace X = 0, then $X^2 = 0$ and so X is nilpotent. If trace X $\neq$ 0, then $X^2 = \lambda X$, $X^4 = \lambda^2 X^2 = X^2$ as $\lambda^2 = 1$ for $0 \neq \lambda \in Z/3Z$; thus $X^3 = \lambda X^2 = \lambda^2 X = X$. Therefore $X^{1+2k} = X$ for $k \geqslant 0$. Therefore for such non-nilpotent matrices X in $Z/3Z$ , $X^{49} \equiv X$.

Note that when n > 2, there may be singular non-nilpotent matrices which do not satisfy an equation of the form $X^k \equiv X$. This happens for instance when the minimal equation is $X^4 - X^2 = 0$. Or even consider a non-nilpotent matrix X which contains a nilpotent block as shown below

$$X = \left(\begin{array}{c|c} \text{nilpotent} & 0 \\ \hline 0 & \text{non-singular} \end{array}\right) = \left(\begin{array}{c|c} N & O \\ \hline O & B \end{array}\right)$$

Then

$$X^k = \left(\begin{array}{c|c} N^k & O \\ \hline O & B^k \end{array}\right)$$

As N is nilpotent for some k, $N^k = 0$ and hence

$$X^k = \left(\begin{array}{c|c} O & O \\ \hline O & B^k \end{array}\right) \neq X$$

Thus in such cases, this would imply that it is not possible to recover the message. These cases need to be avoided if one decides to include such singular non-nilpotent matrices.

From cryptography point of view, the use of such singular non-nilpotent matrices as messages may seem impractical as the sender cannot easily recognize such matrices. In practice, one would like to determine easily whether a message is within the acceptable set or not. Even the restriction of messages to arbitrary non-singular matrices may pose problems as the sender has no control over the matrix elements but must accept what the plaintext dictates. That is, the sender cannot ensure that his messages will always form non-singular matrices (over any modulus). The sender is faced with the problem of determining whether a plaintext message matrix is non-singular or not. This involves finding the determinant of the nxn matrix and then checking whether the determinant is relatively prime to the modulus using the Euclid's algorithm. (Appendix 11). However it will be seen in Section 10.4.6 that for large m, the probability that an arbitrarily selected matrix is non-singular is very much close to 1.

One common approach to obtain an arbitrary non-singular matrix over the reals is to have the diagonal entries of the matrix message much bigger than the corresponding entries in the row and column but this 'diagonal dominance' does not always ensure that the matrix will be non-singular when working over finite rings. For instance, consider the matrix A below which is 'diagonally dominant'

$$A = \begin{pmatrix} p-1 & 1 \\ 1 & p-1 \end{pmatrix}$$

Det $A = p^2 - 2p = 0$ over $Z/pZ$. Hence it is seen that 'diagonal dominance' is not applicable over finite rings.

A simple method to obtain arbitrary non-singular matrices is described in Section 10.4.8 where system implementation is considered.

### 10.4.3 Orthogonal Matrices Over Z/mZ

Now let us consider the use of a special set of non-singular matrices, namely the set of orthogonal matrices, as the message space of the matrix based RSA system.

The set of orthogonal matrices over a finite field of p elements forms a group as shown below.

Let $M_i$ be a member of the set. Then $M_i^t M_i = I$.
As $(M_i M_j)^t = M_j^t M_i^t$, we have

$$(M_i M_j)^t M_i M_j = M_j^t M_i^t M_i M_j = I$$

Hence the closure property is obeyed. Further $(M_i^{-1})^t = (M_i^t)^{-1} = (M_i^{-1})^{-1} = M_i$ and $(M_i^{-1})^t M_i^{-1} = M_i M_i^{-1} = I$. Let $J = M_i^{-1}$ where $J^t J = I$ and J is orthogonal. Hence the set of orthogonal matrices over $Z/pZ$ forms a group.

The order of the group formed by the nxn orthogonal matrices has been worked out by J. MacWilliams [51].
For odd n, ie, $n = 2a + 1$ for some integer a, the order is given by

$$2p^a \prod_{i=0}^{a-1} (p^{2a} - p^{2i})$$

For even n, ie $n = 2a$, the order is given by

$$\frac{2}{p^a + 1} \prod_{i=0}^{a-1} (p^{2a} - p^{2i}) = 2(p^a - 1) \prod_{i=1}^{a-1} (p^{2a} - p^{2i})$$

if -1 is a square in GF (p)
and the order is equal to

$$2(p^a + (-1)^{a+1}) \prod_{i=1}^{a-1} (p^{2a} - p^{2i})$$

if -1 is a non-square in GF (p).

Using the Chinese Remainder Theorem, the order of the group of the orthogonal matrices over $Z/mZ$, where $m = p_1 p_2$, a square free integer, is equal to
(order of orthogonal matrices over $Z/p_1 Z$) x (order of orthogonal matrices over $Z/p_2 Z$). As the factorization of the modulus m is required to calculate the order, one can use the group of orthogonal matrices in the matrix based public key system.

Now the question how easy it is to construct an orthogonal matrix message needs to be looked at. Cayley's theorem [52] gives an easy way of constructing an orthogonal matrix using a skew-symmetric matrix over the reals. If S is a real skew-symmetric matrix then I + S is non-singular because the characteristic roots of S are purely imaginary and the matrix $A = \frac{I-S}{I+S}$ is an orthogonal matrix. But this is not applicable to finite fields as the determinant of I + S can be equal to 0 (mod $p_1 p_2$). Thus one can use the Cayley's technique to construct orthogonal matrix messages provided one ensures that the determinant of (I +S ) is relatively prime to $p_1 p_2$. Note that if I +S is non-singular over $Z/mZ$ ($m = p_1 p_2$), then I − S is also non-singular. This can be seen as follows: Let W be the inverse of I + S (mod m). That is, $(I + S)W \equiv I$ (mod m).
Then $((I + S)W)^t = W^t (I + S)^t = W^t (I - S) \equiv I$. Hence $W^t$ is the inverse of (I - S) (mod m).
Hence to construct an orthogonal matrix message over $Z/mZ$, an arbitrary skew-symmetric matrix S over $Z/mZ$ is chosen and I + S is

formed. If I + S is non-singular over Z/mZ, then (I - S)/(I +S) is constructed to form the message. But again the sender is faced with the problem of determining whether the matrix I + S is non-singular or not.

### 10.4.4 Upper Triangular Matrices Over Z/mZ

Alternatively, let us now consider the set of upper triangular matrices as a possible choice for the message space. If the diagonal entries of an upper triangular matrix are made unity, this ensures that the matrix is invertible over any modulus m as the determinant is equal to 1.

Let M represent such an n x n upper triangular message matrix. One can partition M into I + N where N is a nilpotent matrix and I is the identity matrix. If M is in $U_n$ (Z/pZ) then $(I + N)^p \equiv I$ as $N^p = 0$ assuming $p \geqslant n-1$. The order of the group formed by these upper triangular matrices $U_n$ over Z/pZ is $p^{n(n-1)/2}$. The order becomes $m^{n(n-1)/2}$ when considering matrices $U_n$ over Z/mZ. To determine the exponent EXP of the group formed by such upper triangular matrices over Z/mZ,

let

$$M = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{where a, b, c} \in Z/mZ$$

and let

$$M^k = \begin{pmatrix} 1 & a_k & b_k \\ 0 & 1 & c_k \\ 0 & 0 & 1 \end{pmatrix}.$$

Then

$$M^{k+1} = \begin{pmatrix} 1 & a_k & b_k \\ 0 & 1 & c_k \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & a+a_k & b+ca_k+b_k \\ 0 & 1 & c_k+c \\ 0 & 0 & 1 \end{pmatrix}$$

Therefore $a_{k+1} = a+a_k$ implies that $a_k = ka$

$c_{k+1} = c+c_k$ implies that $c_k = kc$

and

$$b_{k+1} = b+kca+b_k$$

$$\therefore \quad b_N-b_0 = \sum_{k=0}^{N-1} b_{k+1}-b_k$$

$$= \sum_{k=0}^{N-1} (b+kca)$$

$$b_N = Nb+ca \; \frac{(N-1)}{2} \; N$$

Therefore

$$M^k = \begin{pmatrix} 1 & ka & kb + \frac{k(k-1)}{2}\, ac \\ 0 & 1 & kc \\ 0 & 0 & 1 \end{pmatrix}$$

For $M^k \equiv I \pmod{m}$ to be obeyed, the following conditions must be satisfied:

(i) $ka \equiv 0 \pmod{m}$

(ii) $kc \equiv 0 \pmod{m}$

(iii) $kb+\frac{k(k-1)ac}{2} \equiv 0 \pmod{m}$

To satisfy (i) and (ii) $k \equiv 0 \pmod{m}$

To satisfy (iii) $\frac{k(k-1)}{2} \equiv 0 \pmod{m}$

ie , $k(k-1) \equiv 0 \pmod{2m}$

Therefore if m is even, the exponent, EXP, is 2m and if m is odd then EXP is m. Therefore it is seen that both the order and the exponent do not depend on the prime factors of m. Hence this cannot be used in a public key cryptosystem but again one can use it in a conventional cryptosystem with the secret key (e, d, m, n) where $ed \equiv 1 \bmod (2m \text{ or } m)$.

On the other hand, if the message space is altered to contain upper triangular matrices with diagonal entries prime to m, then such messages are invertible modulo m. This is not a serious problem as in practice m is a product of large primes and the diagonal elements can be chosen to be relatively small integers. Now the order of the group formed by such matrices is determined as follows.

Considering a nxn matrix, it is required that all the n diagonal entries must be coprime to m. The number of integers less than m and coprime to m is given by the Euler totient function $\phi(m)$. The remaining $\frac{1}{2} n(n-1)$ superdiagonal entries of the matrix may take any value modulo m. Therefore the order is equal to $m^{n(n-1)/2}\phi(m)^n$. The vital difference between this order and the one calculated above is that now the order of the group is dependent on the prime factors of m. Hence the modulus m needs to be factorized before the decryption exponent d can be calculated using $ed \equiv 1(\bmod(\text{order}))$. As for the set of non-singular matrices, one can determine the exponent of the group formed by these upper triangular matrices with diagonal entries prime to m. The exponent can be used instead of the order in finding e and d.

Initially, consider a square free modulus. Let

$$m = \prod_{j=1}^{s} p_j$$

Consider first a message M in $Z/p_1Z$ whose diagonal elements are relatively prime to $p_1$

$$M = \begin{pmatrix} a_{11} & & ? \\ & \ddots & \\ 0 & & a_{nn} \end{pmatrix} \qquad \text{where } (a_{ii},p_1) = 1, \ \forall i \quad 1 \leqslant i \leqslant n$$

Partitioning the matrix M into a diagonal matrix $D_1$ and an upper triangular nilpotent matrix $U_1$, that is, $M = D_1 + U_1$, then it is seen that $D_1.U_1$, $U_1.D_1$ and $U_1^2$ are also upper triangular nilpotent. Then inductively, if $M^r = D_1^{\ r} + U_r$, we have

$$M^{r+1} = (D_1 + U_1)(D_1^{\ r} + U_r)$$

$$= D_1^{\ r+1} + \underbrace{U_1 D_1^{\ r} + D_1 U_r + U_1 U_r}_{\text{nilpotent upper triangular}}$$

$$= D_1^{\ r+1} + U_{r+1}$$

Hence

$$M^{\phi(p_1)} \equiv I + U_\phi \pmod{p_1}$$

But

$$(I + U_\phi)^{p_1} \equiv I + U_\phi^{\ p_1} \pmod{p_1}$$

$$(I + U_\phi)^{p_1^{\ 2}} \equiv I + U_\phi^{\ p_1^{\ 2}} \pmod{p_1}$$

$$\vdots \qquad\qquad \vdots$$

Thus $M^{p_1^t \phi(p_1)} \equiv I \pmod{p_1}$ for some t.

If $p_1 \geqslant n-1$ then $t = 1$. Therefore the exponent of upper triangular matrices with coprime elements along the diagonal is $\phi(p_1)\cdot p_1$

If $m = \prod_{j=1}^{s} p_j$ then the exponent divides

$$\ell\text{cm}\,\{\phi(p_1)\cdot p_1,\ \phi(p_2)\cdot p_2, \ldots,\ \phi(p_s)\cdot p_s\}$$

If m is made to be a non-square free modulus given by

$$m = \prod_{j=1}^{s} p_j^{\ r_j}$$

then a bound for the exponent can be calculated as follows:

First consider an nxn upper triangular non-singular matrix M over $Z/p^rZ$ , $(p \geqslant n)$. Again let $M = D_1 + U_1$ where $D_1$ is a diagonal matrix and $U_1$ is an upper triangular nilpotent matrix over $Z/p^rZ$ Following the argument given above, it is seen that

$$M^{\phi(p^r)} \equiv I + U_\phi \pmod{p^r}$$

where $U_\phi$ is some upper triangular nilpotent matrix.

Hence

$$\begin{aligned}
(I + U_\phi)^p &\equiv I + p.U_{\phi_1} \pmod{p^r} \\
(I + p.U_{\phi_1})^p &\equiv I + p^2.U_{\phi_2} \pmod{p^r} \\
&\vdots \\
(I + p.U_{\phi_{r-1}})^p &\equiv (I + U_\phi)^{p^r} \equiv I + p^r.U_{\phi_r} \pmod{p^r} \\
&\equiv I \pmod{p^r}
\end{aligned}$$

Therefore

$$M^{\phi(p^r)\,p^r} \equiv I \pmod{p^r}$$

Hence the exponent of the group formed by upper triangular non-singular matrices over Z/mZ, where m is a non-square free modulus given above, divides

$$\ell cm\ \{\phi(p_1^{r_1})p_1^{r_1},\ \phi(p_2^{r_2})p_2^{r_2}\ ,\ \ldots,\ \phi(p_s^{r_s})\ p_s^{r_s}\ \}$$

### 10.4.5 Linear Fractional Group [53]

Finally, let us consider the group of linear fractional substitutions over Z/mZ, where m is equal to the product of r distinct primes to see if such a group is suitable for a trapdoor system. The linear fractional group LF(n,m) is very much similar to the linear homogenous group GL(n,m) considered earlier. Here only the case n = 2 is examined. First consider the group LF(2,p) where p is a prime. These substitutions are of the form

$$u : x \rightarrow \frac{ax + b}{cx + d} \quad \text{over } Z/p*Z = \{\ 0,\ 1,\ \ldots,\ p-1,\ \infty\ \}$$

The symbol $\infty$ is adjoined to represent any formal quotient y/0 where y is a non-zero element of the field. The linear fractional substitutions LF(2,p) can be obtained from the linear homogenous substitution GL(2,p) on variables $x_1$ and $x_2$ by setting $x \equiv x_1/x_2$. The number of transformations of the form u is finite and they form a group as the product of any two substitutions is also a substitution of the same form. The order of the group LF(2,m) can be determined as follows, where $m = \prod_{i=1}^{r} p_i$ .

For u to represent a substitution, it is required that the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be invertible, that is, the determinant ad-bc Prime to m. Now let us consider the two matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } \begin{pmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{pmatrix} \text{ where } \lambda \varepsilon \ Z/mZ$$

The corresponding two substitutions represented by

$$\frac{ax + b}{cx + d} \quad \text{and} \quad \frac{\lambda\ (ax + b)}{\lambda\ (cx + d)} \quad \text{over } Z/mZ$$

are identical if $(\lambda, m) = 1$

Consider the matrix $M_p$ given by

$$M_p = \begin{pmatrix} a_p & b_p \\ c_p & d_p \end{pmatrix} \text{ where } a_p d_p - b_p c_p \not\equiv 0 \pmod p$$

If the condition that det $M_p \not\equiv 0 \pmod p$ can be satisfied by choosing $a_p$, $b_p$, $c_p$ and $d_p$ appropriately, then the Chinese Remainder Theorem (Appendix 10) allows us to find a, b, c and d over Z/mZ such that:

$$a \equiv a_{p_1} \pmod{p_1},\ a \equiv a_{p_2} \pmod{p_2}, \ldots\ a \equiv a_{p_r} \pmod{p_r}$$

and similarly for b, c and d.

Thus the order of LF(2,m) can be found by finding the orders of LF(2,$p_i$) for $1 \leqslant i \leqslant r$. The number of possible values of $\lambda$ which result in identical substitutions is given by the Euler totient function $\phi(p_i)$ for each $p_i$. Thus the order is given by

$$\# \text{LF}(2,m) = \frac{\prod_{i=1}^{r} \# \text{LF}'(2,p_i)}{\prod_{i=1}^{r} \phi(p_i)}$$

But

$$\begin{aligned} \# \text{LF}'(2,p) &= \# \text{GL}(2,p) \\ &= (p^2-1)\ (p^2-p) \text{ from Section 10.4.2 with n=2} \end{aligned}$$

$$\therefore \qquad \# \, LF(2,m) = \frac{\prod_{i=1}^{r} (p^2_i - 1) \cdot (p_i^2 - p_i)}{\prod_{i=1}^{r} (p_i - 1)}$$

$$\# \, LF(2,m) = \prod_{i=1}^{r} (p_i + 1) \, p_i \, (p_i - 1)$$

Thus as the order depends on the factorization of the modulus m, this is suitable for a trapdoor system. The messages are of the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with a, b, c, d $\epsilon$ Z/mZ and ad-bc $\not\equiv 0 \pmod{m}$. The encrypting and decrypting exponents e and d can be determined using $ed \equiv 1$ (mod # LF(2,m)).

### 10.4.6 Proportion of Non-Singular Matrices Over Z/mZ

The extended RSA system described above requires the message space to consist of either arbitrary non-singular matrices or orthogonal matrices or invertible upper triangular matrices including the diagonal elements over Z/mZ. In the first two cases it is required to choose arbitrary message matrices which are non-singular. In the case of orthogonal matrices, recall that one method of forming a message requires the matrix (I+S) to be non-singular. Hence some bound on the fraction of nxn matrices which are invertible modulo m will be useful. If the elements of the nxn matrix are chosen uniformly from the integers modulo m, then these bounds can be taken as the bounds on the probability that the matrix is non-singular.

An nxn matrix M has an inverse modulo m if and only if the rows (columns) of the matrix form a linearly independent set of vectors modulo m. Using the argument given in Section 10.4.2, if m = p, a prime, then the number of invertible matrices (mod p) is given by

$$(p^n - 1)\,(p^n - p) \ldots (p^n - p^{n-1}) = \prod_{i=1}^{n-1} (p^n - p^i)$$

The total number of nxn matrices with elements over Z/pZ is equal to $p^{n^2}$. Thus the proportion of invertible matrices (mod p) is equal to

$$\frac{\prod_{i=1}^{n-1} (p^n - p^i)}{p^{n^2}}$$

$$= \quad (1-1/_{p}n)\ (1-p/_{p}n) \dots (1-p^{n-1}/_{p}n)$$

$$= \quad \prod_{i=1}^{n} (1-p^{-i})$$

Thus as the size of the prime increases, the probability that a chosen matrix is non-singular approaches 1. If the proportion of non-singular matrices over Z/mZ is considered, where $m = \prod_{j=1}^{r} p_j$, then it is equal to

$$\frac{\prod_{j=1}^{r} \prod_{i=1}^{n} (p_j^{\,n} - p_j^{\,i})}{\left(\prod_{j=1}^{r} p_j\right)^{n^2}}$$

$$= \prod_{j=1}^{r} \prod_{i=1}^{n} (1-p_j^{\,-i})$$

Similar expressions can be found for the proportion of non-singular matrices over Z/mZ, where m is a non-square free integer, using the expression for the order $N_m$ derived in Section 10.4.2. The proportion is then given by $N_m/m^{n^2}$. From the cryptography point of view where m is a large integer composed of a few large prime factors, this proportion is very much close to 1. Even for small primes $p_1=13$ and $p_2=23$, the proportion is approximately 0.87, for n=2.

### 10.4.7 System Design and Operation

The designer randomly chooses large primes $p_1$ to $p_r$ for some $r \geqslant 2$ following the guidelines suggested in Section 10.2 and these are kept secret. But the primes need not be necessarily distinct. Then he specifies what his message space is going to be, whether it consists of arbitrary non-singular matrices or orthogonal matrices or upper triangular matrices with invertible elements along the diagonal. This information along with the dimension n of the matrix message is made public. Then he determines the coding exponents e and d using the equation $ed \equiv 1 \bmod$ (Order or Exponent). The order and the exponent expressions for the different message spaces are given in the earlier sections. The public encryption key is therefore given by (e, m, n) and the secret decryption key is (d, m, n). The system is also suitable for authentication purposes like the RSA system.

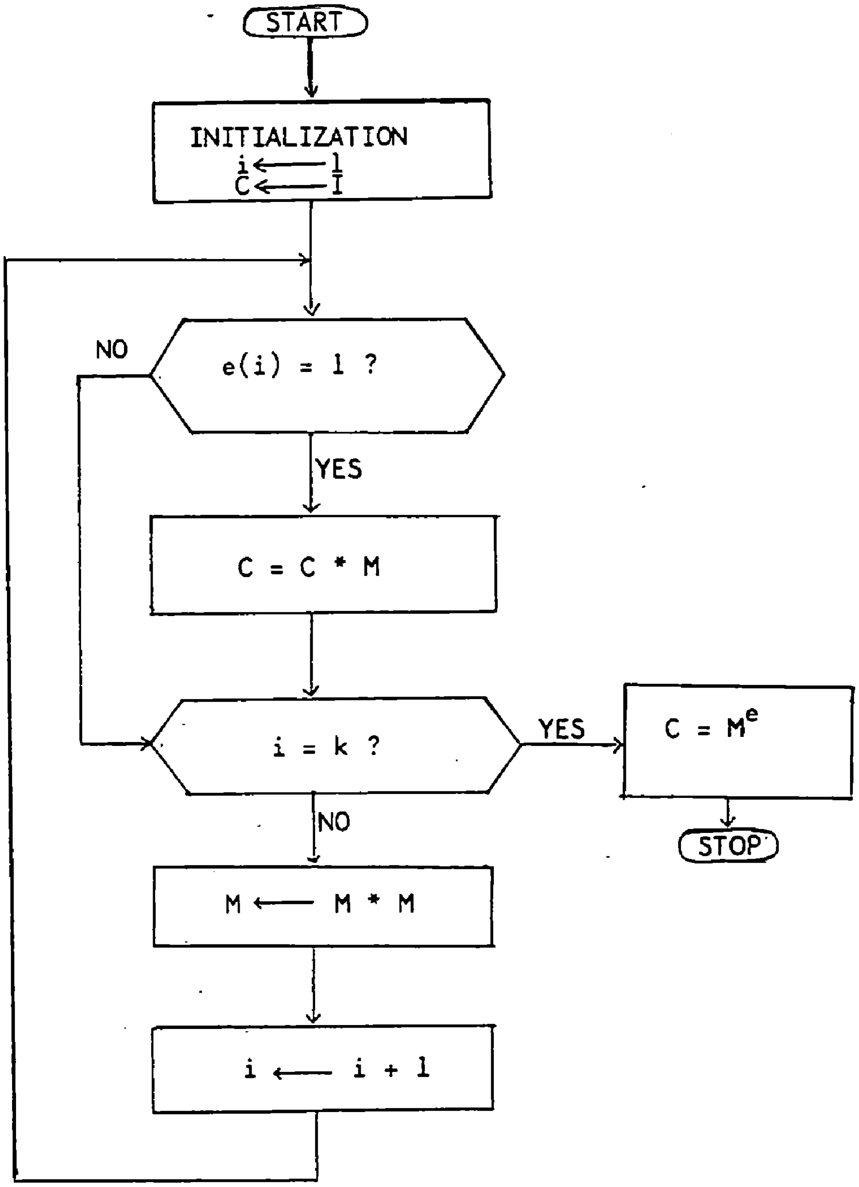
A general procedure to construct a matrix message is as follows. The plaintext message is divided into blocks of integers less than the modulus m and a sequence of nxn message matrices $M_i$ is constructed by arranging the integers in order as they occur, left to right and top to bottom. The encryption procedure consists of raising each of these plaintext matrices to the power e modulo m. The ciphertext matrices, $C_i$, produced are then transmitted to the receiver by sending the ciphertext matrix elements in order as they occur in the matrix, left to right, top to bottom, with a space separating the elements. The receiver recovers the original message by first reconstructing the sequence of nxn ciphertext matrices and then raising each matrix $C_i$ in the sequence to the power d modulo m. Thus the main operation in both the encryption and decryption procedures consists of raising an nxn matrix to some power modulo m. This can be performed using the Square and Multiply Technique [45] which is now briefly described.

### 10.4.7.1 Square and Multiply Technique

Consider the binary representation of the encryption exponent e. Each '1' in the representation is now replaced by a pair of letters SX and each '0' is replaced by the letter S. The symbol SX which appears at the left is now crossed off. The result is a rule for computing $M^e$ if S is interpreted as the squaring operation and X is interpreted as the multiplying operation by M. Each of these operations is performed modulo m. The flowchart of the algorithm is given in Figure 10.2. If the encryption exponent is chosen to have only a few '1' digits in its binary representation, this will make this algorithm run faster than if the exponent has a random binary representation. A similar procedure can be carried out for decryption using the exponent d.

Further, the intermediate computations need not be reduced over Z/mZ but rather over Z/kmZ for any positive integer k. The final answer is computed by reducing the answer over Z/kmZ into Z/mZ. This gives the correct answer because $(a_1 . a_2 \pmod{km}) \equiv a_1 . a_2 \pmod{m}$.

Let $e = (e_k, \ldots, e_2, e_1)$ $e_i \in GF(2)$



All operations are modulo m

Fig. 10.2 - Algorithm for computing $M^e$ (mod m)

### 10.4.8 System Implementation

This extended RSA system using matrix messages has been simulated on the Prime Computer. The encryption and the decryption of message matrices have been performed using the Square and Multiply Technique given above.

#### 10.4.8.1 Non-Singular Matrix Messages

In the case of non-singular matrix message space, the message matrix M is constructed according to the general procedure given in Section 10.4.7 and its determinant is calculated using the program DETMOD.F77 given in Appendix 12. Then Euclid's algorithm is used to test whether the determinant is relatively prime to the modulus m. If so, the message is raised to the power e over Z/mZ using the program MATEXP.FTN given in Appendix 13. The same program is used to decrypt the ciphertext matrices with exponent d. Recall from Section 10.4.6 that the probability of an arbitrarily chosen matrix message M over Z/mZ, where m = pq and p and q are large primes, is non-singular, is very much close to 1.

One can also construct an arbitrary non-singular matrix M over Z/mZ by multiplying together an upper triangular matrix U with unit diagonal and a lower triangular matrix L with unit diagonal over Z/mZ. The elements other than the diagonal ones in U and L can be arbitrarily chosen modulo m. As both U and L are non-singular over Z/mZ, M = LU is non-singular over Z/mZ. Further the non-commutativity property of matrices (UL $\neq$ LU in general) ensures that the opponent still needs to factorize m to be able to calculate the decrypting exponent d, in contrast to the case of upper triangular matrices with unit diagonal considered in Section 10.4.4. That is, $M^e = (UL)^e \neq U^eL^e$. Thus although $U^{ed_1} \equiv U$ (mod m), $L^{ed_1} \equiv L$ (mod m) where $ed_1 \equiv 1$ (mod m or 2m) (see Section 10.4.4), $M^{ed_1} \not\equiv M$ (mod m). However $M^{ed} \equiv M$ (mod m) where $ed \equiv 1$ mod (#GL(n,m) or EXP GL(n,m)) (see Section 10.4.2). The receiver can obtain the matrices L and U uniquely given the matrix M.

#### 10.4.8.2 Upper Triangular Matrix Messages

In the case of upper triangular matrix messages with invertible diagonal elements, the message matrix is again constructed according to the general procedure given in Section 10.4.7. The lower triangular section of the message (excluding the diagonal) is set to zero. The diagonal elements are almost arbitrarily chosen to be relatively prime to the modulus m. This can be done provided they are relatively small, as the prime factors of m are normally very large. Again these messages are encrypted/decrypted using the program MATEXP.FTN given in Appendix 13. Note that in this case, only the upper triangular entries of the ciphertext matrices (ie, $\frac{n(n+1)}{2}$ elements of each ciphertext matrix) need to be transmitted to the receiver.

An example in each of these two message spaces showing the various parameters involved is given below.

10.4.8.3 Example 1 : 2 x 2 Non-Singular Matrix Messages

Let the modulus be $m = p_1^2 p_2 = 3^2 5 = 45$

Exponent of the group formed by 2 x 2 non-singular matrices over Z/45Z divides $\ell cm\{ v_1, v_2\}$

where

$$v_1 = 3^{2-1} \ \ell cm \{3, 3-1, 9-1\} = 3 \ \ell cm \{3, 2, 8\}$$

$$v_2 = \ell cm \{5, 5-1, 25-1\} \quad = \quad \ell cm \{5, 4, 24\}$$

Thus EXP is a divisor of 1080. Hence the coding exponents e and d are given by

$$ed \equiv 1 \pmod{1080}$$

One pair (e, d) which satisfies the above congruence is e = 23, d = 47.

Let us assume that the plaintext message to be encrypted is 81334. Dividing the message as (8 13 3 4), the plaintext message matrix can be constructed as

$$P = \begin{pmatrix} 8 & 13 \\ 3 & 4 \end{pmatrix}$$

The determinant of P is equal to $-7 \equiv 38 \pmod{45}$ and (det P, 45) = 1. Hence P′ is non-singular (mod m).

The ciphertext matrix is then given by

$$C \equiv P^e \equiv \begin{pmatrix} 8 & 13 \\ 3 & 4 \end{pmatrix}^{23} \pmod{45}$$

$$\equiv \begin{pmatrix} 38 & 34 \\ 39 & 31 \end{pmatrix} \pmod{45}$$

This ciphertext matrix is transmitted to the receiver as(38 34 39 31). The receiver reconstructs the matrix C and computes $C^d$ (mod m) to obtain P. That is,

$$C^d \pmod{45} \equiv \begin{pmatrix} 38 & 34 \\ 39 & 31 \end{pmatrix}^{47} \pmod{45}$$

$$\equiv \begin{pmatrix} 8 & 13 \\ 3 & 4 \end{pmatrix} \pmod{45}$$

$$\equiv \quad P$$

10.4.8.4 Example 2 - 3 x 3 Upper Triangular Matrix Messages

Let the modulus m = p.q = 41.29 = 1189

Exponent of the group formed by 3 x 3 upper triangular non-singular matrices over Z/1189Z divides

$$\ell\text{cm} \{40.41,\ 28.29\} = 332920$$

Choosing the encrypting exponent, e = 1317, the decrypting exponent d can be calculated using Euclid's Algorithm (Appendix 11) and is equal to 117293. That is,

$$1317.\ 117293 \equiv 1 \pmod{332920}$$

Let us assume that the plaintext message to be encrypted is 232677205141. In this example, the message is divided into 2-digit blocks of integers less than m starting from right to left as

(23 26 77 20 51 41)

The upper triangular plaintext message matrices then become

$$P_1 = \begin{pmatrix} 90 & 23 & 26 \\ 0 & 50 & 77 \\ 0 & 0 & 48 \end{pmatrix},\ P_2 = \begin{pmatrix} 31 & 20 & 51 \\ 0 & 215 & 41 \\ 0 & 0 & 289 \end{pmatrix}$$

where the diagonal elements are arbitrarily chosen to be relatively prime to m (=1189).

The ciphertext message matrices are then given by

$$C_1 \equiv \begin{pmatrix} 90 & 23 & 26 \\ 0 & 50 & 77 \\ 0 & 0 & 48 \end{pmatrix}^{1317} \equiv \begin{pmatrix} 1105 & 458 & 1070 \\ 0 & 50 & 251 \\ 0 & 0 & 831 \end{pmatrix} \pmod{1189}$$

$$C_2 \equiv \begin{pmatrix} 31 & 20 & 51 \\ 0 & 215 & 41 \\ 0 & 0 & 289 \end{pmatrix}^{1317} \equiv \begin{pmatrix} 843 & 774 & 660 \\ 0 & 592 & 41 \\ 0 & 0 & 405 \end{pmatrix} \pmod{1189}$$

These ciphertext matrices are transmitted to the receiver as (1105 458 1070 50 251 831 843 774 660 592 41 405).

The receiver reconstructs the matrices $C_1$ and $C_2$ and computes $C_1^d \pmod{m}$ and $C_2^d \pmod{m}$ to obtain $P_1$ and $P_2$ and hence the plaintext message. That is,

$$C_1^d \equiv \begin{pmatrix} 1105 & 458 & 1070 \\ 0 & 50 & 251 \\ 0 & 0 & 831 \end{pmatrix}^{117293} \equiv \begin{pmatrix} 90 & 23 & 26 \\ 0 & 50 & 77 \\ 0 & 0 & 48 \end{pmatrix} \pmod{1189}$$

$$C_2^d \equiv \begin{pmatrix} 843 & 774 & 660 \\ 0 & 592 & 41 \\ 0 & 0 & 405 \end{pmatrix}^{117293} \equiv \begin{pmatrix} 31 & 20 & 51 \\ 0 & 215 & 41 \\ 0 & 0 & 289 \end{pmatrix} \pmod{1189}$$

### 10.4.9 Discussion

Thus one can see that the RSA system can be generalized to matrix rings provided the message space is restricted to avoid nilpotent elements. The group of non-singular matrices over Z/mZ, the group of orthogonal matrices over Z/mZ and the group of upper triangular matrices over Z/mZ with diagonal elements coprime to m have been investigated. From a practical implementation point of view, the upper triangular (non-singular) matrix message space seems to be the better candidate as the messages can be constructed in an almost arbitrary manner. In the case of non-singular and orthogonal matrices, an additional procedure to find the determinant of the message matrix is required. From Section 10.4.6, it is seen that for a large enough modulus, the probability that an arbitrarily chosen message matrix is non-singular is very much close to 1. Further if the non-singular matrix is constructed as a product of upper and lower triangular matrices, then it seems that arbitrary non-singular message matrices can be easily constructed.

For a cryptanalyst to break the system by computing the secret decoding exponent d, he needs to find the order (or the exponent) of the group. In all the three cases, it is seen that the factorization of the modulus is required to compute the order (or the exponent) of the group. Provided m is chosen to be a large integer (say 200 decimal digits long, see Figure 10.1), this provides a secure system. Other attacks such as the computation of the order without finding the prime factors of m and determining the secret coding exponent d without factoring m or calculating the order can be shown to be as difficult as factoring m using similar arguments to those given in Section 10.3. Thus this extended matrix RSA system provides a similar level of security as the RSA system over integers.

Further two points are worth mentioning regarding this extended system. Firstly it is seen that a non-square free modulus can be used with this system which is not possible with the RSA system over integers. That is, powers of primes can be used to form the modulus m. Secondly, the use of a matrix as a message allows large amounts of data to be processed within one encryption/decryption cycle. Whether this is an advantage depends upon the ease with which matrix manipulation can be carried out in real time.

Also the use of a matrix as a message may provide some extra features to the system. Consider, for instance, a non-singular message matrix divided into blocks as shown below

$$M = \left(\begin{array}{c|c} M_1 & X_1 \\ \hline X_2 & M_2 \end{array}\right)$$

where $M_1$ and $M_2$ represent genuine message blocks whereas $X_1$ and $X_2$ are redundant random blocks. Encryption of such a message M would truly 'internally' mix the message elements and the random elements in the matrix. In the case of RSA system over integers, a similar effect can be achieved by sending a sequence of ciphers which contains some random elements. For instance, the sequence may be $c_1, x_1, c_2, x_2, \ldots$

where $c_i$ represents genuine cipher and $x_i$ represents random cipher. In the former case,the internal structure of the matrix and in the latter case, the pattern of the sequence need to be prearranged between the sender and the receiver. In the case of the matrix system, the knowledge of the internal structure of the matrix does not give any help to the opponent as the message is enciphered 'internally' and hence this information can be made public. In the latter case, however,the knowledge of the pattern sequence helps the opponent to discard the random ciphers and hence this information cannot be made public. The use of chaining techniques in such matrix systems is considered in Chapter 15.

## 10.5 Extension of RSA System to Polynomial Rings

Another ring of special interest is the polynomial ring R [x] which consists of polynomials whose coefficients are elements of an arbitrary ring R. A possible extension of the factorization trapdoor system in the ring of polynomials R [x] is considered. In particular, to begin with,the ring F [x] where F is a finite field is looked at.

### 10.5.1 Concept of Galois Field

Let P(x) be a given polynomial in x of degree n with integral coefficients not all divisible by a given prime p. Let F(x) be any polynomial in x with integral coefficients. On dividing F(x) by P(x), a quotient Q(x) and a remainder of degree n-1 at most is obtained, where the remainder may be written in the form f(x) + p q(x),and

$$f(x) \equiv a_0 + a_1x + a_2x^2 + \dots + a_{n-1}x^{n-1}$$

Each of the coefficients $a_i$ belong to the set $\{0, 1, 2, \dots, p-1\}$ and q(x) is a polynomial with integral coefficients. That is,

$$F(x) = f(x) + p\,q(x) + P(x).Q(x)$$

The totality of functions F(x) which can be obtained by holding f(x) fixed and varying q(x) and Q(x) in all possible ways, subject to maintaining the named properties of q(x) and Q(x), constitute a class of residues which is completely determined by f(x), the given prime p and P(x). Two polynomials in x with integral coefficients are congruent moduli p and P(x) if and only if they belong to the same class of residues moduli p and P(x). The number of distinct residue classes moduli p and P(x) is equal to the number of functions of the form f(x). Since each of the n independent coefficients $a_i$ can take any one of the p values, there are $p^n$ possible residue classes. These residues constitute a finite field called the Galois field [54]. If P(x) is not irreducible modulo p or if p is not a prime, then the residues do not form a field since at least two non-zero residues can be found whose product is 0 moduli p and P(x).

### 10.5.2 A Polynomial Based RSA System

Using this concept, one can design a conventional cryptographic system as follows. Since the number of residue classes is finite, it must be possible to find two numbers r and s such that

$$(f(x))^r \equiv (f(x))^s \text{ modd } (p, P(x))$$

The non-zero elements f(x) of the extension field GF($p^n$) form a multiplicative group. The order of this group is equal to $p^n$-1, that is,

$$(f(x))^{1 + k(p^n-1)} \equiv f(x) \quad \text{modd } (p, P(x))$$

where k is an arbitrary non-negative integer.

The system designer chooses randomly a large prime number p (see Section 10.2) and a high degree irreducible polynomial P(x) over GF(p) of degree n. (See Section 10.5.2.1) The message space of this cryptographic system consists of polynomials $\{m(x)\}$ belonging to F [x ] where F is the finite field GF(p). The messages are represented as blocks of sequences of p-ary digits each of size n associated with a polynomial m(x) over GF(p) of degree less than n. The encryption and decryption coding exponents are determined using

$$ed \equiv 1 \quad \text{mod (order)}$$

$$\text{ie ,} \quad ed \equiv 1 \quad \text{mod } (p^n-1) \qquad (10.17)$$

The encryption procedure then consists of raising the message polynomial to the power e to form the cipher polynomial, c(x)

$$c(x) \equiv (m(x))^{e} \text{ modd } (p,P(x))$$

The decryption procedure consists of raising the cipher polynomial to the power d to obtain the message polynomial using

$$m(x) \equiv (c(x))^{d} \text{ modd } (p,P(x))$$

The encryption key is (e, p, P(x)) and the decryption key is (d, p, P(x)). Both these keys need to be kept secret since the knowledge of the encryption key would allow the opponent to calculate d using (10.17) and vice versa. Hence this does not give rise to a public key system as such.

Note further that the probability that $c(x) \equiv m(x)$ modd (p, P(x)) is equal to gcd $(e-1, p^{n}-1)/p^{n}$ and this proportion can be made small by the system designer through appropriate choice of e, p and n.

This actually is the generalization of the Pohlig-Hellman secret key scheme over GF(p) to abitrary finite fields $GF(p^{n})$ where p is a prime and n is a positive integer. Note that the field $GF(p^{n})$ is isomorphic to $Z/pZ\ [x]\ /P(x)$ where P(x) is an irreducible polynomial of degree n over Z/pZ.

This can be further extended to the case where the system designer chooses high degree irreducible polynomials $P_i(x)$ and large distinct primes $p_i$ for $1 \leqslant i \leqslant s$. Let the degree of irreducible polynomials be $n_i$ for $1 \leqslant i \leqslant s$. The coding exponents $e_i$ and $d_i$ can then be determined using

$$e_i d_i \equiv 1 \text{ mod}(p^{n}i-1) \text{ for } 1 \leqslant i \leqslant s$$

The message is divided into groups of s blocks where the ith block is of size $n_i$ and consists of a sequence of $p_i$-ary digits. The ith plain block is associated with a polynomial $m_i(x)$ of degree less than $n_i$. The ith cipher block $c_i(x)$ is produced using

$$c_i(x) \equiv (m_i(x))^{e} \text{ modd } (p_i, P_i(x)) \text{ for } 1 \leqslant i \leqslant s$$

Now let $n_{max}$ = maximum of $(n_1,\ldots n_s)$. Each $c_i(x)$ is now extended to form a polynomial of degree $n_{max}-1$ by randomly generating the extra coefficients required over $Z/p_iZ$ for $1\leqslant i\leqslant s$. Thus there are s polynomials each of degree $n_{max}-1$ with coefficients over $Z/p_iZ$ for $1\leqslant i\leqslant s$. These polynomials are combined to form a single polynomial c(x) over Z/mZ where $m = \prod_{i=1}^{s} p_i$ using the Chinese Remainder Theorem (Appendix 10). c(x) is the cipher polynomial which is transmitted to the receiver. The receiver reduces the polynomial c(x) modd $(p_i, x^{n_i})$ to obtain $c_i(x)$ for $1\leqslant i\leqslant s$. Then he uses the coding exponents $d_i$, to obtain $m_i(x)$ using $m_i(x) \equiv (c_i(x))^{d_i} \bmod(p_i, P_i(x))$ for $1\leqslant i\leqslant s$. Hence the encryption key consists of $(e_i, p_i, P_i(x))$ and the decryption key consists of $(d_i, p_i, P_i(x))$ for $1\leqslant i\leqslant s$. Note that this is still a conventional (symmetric) cryptosystem.

The above method can be modified to give a public key system. This has been first proposed by Kravitz [55]. This is based on the difficulty of obtaining the degrees of the irreducible factors of a polynomial of large degree over a finite field, F = Z/pZ, p prime.

Let f(x) be a polynomial of degree s which is equal to the product of r distinct irreducible polynomials over Z/pZ. That is,

$$f(x) = \Pi_1(x).\ \Pi_2(x) \ldots \Pi_r(x) \quad (\text{mod } p)$$

where $\Pi_i(x)$, $1\leqslant i\leqslant s$, are distinct irreducible polynomials of degree $s_i$. Then $s = s_1 + s_2 + \ldots + s_r = \sum_{i=1}^{r} s_i$

Let the message space consists of $\{m(x)\}$ where m(x) is a polynomial over Z/pZ of degree less than s. Then the set of polynomials of degree less than s and relatively prime to f(x) form a multiplicative group modulo f(x). [Two polynomials are said to be relatively prime if their greatest common divisor has degree 0 (ie, a constant polynomial). That is, two polynomials $h_1$ and $h_2$ are relatively prime if $ah_1 + bh_2 = 1$ for some polynomials a, b in Z/pZ [x]].

Hence

$$(m(x))^{order} \equiv 1 \text{ modd}(p, f(x))$$

The order of the group is determined as follows.

First let us consider a slightly different case where

$f(x) = \Pi_1^{g_1}(x)$, a non square free polynomial (although it is said earlier that $f(x)$ consists of distinct irreducible factors mod p). Here the degree of the message polynomial $m(x)$ is less than the degree of $\Pi_1^{g_1}(x)$, that is, less than $g_1 s_1$. In addition, if the greatest common divisor of the degrees of $m(x)$ and $\Pi_1(x)$ is greater than 0, then

$$m(x) = \Pi_1(x).h(x)$$

where the degree of $h(x)$ is less than $g_1 s_1 - s_1 = s_1(g_1-1)$.
Total number of polynomials$\{m(x)\}$of degree less than $g_1 s_1$ is equal to $|F|^{g_1 s_1} = p^{g_1 s_1}$. Within this set, the number of polynomials $m(x)$ which are not relatively prime to $\Pi_1(x)$ (that is, which are multiples of $\Pi_1(x)$ and for which $h(x)$ exists) is equal to the number of such polynomials $h(x)$ possible. There are $|F|^{s_1(g_1-1)}$ such polynomials $h(x)$ and hence there are $|F|^{s_1(g_1-1)}$ polynomials $m(x)$ which are multiples of $\Pi_1(x)$. Therefore, the order of the group is given by

$$|F|^{s_1 g_1} - |F|^{s_1(g_1-1)}$$

Thus if

$$f(x) = \prod_{i=1}^{r} \Pi_i^{g_i}(x) \pmod p$$

then, the order of the group becomes

$$\prod_{i=1}^{r}\left(|F|^{g_i s_i} - |F|^{s_i(g_i-1)}\right) \tag{10.18}$$

In this case $|F| = p$. Substituting this into the above expression gives

$$\prod_{i=1}^{r}\left(p^{g_i s_i} - p^{s_i(g_i-1)}\right) \tag{10.19}$$

In particular, if $g_i = 1$ for all $1 \leq i \leq r$, (that is, $f(x)$ consists of distinct irreducible factors) then the expression (10.19) becomes

$$\prod_{i=1}^{r} (p^{s_i} - 1)$$

As will be seen later in Section 10.5.3,the polynomial $f(x)$ needs to be square free for cryptographic application.

#### 10.5.2.1 System Design and Operation

The system designer randomly chooses a large prime p and r distinct high-degree irreducible polynomials over $Z/pZ$, $\Pi_1(x)$ to $\Pi_r(x)$, with degrees $s_1$ to $s_r$.

The chosen number can be tested to check if it is a prime using the primality tests given in Section 10.2. A chosen polynomial can be tested to see if it is irreducible over $Z/pZ$ as follows: From [56], a polynomial $P(x)$ of degree n is irreducible in $Z/pZ$ if and only if $P(x) \mid (x^{p^n}-x)$ and gcd $(P(x), x^{p^{n_i}}-x) = 1$ for $1 \leq i \leq k$ where $n = u_1 \ldots u_k$ and $u_i$'s are primes and $n_i = n/u_i$. Thus to test whether a chosen polynomial $P(x)$ with degree n is irreducible over $Z/pZ$, the designer computes gcd $(P(x), x^{p^{n_i}}-x)$ and tests whether it is equal to 1 for all i, $1 \leq i \leq k$. Berlekamp [56] shows that a randomly chosen polynomial of degree $s_i$ is irreducible with probability $1/s_i$. Thus an average of $s_i$ tries is required to find an $s_i$th degree irreducible polynomial.

Having selected the irreducible polynomials $\Pi_1(x)$ to $\Pi_r(x)$, the designer forms their product $f(x)$ which is of degree s where $s = \sum_{i=1}^{r} s_i$. The polynomial $f(x)$ and the prime p are then made public information. The r distinct irreducible factors of $f(x)$ are kept secret. The coding exponents e and d are calculated using

$$ed \equiv 1 \mod (\text{order}) \qquad (10.20)$$

$$\text{where order} = \prod_{i=1}^{r} (p^{s_i}-1) \qquad (10.21)$$

The public encryption key is $(e, p, f(x))$ and the secret decryption key is $(d, p, f(x))$. The message is divided into blocks of size s consisting of sequences of p-ary digits. Each such plaintext block is associated with a polynomial $m(x)$ of degree less than s. The encryption procedure then consists of raising the polynomial $m(x)$ to the power e using

$$c(x) \equiv (m(x))^e \text{ modd } (p, f(x))$$

and the decryption procedure is given by

$$m(x) \equiv (c(x))^d \text{ modd } (p, f(x))$$

The above scheme can be used as a public key system as the order is found to depend upon the degrees of the irreducible polynomial factors of the composite modulus polynomial thus providing the trapdoor property.

10.5.2.2 Security of The System

The above scheme is not as secure as the RSA system over integers or the matrix based RSA system considered earlier due to the following reasons.

Following Section 10.3, the main cryptanalytical attacks consist of finding the order (expression (10.21)). Once the order is found, the cryptanalyst can determine the secret decoding exponent d using (10.20). The order expression contains the degrees of the irreducible factors of f(x) and the prime p. As the prime p is made public, the security of the system relies on the difficulty of factorizing the polynomial f(x) into its irreducible factors. Furthermore, the same decoding exponent d works for all sets of $\Pi_i(x)$ for $i = 1$ to r with same degrees $s_i$, $1 \leq i \leq r$.

It is generally true that factorization of a polynomial over a finite field is not a hard problem in sharp contrast with the factorization problem of a large integer. Berlekamp [57] proposed an efficient algorithm for factoring polynomials in Z/pZ where p is prime. For large prime p, Knuth [45] suggests some modifications to the Berlekamp's procedure. Here the Berlekamp's procedure is briefly described. The basic strategy of Berlekamp's method of factoring a polynomial in Z/pZ is to translate the problem into that of solving a system of linear equations with coefficients in Z/pZ and finding greatest common divisors. Both of these steps can be done using known methods in a finite number of steps. The idea behind the algorithm is now briefly outlined.

Suppose that f(x) has degree s and suppose that a polynomial h(x) can be found in Z/pZ of degree greater than or equal to 1 but less than s such that

$$f(x) \mid (h(x))^p - h(x)$$

If degree $(h(x)) = k \geq 1$, then $(h(x))^p - h(x) \neq 0$ for the coefficient of $x^{pk}$ is non-zero. By Fermat's theorem, the polynomial $u^p - u$ has p roots in Z/pZ namely u = 0, 1, 2,..., p-1. Thus $u^p - u$ factors (mod p) into

$$u^P - u = u(u-1)\ (u-2)\ \ldots\ (u-(p-1))$$

Setting $u = h(x)$ gives

$$(h(x))^P - h(x) = h(x)\ (h(x)-1)\ (h(x)-2)\ \ldots\ (h(x)-(p-1)) \text{ in } Z/pZ \qquad (10.22)$$

Since $f(x)$ divides $(h(x))^P - h(x)$, $f(x)$ is the greatest common divisor of $f(x)$ and $(h(x))^P - h(x)$. Since $h(x)-v_1$ and $h(x)-v_2$ are relatively prime if $v_1 \neq v_2$, from (10.22) it is seen that

$$\begin{aligned} f(x) &= \gcd\ (f(x),\ (h(x))^P - h(x)) \\ &= \prod_{j=o}^{p-1} \gcd\ (f(x),\ h(x) - j) \end{aligned} \qquad (10.23)$$

Each factor on the right hand side has degree at most that of $h(x)$ which is in turn less than $s$, the degree of $f(x)$. Thus there must be at least two non-trivial factors of $f(x)$ on the right hand side of (10.23),that is, at least two factors of $f$ which have degree $\geqslant 1$ and hence (10.23) is a non-trivial factorization of $f(x)$.

To factor $f(x)$, thus one needs to find a polynomial $h(x)$ such that $f(x)$ divides $(h(x))^P - h(x)$. This is done by solving a set of linear equations for the coefficients of $h(x)$ as follows.

Suppose $h(x) = b_0 + b_1 x + \ldots + b_{s-1} x^{s-1}$ where $b_0, \ldots, b_{s-1}$ are the coefficients to be determined. To see whether $f(x)$ divides $(h(x))^P - h(x)$, consider first $(h(x))^P$.

Using the equality

$$(a + b)^P = a^P + b^P \text{ over } Z/pZ$$

we have

$$(h(x))^P = b_o{}^P + b_1{}^P x^P + \ldots + b^p_{s-1}\ x^{(s-1)p}$$

Applying Fermat's theorem to each of $b_0$, $b_1, \ldots, b_{s-1}$, $b_i{}^P = b_i$ in $Z/pZ$ for all $i$, and hence

$$(h(x))^P = b_0 + b_1\ x^P + \ldots + b_{s-1}\ x^{(s-1)p} \qquad (10.24)$$

Now using the division algorithm and dividing $f(x)$ into $x^{iP}$ for $i = 0, 1, \ldots, s-1$, gives

$$x^{ip} = f(x)\, q_i(x) + r_i(x) \tag{10.25}$$

where

$$r_i(x) = r_{i,0} + r_{i,1}\, x + \ldots + r_{i,s-1} x^{s-1} \tag{10.26}$$

Substituting (10.25) into (10.24) yields

$$(h(x))^p = b_0 r_0(x) + b_1 r_1(x) + \ldots + b_{s-1} r_{s-1}(x) + (\text{multiple of } f(x))$$

Thus f(x) divides $(h(x))^p - h(x)$ if and only if f(x) divides the polynomial

$$b_0 r_0(x) + b_1 r_1(x) + \ldots + b_{s-1} r_{s-1}(x) - (b_0 + b_1 x \ldots + b_{s-1} x^{s-1})$$

But this polynomial has degree ≤ s-1 hence is divisible by f(x) (of degree s) if and only if it is equal to 0.

Thus f(x) divides $(h(x))^p - h(x)$ if and only if the coefficients $b_0, b_1, \ldots b_s$ of h(x) satisfy

$$b_0 r_0(x) + b_1 r_1(x) + \ldots + b_{s-1} r_{s-1}(x) - (b_0 + b_1 x + \ldots + b_{s-1}\, x^{s-1}) = 0 \tag{10.27}$$

Collecting coefficients of $1, x, x^2, \ldots, x^{s-1}$ in (10.27), s simultaneous linear equations in the s unknowns $b_0, b_1, \ldots, b_{s-1}$ are obtained. These equations can be solved for $b_0, \ldots, b_{s-1}$ and coefficients of a polynomial h(x) such that f(x) divides $(h(x))^p - h(x)$ can be obtained.

To calculate the decoding exponent d in (10.20), one does not need to obtain the irreducible factors but only need to obtain their degrees. In particular, in a system where the modulus f(x) consists of two distinct irreducible factors $\Pi_1(x)$ and $\Pi_2(x)$ (analogous to the RSA system over integers), one can follow a simple technique to obtain the degrees of the irreducible factors. Without loss of generality, it can be assumed that polynomials are all monic. From finite fields theory [56], it is known that in a field of order p, $x^{p^\ell} - x$ factors into the product of all monic irreducible polynomials whose degrees divide $\ell$. Thus the cryptanalyst computes $h_\ell(x) = \gcd(f(x), x^{p^\ell} - x)$ for $\ell = 1, 2, \ldots, s/2$, successively until he finds a $h_\ell(x) \neq 1$. In such an instance, $h_\ell(x) = \Pi_1(x)$ or $\Pi_2(x)$ or f(x).

The first two cases give the degrees $\ell = s_1$ or $s_2$ and in the third case $\ell = s_1 = s_2$. Thus in all the three cases, the opponent has obtained the degrees of irreducible factors even though in the third case, he has not actually factorized $f(x)$. Thus he can calculate the order (10.21) and can determine the decoding exponent using (10.20).

To compute the gcd $(f(x), x^{p^{\ell}} - x)$ for $\ell = 1, 2, \ldots, s/2$ requires approximately

(i) $s/2$ operations of gcd $(f(x), x^{p^{\ell}} - x)$

(ii) each such operation requires approximately $\log_2 p^{s/2}$ multiplications mod $f(x)$, giving a total of $O(s^2 \log_2 p)$ multiplications mod $f(x)$ in $Z/pZ$.

The encryption (or decryption) procedure requires a maximum of $\log_2 w$ multiplications mod $f(x)$ where $w = (p^{s_1}-1)(p^{s_2}-1) - 1$ because the maximum value of e (or d) is $(p^{s_1}-1)(p^{s_2}-1) - 1$. As $w < p^s$, the computational effort is $O(s \log_2 p)$ multiplications mod $f(x)$. Thus the work factor is approximately $O(s)$. Thus if the cryptanalyst is required to make about $2^{60}$ trials before finding the decoding exponent d, then the degree of $f(x) = s_1 + s_2 = s = 2^{60}$. In such a case, the length of the cipher polynomial also becomes $2^{60}$ introducing a long delay in both encrypting and decrypting devices.

Further Gait [59] points out a short cycling attack based on superenciphering any non-trivial polynomial $t(x)$ with encrypting exponent e equal to any non-trivial power of p. He shows that the period of the sequence of powers of $t(x)^e$ mod $f(x)$ is equal to $s_1 s_2$. It seems then the work required to recover the decoding exponent depends on the difficulty of factorizing $s_1 s_2$. Hence for the system to be secure $s_1 s_2$ needs to be very large. Short cycling attacks against public key systems are considered in Chapter 13.

### 10.5.3 An Improved Polynomial Based RSA System

Now let us extend this polynomial scheme further to the case where $f(x)$ is a composite polynomial over $Z/mZ$, where m is a composite number, in an attempt to increase the security of the above mentioned public key system.

Let

$$m = \prod_{i=1}^{\ell} p_i^{\alpha_i} \quad \text{where } p_i \text{ are primes} \tag{10.28}$$

and the degree of the polynomial f(x) be s.

Then the message space consists of polynomials $\{m(x)\}$ whose degrees are less than s and whose coefficients are elements of the ring Z/mZ. The first step is to find the order of the group formed by the polynomials of degree less than s and relatively prime to f(x) over Z/mZ. Let us denote this order using the Euler totient function symbol $\phi_m$ (f(x)). From the expression derived for the finite field F = Z/pZ in Section 10.5.2, the order of the group formed by the polynomials which are relatively prime to f(x) in Z/pZ is given by

$$\phi_p\,(f(x)) = \prod_{i=1}^{r} (p^{s_i b_i} - p^{s_i(b_i-1)}) \tag{10.29}$$

where $f(x) \equiv \prod_{i=1}^{r} g_i^{b_i}(x) \pmod{p}$ and $g_i(x)$ are irreducible polynomials of degrees $s_i$ in Z/pZ.

Note that the commutative ring R=Z/pZ[x]/f(x) under consideration is isomorphic to Z[x]/(p,f(x)). Now let us consider the order of the group formed by the units in the ring $Z/p^tZ[x]/f(x) \cong Z[x]/(p^t,f(x))$. Let Θ be a surjective ring homomorphism between the rings given below

$$\Theta : \frac{Z[x]}{(p^{t+1},f(x))} \twoheadrightarrow \frac{Z[x]}{(p^t,\ f(x))}$$

If restricted to units, then there is a surjective homomorphism Θ′ such that

$$\Theta' : U\left(\frac{Z[x]}{(p^{t+1},f(x))}\right) \twoheadrightarrow U\left(\frac{Z[x]}{(p^t,\ f(x))}\right)$$

The kernel of the mapping Θ′ consists of those sets of units of the left hand side which are congruent to 1 mod $(p^t,f(x))$, that is,

kernel $\Theta' = \{h(x)\}$ such that $h(x) \equiv 1 \bmod (p^t,\ f(x))$

or,

$$h(x) = 1 + p^t\ v(x) + f(x)\ k(x)$$

and h(x) is a unit in $\frac{Z[x]}{(p^{t+1},f(x))}$. Let us now calculate the number of such classes mod $(p^{t+1},\ f(x))$. The standard form for representatives of classes mod $(p^t,\ f(x))$ is given by

$$u_0 + u_1 x + \ldots + u_{s-1} x^{s-1} \text{ where } 0 \leqslant u_i < p^t.$$

The standard form for representatives of classes mod $(p^{t+1}, f(x))$ is given by

$$u_0 + u_1 x + \ldots + u_{s-1} x^{s-1} \text{ where } 0 \leqslant u_i < p^{t+1}$$

As $f(x)$ is same in both cases, $h(x) \equiv 1 + p^t\, v(x) \pmod{f(x)}$.
The number of such $v(x)$ possible in mod $(p^{t+1}, f(x))$ is $p^s$ because: $v(x) = v_0 + v_1 x + \ldots + v_{s-1} x^{s-1}$ and each coefficient $v_i$ can take values in the range 0 to $p-1$. If $v_i$ is greater than $p-1$ then it will get reduced mod $p^{t+1}$ as $p^t . p = p^{t+1}$. Thus the total number of such $v(x)$ possible is equal to $p^{\deg f(x)} = p^s$.

Therefore

$$\# \text{ kernal } \theta' = p^s$$

Using group theory ,

$$\frac{U\left(\frac{Z[x]}{(p^{t+1}, f(x))}\right)}{\text{Kernel } \theta'} \cong U\left(\frac{Z[x]}{(p^t, f(x))}\right)$$

and

$$\# U\left(\frac{Z[x]}{(p^{t+1}, f(x))}\right) = \# (\text{kernel } \theta') \; \# U\left(\frac{Z[x]}{(p^t, f(x))}\right)$$

where # denotes the order.

Therefore,

$$\phi_{p^{t+1}}(f(x)) = p^s \phi_{p^t}(f(x)) \qquad t \geqslant 1$$

$$= p^{2s} \phi_{p^{t-1}}(f(x))$$

$$= \ldots = \ldots$$

and $\therefore$, $$\phi_{p^t}(f(x)) = p^{(t-1)s} \phi_p(f(x)) \qquad (10.30)$$

The next stage is to consider the case where $m = \prod_{i=1}^{\ell} p_i^{\alpha_i}$ and to evaluate the required expression $\phi_m(f(x))$.

Using the Chinese Remainder Theorem it is seen that

$$\frac{Z[x]}{(m, f(x))} \cong \frac{Z[x]}{(p_1^{\alpha_1}, f(x))} \oplus \frac{Z[x]}{(p_2^{\alpha_2}, f(x))} \oplus \ldots \oplus \frac{Z[x]}{(p_\ell^{\alpha_\ell}, f(x))}$$

where ⊕ denotes the direct sum.

Thus

$$\phi_m(f(x)) = \prod_{i=1}^{\ell} \phi_{p_i^{\alpha_i}}(f_i(x)) \qquad (10.31)$$

where $f_i(x) \equiv f(x) \pmod{p_i}$ for $i=1, 2...,\ell$.

Considering the factorization of $f_i(x)$ rather than $f(x)$ over Z, let the factorization of $f_i(x)$ be

$$f_i(x) = \prod_{j=1}^{r_i} g_{ij}^{b_{ij}}(x) \pmod{p_i} \qquad (10.32)$$

where the degree of irreducible polynomial $g_{ij}^{b_{ij}}(x)$ is $s_{ij}\ b_{ij}$. Note that the upper limit of the product term in the expression (10.32) goes up to $r_i$, that is, it is a function of to which prime $p_i$ the polynomial $f(x)$ is being factored. This is because in general $f(x) \pmod{p_i}$ will factorize into some $r_i$ distinct irreducible polynomials as i varies.

Subsituting (10.30) into (10.31) yields

$$\phi_m(f(x)) = \prod_{i=1}^{\ell} p_i^{(\alpha_i-1)s}\ \phi_{p_i}(f_i(x)) \qquad (10.33)$$

From (10.29), $\phi_{p_i}(f_i(x))$ is given by

$$\phi_{p_i}(f_i(x)) = \prod_{j=1}^{r_i} (p_i^{s_{ij}b_{ij}} - p_i^{s_{ij}(b_{ij}-1)}) \qquad (10.34)$$

Substituting (10.34) into (10.33) gives

$$\phi_m(f(x)) = \prod_{i=1}^{\ell} p_i^{(\alpha_i-1)s} \prod_{j=1}^{r_i} (p_i^{s_{ij}b_{ij}} - p_i^{s_{ij}(b_{ij}-1)})$$

Substituting $m = \prod_{i=1}^{\ell} p_i^{\alpha_i}$, the above expression becomes

$$\phi_m(f(x)) = \left(\frac{m}{\prod_{i=1}^{\ell} p_i}\right)^s \prod_{i=1}^{\ell} \prod_{j=1}^{r_i} (p_i^{s_{ij}b_{ij}} - p_i^{s_{ij}(b_{ij}-1)}) \qquad (10.35)$$

Although the general expression (10.35) for the order of the group formed by polynomials $\{m(x)\}$ which are relatively prime to $f(x)$ over Z/mZ has been derived, from cryptography point of view, only square free modulus m and square free $f(x)$ are allowed. If for instance, m is chosen to be a non-square free integer then there will be nilpotent residue classes mod $f(x)$ for any $f(x)$ and proper decryption will not be possible in such cases. Similar arguments also apply to the case where f(x) is non-square free. Consider the two examples given below which illustrate the two cases m and f(x) non-square free.

Example 1 : non-square free m

Let $m = p_1^2 . p_2 = 3^2 .5 = 45$ and let $f(x) = x$
$f(x) \pmod 3 \equiv f_1(x) = x$
$f(x) \pmod 5 \equiv f_2(x) = x$
$\phi_m(f(x)) = (\frac{45}{15}).2.4 = 24$
choosing $e = d = 5$ so that $ed \equiv 1 \pmod{24}$

Let the message m(x) = 6. Then $(m(x))^{ed} \equiv 6^{ed} \equiv 36 \not\equiv 6$ (mod(45,x)).
The message m(x) =36 gives $36^{ed} \equiv 36$ (mod(45,x)).
Thus there is ambiguous decryption as both messages 6 and 36 produce identical output after decryption.

Example 2 : non-square free f(x)

Let $m = p_1 . p_2 = 3.5 = 15$ and let $f(x) = 4x^2 + 3$
$f(x) \pmod 3 \equiv f_1(x) \equiv x^2 \equiv (\Pi_{11}(x))^2$
$f(x) \pmod 5 \equiv f_2(x) \equiv 4x^2 + 3 \equiv \Pi_{21}(x)$
$\phi_m(f(x)) = 24$
choosing $e = d = 5$ so that $ed \equiv 1 \pmod{24}$

Let the message m(x) = x. Then $(m(x))^{ed} \equiv 6x \not\equiv x$ (mod(15, $4x^2 + 3$)).
The message m(x) = 6x gives $(6x)^{ed} \equiv 6x$ (mod(15,$4x^2 + 3$)).
Thus there is ambiguous decryption as both messages x and 6x produce identical output after decryption.
Thus if m and f(x) are made square free, then the expression (10.35) becomes

$$\phi_m(f(x)) = \prod_{i=1}^{\ell} \prod_{j=1}^{r_i} (p_i^{s_{ij}} - 1) \qquad (10.36)$$

Thus the order $\phi_m$ (f(x)) not only depends on the degrees of the irreducible factors ($s_{ij}$) of f(x) but also on the prime factors of the integer m. So if the original polynomial scheme is operated over Z/mZ, then this seems to give rise to a secure public key cryptosystem like the RSA system over integers.

10.5.3.1 System Design and Operation

The designer randomly chooses large primes $p_1$ to $p_\ell$ for some $\ell, \ell \geqslant 2$ following the guidelines suggested in Section 10.2. He then needs to choose a modulus polynomial f(x) with the property that $f_i(x) \equiv f(x) \pmod{p_i}$, $1 \leqslant i \leqslant \ell$, are square free polynomials of degree s. Two methods for designing such a polynomial f(x) are now outlined.

In method 1, the designer chooses randomly irreducible polynomials $h_{ij}(x)$ of degrees $s_{ij}$ over $Z/p_iZ$ for $i = 1$ to $\ell$. As noted in Section 10.5.2.1, an average of $s_{ij}$ tries is required to find an irreducible polynomial of degree $s_{ij}$ in $Z/p_iZ$. The polynomials are chosen in such a way that the degrees of the factors modulo $p_i$ for a given i add up to form s. That is, the distinct factors $h_{ij}(x)$, $1 \leqslant j \leqslant r_i$, are combined to form $f_i(x) \pmod{p_i}$ of degree s. Thus the designer now has the values $s_{ij}$ and $p_i$ for all $1 \leqslant i \leqslant \ell$ and $1 \leqslant j \leqslant r_i$.

In method 2, the designer randomly chooses polynomials $f_i(x)$ of degree s and accepts them if they are square free over $Z/p_iZ$ respectively. This is done by choosing a polynomial $f_i(x)$ and computing the gcd $(f_i(x), f_i'(x))$. If the gcd $(f_i(x), f_i'(x)) = 1$, then $f_i(x)$ is square free. Knuth [45] estimates that a randomly chosen polynomial $f_i(x)$ will be square free over $Z/p_iZ$ with a probability of $1-1/p_i$. Hence the expected number of trials for finding a square free polynomial $f_i(x)$ is less than 2 for each i. An alternative way of finding square free polynomials is to start off with irreducible polynomials f(x) over Z. The discriminant of f(x) can be calculated using standard formula [85, p 451] and the prime p is then chosen so that p does not divide the discriminant. Then f(x) is square free over Z/pZ. This procedure can be repeated for $p_i, 1 \leqslant i \leqslant \ell$. Having obtained the square free polynomials $f_i(x)$ over $Z/p_iZ$, $1 \leqslant i \leqslant \ell$, through one way or other, the designer can use the Chinese Remainder Theorem to form a unique sth degree composite polynomial f(x) over Z/mZ where $m = \prod_{i=1}^{\ell} p_i$ and $f(x) \equiv f_i(x) \pmod{p_i}$ for $1 \leqslant i \leqslant \ell$. The degrees of the irreducible factors of $f_i(x)$ over $Z/p_iZ$ can be determined using the polynomial factorization algorithm of Berlekamp (Section 10.5.2.2) or one of the modified techniques given by Knuth [45]. Thus the designer now has the values $s_{ij}$ and $p_i$ for all $1 \leqslant i \leqslant \ell$ and $1 \leqslant j \leqslant r_i$.

These $s_{ij}$ and $p_i$ values can be substituted into the expression of $\phi_m(f(x))$ given by (10.36). Recall that for the encryption and decryption to work properly,

$$(m(x))^{1+k\phi_m(f(x))} \equiv m(x) \quad \text{modd } (m,f(x))$$

for any m(x) with coefficients in Z/mZ and whose degree is less than s. The coding exponents e and d can therefore be determined using

$$ed \equiv 1 \quad \text{mod } \phi_m \ (f(x))$$

where e and d are multiplicative inverse of each other mod $\phi_m$ (f(x)). The public encryption key is given by (e, m, f(x)) and the secret decryption key is equal to (d, m, f(x)).

The message is divided into blocks of size s (degree of f(x)), each block consisting of integers over Z/mZ. Each such block is associated with a polynomial m(x) of degree less than s. The encryption procedure then consists of raising m(x) to the power e to form the cipher polynomial c(x) using

$$c(x) \equiv (m(x))^e \quad \text{modd } (m, f(x))$$

The decryption procedure uses

$$m(x) \equiv (c(x))^d \quad \text{modd } (m, f(x))$$

to obtain the message polynomial back.

10.5.3.2 System Implementation

This extended polynomial based RSA system has been simulated on the Prime Computer. The encryption/decryption of message/cipher polynomials are performed using the Square and Multiply technique given in Section 10.4.7.1. The listing of the program is given in Appendix 14. The cipher polynomial is transmitted to the receiver by the sender in the form of a vector. The coefficients of the cipher polynomial are sent to receiver, separated by a space starting from the lowest power coefficient. The receiver reconstructs the cipher polynomial and decrypts it to recover the message. An example showing the various parameters involved is given below:

Example

Let $m = p_1 p_2 = 5.7 = 35$. Hence $\ell = 2$

Let $f(x) = x^4 + x^2 + 1$

$f(x) \pmod 5 \equiv f_1(x) = (x^2 + 4x + 1)(x^2 + x + 1)$

$f(x) \pmod 7 \equiv f_2(x) = (x^2 + 5)(x^2 + 3)$

Therefore, $\emptyset_m(f(x)) = (5^2 - 1)^2 (7^2 - 1)^2 = 1327104$

The coding exponents e and d are chosen to be $e = 6005$ and $d = 221$ where $ed \equiv 1 \pmod{1327104}$

Let the 35-ary representation of the message to be encrypted be (18 9 30 23), (4 21 13 7) where the message is broken into blocks of size 4. That is,

$m_1(x) = 4x^3 + 21x^2 + 13x + 7$ and

$m_2(x) = 18x^3 + 9x^2 + 30x + 23$

Encryption:

cipher polynomial $c_1(x) \equiv (4x^3 + 21x^2 + 13x + 7)^{221} \text{ modd}(35, x^4+x^2+1)$

$\equiv x^3+14x^2+12x+21$

cipher polynomial $c_2(x) \equiv (18x^3+9x^2+30x+23)^{221} \text{ modd}(35, x^4+x^2+1)$

$\equiv 8x^3+6x^2+20x+14$

Decryption:

Message polynomial $m_1(x) \equiv (x^3+14x^2+12x+21)^{6005} \text{ modd}(35, x^4+x^2+1)$

$\equiv 4x^3+21x^2+13x+7$

Message polynomial $m_2(x) \equiv (8x^3+6x^2+20x+14)^{6005} \text{ modd}(35, x^4+x^2+1)$

$\equiv 18x^3+9x^2+30x+23$

10.5.4 Discussion

Thus it is seen that the RSA cryptosystem can be extended to polynomial rings. The difficulty of factorization of a polynomial into its irreducible factors over a finite field does not in itself provide the necessary security required for a public key cryptosystem. It is necessary to incorporate the factorization of an integer into its primes by considering the modulus polynomial f(x) over $Z/mZ$ (m-square free composite integer) to enhance the security of the system. This gives a system which is comparable in strength to that of the RSA system over integers. Furthermore, from cryptography point of view, it is required that both the modulus polynomial f(x) and the modulus integer m be square free. This is very much like the RSA system over integers in contrast to the matrix RSA system discussed in Section 10.4.

## 10.6 Extension of RSA System to Matrix Rings with Polynomial Elements

In this section, the aim is to combine the two systems discussed in Sections 10.4 and 10.5, thus extending the factorization trapdoor concept to matrix messages containing polynomials over a ring as their elements. The approach adopted in this section parallels with that used in Section 10.4.2. If the ring of all nxn matrices over R is considered, then it is seen that the ring contains nilpotent elements when $n > 1$. As mentioned in Section 10.4.2, from cryptography point of view, it is necessary to avoid such nilpotent elements to satisfy the condition $M^{r+1} \equiv M$ for some $r > 0$. As in Section 10.4.2, to begin with the set of non-singular matrices is investigated; then the set of upper triangular matrices is considered.

### 10.6.1 Non-singular Matrices Over $R = Z[x]/(m,f(x))$

The matrix messages are represented as M(x) instead of M, to indicate that their elements are polynomials in a single indeterminate x. Hence M(x) is written as

$$M(x) = \begin{bmatrix} m_{11}(x) & \cdots & m_{1n}(x) \\ m_{12}(x) & \cdots & m_{2n}(x) \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ m_{n1}(x) & \cdots & m_{nn}(x) \end{bmatrix} \qquad (10.37)$$

The complete public key system using such messages is developed in four separate stages.

#### 10.6.1.1 Stage 1

In this stage, the elements $m_{ij}(x)$ in (10.37) for all i, j

$1 \leq i \leq n$ and $1 \leq j \leq n$, are to belong to the ring $R_1 = \frac{Z/pZ[x]}{(\Pi_1(x))}$ where $\Pi_1(x)$ is an irreducible polynomial in Z/pZ of degree $s_1$ (more precisely, here $R_1$ is a field). That is, the group $M_n$ formed by non-singular message matrices of order n with elements in $R_1$ is considered. In this instance, non-singularity implies that the determinant of M(x) is relatively prime to p and $\Pi_1(x)$. The order of the group formed by such nxn non-singular matrices can be obtained in a similar fashion as in the case of non-singular matrices over Z/pZ discussed in Section 10.4.2.

The order of the group formed by these matrices is given by the expression below, where # denotes the order.

$$\# GL(n, \frac{Z/pZ[x]}{(\Pi_1(x))}) = \# GL(n, F_{p^{s_1}})$$

because

$$\frac{Z/pZ[x]}{(\Pi_1(x))} \text{ is a finite field of } p^{s_1} \text{ elements}$$

And

$$\# GL(n, F_{p^{s_1}}) = \left[(p^{s_1})^n - 1\right] \left[(p^{s_1})^n - (p^{s_1})\right] \ldots \left[(p^{s_1})^n - (p^{s_1})^{n-1}\right] \qquad (10.38)$$

using (10.4) in Section 10.4.2.

If such non-singular matrices are used as messages then one can form a conventional cryptographic system where the secret key contains the modulus p itself. The encrypting (e) and decrypting (d) exponents can then be determined using

$$ed \equiv 1 \bmod (\# GL(n, F_{p^{s_1}}))$$

The encrypting key is therefore $(e, p, \Pi_1(x), n)$ and the decrypting key is $(d, p, \Pi_1(x), n)$. None of these keys can be made public and the encryption and the decryption procedures are given by

$$C(x) = (M(x))^e \text{ modd } (p, \Pi_1(x))$$

and

$$M(x) = (C(x))^d \text{ modd } (p, \Pi_1(x))$$

respectively where $M(x)$, $C(x) \varepsilon M_n(R_1)$

10.6.1.2 Stage 2

The above system can be modified to include the public key property as follows:
To begin with, the modulus polynomial is allowed to be equal to the product of distinct irreducible polynomials. This will result in a public key system analogous to the original polynomial scheme considered in Section 10.5.2 whose security depends upon the difficulty of factorizing a composite polynomial to irreducible factors. Let

$$f(x) = \prod_{i=1}^{r} g_i(x) \pmod{p}$$

where the polynomials $g_i(x)$ for $1 \leqslant i \leqslant r$ are irreducible in $Z/pZ$. The polynomials $g_i(x)$ are of degrees $s_i$ for $1 \leqslant i \leqslant r$ respectively.

The order of the group formed by the non-singular message matrices $M(x)$ with elements in the ring $R_2 = \frac{Z/pZ[x]}{(f(x))}$ can be obtained by the application of the Chinese Remainder Theorem as follows:

$$\begin{aligned} \# GL(n, \frac{Z/pZ[x]}{(f(x))}) &= \# GL(n, \frac{Z/pZ[x]}{(g_1(x))} \oplus \ldots \oplus \frac{Z/pZ[x]}{(g_r(x))}) \\ &= \# GL(n, F_{p^{s_1}} \oplus \ldots \oplus F_{p^{s_r}}) \\ &= \prod_{i=1}^{r} \# GL(n, F_{p^{s_i}}) \end{aligned} \qquad (10.39)$$

Substituting (10.38) into (10.39) gives

$$\# GL(n, \frac{Z/pZ[x]}{(f(x))}) = \prod_{i=1}^{r} \left[(p^{s_i})^n - 1\right]\left[(p^{s_i})^n - (p^{s_i})\right] \ldots \left[(p^{s_i})^n - (p^{s_i})^{n-1}\right]$$

Thus

$$(M(x))^{order} \equiv 1 \quad \text{modd } (p, f(x))$$

and hence the coding exponents e and d can be calculated using

$$ed \equiv 1 \pmod{(\# GL(n, \frac{Z/pZ[x]}{(f(x))}))}$$

and

$$(M(x))^{ed} \equiv M(x) \quad \text{modd}(p, f(x))$$

It is seen that in this system, the order depends on the degrees of the irreducible factors $g_i(x)$. (cf. Section 10.5.2). Thus this system can be used as a public key system with the public encryption key $(e,p,f(x),n)$ and the secret decryption key $(d,p,f(x),n)$. The encryption and decryption procedures are given by

$$C(x) \equiv (M(x))^e \text{ modd } (p,f(x))$$

and

$$M(x) \equiv (C(x))^d \text{ modd } (p,f(x))$$

respectively, where $M(x), C(x) \varepsilon M_n (R_2)$.

The security of this system is dependent on the difficulty of factorizing the modulus polynomial $f(x)$ in $Z/pZ$. This is so because if the degrees of the irreducible factors of $f(x)$ can be found then the order of the group can be evaluated and hence the secret decoding exponent d can be determined. For the reasons mentioned in Section 10.5.2.2, this does not give rise to a secure public key system.

10.6.1.3 <u>Stage 3</u>

The next logical step is to consider the case where the modulus polynomial $f(x)$ consists of powers of irreducible polynomials as its factors, that is, $f(x)$ is a non-square free polynomial. Note that in contrast to the system considered in Section 10.5, here it is allowed to have a non-square free $f(x)$ because the nilpotent elements are being eliminated by considering only the non-singular matrices over the ring $R_3 = \frac{Z/pZ[x]}{(f(x))}$. Considering the factors of $f(x)$ in $Z/pZ[x]$rather than in $Z[x]$,

let

$$\bar{f}(x) \equiv f(x) \pmod p$$

and

$$\bar{f}(x) \equiv \prod_{i=1}^{r} g_i^{b_i}(x) \quad \pmod p$$

where $g_i(x)$, $1 \leqslant i \leqslant r$, are irreducible polynomials in $Z/pZ$ with degrees $s_i$ respectively.

The order of the group formed by non-singular matrices $M(x)$ over the ring $R_3$ is evaluated as follows:

Using the Chinese Remainder Theorem,

$$\frac{Z/pZ[x]}{(f(x))} \cong \frac{Z/pZ[x]}{(g_1^{b_1}(x))} \oplus \cdots \oplus \frac{Z/pZ[x]}{(g_r^{b_r}(x))}$$

Letting $T = Z/pZ[x]$ and $\rho_i = (g_i(x))$
then,

$$GL(n, T/_{f(x)}) \cong GL(n, T/\rho_1^{b_1}) \times \cdots \times GL(n, T/\rho_r^{b_r})$$

where x denotes direct product.

To determine the order, $\# GL(n, T/\rho_i^{b_i})$, consider the sequence of surjective homomorphisms, $\theta_i$.

$$\theta_i : GL(n, T/_{\rho}i) \twoheadrightarrow GL(n, T/_{\rho}i-1) \qquad i \geq 2$$

Under such a mapping an nxn matrix $M(x) \pmod{\rho^i}$ becomes $M'(x)$ $\pmod{\rho^{i-1}}$ as shown below:

$$M(x) \pmod{\rho^i} \longrightarrow M'(x) \pmod{\rho^{i-1}}$$

Using group theory, for such an onto mapping $\theta_i$,

$$\# GL(n, T/_{\rho}i) = \# GL(n, T/_{\rho}i-1) . \# (\text{Kernel } \theta_i)$$

The kernel $\theta_i$ consists of the set of matrices which are mapped to the identity matrix I in $\pmod{\rho^{i-1}}$, that is,

$$m_{\ell j}(x) \equiv 0 \pmod{\rho^{i-1}} \qquad \ell \neq j \qquad (10.40)$$

$$m_{\ell\ell}(x) \equiv 1 \pmod{\rho^{i-1}} \qquad 1 \leq \ell \leq n \qquad (10.41)$$

There are $p^{\deg g_i(x)} = p^{s_i}$ possibilities for each of the equations (10.40) and (10.41) giving rise to $(p^{s_i})^{n^2}$ total possibilities. Hence,

$$\# GL(n, T/_{\rho}i) = (p^{s_i})^{n^2} \; \# GL(n, T/_{\rho}i-1)$$

$$= \ldots \qquad = \ldots$$

Therefore,

$$\#GL(n,T/_{\rho}b_{i}) = (p^{s}i)^{n^{2}(b_{i}-1)} \quad \# GL(n,T/\rho) \qquad (10.42)$$

But $T/\rho = \frac{Z/pZ[x]}{(g_{i}(x))}$ is a finite field of $p^{s}i$ elements.

Therefore using (10.38)

$$\# GL(n,T/\rho) = [(p^{s}i)^{n}-1]\, [(p^{s}i)^{n}-(p^{s}i)] \, \ldots \, [(p^{s}i)^{n}-(p^{s}i)^{n-1}]$$

Substituting this into (10.42) yields

$$\# GL\left(n, \frac{Z/pZ[x]}{(g_{i}{}^{b}i(x))}\right) = (p^{s}i)^{n^{2}(b_{i}-1)} \left[(p^{s}i)^{n}-1\right] \ldots \left[(p^{s}i)^{n}-(p^{s}i)^{n-1}\right] \qquad (10.43)$$

Thus

$$\# GL(n,\frac{Z/pZ[x]}{(f(x))}) = \prod_{i=1}^{r} \# GL(n,\frac{Z/pZ[x]}{(g_{i}{}^{b}i(x))}) \qquad (10.44)$$

It is necessary to consider one further stage to complete the design of a secure public key system using such non-singular matrices as messages.

### 10.6.1.4 Stage 4

The final stage is to consider the elements $m_{ij}(x)$ in the message matrix $M(x)$ in (10.37) to belong to the ring $R_{4} = \frac{Z/mZ[x]}{(f(x))}$

where

$$m = \prod_{i=1}^{t} p_{i} \quad \text{and } p_{i},\ 1\leqslant i\leqslant t, \text{ are distinct primes}$$

and the factorization of $f(x)$ is given by

$$f(x) \pmod{p_{i}} \equiv f_{i}(x) = \prod_{j=1}^{r_{i}} g_{ij}{}^{b}ij(x) \quad \text{for } 1\leqslant i\leqslant t$$

The degrees of irreducible polynomials $g_{ij}(x)$ over $Z/p_{i}Z$ are $s_{ij}$ respectively.

The order of the group formed by the nxn non-singular matrices $M(x)$ over the ring $R_{4}$, denoted by $\# GL(n, \frac{Z/mZ[x]}{(f(x))})$, is evaluated as follows:

The ring $\frac{Z/mZ[x]}{(f(x))}$ is isomorphic to the ring $\frac{Z[x]}{(m,f(x))}$. Using the Chinese Remainder Theorem,

$$\frac{Z[x]}{(m,f(x))} \cong \frac{Z[x]}{(p_1,f(x))} \oplus \frac{Z[x]}{(p_2,f(x))} \oplus \dots \oplus \frac{Z[x]}{(p_t,f(x))}$$

Therefore ,

$$GL(n, \frac{Z[x]}{(m,f(x))}) \cong GL(n, \frac{Z[x]}{(p_1,f(x))}) \times \dots \times GL(n, \frac{Z[x]}{(p_t,f(x))})$$

But

$$GL\left(n, \frac{Z[x]}{(p_i, f(x))}\right) \cong GL\left(n, \frac{F_{p_i}[x]}{(f(x))}\right)$$

where $F_{p_i}$ is a finite field of $p_i$ elements.

The order $\# GL\left(n, \frac{F_{p_i}[x]}{(f(x))}\right)$ has already been evaluated and is given by (10.44) as

$$\# GL(n, \frac{F_{p_i}[x]}{(f(x))}) = \prod_{j=1}^{r_i} (p_i^{s_{ij}})^{n^2(b_{ij}-1)} \left[(p_i^{s_{ij}})^n-1\right] \dots \left[(p_i^{s_{ij}})^n - (p_i^{s_{ij}})^{n-1}\right]$$

Hence

$$\# GL (n, \frac{Z[x]}{(m,f(x))}) = \prod_{i=1}^{t} \prod_{j=1}^{r_i} (p_i^{s_{ij}})^{n^2(b_{ij}-1)} \left[(p_i^{s_{ij}})^n-1\right] \dots \left[(p_i^{s_{ij}})^n - (p_i^{s_{ij}})^{n-1}\right] \quad (10.45).$$

The order given by the expression (10.45) above shows that it is dependent on the degrees of the irreducible factors of $f(x)$, namely, $s_{ij}$ $b_{ij}$, as well as on the prime factors of the modulus m, namely $p_i$, for all i and j. Thus when used as a public key system, this provides considerably more security than the system considered in Stage 2, provided that the modulus m is chosen large enough. The encryption and the decryption of messages are carried out in the normal fashion using

$$C(x) \equiv (M(x))^e \quad \text{modd } (m,f(x))$$

and

$$M(x) \equiv (C(x))^d \text{ modd } (m,f(x))$$

where $M(x),\ C(x) \in M_n\ (R_4)$

and $ed \equiv 1 \pmod{\text{order}}$

and the order is given by (10.45).

Note that this system is a combined version of the systems developed in Section 10.4 and 10.5.

### 10.6.2 Upper Triangular Matrices Over $R = \frac{Z[x]}{(m,f(x))}$

For the systems developed in the previous section, the message space is restricted to contain only non-singular matrices. But as mentioned in Section 10.4.2, the sender is faced with the problem of determining whether his message matrix is non-singular or not, to find whether it belongs to the message space. This poses problems as the sender has no control over the matrix elements but must accept what the plaintext dictates. Alternatively,as in Section 10.4.4,one can consider the set of upper triangular matrices with invertible elements along the diagonal as the message space. This makes the construction of non-singular message matrices easier especially when m is a product of a few large primes and f(x) is a product of a few high degree irreducible polynomials. In such a situation one can almost arbitrarily choose the diagonal elements of the upper triangular message matrix to satisfy the condition that they are relatively prime to f(x) modulo m. In particular, the diagonal elements can be chosen to be elements in the ring Z/mZ which are relatively prime to m. If necessary, one can also use the Euclid's algorithm to evaluate the gcd to test the relative primeness.

In this section, the set of upper triangular matrices whose elements are polynomials over a chosen ring R forms the message space of the developed cryptosystem. The four stages involved in developing the complete public key system, similar to those discussed in the previous section, are now considered in turn. Note that in all these stages, it is assumed that the diagonal elements are invertible elements over the chosen ring R. This ensures that the message matrix is invertible. A typical message matrix $M_\Delta(x)$ is written as

$$M_{\Delta}(x) = \begin{bmatrix} m_{11}(x) & \cdot & \cdot & \cdot & & m_{1n}(x) \\ 0 & \cdot & & & & \cdot \\ \cdot & & \cdot & & & \cdot \\ \cdot & \bigcirc & & \cdot & & \cdot \\ \cdot & & & & & \\ 0 & \cdot & \cdot & \cdot & 0 & m_{nn}(x) \end{bmatrix}$$

where $\Delta$ indicates that the matrix is triangular.

10.6.2.1 Stage 1

In this stage, the message space consists of upper triangular matrices whose non-zero elements $m_{ij}(x)$ belong to the ring $R_1 = \frac{Z/pZ[x]}{(\Pi_1(x))}$ where $\Pi_1(x)$ is an irreducible polynomial in Z/pZ of degree $s_1$. (That is, $R_1$is actually a field).

The order of the group formed by such upper triangular matrices over $R_1$ can be found as follows:

Each diagonal entry may be any one of the polynomials which is relatively prime to $\Pi_1(x)$ over Z/pZ. The number of such polynomials has already been evaluated in Section 10.5.2 and is given by the Euler totient function $\phi_p(\Pi_1(x))$ where

$$\phi_p(\Pi_1(x)) = (p^{s_1}-1)$$

The remaining $\frac{1}{2}n(n-1)$ superdiagonal entries of the upper triangular matrix may take any value in the field $\frac{Z/pZ[x]}{(\Pi_1(x))}$. Thus each of these entries has $p^{s_1}$ possibilities as $\frac{Z/pZ[x]}{(\Pi_1(x))}$ is a finite field of $p^{s_1}$ elements.

Thus the order of this group $U(n, \frac{Z/pZ[x]}{(\Pi_1(x))})$ is given by

$$\# U\left(n, \frac{Z/pZ[x]}{(\Pi_1(x))}\right) = (p^{s_1}-1)^n (p^{s_1})^{n(n-1)/2}$$

This can be used to form a conventional cryptographic system where the secret coding exponents e and d can be found using

$$ed \equiv 1 \left(\bmod \# U\left(n, \frac{Z/pZ[x]}{(\Pi_1(x))}\right)\right)$$

and the encryption and the decryption procedures are given by

$$\underset{\Delta}{C}(x) \equiv (\underset{\Delta}{M}(x))^{e} \quad \text{modd } (p, \Pi_1(x))$$

and

$$\underset{\Delta}{M}(x) \equiv (\underset{\Delta}{C}(x))^{d} \quad \text{modd } (p, \Pi_1(x))$$

respectively, where $\underset{\Delta}{M}(x)$, $\underset{\Delta}{C}(x) \in U_n(R_1)$

10.6.2.2 Stage 2

The above system can be modified to include the public key property as follows:

Initially, it is assumed that the modulus polynomial consists of a product of distinct irreducible polynomials modulo p. That is,

$$f(x) = \prod_{i=1}^{r} g_i(x) \quad (\text{mod } p)$$

where $g_i(x)$, $1 \leqslant i \leqslant r$ are irreducible in Z/pZ.

The polynomials $g_i(x)$ are of degrees $s_i$, $1 \leqslant i \leqslant r$, respectively.

The order of the group formed by upper triangular matrices over the ring $R_2 = \frac{Z/pZ[x]}{(f(x))}$ with invertible diagonal elements is evaluated as follows:

Using the Chinese Remainder Theorem,

$$\# U (n, \frac{Z/pZ[x]}{(f(x))}) = \# U (n, \frac{Z/pZ[x]}{(g_1(x))} \oplus \dots \oplus \frac{Z/pZ[x]}{(g_r(x))})$$

$$= \# U (n, F_{p^{s_1}} \oplus \dots \oplus F_{p^{s_r}})$$

Each of the diagonal entries can be any one of the polynomials relatively prime to f(x) over Z/pZ and is given by $\phi_p(f(x))$ where

$$\phi_p(f(x)) = \prod_{i=1}^{r} (p^{s_i} - 1) \quad \text{(See Section 10.5.2)}$$

The remaining $\frac{n(n-1)}{2}$ superdiagonal entries have $p^s$ possibilities each where

$$s = \sum_{i=1}^{r} s_i$$

Therefore the order is given by

$$\# U\left(n, \frac{Z/pZ[x]}{(f(x))}\right) = \left[\phi_p\,(f(x))\right]^n \left[(p^s)\right]^{n(n-1)/2}$$

It is seen that in this system, the order depends on the degrees of the irreducible factors of the modulus polynomial f(x) (cf Sections 10.5.2 and 10.6.1.2). Thus this system can be used as a public key system with the public encryption key (e,p,f(x),n) and the secret decryption key (d,p,f(x),n). The coding exponents e and d are calculated using

$$ed \equiv 1 \ (\text{mod}\, \# U\ (n, \frac{Z/pZ[x]}{(f(x))}))$$

and

$$(\underset{\Delta}{M}\ (x))^{ed} \equiv \underset{\Delta}{M}\ (x) \quad \text{modd}\ (p,f(x))$$

where $\underset{\Delta}{M}\ (x) \in U_n\ (R_2)$

For the reasons mentioned in Section 10.5.2.2, this does not offer a secure public key system.

10.6.2.3 Stage 3

The next step is to consider the case where the modulus polynomial f(x) consists of powers of irreducible polynomials as its factors, that is, f(x) is a non-square free polynomial in Z/pZ.

Let

$$f(x) \equiv \prod_{i=1}^{r} g_i^{b_i}(x) \quad (\text{mod } p)$$

where $g_i(x)$, $1 \leq i \leq r$, are irreducible polynomials in Z/pZ with degrees $s_i$ respectively.

The order of the group formed by upper triangular matrices over the ring $R_3 = \frac{Z/pZ[x]}{(f(x))}$ is now evaluated.

Using the Chinese Remainder Theorem,

$$\frac{Z/pZ[x]}{(f(x))} \cong \frac{Z/pZ[x]}{(g_1^{b_1}(x))} \oplus \ldots \oplus \frac{Z/pZ[x]}{(g_r^{b_r}(x))}$$

and hence

$$\# U \left(n, \frac{Z/pZ[x]}{(f(x))}\right) = \# U \left(n, \frac{Z/pZ[x]}{(g_1^{b_1}(x))}\right) \oplus \ldots \oplus \frac{Z/pZ[x]}{(g_r^{b_r}(x))} )$$

Using arguments similar to the ones given in Section 10.6.2.2,

$$\# U \left(n, \frac{Z/pZ[x]}{(f(x))}\right) = \left[\phi_p (f(x))\right]^n (p^s)^{n(n-1)/2}$$

where

$$s = \sum_{i=1}^{r} s_i b_i$$

and

$$\phi_p(f(x)) = \prod_{i=1}^{r} \left[p^{s_i b_i} - p^{s_i(b_i-1)}\right] \qquad \text{(see Section 10.5.2)}$$

### 10.6.2.4 Stage 4

Finally the group formed by the upper triangular matrices with invertible diagonal elements over the ring $R_4 = \frac{Z/mZ[x]}{(f(x))}$ is considered. The modulus $m = \prod_{i=1}^{t} p_i^{\alpha_i}$, where $p_i$, $1 \leqslant i \leqslant t$, are distinct primes and the factorization of $f(x)$ is given by

$$f(x) \pmod{p_i} \equiv f_i(x) = \prod_{j=1}^{r_i} g_{ij}^{b_{ij}}(x) \text{ for } 1 \leqslant i \leqslant t$$

where the degreesof the irreducible polynomials $g_{ij}(x)$ are $s_{ij}$ respectively. Using arguments similar to the ones given in Section 10.6.2.2, the order of the group is given by

$$\# U \left(n, \frac{Z/mZ[x]}{(f(x))}\right) = \left[\phi_m(f(x))\right]^n (m^s)^{n(n-1)/2}$$

where $\phi_m(f(x))$ is obtained from Section 10.5.3 as

$$\phi_m(f(x)) = \left(\frac{m}{\prod_{i=1}^{t} p_i}\right)^s \prod_{i=1}^{t} \prod_{j=1}^{r_i} \left(p_i^{s_{ij} b_{ij}} - p_i^{s_{ij}(b_{ij}-1)}\right)$$

Note also that by substituting n=1 in the expression (10.45) gives

$$\# GL \left(1, \frac{Z[x]}{(m, f(x))}\right) = \phi_m (f(x)) \qquad (\alpha_i = 1, \forall\, i)$$

Thus the order is seen to be dependent on the degrees of the irreducible factors of f(x) as well as on the prime factors of m. Thus this system can be used to provide a secure public key system provided the modulus m is chosen large enough (say 200 decimal digits, see Section 10.2). The encryption and the decryption procedures are given by

$$C_{\Delta}(x) \equiv (M_{\Delta}(x))^{e} \quad \text{modd } (m,f(x))$$

and

$$M_{\Delta}(x) \equiv (C_{\Delta}(x))^{d} \quad \text{modd } (m,f(x))$$

where $M_{\Delta}(x), C_{\Delta}(x) \in U_n(R_4)$

and $ed \equiv 1$ (mod order).

The public encryption key is given by (e,m,f(x),n) and the secret decryption key is equal to (d,m,f(x),n).

## 10.7 Discussion

In this chapter, the factorization trapdoor concept has been extended to some matrix and polynomial rings which are isomorphic to a direct sum of finite fields. This has resulted in a generalization of the RSA cryptostructure to matrix and polynomial ring message space. It is seen that some of these extended systems can be made at least as secure as the original RSA system over integers modulo m. Other features such as the use of non-square free moduli seem to be possible with some of these extended systems in contrast to the original RSA system over Z/mZ. Investigation of such systems indicate that factorization trapdoor structures required for the design of public key system can be found in rings other than the ring of integers modulo m. From a practical point of view, it seems that the high complexity of such systems may favour the implementation of the factorization trapdoor in the ring of integers.

# CHAPTER 11

## FACTORIZATION TRAPDOOR FROM IDEAL POINT OF VIEW

### 11.1 General

In addition to considering the factorization trapdoor system from an 'element' point of view, the trapdoor concept can also be treated from the more general 'ideal' point of view. In particular the integer, polynomial and matrix based RSA factorization trapdoor schemes considered in the previous chapter are briefly re-examined from the ideal point of view. Some of the principles of the ideal theory are used in the next chapter in further extending the trapdoor concept to such algebraic number fields as the ring of Gaussian integers and some other quadratic fields.

To begin with, some basic definitions and principles of the ideal theory are stated (without proofs) which will be required in subsequent sections. A detailed treatment of ideal theory can be found in a number of mathematical textbooks in particular in [60, 61].

### 11.2 Basic Concepts

#### 11.2.1 Ideal

A set J of one or more elements of a ring R is called an ideal in R if and only if it has the following properties:

(i) If i and j are elements of J, then i ± j is an element of J.

(ii) If i is an element of J, then for every element r of R, ir and ri are elements of J.

#### 11.2.2 Congruence

Let J be an ideal. Two elements a and b are defined to be congruent modulo the ideal J if a-b is in J, denoted by $a \equiv b \pmod{J}$.

#### 11.2.3 Principal Ideal

Let R be a commutative ring with 1 and 'a' be a non-zero element of R. If A is any ideal which contains the element 'a', then

A must also contain all the elements of the form ra where $r \in R$. An ideal generated by a single element 'a' of R is called a principal ideal and is denoted by <a> . An integral domain in which all ideals are principal is called a principal ideal domain (PID).

### 11.2.4 Prime Ideal

If R is a commutative ring with 1, an ideal $\rho$ in R is said to be a prime ideal if and only if $ab \equiv 0 \pmod{\rho}$ implies that $a \equiv 0 \pmod{\rho}$ or $b \equiv 0 \pmod{\rho}$. An alternative definition of prime ideal is that it is an ideal $\rho$ other than the unit ideal with the property that for any two ideals A and B if $\rho | AB$ then $\rho | A$ or $\rho | B$. Note that an ideal $\rho$ is said to 'divide' an ideal A if there exists an ideal C such that $A = \rho C$.

### 11.2.5 Product of Ideals

The product AB of two ideals A and B is defined as the ideal C 'generated by all products' ab where $a \in A$ and $b \in B$.

### 11.2.6 Unique Factorization of Ideals

Every ideal in a Dedekind domain [60] can be factored into the product of a finite number of prime ideals and this representation is unique.

### 11.2.7 Factorization Trapdoor

Using the ideal factorization theorem (Section 11.2.6), let the decomposition of a non-square free ideal A be

$$A = \rho_1^{s_1} \ldots \rho_r^{s_r}$$

The number of residues modulo the ideal A is given by the norm of the ideal, N(A). The number of invertible residue classes modulo the ideal A is denoted by $\Phi(A)$ in a similar manner to the Euler totient function $\phi$.

**Theorem 1**

If A and B are relatively prime ideals then

$$\Phi(AB) = \Phi(A) \cdot \Phi(B)$$

The congruences $a \equiv c \pmod{A}$ and $a \equiv d \pmod{B}$ establish a one to one correspondence $a \rightleftarrows (c,d)$ between the residues 'a' prime to AB and the pairs (c,d) whose two members c,d range over the residues prime to A and B respectively. Hence the theorem 1 follows.

Theorem 2

If $\rho$ is a prime ideal then,

$$\begin{aligned}\overline{\phi}(\rho^s) &= (N\rho)^s \quad \{1 - 1/N\rho\} \\ &= (N\rho-1)\ (N\rho)^{s-1}\end{aligned}$$

A complete proof is given in [61]. But it can be proved using a method similar to that which is employed to calculate the Euler totient function $\phi(p^s)$ where p is a prime in Z. The complete system of residues with respect to $\rho$ is represented using $N\rho$ integers $0,1,\ldots p-1$. Of these only 0 is not prime to p. Hence $\overline{\phi}(\rho) = N\rho-1 = N\rho(1-1/N\rho)$. As $N(AB) = N(A)\,N(B)$ for any two ideals A and B, it is seen that $N(\rho^2) = (N\rho)^2$. With respect to $\rho^2$ there are $p^2-p$ incongruent classes that are relatively prime to $\rho$ and hence $\overline{\phi}(\rho^2) = (N\rho)^2\ [1-\frac{1}{N\rho}]$. Use of induction, gives Theorem 2.

Using theorems 1 and 2, if

$$A = \rho_1^{s_1} \ldots \rho_r^{s_r}$$

then

$$\overline{\phi}(A) = N(A) \prod_{i=1}^{r} (1 - \frac{1}{N\rho_i}) \qquad (11.1)$$

The expression (11.1) can be used to give the generalized version of Fermat's theorem for ideals, namely,

if 'a' is prime to an ideal A then $a^{\overline{\phi}(A)} \equiv 1 \pmod{A}$

If A is a prime ideal, ie, $A = \rho$, then

$$a^{N\rho-1} \equiv 1 \pmod{\rho}$$

Thus the relatively prime residues modulo A form a group of order $\overline{\phi}(A)$.

Some of these concepts are now applied to the specific cases already looked at such as the ring integers, the ring of polynomials and matrix rings.

## 11.3 Ring of Integers

Let Z be the ring of integers. If m is a fixed integer in the ring Z then the ideal $<m>$ contains just the integers which are divisible by m.

**Theorem 3**

In the ring Z, every ideal is a principal ideal.

Let A be an ideal in Z. If A is the zero ideal then A is the principal ideal $<0>$ . If $A \neq <0>$ let m be the smallest integer in A. If n is an integer then using Euclid's algorithm, we have

$$n = qm + r$$

where q and r are integers and $0 \leq r < m$.

Now if n is an element of A, then n-qm is also in A. That is, r is in A. If r is greater than zero, then it contradicts the assumption that m is the least positive integer. Therefore r = 0, that is, n = qm. So all elements of A are of this form and hence $A = <m>$.

The prime ideals in Z are therefore precisely the ideals $<p>$ where p is a prime (together with the ideal $<0>$ and the ideal Z).

If the unique factorization of m into primes over Z is given by

$$m = p_1 \ldots p_r \quad \text{for some } r \geq 2 \quad \text{(m assumed to be square free)}$$

then one can view the ideal decomposition of $<m>$ into prime ideals $<p_i>$ as

$$<m> = <p_1> \ldots <p_r>$$

The order of the group formed by relatively prime residues modulo $<m>$ is given by the Euler totient function for the ideal $<m>$, $\emptyset<m>$, where

$$\phi\langle m\rangle = \phi\langle p_1\rangle \cdots \phi\langle p_r\rangle$$

and

$$\phi\langle p_i\rangle = N\langle p_i\rangle - 1$$

$N\langle p_i\rangle$ is the norm of the ideal $\langle p_i\rangle$ and is equal to the number of residue classes modulo $\langle p_i\rangle$ given by $p_i$ . Hence

$$\phi\langle m\rangle = (p_1-1) \cdots (p_r-1)$$

and

$$a^{\phi\langle m\rangle} \equiv 1 \pmod{\langle m\rangle}$$

This gives the familiar RSA system over the integers looked at from the ideal point of view.

11.4 Polynomial Rings

If F is a field and x is an indeterminate, then again every ideal in F[x] is a principal ideal. The proof is very much similar to the one given for the ring of integers Z (cf. Theorem 3). In this case, m(x) is chosen to be a polynomial of least degree in a given ideal and the Euclidean division algorithm for polynomials is used. Hence a non-trivial ideal A of F[x] is of the form A = <f(x)> where f(x) is a non-zero monic polynomial of minimal degree in A.

The prime ideals in this ring are those which are generated by irreducible polynomials f(x) over F. If f(x) is composite, then let the unique factorization of f(x) into irreducible polynomials $g_i(x)$ be

$$f(x) = g_1^{b_1}(x) \cdots g_r^{b_r}(x)$$

where the degree of polynomial $g_i(x)$ is $s_i$.

Then one can view the decomposition of the ideal generated by f(x) into ideals generated by $g_i^{b_i}(x)$ as

$$< f(x)> = < g_1^{b_1}(x)> \cdot < g_2^{b_2}(x)> \dots < g_r^{b_r}(x)>$$

From Theorem 2, we have

$$\phi(\rho^b) = (N_\rho)^{b-1} \quad (N_\rho - 1)$$

Here

$$N<g_i(x)> = \text{number of elements in the residue class ring } F[x]/<g_i(x)>$$

$$= \text{number of polynomials in } F[x] \text{ of degree less than } s_i$$

$$= |F|^{s_i}$$

Hence

$$\phi<g_i^{b_i}(x)> = |F|^{s_i(b_i-1)} \left\{ |F|^{s_i} - 1 \right\}$$

$$= |F|^{s_i b_i} - |F|^{s_i(b_i-1)}$$

This expression is same as the one obtained earlier in Section 10.5.2 by considering elements in the field.

The order of the group formed by residue classes modulo $<f(x)>$ is obtained using Theorem 1,

$$\phi<f(x)> = \phi<g_1^{b_1}(x)> \dots \phi<g_r^{b_r}(x)>$$

$$= \prod_{i=1}^{r} \left\{ |F|^{s_i b_i} - |F|^{s_i(b_i-1)} \right\}$$

## 11.5 Matrix Rings

Let us now reconsider the matrix system discussed in Section 10.4 to see whether it is possible to improve it using some ideal theory principles.

The ring of nxn matrices over a field F, $M_n^*(F)$, has no non-trivial ideals. The only ideals are the zero ideal and the whole ring itself. Hence one cannot consider the ring $M_n^*(F)$ for our purpose. One can consider two possible alternatives.

(a) The first approach is to consider a subring, say the ring of nxn upper triangular matrices (including the diagonal

$M_n^*$ - See note on p. 243

elements) over F, $U_n(F)$. This ring has many ideals. For instance,

$$J = \left\{ \begin{pmatrix} 0 & ? \\ 0 & 0 \end{pmatrix} \right\} \text{ is an ideal in } U_n(F)$$

? - denotes any arbitrary elements in F.

(b) The second approach is to consider matrices over a commutative ring R rather than a field. Then there is one to one correspondence between the two-sided ideals in R and the two-sided ideals in $M_n^*(R)$. This may give rise to a trapdoor coding system in $M_n^*(R)$. As will be seen later, the system obtained using this approach is same as that already discussed in Section 10.4.2.

11.5.1 Approach (a)

Consider the ring of 2x2 upper triangular matrices with arbitrary diagonal elements over a finite field, $R = U_2(F_p)$ where p is a prime. Let J be an ideal generated by an element M. Then J is the smallest ideal containing the element and by definition is equal to

$$J = \left\{ \sum_i r_i M r_i' \quad \begin{matrix} \text{all finite combinations} \\ \text{as } r_i, r_i' \text{ vary over } R \end{matrix} \right\}$$

The element M must be chosen to be non-invertible otherwise it is possible to choose $r_i = M^{-1}$ and $r_i' = I$ (identity). This results in the ideal J containing $(M^{-1} M I) = I$. Hence the ideal must contain $r_i I$ for all $r_i \varepsilon R$. That is, J = R. This contradicts the assumption that J is a proper subset of R. Hence no element of the ideal must possess multiplicative inverse. Let us consider X to be a generator matrix in $U_2(F_p)$ where X is equal to

$$X = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

The ideal J is

$$J = \left\{ \sum \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u & v \\ 0 & w \end{pmatrix} \right\}$$

where a, b, c, u, v, w $\varepsilon$ Z/pZ, p prime.

That is,

$$J = \left\{ \sum \begin{pmatrix} au & av \\ 0 & 0 \end{pmatrix} \right\}$$

Let $x = au$ and $y = av$, $x, y \in Z/pZ$, then

$$J = \left\{ \sum \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} \right\} \qquad (11.2)$$

It is seen that if $r \in R$ and $j \in J$, then

$$rj = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} \in J$$

$$jr = \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u & v \\ 0 & w \end{pmatrix} \in J$$

The number of elements in the ring $R = U_2(Z/pZ)$ is equal to $p^3$ as each of a, b and c has p possible choices. The order of the ideal J is $p^2$. Hence the order of the quotient ring R/J is equal to p. Thus the ring R/J is isomorphic to Z/pZ. From group theory, in general, the order of the ideal J is a factor of the order of the ring R. If the ring $R = U_2(Z/mZ)$ where m is equal to product of primes is now considered, then R/J is isomorphic to the ring Z/mZ. If m is square free, R/J forms a trapdoor system.

Let X be a generator matrix of the form given below

$$X = \begin{pmatrix} g & 0 \\ 0 & 0 \end{pmatrix} \qquad (11.3)$$

If gcd $(g,m) = 1$, then the ideal J becomes $\{ \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} \}$ which is same as the one given in (11.2). Hence assume that gcd $(g,m) = \ell \neq 1$. For instance, m could be equal to the product of two large primes p and q and some $\ell$, that is, $m = \ell.p.q$. Then g could be equal to $\ell.s$ where gcd $(s,p) = 1$ and gcd $(s,q) = 1$. The ideal J then consists of elements of the form given below

$$J = \left\{ \begin{pmatrix} gx & gy \\ 0 & 0 \end{pmatrix} \right\}$$

where x, y $\varepsilon$ T = Z/mZ

The residue classes of the quotient ring R/J are of the form

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a \pmod{g} & b \pmod{g} \\ 0 & c \pmod{m} \end{pmatrix}$$

Proposition 1

The number of elements in the quotient ring R/J is given by

$$\# R/J = m\ell^2$$

Proof

If two members of the ring R are congruent to each other modulo J, that is,

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \equiv \begin{pmatrix} a' & b' \\ 0 & c' \end{pmatrix} \pmod{J}$$

where a, b, c, a´, b´, c´ $\varepsilon$ T = Z/mZ

then this implies that

$$a \equiv a' \pmod{g}$$
$$b \equiv b' \pmod{g}$$
$$c \equiv c'$$

The number of possible choices for the element c is equal to m. Let k be equal to the number of possible choices for elements a or b. Then k is equal to the number of residue classes mod gT which are distinct in T. In other words, k is equal to #T/gT. Consider the mappings

$$Z \xrightarrow{\alpha} T \longrightarrow T/gT$$

and

$$Z \xrightarrow{\Theta} T/gT$$

Then

$$T/gT \cong Z/\text{Kernel } \Theta$$
$$Z/\text{Kernel } \Theta = Z/\{x \varepsilon Z;\ \alpha(x) \varepsilon gT\}$$
$$= Z/\{x \varepsilon Z;\ x \equiv gy \pmod{m} \text{ is soluble for } y \varepsilon Z\}$$
$$= Z/\{x \varepsilon Z;\ x \equiv 0 \bmod (\gcd(g,m))\}$$

That is,

$$T/gT \cong Z/\ell Z \quad \text{where } \ell = \gcd(g,m)$$

$$\therefore \quad \#T/gT = \ell$$

Hence the number of residue classes of the ring $R/J$ is $m\ell^2$.

As the order does not depend on the factorization of the modulus m, there is no trapdoor as such yet. But let us now consider the invertible residue classes modulo the ideal J, that is, the diagonal elements a and c are chosen to be invertible in the ring T/gT and T respectively.

Proposition 2

The number of invertible residue classes modulo J is given by

$\phi(\ell) \cdot \ell \cdot \phi(m)$

Proof

The invertible residue classes of the ring R/J are of the form

$$\begin{pmatrix} a \pmod g & b \pmod g \\ 0 & c \end{pmatrix}$$

where a, b, c $\varepsilon$ $T = Z/mZ$
and such that a is invertible (mod g) and c is invertible (mod m). The number of invertible residue classes mod m is given by the Euler totient function $\phi(m)$. Hence the number of possible choices for the element c is equal to $\phi(m)$. The element b can be arbitrarily chosen (mod g) and the number of possible choices is equal to $\ell$ from Proposition 1 above. Let k be equal to the number of possible choices for the element a. From above, it is known that $T/gT \cong Z/\ell Z$. Therefore, k is equal to the number of invertible elements in $Z/\ell Z$, that is, $\phi(\ell)$. Hence the number of invertible residue classes in $R/J = \phi(\ell) \cdot \ell \cdot \phi(m)$.

Thus in practice if m is equal to the product of a few large primes and $\ell$ which is itself equal to the product of a few reasonable size prime integers, then the elements a and c ($\neq 0$) can be chosen arbitrarily provided they are relatively small compared to m and $\ell$ .

The above arguments can be extended to the case where the ideal J is of the form

$$J = \left\{ \begin{pmatrix} gx_1, & gx_2, & \dots & gx_n \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \end{pmatrix} \right\} \qquad (11.4)$$

The order of the quotient ring R/J,that is, the number of residue classes modulo J is now given by

$$\ell^{n} \, m^{n\,(n-1)/2}$$

The number of invertible residue classes modulo J is given by

$$\text{Ord.} = \phi(\ell) \ \{\phi(m)\}^{\,n-1} \ \ \ell^{n-1} \ \ \ m^{(n-2)(n-1)/2} \qquad (11.5)$$

11.5.1.1 <u>Choice of Generator Matrix</u>

When deciding on a generator matrix X, the following points must be taken into account.

(i) The smaller the generated ideal J, the larger the quotient ring R/J. That is, the smaller the order of the ideal J is, the greater the number of residue classes in R/J thus giving rise to a larger number of possible messages that can be used in the system.

(ii) For the encryption and decryption procedures to be relatively easy, the generator matrix must have a simple form; hence the choice of the matrix X in (11.3). For instance, another simple generator matrix X is given by

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

The ideal J is then given by

$$J = \left\{ \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \right\}$$

where $x \in Z/mZ$

Hence the residue classes of R modulo J are of the form

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \equiv \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix} \pmod{J}$$

That is,

$$R/J \cong Z/mZ \oplus Z/mZ$$

This is equivalent to 2 scalar RSA prototype systems.

(iii) More generally, one can use more than one generator matrix to generate the ideal J but this makes the whole system more complicated without increasing the security.

$$J = \left\{ \sum \begin{pmatrix} \text{arbitrary} \\ \text{element} \\ \text{in R} \end{pmatrix} \underset{\text{Generator matrices}}{\left( X_1 \dots X_k \right)} \begin{pmatrix} \text{arbitrary} \\ \text{element} \\ \text{in R} \end{pmatrix} \right\}$$

11.5.1.2 <u>System Design</u>

A factorization trapdoor system can therefore be constructed as follows:

The message space consists of nxn invertible upper triangular matrices (including the diagonal elements) modd (m,J), where $m = \prod_{i=1}^{r} p_i^{a_i}$ and the ideal J is given by (11.4). The encryption procedure raises the message matrix to the power e giving the cipher matrix C as

$$C \equiv M^{e} \ (\text{modd}(m,J))$$

The decryption procedure is given by

$$M \equiv C^{d} \ (\text{modd }(m,J))$$

The coding exponents e and d can be determined using

$$ed \equiv 1 \ (\text{mod ord.})$$

where ord. is given by (11.5).

The public encryption key is given by (e,m,n,J) and the secret decryption key is (d,m,n,J).

As the order depends on the structure of m, the security of the system again lies in the difficulty of factorization of the modulus m. But note that if the size of $\ell$ is is made large then although the opponent needs to factorize $\ell$ to be able to calculate $\phi(\ell)$ in (11.5), the factorization of m and hence the computation of $\phi(m)$ is made that much easier. This is because as J is made public, the opponent knows $\ell$, the gcd (m,g) and hence he achieves partial factorization of $\frac{m}{\ell} = pq$. Larger the size of $\ell$ smaller the value of $\frac{m}{\ell}$.

Thus it seems that the process of performing modulo an ideal as indicated above does not appear to increase the security of the trapdoor system but instead disguises further the basic trapdoor system.

### 11.5.2 Approach (b)

Let us first state an important result about two-sided ideals in complete matrix rings. For proof refer to [62].

If M is a two-sided ideal in the ring R, then the ring $M_n^*$ of all matrices of order n over M is a two-sided ideal in the ring $M_n^*$ of all matrices of order n over R.

In our case, R = Z and assume m = pq where p and q are distinct primes in Z. Then <m> , <p> and <q >are ideals in Z and $M_n^*(mZ)$, $M_n^*(pZ)$ and $M_n^*(qZ)$ are all ideals of $M_n^*(Z)$ using the above result. Further the ideal factorization of $< m >$ in R is given by

$$< m > = < p > . < q >$$

Using the one-to-one correspondence between the two-sided ideals in R and in $M_n^*(R)$, gives

$$J_m = J_p \cdot J_q$$

where $J_p$, $J_q$ and $J_m$ denote the ideals formed by $M_n^*(pZ)$, $M_n^*(qZ)$ and $M_n^*(mZ)$ in the ring $M_n^*(Z)$.

Although a trapdoor system is possible in $Z/mZ$ when m is square free,there is no corresponding system in the case of $M_n^*(R)$. This is due to the fact that the quotient ring given by $M_n^*(Z/mZ)$,

$$\frac{M_n^*(Z)}{M_n^*(mZ)} \cong M_n^*(Z/mZ) \cong M_n^* (Z/pZ) \oplus M_n^* (Z/qZ)$$

is not isomorphic to a direct sum of finite fields whereas the quotient ring $Z/<m>$ is isomorphic to $Z/<p>$ and $Z/<q>$. As seen earlier in Section 10.4.2,$M_n^*(Z/pZ)$ is not a finite field as it contains nilpotent elements. Thus a corresponding trapdoor system is possible if and only if the nilpotent elements are eliminated. This has already been considered and it gave rise to the trapdoor system in matrix rings discussed in Section 10.4.2.

Note: $M_n^*(R)$ - Ring of all nxn matrices over R
$M_n(R)$ - Ring of non-singular nxn matrices over R

# CHAPTER 12

## FACTORIZATION TRAPDOOR IN ALGEBRAIC NUMBER FIELDS

### 12.1 General

The ring of Gaussian integers is initially considered with a view to extending the factorization trapdoor concept to algebraic number fields other than rational integers Z. Then the more general quadratic fields are briefly investigated.

### 12.2 Factorization Trapdoor System in Gaussian Integers

### 12.2.1 Ring of Gaussian Integers

Before considering the design of the factorization trapdoor system in this ring, it is useful to describe very briefly some of the properties of Gaussian integers. A detailed treatment of Gaussian integers can be found in [63, 64, 65].

Let $i = \sqrt{-1}$ and consider the set of complex numbers $Z[i]$ defined by $\{a+bi \mid a,b \in Z\}$. This set is closed under addition and subtraction. Moreover if $a+bi$, $c+di \in Z[i]$, then $(a+bi)(c+di) = ac+adi+bci+bdi^2 = (ac-bd) + (ad+bc)i \in Z[i]$. Thus $Z[i]$ is closed under multiplication and is a ring.

The norm of an element, $\alpha = a+bi$, in $Z[i]$ is defined to be equal to $\alpha . \bar{\alpha}$ where $\bar{\alpha}$ is the complex conjugate of $\alpha$. That is, $N\alpha = (a+bi)(a-bi)$ and hence $N\alpha = a^2+b^2$. Further it is seen that $N(\alpha\beta) = (N\alpha).(N\beta)$ as $N(\alpha\beta) = (\alpha\beta).(\overline{\alpha\beta}) = (\alpha\bar{\alpha})(\beta\bar{\beta})$.

An integer in $Z[i]$, $\alpha$, is called a unit if 1 is divisible by $\alpha$. Hence if $\alpha$ is a unit both $\alpha$ and $1/\alpha$ are integers in $Z[i]$.

Lemma 1

The norm of a unit is 1 and any integer whose norm is 1 is a unit.

If $\alpha$ is a unit then $\alpha \mid 1$, that is, $1 = \alpha\beta$ and so $1 = N\alpha N\beta$. This means that $N\alpha \mid 1$ and hence $N\alpha = 1$. If $\alpha = a+bi$, then $1 = a^2+b^2 = (a+bi)(a-bi)$, that is, $(a+bi) \mid 1$. So $(a+bi)$ is a unit. The units in $Z[i]$ are $\pm 1$, $\pm i$ as the only solutions of $a^2+b^2=1$ are $a = \pm 1$, $b = 0$; $a = 0$, $b = \pm 1$.

A prime $\Pi$ is an integer, not 0 or a unit, divisible only by the numbers associated with itself or with 1. In $Z[i]$, a prime has no divisors except the eight trivial ones, namely, $1, \Pi$, $-1, -\Pi$, $i$, $i\Pi$, $-i$ and $-i\Pi$.

Lemma 2

An integer whose norm is a rational prime is also a prime. For suppose that $N\alpha = p$ and that $\alpha = \beta_1 \beta_2$ where $\alpha, \beta_1, \beta_2 \varepsilon Z[i]$. Then $p = N\alpha = N\beta_1 N\beta_2$. Hence either $N\beta_1 = 1$ or $N\beta_2 = 1$ and either $\beta_1$ or $\beta_2$ is a unit; and therefore $\alpha$ is a prime.

Lemma 3

If $\Pi$ is a prime in $Z[i]$, then it divides exactly one positive rational prime p.

$N\Pi = \Pi\bar{\Pi}$ and so $\Pi | N\Pi$. Let the prime decomposition of $N\Pi$ in Z be $N\Pi = p_1 \dots p_r$ where $p_i$'s $(1 \leq i \leq r)$ are distinct positive primes. Then $\Pi | p_1 \dots p_r$. That is, $\Pi$ divides one of the primes $p_i$. It cannot divide two primes $p_j$ and $p_k$. If so, then one can find two rational integers $\ell_j$ and $\ell_k$ using Euclid's algorithm such that $\ell_j p_j + \ell_k p_k = 1$. If $\Pi | p_j$ and $\Pi | p_k$ then $\Pi | 1$. So $\Pi$ is a unit not a prime contrary to the hypothesis.

Lemma 4

Any integer, not zero or unit, is divisible by a prime and can be expressed as a product of primes.

If $\alpha$ is an integer, not a prime, then

$$\alpha = \beta_1 \beta_2, \quad N\beta_1 > 1, \ N\beta_2 > 1, \ N\alpha = N\beta_1 N\beta_2 \text{ and } 1 < N\beta_1 < N\alpha$$

If $\beta_1$ is not a prime, then

$$\beta_1 = \beta_3 \beta_4, \ N\beta_3 > 1, \ N\beta_4 > 1, \ N\beta_1 = N\beta_3 \ N\beta_4 \text{ and } 1 < N\beta_3 < N\beta_1$$

This process can be continued so long as $\beta_r$ is not a prime. Since $N\alpha$, $N\beta_1$, $N\beta_2$ ... form a decreasing sequence of positive rational integers, this must come to a prime $\beta_r$. Then $\alpha = \beta_2\beta_1 = \beta_2\beta_3\beta_4 = \dots = \beta_2 \beta_3 \dots \beta_r$ and so $\beta_r | \alpha$. Therefore now it is assumed that $\alpha$ is divisible by a prime $\Pi_1$ and $\alpha = \Pi_1 \alpha_1$ and the above process can be repeated for $\alpha_1$.

In Z[i], the representation of $\alpha$ as a product of primes is unique as in Z (except from trivial variations).

The unique factorization of integers in Z[i] is equivalent to the principality of all ideals in it [63]. That is, Z[i] is a principal ideal domain. The argument to show this is very much similar to the one used to prove all ideals are principal in the ring of integers Z. Here instead of choosing the least positive number, the element of least positive norm is used. The prime ideals in Z[i] are therefore the ideals generated by primes in Z[i].

### 12.2.2 Design of Trapdoor Coding System in Z[i]

As before, $\alpha \equiv \beta \mod <\gamma>$, where $\alpha, \beta, \gamma \in Z[i]$, is defined to imply that $\alpha - \beta$ is in the principal ideal $<\gamma>$, that is, $\alpha - \beta$ is divisible by $\gamma$.

Fermat's theorem in Z[i] can be stated as follows:
If $\Pi_1$ and $\Pi_2$ are relatively prime then

$$\Pi_1^{\phi<\Pi_2>} \equiv 1 \mod <\Pi_2> \tag{12.1}$$

where the Euler totient function $\phi<\Pi_2> = N\Pi_2 - 1$

Now if an ideal $<m>$ generated by a Gaussian integer m is considered whose decomposition is

$$m = \Pi_1 \ldots \Pi_r$$

where $\Pi_i$ for $1 \leq i \leq r$ are distinct primes in Z[i], then the number of invertible residue classes modulo $<m>$ is given by $\phi<m>$

$$\phi<m> = \phi<\Pi_1> . \phi<\Pi_2> \ . \ . \ . \ \phi<\Pi_r> \tag{12.2}$$

In order to calculate the order $\phi<m>$, one needs to compute $N\Pi_i$ for all i.

Using Lemma 3, let the Gaussian prime $\Pi$ divide a rational prime p, ie, $\Pi | p$. Then $N\Pi | Np$. But $Np = p^2$. Therefore $N\Pi = p$ or $N\Pi = p^2$. That is, if $\Pi = a + bi$, then $a^2 + b^2 = p$ or $p^2$.

Case 1 : $p \equiv 3 \pmod 4$

Since p is odd, one of a or b must be even, the other odd. Otherwise, the sum of their squares would be even. Let a = 2x and b = 2y + 1. If $a^2 + b^2 = p$ then

$$\begin{aligned} p &= 4\,x^2 + (2y + 1)^2 \\ &= 4(x^2 + y^2 + y) + 1 \\ &\equiv 1 \pmod{4} \end{aligned}$$

whereas to begin with,the assumption was $p \equiv 3 \pmod{4}$. So in this case, only $a^2 + b^2 = p^2$ is possible and $N\Pi = Np = p^2$. That is, the rational prime p stays prime in Z[i].

Case 2 : $p \equiv 2 \pmod{4}$

p = 2 is the only prime which falls into this class and from cryptography point of view, this case is not interesting. ($N\Pi = 2$).

Case 3 : $p \equiv 1 \pmod{4}$

Here p is of the form 4k + 1 where k is any rational integer. Then $p | n^2 + 1$ for some rational integer n. But $(n^2 + 1) = (n + i)(n - i)$ and as $\Pi | p$, $\Pi$ divides $(n + i)(n - i)$. But p does not divide $(n + i)$ or $(n - i)$ for otherwise one of $n/p \pm 1/p\, i$ would be a Gaussian integer; this cannot be possible as 1/p is not a rational integer. Hence $\Pi$ and p are not associated and $N\Pi \neq Np$. So $a^2 + b^2 \neq p^2$ and hence only $a^2 + b^2 = p$ is possible, ie $N\Pi = p$.

The system designer first chooses primes $p_1, p_2, \ldots, p_r$ randomly for some $r \geqslant 2$ such that $p_j \equiv 1 \pmod{4}$ or $p_j \equiv 3 \pmod{4}$, for $1 \leqslant j \leqslant r$. Then he computes the norm of each of the Gaussian primes $\Pi_j$ using $N\Pi_j = p_j$ or $p_j^2$. The order $\bar{\emptyset}\langle m\rangle$ can then be calculated using

$$\bar{\emptyset}\langle m\rangle = \prod_{j=1}^{r} (N\Pi_j - 1) \qquad (12.3)$$

The coding exponents e and d required for encryption and decryption procedures can be determined using

$$ed \equiv 1 \pmod{\bar{\emptyset}\langle m\rangle} \qquad (12.4)$$

The public encryption key consists of (e, m) and the secret decryption key is (d, m) where $m \in Z[i]$. As the designer is required to make m public, to obtain m, he needs to solve equations of the form

$$a_j^2 + b_j^2 = p_j \text{ or } p_j^2 \quad \text{for } 1 \leq j \leq r$$

and $\Pi_j = a_j + b_j i$.

Consider the problem of finding a and b in $a^2 + b^2 = p$ or $p^2$ given p (system designer knows p). The subscript j has been dropped for convenience.

<u>Case 1 :</u> $a^2 + b^2 = p^2$ where $p \equiv 3 \pmod 4$

This implies that

$$a^2 + b^2 \equiv 0 \pmod p \qquad (12.5)$$

Suppose $a \not\equiv 0 \pmod p$, then

$$1 + \left(\frac{b}{a}\right)^2 \equiv 0 \pmod p$$

Letting $X = b/a$, then

$$1 + X^2 \equiv 0 \pmod p$$

ie $$X^2 \equiv -1 \pmod p \qquad (12.6)$$

For the congruence (12.6) to have a solution, -1 must be a quadratic residue modulo p (Section 12.3.2). That is, the Legendre Symbol $\left(\frac{-1}{p}\right)$ must be equal to 1.

$$\left(\frac{-1}{p}\right) = (-1)^{(p-1)/2} = 1$$

As $p \equiv 3 \pmod 4$, $p = 4k + 3$ for some rational integer k

ie, $p-1 = 4k + 2$

Therefore,

$$(-1)^{(p-1)/2} = (-1)^{2k+1} = -1$$

Thus when $p \equiv 3 \pmod 4$, the congruence (12.5) cannot be solved for a and b if $a \not\equiv 0 \pmod p$. Thus 'a' must be equal to 0 (mod p). This implies that the only solutions of (12.5) that are possible are:

$$a = 0,\ b^2 = p^2;\ b = 0,\ a^2 = p^2$$

Case 2 : $a^2 + b^2 = p$ where $p \equiv 1 \pmod 4$ (12.7)

Method 1

Again assuming $a \not\equiv 0 \pmod p$ and letting $X = b/a$,

$$1 + X^2 \equiv 0 \pmod p \qquad (12.8)$$

For the above congruence to have a solution, -1 must be a quadratic residue modulo p , ie,

$$(-1)^{(p-1)/2} = 1$$

As $p \equiv 1 \pmod 4$, $p - 1 = 4k$ for some rational integer k. Therefore, $(-1)^{(p-1)/2} = 1$.

Hence there are solutions of (12.7) for which $a, b \not\equiv 0 \pmod p$. The solutions to the congruence (12.8) are given by

$$X = \pm \left[\frac{(p-1)}{2}\right]! \pmod p$$

This can be seen as follows:

$$\left\{\left[\frac{p-1}{2}\right]!\right\}^2 \equiv (1.2 \ldots \frac{p-1}{2})(1.2 \ldots \frac{p-1}{2}) \pmod p$$

$$\equiv (1.2 \ldots \frac{p-1}{2})(-(p-1) \ldots -(\frac{p+1}{2})) \pmod p$$

$$\equiv (1.2 \ldots \frac{p-1}{2})(\frac{p+1}{2} \ldots p-1)(-1)^{(p-1)/2}$$

$$\left\{\left[\frac{p-1}{2}\right]!\right\}^2 \equiv (p-1)! \equiv -1 \pmod p \text{ using Wilson's theorem [38].}$$

Hence $\{[\frac{p-1}{2}]!\}$ is a solution of (12.8).

As $b \equiv a X \pmod p$, 'a' is allowed to vary in the range 0 to $\sqrt{p}$ and the least positive residue $aX \pmod p$ is tested to see whether it is less than $\sqrt{p}$. If so, then this value can be used for b because then $0 < a^2 + b^2 < 2p$ and $a^2 + b^2 \equiv 0 \pmod p$ imply that $a^2 + b^2 = p$.

Example

Using the above method to find a and b such that

$$a^2 + b^2 = p = 29 \qquad (12.9)$$

Then

$$X = \pm \left(\frac{29-1}{2}\right)!$$

$$X \equiv 12 \pmod{29}$$

Considering $0<a<\sqrt{29}$, ie, $0<a\leqslant 5$, gives

$$b_1 \equiv 1.12 \equiv 12 \pmod{29}$$
$$b_2 \equiv 2.12 \equiv 24 \pmod{29}$$
$$b_3 \equiv 3.12 \equiv 7 \pmod{29}$$
$$b_4 \equiv 4.12 \equiv 19 \pmod{29}$$
$$b_5 \equiv 5.12 \equiv 2 \pmod{29}$$

Hence $b = b_5 = 2$ is the only one which is less than 5 and therefore $a = 5$, $b = 2$ would satisfy (12.9).

For large prime p, this method does not appear to be feasible as one needs to evaluate $(\frac{p-1}{2})!$ and test values of 'a' upto $\sqrt{p}$. This requires of the order of $(\frac{p}{2}+\sqrt{p})$ multiplications (mod p) in the worst case.

Method 2

This method involves the use of Jacobsthal sum, $S(c)$,[66] in determining a and b in (12.7). $S(c)$ is given by

$$S(c) = \sum_{n=1}^{p} \left(\frac{n}{p}\right) \left(\frac{n^2-c}{p}\right)$$

where $p \equiv 1 \pmod 4$ and $c \not\equiv 0 \pmod p$. (Note that $S(c) = 0$ if $p \equiv 3 \pmod 4$).

Considering

$$\sum_{c=1}^{p} S^2(c) = \sum_{c=1}^{p} \sum_{m=1}^{p} \left(\frac{m}{p}\right) \left(\frac{m^2-c}{p}\right) \sum_{n=1}^{p} \left(\frac{n}{p}\right) \left(\frac{n^2-c}{p}\right)$$
$$= \sum_{m=1}^{p} \sum_{n=1}^{p} \left(\frac{m}{p}\right) \left(\frac{n}{p}\right) \left\{ \sum_{c=1}^{p} \left(\frac{m^2-c}{p}\right) \left(\frac{n^2-c}{p}\right) \right\}$$

Consider the inner sum { }.

if $m^2 \equiv n^2$ (mod p), then it is equal to $\sum_{c=1}^{p} \left(\frac{m^2-c}{p}\right)^2 = p-1$

if $m^2 \not\equiv n^2$ (mod p), then it is equal to $-1$.

$$\therefore \sum_{c=1}^{p} s^2(c) = (p-1) \underset{m^2 \equiv n^2}{\sum_{m=1}^{p} \sum_{n=1}^{p}} \left(\frac{m}{p}\right) \left(\frac{n}{p}\right) - \underset{m^2 \not\equiv n^2}{\sum_{m=1}^{p} \sum_{n=1}^{p}} \left(\frac{m}{p}\right) \left(\frac{n}{p}\right)$$

That is,

$$\sum_{c=1}^{p} s^2(c) = 2(p-1)(p-1) - \sum_{n=1}^{p} \left(\frac{n}{p}\right) \left(0-\frac{n}{p} - \left(\frac{-n}{p}\right) \right)$$

$$= 2(p-1)^2 + 2(p-1)$$

$$\sum_{c=1}^{p} s^2(c) = 2p\ (p-1)$$

But

$$\sum_{c=1}^{p} s^2(c) = \left(\frac{p-1}{2}\right) s^2(1) + \left(\frac{p-1}{2}\right) s^2(k)$$

where k is any quadratic non-residue [66].

That is, $2p\ (p-1) = \frac{p-1}{2}\ s^2(1) + \frac{p-1}{2}\ s^2(k)$

$$p = \left(\frac{s(1)}{2}\right)^2 + \left(\frac{s(k)}{2}\right)^2$$

Thus to calculate a and b in $a^2 + b^2 = p$, one finds s(1) where

$$s(1) = \sum_{n=1}^{p} \left(\frac{n}{p}\right) \left(\frac{n^2-1}{p}\right)$$

and then computes $p - \left(\frac{s(1)}{2}\right)^2$ giving

$$a^2 = \left(\frac{s(1)}{2}\right)^2 \quad \text{and } b^2 = p - \left(\frac{s(1)}{2}\right)^2$$

Considering the same example as before to find a and b in $a^2 + b^2 = p = 29$, using this method

$$s(1) = \sum_{n=1}^{29} \left(\frac{n}{29}\right) \left(\frac{n^2-1}{29}\right) = 10$$

$(\frac{s(1)}{2})^2 = 25$ hence $a = 5$ and $b = \sqrt{p - (\frac{s(1)}{2})^2} = 2$

Again this method may not seem to be attractive when p is large.

Method 3

Legendre's method [86] is based on the continued fraction expansion of $\sqrt{p}$. He showed that if p is of the form 4k + 1, then the expansion of $\sqrt{p}$ is of the form

$$\sqrt{p} = q_0 + \cfrac{1}{q_1 + \cfrac{1}{q_2 + \dots \cfrac{1}{q_2 + \cfrac{1}{q_1 + \cfrac{1}{2q_0 + \dots}}}}}$$

It is seen that there is a symmetrical part $q_1, q_2, \dots, q_2, q_1$ followed by $2q_0$ and there is no central term. Now let c be the particular complete quotient which begins at the middle of the period, that is,

$$c = c_m = q_m + \cfrac{1}{q_{m-1} + \dots \cfrac{1}{q_1 + \cfrac{1}{2q_0 + \cfrac{1}{q_1 + \dots}}}}$$

This is a purely periodic continued fraction whose period consists of $q_m, \dots, q_1, 2q_0, q_1, \dots, q_m$. Since this period is symmetrical, $\bar{c} = -\frac{1}{c}$, where $\bar{c}$ denotes the conjugate of c. c is now expressible in the form $c = \frac{b + \sqrt{p}}{a}$ where a and b are integers. The equation $c\bar{c} = -1$ gives $(\frac{b + \sqrt{p}}{a}) \cdot (\frac{b - \sqrt{p}}{a}) = -1$ or $a^2 + b^2 = p$.

Again consider the example p = 29. The process for developing $\sqrt{29}$ in a continued fraction is

$$\sqrt{29} = 5 + \frac{1}{c_1}$$

$$c_1 = \frac{1}{4}(5 + \sqrt{29}) = 2 + \frac{1}{c_2}$$

$$c_2 = \frac{1}{5}(3 + \sqrt{29}) = 1 + \frac{1}{c_3}$$

$$c_3 = \frac{1}{5}(2 + \sqrt{29}) = 1 + \frac{1}{c_4}$$

$$c_4 = \frac{1}{4}(3 + \sqrt{29}) = 2 + \frac{1}{c_5}$$

$c_5 = 5 + \sqrt{29}$

The continued fraction is $5, \overline{2, 1, 1, 2, 10}$. The appropriate complete quotient to take is $c = c_3$ giving $a = 5$ and $b = 2$.

Method 4

The cryptographer can also design the system starting with primes $\Pi_j$ in $Z[i]$ rather than primes $p_j$ in $Z$. That is, he selects Gaussian integers $\Pi_j$, $1 \leq j \leq r$, arbitrarily by choosing say $a_j$ to be an even integer and $b_j$ to be an odd integer in $\Pi_j = a_j + b_j i$ so that $a_j^2 + b_j^2$ is always odd. Then he can test the norms $N\Pi_j$ for $1 \leq j \leq r$, to find whether they are primes or squares of primes in $Z$. The primality testing can be done using the probabilistic algorithm mentioned in Section 10.2. If $N\Pi_j$ is a prime or a square of a prime in $Z$, then $\Pi_j$ must be a prime in $Z[i]$. If $N\Pi_j$ is not a prime or a prime square in $Z$, then he chooses another pair of $a_j$ and $b_j$. As this procedure needs to be done only once by each user at the beginning, this could be a feasible approach especially when $r = 2$ as in the prototype RSA system.

Choice of primes

Although the designer can choose any primes of the form $p \equiv 3 \pmod 4$ or $p \equiv 1 \pmod 4$ (except the special ones such as the Mersenne primes), now it is shown that certain combinations will result in the easy factorization of the composite Gaussian integer m. If m is considered to be a product of two Gaussian primes $\Pi_1$ and $\Pi_2$ where $\Pi_1 | p_1$ and $\Pi_2 | p_2$, then there are four possible combinations. They are:

(i) $p_1 \equiv 1 \pmod 4$, $p_2 \equiv 1 \pmod 4$

(ii) $p_1 \equiv 3 \pmod 4$, $p_2 \equiv 3 \pmod 4$

(iii) $p_1 \equiv 1 \pmod 4$, $p_2 \equiv 3 \pmod 4$

(iv) $p_1 \equiv 3 \pmod 4$, $p_2 \equiv 1 \pmod 4$

Case (i) results in $\Pi_1 = a_1 + b_1 i$ and $\Pi_2 = a_2 + b_2 i$ where $a_1, a_2, b_1, b_2 \neq 0$ in $Z$ and $m = \Pi_1 \Pi_2 \in Z[i]$. The order of the group formed by the invertible residue classes modulo $<m>$ is therefore equal to $\phi<m> = (N\Pi_1 - 1)(N\Pi_2 - 1) = (p_1 - 1)(p_2 - 1)$.

Case (ii) yields $\Pi_1 = a_1$ or $b_1 i$ and $\Pi_2 = a_2$ or $b_2 i$ as one of each pair (a,b) is equal to zero. Letting $m = a_1 a_2$, it is seen that m is a rational integer and the order $\bar{\phi}\langle m\rangle = (p_1^2-1)(p_2^2-1)$.

Case (iii) and (iv) will yield an m of the form cd+cfi. As m is made public, the opponent can easily spot the factor c and hence factorize m.

Hence it is seen from cryptographic considerations that, only the schemes where both $p_1$ and $p_2$ are chosen to be of same type (either 1(mod 4) or 3(mod 4)) will provide secure systems. This idea extends to all primes $p_j$, $1 \leq j \leq r$, when m is a product of r Gaussian primes.

### 12.2.3 Security of the System in Z[i]

As in the case of the prototype RSA system over Z, the publicly available information for the opponent consists of the encrypting exponent e and the composite modulus m. If the rational primes are chosen such that $p_j \equiv 3 \pmod 4$ for all j, then the modulus m can be made to be a rational integer and the norm $Nm = m^2$. If the primes are chosen such that $p_j \equiv 1 \pmod 4$ for all j, then m is of the form $c+di \in Z[i]$. In this case, the opponent can easily calculate the norm $Nm = c^2+d^2$. Hence in either case, the security of the system essentially lies in the difficulty of factoring a large rational integer, the norm Nm. The problem of factoring Nm is similar to that of factoring the modulus m ($\epsilon Z$) in the prototype RSA system. Thus the security of this system is same as that of its predecessor. Once the norm Nm is factorized into r primes $q_j$, then the order $\bar{\phi}\langle m\rangle$ can be found using $\prod_{j=1}^{r}(N\Pi_j-1)$ where $N\Pi_j = q_j$ or $q_j^2$. Then the opponent can compute the secret decoding exponent d using (12.4). Note that the cryptanalyst does not need to know the Gaussian primes $\Pi_1,\ldots,\Pi_r$ but only needs to know their respective norms. In other words, the opponent will be working over Z not over Z[i] and does not need to solve the equation $a^2+b^2 = p$ for a and b.

### 12.2.4 Representation of Messages and System Operation

The messages are to be represented using the residue classes modulo the ideal generated by the modulus m. The number of distinct messages possible is equal to the number of incongruent residues mod $\langle m\rangle$ and is given by the norm of the ideal $\langle m\rangle$. In the case of a principal ideal domain $N\langle m\rangle = Nm$, the norm of the element m.

In performing encryption or decryption using this system in

$Z[i]$, there is a problem with regard to the message representation. First consider the process of reduction mod<m> which is required in both encryption and decryption procedures. This can be carried out using the Euclidean algorithm as $Z[i]$ is a PID. Let $y = x^e$ where e is the encrypting exponent and consider the operation y mod<m> where x, y and $m \in Z[i]$.

Using Euclid's algorithm, there exists two integers u and v in $Z[i]$ such that

$$y = um + v \quad \text{where } Nv < Nm$$

Consider

$$y/m = A + Bi \quad \text{where A and B are rational numbers.}$$

Choose rational integers s and t such that

$$|A-s| \leqslant \tfrac{1}{2}, \ |B-t| \leqslant \tfrac{1}{2}$$

This can be done by choosing s and t as rational integers nearest to A and B respectively (see Figure 12.1). Now let u = s+ti and v = y-um, then it is seen that Nv< Nm as

$$|v| = |y-um| = |y-(s+ti)\,m|$$
$$= |m|\,\left|\frac{y}{m} - s - ti\right|$$

$$|v| = |m|\,|(A-s) + (B-t)i|$$
$$= |m|\,\{(A-s)^2 + (B-t)^2\}^{\frac{1}{2}}$$

$$\therefore\ |v| \leq |m|\,\{\frac{1}{2^2} + \frac{1}{2^2}\}^{\frac{1}{2}} < |m|$$

Since $Nv = |v|^2 < |m|^2 = Nm$, the inequality is established .

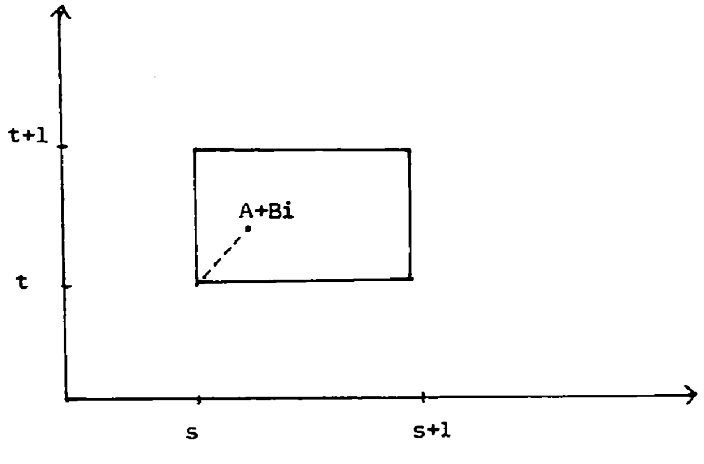


Fig 12.1

But the u and v are not uniquely determined. Thus if a message $M = x+iy$ is encrypted to form $g+hi \equiv (x+iy)^e \bmod \langle m\rangle$, the decrypted message, $M'$ is given by

$$M' \equiv (g+hi)^d \bmod \langle m\rangle$$
$$\equiv k+\ell i \quad \bmod \langle m\rangle$$

The decrypted message, $k+\ell i \equiv x+iy \bmod \langle m\rangle$ but it is of 'different representation'. For instance, consider the example given below.

Let the modulus $m = 8+i \in Z[i]$

The norm, $Nm = 65$

$$= N\Pi_1\, N\Pi_2 = 5.13$$

As $p_1 \equiv 1 \pmod 4$ and $p_2 \equiv 1 \pmod 4$, $\phi\langle m\rangle$ is given by

$$\phi\langle m\rangle = 4.12$$
$$= 48$$

Choosing $e = d = 7$ where $ed \equiv 1 \pmod{48}$, consider the message $M = 5+3i$. Using the Euclidean algorithm,

$$(5+3i)^7 \equiv -2-i \bmod \langle m\rangle \equiv g+hi$$

Thus

$$M' \equiv (-2-i)^7 \quad \bmod \langle m\rangle$$

ie, $M' \equiv (-2-i)^7 \equiv -2-6i \bmod \langle m\rangle \equiv k+\ell i$

But

$$(-2-6i) \equiv 5+3i \equiv 2i-3 \bmod \langle m\rangle$$

The receiver would not be able to differentiate say between $-2-6i$ or $5+3i$ or $2i-3$ although they are congruent to each other mod $\langle m\rangle$. Thus there is not a unique representation of messages unlike in the case of the RSA system over Z where a message is uniquely represented by taking the least positive residue modulo m. In Z[i], one approach could be to use the norm of the modulus, Nm, to construct a standard set of representatives. But this would not work as two elements $\alpha$ and $\beta$ can be congruent to each other and their norms $N\alpha$ and $N\beta$ be less than Nm. Thus some form of standard set of representatives is essential.

Case 1

First consider the case where the primes $\Pi_j$ which form m divide rational primes $p_j$ of the form $p_j \equiv 1 \pmod 4$. Then the norm is a square free rational integer given by

$$Nm = \prod_{j=1}^{r} p_j \qquad \text{where } m = \prod_{j=1}^{r} \Pi_j$$

The residue class ring $Z[i]/\langle m\rangle$ is isomorphic to a direct sum of finite fields $Z[i]/\langle\Pi_j\rangle$ , ie,

$$Z[i]/\langle m\rangle \cong Z[i]/\langle\Pi_1\rangle \oplus \ldots \oplus Z[i]/\langle\Pi_r\rangle$$

The field $Z[i]/\langle\Pi_j\rangle$ contains $N\Pi_j = p_j$ elements. Therefore one standard method of representing the messages mod $\langle m\rangle$ would be to use the integers in the ring $Z/NmZ$, that is, 0 to $Nm-1$. And every element of $Z[i]/\langle m\rangle$ is congruent to an element in $Z/NmZ$. This is same as the message space of the RSA system over the rationals.

For example if $p_1 = 13$ and $p_2 = 5$, $p_1 \equiv p_2 \equiv 1 \pmod 4$ then $Nm = 65$ and $\phi\langle m\rangle = 48$. The message space is therefore equal to $\{0, 1, \ldots 64\}$. The encryption and decryption processes are carried out using $C \equiv M^e \pmod{Nm}$ and $M \equiv C^d \pmod{Nm}$ where $ed \equiv 1 \pmod{\phi\langle m\rangle}$. The coding exponent e and Nm are made public.

Now consider the case where the message space still consists of the integers in $Z/NmZ$ but now the encryption and decryption procedures are calculated modulo m where $m = a+bi \in Z[i]$. In this case, e, m and Nm are made public. Let the message be $M \in Z/NmZ$, then the encryption procedure results in

$$M^e \bmod (a+bi) \equiv g+hi = \text{cipher}$$

Decryption produces

$$(g+hi)^d \bmod (a+bi) \equiv k+\ell i$$

That is, the recovered message M is equal to $k+\ell i$

$$M \equiv k+\ell i \pmod{a+bi} \tag{12.10}$$

Conjugating both sides of (12.10)

$$M \equiv k-\ell i \pmod{a-bi} \tag{12.11}$$

Using Chinese Remainder Theorem, the original M can be formed where

$$M \equiv \gamma_1 (k-\ell i) + \gamma_2 (k+\ell i) \pmod{(a+bi)(a-bi)}$$

where $\gamma_1 + \gamma_2 = 1$ and $M \in Z/NmZ$, $\gamma_1, \gamma_2 \in Z[i]$

A standard set of representatives of the ring $Z[i]/\langle m\rangle$ can also be obtained using elementary divisor theory [65].

The ideal $\langle m\rangle$ in $Z[i]$ is generated by the element $m = a+bi$

and it consists of the set of Gaussian integers

$$<m> \;=\; \{(a+bi)\lambda\} \qquad \text{where } \lambda \in Z[i]$$

The ideal <m> can be regarded as a Z-module generated by [a+bi, ai-b] and the Z-basis of the ideal <m> is therefore given by (a+bi, ai-b). The integral basis of Z[i]is (1,i) [65].

Hence one can associate a matrix A with the pair of bases as follows

Z-basis of ideal <m> = A (Z-basis of Z[i] )

ie,

$$\begin{pmatrix} a + bi \\ ai - b \end{pmatrix} = A \begin{pmatrix} 1 \\ i \end{pmatrix}$$

where $A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$

Let

$$\underset{\sim}{t} = \begin{pmatrix} a + bi \\ ai - b \end{pmatrix} \text{ and } \underset{\sim}{s} = \begin{pmatrix} 1 \\ i \end{pmatrix}$$

Therefore

$$\underset{\sim}{t} = A \underset{\sim}{s} \qquad (12.12)$$

$\underset{\sim}{s}$ and $\underset{\sim}{t}$ can be replaced by $\underset{\sim}{w}$ and $\underset{\sim}{v}$ where $\underset{\sim}{w}$ and $\underset{\sim}{v}$ are equal to $\underset{\sim}{s}$ and $\underset{\sim}{t}$ multiplied by some unimodular matrices respectively.

Let

D = VAU where U and V are suitably chosen unimodular transformations such that D = diag($d_1$, $d_2$) and $d_1 | d_2$. That is,

$$\begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} = VAU$$

Multiplying $\underset{\sim}{s}$ by $U^{-1}$, gives

$$\underset{\sim}{w} = U^{-1} \underset{\sim}{s}$$

and hence

$$\underset{\sim}{t} = AU \underset{\sim}{w} \qquad (12.13)$$

Multiplying (12.13) by V gives

$$V\underset{\sim}{t} = VAU \underset{\sim}{w}$$

ie $$\underset{\sim}{v} = D \underset{\sim}{w}$$

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

The new basis of $Z[i]$ is $(w_1, w_2)$ and the new basis of $<m>$ is $(d_1\ w_1, d_2 w_2)$.
Therefore, the residue classes mod $<m>$ are represented as $x_1 w_1 + x_2 w_2$ where $x_i$ goes from 0 to $d_i-1$ for $i=1,2$.

Hence the system designer is required to reduce the matrix A to its diagonal form. The steps involved in this reduction algorithm of an nxn matrix A are now considered [67].

First Stage of Reduction

The aim is to reduce the nxn matrix A to an equivalent nxn matrix C of the special form

$$C = \left[\begin{array}{c|c} d_1 & 0 \ldots 0 \\ \hline 0 & \\ \vdots & C^* \\ 0 & \end{array}\right] \qquad (12.14)$$

where $d_1$ divides each entry of $C^*$.
A finite sequence of elementary row and column operations is considered which when performed on A either yields a matrix of the form (12.14) or else leads to an nxn matrix $B = (b_{ij})$ satisfying the condition

$$b_{11} < a_{11} \qquad (12.15)$$

In the latter case, one goes back to the beginning and applies the sequence of operations again. Either the form (12.14) is achieved in which case, this stage ends or (12.15) is reached in which case the leading entry is reduced still further and the process continues. After a finite number of steps, the form (12.14) will be reached.

The sequence of operations is as follows:
If A is the zero matrix, then it is already of the form (12.14); otherwise, A has a non-zero entry and by suitable interchanges of rows and columns this can be moved to the leading position. Therefore assume $a_{11} \neq 0$ and consider the following three possibilities:

Case (i)

There is an entry $a_{1j}$ in the first row such that $a_{11} \nmid a_{1j}$.
By the properties of Euclidean domain,

$$a_{1j} = a_{11}\, q + r$$

where either $r = 0$ or $r < a_{11}$
Since $a_{11} \nmid a_{1j}$, one must have $r \neq 0$ and so $r < a_{11}$. Subtracting q times the first column from the jth column and then interchanging the first and the jth columns, the leading entry $a_{11}$ is replaced by r and (12.15) is achieved.

Case (ii)

There is an entry $a_{11}$ in the first column suchthat $a_{11} \nmid a_{i1}$. In this case, proceeding as in Case(i) but operating with rows instead of columns, (12.15) is reached.

Case (iii)

$a_{11}$ divides every entry in the first row and first column. In this case, by subtracting suitable multiples of the first column from the other columns, one can replace all the entries in the first row other than $a_{11}$ itself by zeroes. Similarly, subtracting multiples of the first row from the others, a matrix is obtained which is of the form,

$$D = \left[\begin{array}{c|c} a_{11} & 0 \ldots 0 \\ \hline 0 & D^* \\ \vdots & \\ 0 & \end{array}\right]$$

If $a_{11}$ divides every entry of $D^*$, (12.14) has been reached; if not there is an entry, say, $d_{ij}$ such that $a_{11} \nmid d_{ij}$. In that case, by adding the ith row to the top row leads to case (i).

Repeated application of these procedures will result in the form (12.14) after a finite number of steps, thereby completing the first stage of reduction.

End of Reduction

By applying the above process to the submatrix $C^*$, one can reduce its size still further, leaving a trail of diagonal elements. Any elementary operation on $C^*$ corresponds to an elementary operation on C which does not affect the first row and column. Also any elementary operation on $C^*$ gives a new matrix whose entries are linear combinations of the old ones and hence the new entries will therefore still be divisible by $d_1$. Thus $d_1 | d_2 | \ldots | d_n$.

In particular in the case of a 2x2 matrix A, the unimodular transformations can be found as follows

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $b > a$ $\quad a \nmid \gcd(b,c,d)$

It is first shown that the matrix A is equivalent to a matrix of the form

$$\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \quad \text{where} \quad a' \mid \gcd(b',c',d')$$

Using induction on a: the case $a = 1$ is trivial; when $a > 1$ and $a \nmid b$, choosing q so that $0 < aq + b < a$

and consider

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} q & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} aq+b & * \\ * & * \end{pmatrix}$$

where the leading element is a positive integer less than a. If $a \mid b$ and $a \nmid c$, then choosing $q'$ such that $0 < aq'+c < a$ and consider

$$\begin{pmatrix} q' & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} aq'+c & * \\ * & * \end{pmatrix}$$

where the leading element is once again a positive integer less than a. Finally if $a \mid \gcd(b,c)$ but $a \nmid d$, let $c = c_1 a$ so that

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -c_1 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & (1-c_1)b+d \\ * & * \end{pmatrix}$$

and $a \nmid \{(1-c_1)\, b+d\}$ which reduces back to the case when $a \nmid b$. The inductive argument is now complete. Now $a' \mid \gcd(b',c',d')$. Letting $b' = a'b''$, $c' = a'c''$ and $d' = a'd''$ and considering

$$\begin{pmatrix} 1 & 0 \\ -c'' & 1 \end{pmatrix}\begin{pmatrix} a' & a'b'' \\ a'c'' & a'd'' \end{pmatrix}\begin{pmatrix} 1 & -b'' \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a' & 0 \\ 0 & a'(d''-b''c'') \end{pmatrix} \quad (12.16)$$

the desired result is obtained.

With $A = \begin{pmatrix} a & +b \\ -b & a \end{pmatrix}$, (12.16) reduces to

$$D = \begin{pmatrix} a' & 0 \\ 0 & a'(1+b''^2) \end{pmatrix}$$

Further as $d_1 \mid d_2$ and $Nm = d_1 d_2$ where $Nm = \prod_{j=1}^{r} p_j$, the only allowed values for $d_1$ and $d_2$ are 1 and Nm respectively. So the messages can be represented as $x_2 w_2$ where $x_2$ goes from 0 to Nm-1. In this case, the elementary divisor $d_2$, the encrypting exponent e and the matrix U (see 12.13) are made public. The decrypting exponent d is kept secret. The security lies in the difficulty of factoring $d_2$ = Nm. The encryption procedure is carried out as follows:

Let $M = x_2 w_2$

$$\begin{aligned} C &\equiv M^e \bmod \langle m \rangle \\ &\equiv (x_2 w_2)^e \bmod \langle m \rangle \\ &\equiv x_2^e \, w_2^e \bmod \langle m \rangle \end{aligned}$$

Using

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = U^{-1} \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} \quad \text{where} \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} 1 \\ i \end{pmatrix}$$

and

$$i^2 + 1 = 0$$

$C' \equiv y_2' w_2$ is obtained. Now reducing $y_2'$ (mod $d_2$) the cipher C is obtained

$$C \equiv y_2 w_2 \bmod \langle m \rangle$$

A similar procedure can be carried out for decryption to recover the original message using the decrypting exponent d. An example using this method is now given below.

Let $\Pi_1 = 2+3i$ and $\Pi_2 = 1+2i$ where $\Pi_1$ and $\Pi_2$ are primes in $\mathbb{Z}[i]$. The modulus m = 7i-4 and Nm = 65. $\Phi\langle m \rangle = 48$. Let e = d = 7 where ed $\equiv$ 1 (mod 48).

$$A = \begin{pmatrix} -4 & 7 \\ -7 & -4 \end{pmatrix}$$

Using the method given above to reduce A to diagonal matrix D,

$$D = \begin{pmatrix} 1 & 0 \\ -18 & 1 \end{pmatrix} \begin{pmatrix} -4 & 7 \\ -7 & -4 \end{pmatrix} \begin{pmatrix} 2 & -7 \\ 1 & -4 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 65 \end{pmatrix}$$

$$\underset{\sim}{w} = \begin{pmatrix} 4 - 7i \\ 1 - 2i \end{pmatrix} = \begin{pmatrix} 4 & -7 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ i \end{pmatrix}$$

where

$$U^{-1} = \begin{pmatrix} 4 & -7 \\ 1 & -2 \end{pmatrix}$$

Hence

$$\begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} 2 & -7 \\ 1 & -4 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \qquad (12.17)$$

Let the message be $M = 4w_2$

The cipher is given by

$$C \equiv M^7 \bmod <7i-4>$$
$$\equiv 4^7 w_2^{\,7} \bmod < 7i-4>$$

Using (12.17), $w_2 = 1-2i$

and $w_2^7 = 29-278i$

Using (12.17),

$$29-278i = -220w_1 + 909w_2 \qquad (12.18)$$

Now reducing (12.18) (mod $d_1$) and (mod $d_2$), the cipher is given by

$$C \equiv 61w_2 \bmod <7i-4>$$

A similar process for decryption with decoding exponent $d = 7$ gives back the original message $M$.

Case 2

Now consider the case where the primes $\Pi_j$ which form the modulus m divide the rational primes $p_j$ of the form $p_j \equiv 3 \pmod 4$. Then the norm Nm is a non-square free rational integer given by

$$Nm = \prod_{j=1}^{r} p_j^{\,2}$$

In this case although $Z[i]/<\Pi_j>$ is a finite field of $p_j^{\,2}$ elements, one cannot represent the residue classes modulo $<\Pi_j>$ using the integers $Z/p_j^{\,2}Z$ as the latter does not form a field. On the other hand, one can represent the messages in the form $x+iy$ where $x, y \in Z$. As mentioned in 12.2.1, the Nm can be made to equal $m^2$ by appropriately choosing m to be a rational integer. Therefore, one can represent the distinct residue classes mod $<m>$ as $x+iy$ where $0\leq x,y \leq \lfloor\sqrt{Nm}\rfloor-1$, thus giving rise to Nm residue classes. Using the elementary divisor theory described above, this corresponds to the case where $d_1 = \prod_{j=1}^{r} p_j$ and $d_2 = \prod_{j=1}^{r} p_j$. Again these are the only allowed values for $d_1$ and $d_2$ as the following conditions must hold:

(i) $d_1 \mid d_2$

(ii) $d_1 . d_2 = Nm$

(iii) $Z/d_iZ$ is isomorphic to a direct sum of finite fields, for $i = 1,2$.

The divisors $d_1$ and $d_2$ and the exponent e are made public. The encryption and decryption procedures are very much simplified in this case. If a message M is raised to the power e then to perform mod <m> , it is only necessary to reduce the coefficients by the corresponding elementary divisors. Let M=x+iy be the message then, the cipher C is obtained by

$$\begin{aligned} C &\equiv M^e \bmod \langle m \rangle \\ &\equiv (x+iy)^e \bmod \langle m \rangle \\ &\equiv g(\bmod d_1) + h\ (\bmod d_2)\ i \end{aligned}$$

where $d_1 = d_2 = |\sqrt{Nm}|$

A similar procedure can be used for decryption.

## 12.3 Factorization Trapdoor System in Other Quadratic Fields

### 12.3.1 Quadratic Fields R ($\sqrt{D}$)

Note that when D=-1, $R(\sqrt{D})$, where R is the set of rational numbers, includes the ring of Gaussian integers considered in the previous section. Initially some of the properties of the integers in $R(\sqrt{D})$ which are required in the design of a trapdoor system are briefly examined. A detailed treatment of quadratic fields can be found in [61, 63, 65] .

A quadratic field is a field of degree 2 over the rationals. Such a field is of the form $R(\theta)$ where $\theta$ is a root of a quadratic polynomial which is irreducible over the rationals. Let $\theta$ satisfy an equation

$$x^2 + 2ax + b = 0 \quad \text{where } a, b \in Z$$

Then $\theta = -a \pm \sqrt{a^2-b}$. Removing from $a^2-b$ all square factors so that $a^2-b = s^2D$ where D has no factor higher than the first power, then $R(\theta)$ is equivalent to $R\ (\sqrt{D})$. That is, every quadratic field is of the form $R(\sqrt{D})$ where D is a rational integer free of square factors.

The ring of integers of $R(\sqrt{D})$ for square free D depends on the arithmetic properties of D. It is shown in [61] that the integers in $R(\sqrt{D})$ fall into two categories, namely,

(i) if $D \not\equiv 1 \pmod 4$, then the integers are of the form $x+y\sqrt{D}$

where x, y $\varepsilon$ Z, ie, Z $[\sqrt{D}]$

(ii) if $D \equiv 1 \pmod 4$, then the integers are of the form x+y $(\frac{1+\sqrt{D}}{2})$ where x, y $\varepsilon$ Z, ie, Z $\left[\frac{1+\sqrt{D}}{2}\right]$

Note that the Gaussian integers considered in the last section falls into the first category.

Using Lemma 1 (Section 12.2.1), it is seen that an integer in $R(\sqrt{D})$ is a unit if its norm is $\pm 1$. When $D \not\equiv 1 \pmod 4$, the norm of $\alpha$ is $N\alpha = \alpha\,\bar{\alpha} = (a+b\sqrt{D})\,(a-b\sqrt{D}) = a^2-Db^2$.

When $D \equiv 1 \pmod 4$, the norm of $\alpha$ is given by

$$N\alpha = [a + \frac{b}{2}(1+\sqrt{D})] \quad [a + \frac{b}{2}(1-\sqrt{D})] = (a + \tfrac{1}{2}b)^2 - \frac{D}{4}b^2.$$

Note that the norms are positive in complex quadratic fields (ie, D is negative) but not necessarily positive in real quadratic fields (ie, D is positive). Thus $\alpha$ is a unit if

$$\left.\begin{array}{l} a^2 - Db^2 = \pm 1 \\ \text{or} \\ (a + \frac{1}{2}b)^2 - \frac{D}{4}\, b^2 = \pm 1 \end{array}\right\} \qquad (12.19)$$

When D<0, the equations (12.19) have only a finite number of solutions [63]. When D = -1, as seen in Section 12.2.1 the ring of Gaussian integers Z[i] has four units namely $\pm 1$, $\pm i$. When D = -3, there are six solutions to the equation (12.19) namely $\pm 1$, $\pm w$, $\pm w^2$ where w is the cube root of unity, $w = (-1 + \sqrt{-3})/2$. For all other complex fields, the only units are $\pm 1$. In the case of real fields, there exists an infinite number of solutions to the equation (12.19) and hence an infinite number of units [63]. These units however may be expressed in the form of $\pm \epsilon^n$ where n takes all positive and negative rational values. The quantity $\epsilon$ is called the fundamental unit.

Using the definition of a prime element given in Section 12.2.1 it is seen that the Lemmas 2, 3 and 4 are also applicable in $R(\sqrt{D})$. Although every integer in $R(\sqrt{D})$ can be expressed as a product of Primes, it does not necessarily imply that the factorization is unique like in Z or Z[i]. Consider for instance the factorizations of the integer 6 in $R(\sqrt{10})$ expressed as

$$6 = 2.3 = (4 + \sqrt{10})\,(4 - \sqrt{10})$$

or of the integer 21 in $R(\sqrt{-5})$

$$21 = 3.7 = (1 + 2\sqrt{-5})\,(1 - 2\sqrt{-5}) = (4 + \sqrt{-5})\,(4 - \sqrt{-5}) \qquad (12.20)$$

Considering (12.20), we have the following situation: the integer 3, a Prime, divides $(1 + 2\sqrt{-5})(1 - 2\sqrt{-5})$ but fails to divide either factor in $R(\sqrt{-5})$. Such a situation does not arise for instance in Z or Z[i]. Hence it is seen that in $R(\sqrt{-5})$, Prime integers which are not associated can have a common factor which is not in $R(\sqrt{-5})$. It appears then that in such algebraic number fields the Primes are not necessarily the atoms from which all the integers are constructed.

It is in such cases the factorization of an ideal into a unique set of prime ideals (see Section 11.2.6) comes into use. The rings where the unique factorization of integers fails, correspond to non-principal ideal domains. The theory of non-principal ideal domains is considered to be beyond the scope of this thesis and hence the design of factorization trapdoor systems has been confined to principal ideal domains.

### 12.3.2 Design of Trapdoor Coding System: Complex Euclidean Quadratic Fields

The fields which possess the unique factorization of elements property obey Euclidean algorithm of one form or other. There are just five complex Euclidean fields namely when D =-1, -2, -3, -7 and -11. (There are 4 other complex fields which have the property of unique factorization of elements but obey a slightly different form of Euclidean algorithm [38]). For these nine cases the ring $R(\sqrt{D})$ is a principal ideal domain. The prime ideals are therefore the ideals generated by the prime integers in $R(\sqrt{D})$. From the point of view of designing a factorization trapdoor system, the primes in $R(\sqrt{D})$ and the relationship between the primes in Z and the primes in $R(\sqrt{D})$ need to be considered. More exactly, it is necessary to know whether a rational prime splits in $R(\sqrt{D})$ and if so how does it split.

From Kummer's theorem [68], the decomposition of an ideal <p>, where p is a prime in Z, into prime ideals in $R(\sqrt{D})$ is determined by the factorization of the polynomial $f(x) = x^2-D$ in Z/pZ. Over Z/pZ, the factorizations of f(x) are

$$x^2-D = \begin{cases} x^2 & \text{if } p \mid D \text{ or } 4D \\ x^2-D & \text{if } D \text{ is not a square (mod } p) \\ (x-a)(x+a) & \text{if } D \equiv a^2 \text{ (mod } p),\ a \in Z \end{cases}$$

The three cases therefore correspond to:

1. If a prime p in Z divides the discriminant [68] of the field $R(\sqrt{D})$ (if $D \equiv 1 \pmod 4$), discriminant = D and if $D \not\equiv 1 \pmod 4$), discriminant = 4D), then the ideal $<p>$ is factorized into the square of a prime ideal in $R(\sqrt{D})$. That is, $<p> = \rho^2$ where $\rho$ denotes a prime ideal in $R(\sqrt{D})$, and $N\rho = p$. (12.21)

2. An odd prime p which does not divide the discriminant generates a prime ideal of degree 2 in $R(\sqrt{D})$ if $x^2 - D \equiv 0 \pmod p$ does not have an integral solution. That is, p is irreducible in $R(\sqrt{D})$ and $<p> = \rho$ and $N\rho = p^2$. (12.22)

3. On the other hand, if $x^2 - D \equiv 0 \pmod p$ (or $y^2 - 4D \equiv 0 \pmod p$) has a solution then $<p>$ decomposes into two distinct conjugate prime ideals, $\langle p \rangle = \rho_1 \rho_2$ where $N\rho_1 = N\rho_2 = p$ (12.23)

(Note that the prime p = 2 is of no cryptographic significance)

As an example, consider the non-principal ideal domain $R(\sqrt{-5})$ whose discriminant is equal to 4D = -20. 2 and 5 are the only prime factors of the discriminant and consequently are factorable into squares of prime ideals as $<2> = <2, 1 + \sqrt{-5}>^2$ and $<5> = <\sqrt{-5}>^2$

The congruence $x^2 + 5 \equiv 0 \pmod p$ has solution for p = 3, 7, 23,... but cannot be solved for p = 11, 13, 17, ... Therefore

$<3> = <3, 1 + \sqrt{-5}> \; <3, 1 - \sqrt{-5}>$

$<7> = <7, 3 + \sqrt{-5}> \; <7, 3 - \sqrt{-5}>$ ...

while <11>, <13> ... are prime ideals.

The designer chooses primes $p_1, \ldots, p_r$ in Z which give rise to prime ideals in $R(\sqrt{D})$. If the primes $p_i$ are chosen so that $x^2 - D \equiv 0 \pmod{p_i}$ have integral solutions then letting

$$<p_i> = \rho_{i1} \cdot \rho_{i2} \quad \text{where } \rho_{i1}, \rho_{i2} \text{ are prime ideals in } R(\sqrt{D})$$

the composite ideal <m> is equal to, say,

$$<m> = \prod_{i=1}^{r} \rho_{i1}$$

The number of incongruent residues with respect to the ideal <m> and relatively prime to <m> is given by

$$\begin{aligned} \phi\langle m\rangle &= \prod_{i=1}^{r} \phi(\rho_{i1}) \\ &= \prod_{i=1}^{r} N\rho_{i1} - 1 \end{aligned} \qquad (12.24)$$

And hence using (12.23),

$$\phi\langle m\rangle = \prod_{i=1}^{r} (p_i-1) \qquad (12.25)$$

If the primes $p_i$, $1\leqslant i\leqslant r$, are chosen such that $x^2-D \equiv 0 \pmod{p_i}$ do not have an integral solution for all i, then the ideals $\langle p_i\rangle$ are irreducible in $R(\sqrt{D})$ and

$$\phi\langle m\rangle = \prod_{i=1}^{r} (p_i^2-1) \qquad (12.26)$$

using (12.22).

The designer can decide whether $x^2-D \equiv 0 \pmod{p}$ for a chosen prime p has solution or not using the law of quadratic reprocity. In general if $x^2 \equiv a \pmod{n}$, where $\gcd(a,n) = 1$, has a solution then 'a' is said to be a quadratic residue modulo n. Here n=p and a=D. To determine whether D is a quadratic residue or not modulo p, the designer computes the Legendre Symbol $(\frac{D}{p})$. If $(\frac{D}{p}) = 1$ then D is a quadratic residue and if it is equal to -1 then D is a quadratic non-residue. Calculating the Legendre symbol is not much different from evaluating gcd of two numbers using Euclid's algorithm and can be done by repeated divisions in polynomial time. (Note that $|D|$ is of small value).

The Fermat theorem in this case is given by

$$\alpha^{\phi\langle m\rangle} \equiv 1 \pmod{\langle m\rangle}$$

where $\alpha$ is an arbitrary integer in $R(\sqrt{D})$ relatively prime to $\langle m\rangle$.

The encryption and decryption exponents e and d can be calculated using

$$ed \equiv 1 \pmod{\phi\langle m\rangle}$$

where $\phi\langle m\rangle$ is given by (12.25) or (12.26)

The public encryption key is (e,m) and the secret decryption key is (d,m). Having generated the rational primes $p_i$, calculated $\phi\langle m\rangle$, e and d, the designer needs to obtain m to make it public. That is, given the primes $p_1,\ldots, p_r$ in Z, he needs to calculate m where $m = \Pi_1 \ldots \Pi_r$ and $\Pi$'s are primes in $R(\sqrt{D})$, and $\Pi_i$ is of the form $\Pi_i = a_i + b_i\sqrt{D}$ if $D \not\equiv 1 \pmod 4$ and $\Pi_i = a_i + b_i(\frac{1+\sqrt{D}}{2})$ if $D \equiv 1 \pmod 4$. To obtain m, he needs to find $\Pi_1,\ldots, \Pi_r$, that is, he needs to calculate $a_i$ and $b_i$ for all i, $1\leqslant i\leqslant r$ by solving either

$$N\Pi_i = a_i^2 - b_i^2 D = p_i \text{ or } p_i^2, \text{ if } D \not\equiv 1 \pmod 4 \qquad (12.27)$$

or

$$N\Pi_i = a_i^2 + a_i b_i - \frac{D-1}{4} b_i^2 = p_i \text{ or } p_i^2, \text{if } D \equiv 1 \pmod 4 \qquad (12.28)$$

Note that (12.28) can be written as

$$A_i^2 - Db_i^2 = 4p_i \text{ or } 4p_i^2 \qquad (12.29)$$

where $A_i = (2a_i + b_i)$

For solving (12.27) or (12.29), the method 4 of Section 12.2.2, which consists of arbitrarily choosing $a_i$ and $b_i$ to form integers $\Pi_i$ and then checking whether their norms are primes or squares of primes, seems to be the most attractive one for the system designer.

Again it may be advisable for the designer to choose all the primes $p_i$, $1 \leq i \leq r$, to be of the same type, that is, either they all decompose into distinct conjugate prime ideals in $R(\sqrt{D})$ or they are irreducible in $R(\sqrt{D})$. If one of $a_i(A_i)$ or $b_i$ is zero and the other is $p_i(2p_i)$, then factorization of m would be made easy as seen in Section 12.2.2.

### 12.3.3 Security of The System in $R(\sqrt{D})$

The security of the system seems to be the same as that of the RSA system. The opponent needs to find the decrypting exponent d to break the system. One way of finding this is to obtain $\emptyset\langle m\rangle$. To calculate $\emptyset\langle m\rangle$, he needs to factorize Nm given the modulus m. If the designer had chosen the rational primes such that congruence $x^2-D \equiv 0 \pmod{p_i}$ have a solution for all i, then the norm Nm is given by

$$Nm = \prod_{i=1}^{r} p_i$$

On the other hand, if the primes have been chosen such that $x^2-D \equiv 0 \pmod{p_i}$ have no solution for all i, then

$$Nm = \prod_{i=1}^{r} p_i^2$$

As in the case of Gaussian integers, note that the cryptanalyst does not require to find the primes $\Pi_i$ in $R(\sqrt{D})$ to break the system. Thus unlike the designer, he is not faced with the problem of solving (12.27) or (12.29).

### 12.3.4 Representation of Messages and System Operation

If the primes $p_i$ are chosen such that $x^2-D \equiv 0 \pmod{p_i}$ have solution for all i, then every integer of $R(\sqrt{D})/\langle m\rangle$ is congruent to an element in the range 0 to $N\langle m\rangle -1$ as in Section 12.2.4. Thus the messages can be represented using integers in Z in the range 0 to $N\langle m\rangle-1$. As in Section 12.2.4, the encryption and decryption procedures can be performed modulo Nm or m, where $Nm \varepsilon Z$ and $m \varepsilon R(\sqrt{D})$. The messages can also be represented in the form $x + y\sqrt{D}$ ($D \not\equiv 1 (\bmod 4)$) or $x+y\ (\frac{1+\sqrt{D}}{2})$ ($D \equiv 1\ (\bmod\ 4)$), $x,y \varepsilon Z$, using the elementary divisor method given in Section 12.2.4.

If $D \not\equiv 1\ (\bmod\ 4)$ then the integral basis of $R(\sqrt{D})$ is $(1,\sqrt{D})$ and the matrix A in Section 12.2.4 is equal to

$$A = \begin{pmatrix} a & b \\ bD & a \end{pmatrix}$$

If $D \equiv 1\ (\bmod\ 4)$, then the integral basis of $R(\sqrt{D})$ is $(1, \frac{1+\sqrt{D}}{2})$ and the matrix A is equal to

$$A = \begin{pmatrix} a & b \\ \frac{b(D-1)}{4} & a+b \end{pmatrix}$$

Using the method given in Section 12.2.4, the matrix A can be reduced to diagonal form $(d_1, d_2)$ and the modified basis of the ideal $<m>$ is obtained as $(d_1\ w_1,\ d_2\ w_2)$. The residue classes mod $<m>$ are therefore given by $x_1\ w_1 + x_2\ w_2$ where $0 \leqslant x_1 < d_1$ and $0 \leqslant x_2 < d_2$. As Nm is equal to $\prod_{i=1}^{r} p_i$ in this case, the only allowed values for $d_1$ and $d_2$ are given by $d_1 = 1$ and $d_2 = \prod_{i=1}^{r} p_i$. The operation modulo $<m>$ required in encryption and decryption is performed in the same way as that given in Section 12.2.4.

If the primes $p_i$ are chosen such that $x^2 - D \equiv 0\ (\bmod\ p_i)$ have no solution for all i, then the norm is a non-square free rational integer and as in Section 12.2.4, one cannot represent the residue classes modulo $<\Pi_i>$ using the integers $Z/p_i^2\ Z$. This case corresponds to the situation when $d_1 = d_2 = \prod_{i=1}^{r} p_i$ in the elementary divisor method. The messages are hence represented in the form $x_1 + x_2 \sqrt{D}$ for $D \not\equiv 1\ (\bmod\ 4)$ and $x_1 + x_2\ (\frac{1+\sqrt{D}}{2})$ for $D \equiv 1\ (\bmod\ 4)$ where $0 \leqslant x_1, x_2 < \sqrt{Nm} = d_1 = d_2$. The reduction mod $<m>$ is performed by simply reducing each of the components $x_1$ and $x_2$ mod $\sqrt{Nm}$.

Again, the elementary divisors $d_1$ and $d_2$ are made public together with the encrypting exponent, e.

### 12.3.5 Real Quadratic Fields

There are 16 real quadratic fields which obey the Euclid's algorithm with respect to the field norm and hence possess the unique factorization of integers property [65]. They occur when D is equal to

2,3,5,6,7,11,13,17,19,21,29,33,37,41,57 and 73. One major difference between the complex quadratic fields and real quadratic fields is that the latter has infinitely many units (Section 12.3.1). However this does not cause any serious problems provided we choose the message representation within the allowed standard set mod <m>. This is done in any case, by either using the elementary divisor method or using the standard set of messages modulo Nm. Thus it seems that in this case, there are not any major changes to the trapdoor system described above for the Euclidean complex quadratic fields.

## 12.4 Discussion

The design of factorization trapdoor systems in some quadratic fields which are principal ideal domains has been considered. However majority of the quadratic fields are non-principal ideal domains and they do not possess the unique factorization of elements property. But the unique factorization of a non-zero ideal into prime ideals still applies in such fields. Factorization trapdoor system seems possible if the chosen ring modulo the ideal is isomorphic to a direct sum of finite fields. Choosing a square free ideal A, not necessarily a principal ideal, in $R(\sqrt{D})$ and let

$A = \rho_1 \ldots \rho_r$ where $\rho_i$ are prime ideals in $R(\sqrt{D})$

Then $R(\sqrt{D})/A \cong R(\sqrt{D})/\rho_1 \oplus \ldots \oplus R(\sqrt{D})/\rho_r$

where $R(\sqrt{D})/\rho_i$ is a finite field of $N\rho_i$ elements

The order of the group formed by residues relatively prime to A is given by

$$\emptyset(A) = \prod_{i=1}^{r} \emptyset(\rho_i) = \prod_{i=1}^{r} (N\rho_i - 1) \quad \text{using (11.1)}$$

The coding exponents e and d can be found using $ed \equiv 1 \pmod{\emptyset(A)}$.

From Lemma 3, every prime ideal divides a rational prime p which is unique [63]. If the rational primes $p_i$ are chosen such that $x^2-D\equiv 0 \pmod{p_i}$ have solution for all i, then the residue classes mod $\rho_i$ can be represented using rational integers mod $p_i$. Hence the messages modulo A can be represented using integers 0 to N(A)-1. The elementary divisor method of Section 12.2.4 can also be employed to represent the messages. If the primes $p_i$ are chosen such that $x^2-D\equiv 0 \pmod{p_i}$ have no solution for all i, then both elementary divisors $d_1$ and $d_2$ are equal to $\sqrt{N(A)}$.

Let us conclude this chapter by considering a small example which shows some of the calculations involved in the design of a trapdoor system in $R(\sqrt{-5})$, a non-principal ideal domain.

Choosing rational primes $p_1=3$ and $p_2=7$, their decomposition in $R(\sqrt{-5})$ is given by

$$\langle 3\rangle = (3,\ 1+\sqrt{-5})\ (3,\ 1-\sqrt{-5}) = \rho_{11}\ \rho_{12}$$

$$\langle 7\rangle = (7,\ 3+\sqrt{-5})\ (7,\ 3-\sqrt{-5}) = \rho_{21}\ \rho_{22}$$

Let

$$A = \rho_1\ \rho_2 = (3,\ 1+\sqrt{-5})\ (7,3-\sqrt{-5})$$

Then

$$A = (21,\ 7+7\sqrt{-5},\ 9-3\sqrt{-5},\ 8+2\sqrt{-5})$$

Any ideal can be represented using a two-element basis over the ring [65]. Using standard rules for transforming the ideal basis [65],

$$A = (21,\ 4+\sqrt{-5})$$

$$N(A) = 21$$

The integral basis of $R(\sqrt{-5})$ is $(1,\sqrt{-5})$ as $-5 \not\equiv 1 \pmod 4$

Representing the ideal A as a Z-module

$$[21,\ 21\theta,\ 4+\theta,\ -5+4\theta\ ] \qquad \text{where } \theta = \sqrt{-5}$$

or

$$\begin{pmatrix} 21 \\ 21\theta \\ 4+\theta \\ -5+4\theta \end{pmatrix} = \begin{pmatrix} 21 & 0 \\ 0 & 21 \\ 4 & 1 \\ -5 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ \theta \end{pmatrix}$$

Let

$$B = \begin{pmatrix} 21 & 0 \\ 0 & 21 \\ 4 & 1 \\ -5 & 4 \end{pmatrix}$$

Using the algorithm given in Section 12.2.4, this matrix is reduced to a diagonal form $(d_1,d_2)$ where $d_1 | d_2$.

$$VBU = \begin{pmatrix} 1 & 0 \\ 0 & 21 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

where

$$V = \begin{bmatrix} 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & -4 & 1 \\ 0 & 1 & -5 & -4 \end{bmatrix} \quad \text{and } U = \begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix}$$

Now the messages can be represented as

$$M = x_1 w_1 + x_2 w_2 \qquad 1 \leqslant x_i < d_i \qquad i=1,2 \qquad (12.30)$$

where

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = U^{-1} \begin{pmatrix} 1 \\ \theta \end{pmatrix} = \begin{pmatrix} 1 & -5 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \theta \end{pmatrix}$$

In this case, as $d_1 = 1, M = x_2 w_2$, $1 \leqslant x_2 < d_2$. However if the rational primes are chosen such that they stay as primes in the higher field $R(\sqrt{D})$, then $d_1 = d_2 = \sqrt{N(A)}$ and the messages can be represented using (12.30). $\phi(A) = (N\rho_1 - 1)(N\rho_2 - 1) = 12$. One set of coding exponents e and d is e = 5, d = 5. The messages can be encrypted using a similar procedure to that given in Section 12.2.4 for Gaussian integers, except in this case, the recursive equation is $f(\theta) = \theta^2 + 5 = 0$ instead of $i^2 + 1 = 0$. The elementary divisors $d_1$, $d_2$, the encrypting exponent e and the matrix U are made public and the decrypting exponent d is kept secret.

# CHAPTER 13

## CONVENTIONAL CRYPTOSYSTEM WITH PUBLIC KEY DISTRIBUTION

### 13.1 General

This chapter focusses on the concept of public key distribution (PKD) mentioned in Section 9.6, whereby the 'public key idea' is solely used to transfer a key between two users over an insecure channel. In this case, there is no message as such which gets encrypted at the sending end and decrypted at the receiving end. In such a system, two users who wish to exchange a key communicate back and forth until they arrive at a key in common. The opponent eavesdropping on this exchange must therefore find it computationally infeasible to derive the key from the information overheard. This type of arrangement is used in conjunction with the conventional cryptographic DES interface unit to form a DES/PKD hybrid system. An implementation of such a hybrid system is discussed in Section 13.5.

Diffie and Hellman [35] proposed such a key distribution system based on exponentiation over finite field. This technique briefly described in Section 9.5.3 makes use of the apparent difficulty of computing logarithms over a finite field GF(q) where q is a very large prime number. Each user generates an independent random number $x_i$ chosen uniformly from the set of integers $\{1,2,...q-1\}$, and computes $y_i = a^{x_i} (\text{mod } q)$ where 'a' is a primitive element of GF(q). The number $y_i$ is made public and the number $x_i$ is kept secret. When users i and j wish to communicate privately, they can use the common key $K_{ij}$, given by $K_{ij} \equiv a^{x_i x_j} \equiv (y_i)^{x_j} \equiv (y_j)^{x_i} (\text{mod } q)$.

For the system to be secure, the key $K_{ij}$ must be difficult to compute for anyone who knows $y_i$ and $y_j$ but does not know either $x_i$ or $x_j$. In order to ensure that this computation is difficult, it is necessary that logarithms over GF(q) be difficult to compute. Otherwise an opponent could compute $x_i$ from $y_i$ and impersonate user i. Some currently known algorithms for calculating the logarithms over finite fields are now briefly looked at.

### 13.2 Logarithms Over Finite Fields

Knuth's algorithm [45] to compute logarithm over GF(q) requires 2 $\lceil q^{1/2} \rceil$ multiplications (mod q) in addition to other operations of comparable complexity. This algorithm requires 2 $\lceil q^{1/2} \rceil$ words of memory, each $\lceil \log_2 q \rceil$ bits long. This algorithm can be generalized to allow a time-memory tradeoff in which time is almost proportional to $q^r$ and memory to $q^{1-r}$ for any $0 \leq r \leq 1$. Knuth's algorithm corresponds to the case where $r = \frac{1}{2}$. Allowing arbitrary values of r enables the algorithm to be adjusted to particular time-memory requirements. The algorithm to compute x from $y \equiv a^x$ (mod q) works as follows:

Let $m = \lceil q^r \rceil$

Then there exist integers c and d such that

$x = cm+d$ with $0 \leq c < \lceil q/m \rceil \doteq q^{1-r}$

and $0 \leq d < m \doteq q^r$

Substituting for x in $a^x$, gives

$y = a^{cm+d}$ (mod q)

ie, $a^d \equiv y\ a^{-cm}$ (mod q)

In order to determine c and d, the values of $a^d$ (mod q) are precomputed for d=0,1,...m-1 in $O(q^r)$ operations and the results stored in a table in $O(q^r \log_2 q^r)$ operations. Then $y, ya^{-m}, ya^{-2m} \ldots$ (mod q) are each computed and compared with the sorted table of $\{a^d\}$ until a match is found. Each value of c tried requires 1 multiplication (mod q) and $\log_2 q^r$ comparisons, thus giving a total of $(1+\log_2 q^r)$ operations. There are $O(q^{1-r})$ values of c to be tried. When r=1, this algorithm is a look up table and when r=0, the algorithm reduces to an exhaustive search. Neglecting the logarithmic factors it is seen that the time-memory product is constant (since $q^r.q^{1-r} = q$) as the algorithm ranges between the extremes of a look up table and an exhaustive search.

Pohlig and Hellman [69] proposed an improved algorithm to compute logarithms over GF(q) when the prime q is chosen such that q-1 has only small prime factors. On the other hand, it has long been known that a disproportionately large portion of numbers are entirely composed of small prime factors and it is precisely this fact on which Adleman based his very recent discrete logarithm algorithm [70]. Assuming a microsecond per operation machine, this new algorithm could be expected to compromise a system based on a 200-bit prime q in 2.6 days rather than the $3 \times 10^{16}$ years using the best previously published method due to Shanks [45]. For a large enough prime q, however, Adleman's algorithm is also infeasible. A sketch of this subexponential Adleman's solution to the logarithm problem is now given following [6].

Problem: Given a, y and q (a prime) find x that satisfies $y \equiv a^{x} \pmod{q}$ (where 'a' is a primitive element in GF(q) in the PKD system).

The algorithm requires the following preliminary definition: a number $\alpha$ is said to be 'smooth' with respect to a bound BD, if the factorization of $\alpha$ into primes, $\alpha = \prod_{i=1}^{r} p_i^{e_i}$ is such that all $p_i$, $1 \le i \le r$, satisfy $p_i \le BD$.

Step 1 Find by random sampling (and checking) a positive integer R such that $B \equiv y^{R} \pmod{q}$, $1 \le B \le q-1$ and B is smooth with respect to the bound $BD(q) = e^{(\ln q \ln \ln q)^{\frac{1}{2}}}$ and gcd $(R, q-1) = 1$.

Step 2 Let $q_1 \ldots q_m$ be all the primes $\le BD(q)$. Find by random sampling (and checking) positive integers $R_i$ for $1 \le i \le m$ such that $A_i \equiv a^{R_i} \pmod{q}$ where $1 \le A_i \le q-1$ and $A_i$ is smooth with respect to BD(q) and the vectors $\vec{A}_i = (e_{i1}, \ldots e_{im})$, where $A_i = \prod_{j=1}^{m} q_j^{e_{ij}}$, span the m-dimensional module over $Z/(q-1)Z$, the ring of integers modulo $(q-1)$.

Step 3 By Gaussian elimination express $\vec{B} = (f_1, \ldots, f_m)$, where $B = \prod_{j=1}^{m} q_j^{f_j}$, as a linear combination of the $\vec{A}_i$, namely $\vec{B} = n_1\vec{A}_1 + n_2\vec{A}_2 + \ldots + n_m\vec{A}_m \pmod{(q-1)}$ where $0 \le n_i \le q-2$ for $1 \le i \le m$.

Now
$$B = \prod_{j=1}^{m} q_j^{f_j}$$
and
$$\ln.\ B = \sum_{j=1}^{m} f_j \ln q_j$$
$$\ln.\ B = \sum_{j=1}^{m} \Big( \sum_{i=1}^{m} n_i e_{ij} + k_j(q-1) \Big) \ln q_j \qquad (13.1)$$
as
$$f_j \equiv \sum_{i=1}^{m} n_i e_{ij} \pmod{(q-1)}$$

Exponentiating (13.1) gives,
$$B \equiv \prod_{i=1}^{m} \left( \prod_{j=1}^{m} q_j^{e_{ij}} \right)^{n_i} \pmod{q}$$
That is,
$$B \equiv \prod_{i=1}^{m} A_i^{n_i} \pmod{q}$$
Where $A_i$ can be replaced by $a^{R_i}$.

Step 4 Calculate $R^{-1} \pmod{q-1}$. Then $y = \prod_{i=1}^{m} a^{n_i R_i R^{-1}} \pmod{q}$

That is, $y \equiv a^{R^{-1}(\sum_{i=1}^{m} n_i R_i)} \pmod{q}$. That is, $x \equiv R^{-1} \sum_{i=1}^{m} n_i R_i \pmod{q}$

Now briefly consider the execution time of this algorithm.

Let $\psi(x,y)/x$ = Prob($z$ is smooth with respect to $y/z \leqslant x$). Using Erdos's result [6], $\psi(x,y)/x = e^{-(\ln x/\ln y)(\ln\ln x-1)}$. Thus the expected number of tries needed to obtain a smooth number is approximately $e^{(\ln x/\ln y)(\ln\ln x-1)}$. This search must be carried out $\delta(y)$ times where $\delta(y)$ is the number of primes less than or equal to y and $\delta(y) \doteq y/\ln y = e^{\ln y-\ln\ln y}$. Thus the total computational effort is approximately $e^{(\ln x/\ln y)(\ln\ln x-1)+\ln y-\ln\ln y}$ which is approximately $e^{c/d+d}$ with $c = \ln x\ln\ln x$ and $d = \ln y$. But $e^{c/d+d}$ $(c>0)$ is maximized when $d = c^{\frac{1}{2}}$, yielding a computational effort of order $e^{2(\ln x\ln\ln x)^{\frac{1}{2}}}$. A sharper analysis [6] results in an upper bound for the logarithm problem of $e^{(\ln q\ln\ln q)^{\frac{1}{2}}}$ thus removing the factor 2 from the exponent.

System designer's avoidance of q such that q-1 has only small prime factors cannot be accomplished by first choosing a prime q such that $e^{(\ln q\ln\ln q)^{\frac{1}{2}}}$ is prohibitively large from the cryptanalyst point of view and then determining the prime power factorization of q-1. This is because the most efficient known factorization algorithm due to Schroeppell (unpublished) also makes use of the concept of smoothness. It has an expected running time for factoring q-1 of $e^{(\ln(q-1)\ln\ln(q-1))^{\frac{1}{2}}}$ One way to overcome this problem is to generate a large random prime number u and let q to be the first prime in the sequence iu+1 for i=2,4,6... as mentioned in Section 10.2.

## 13.3 Public Key Distribution in GF($2^n$)

While it is possible to implement the exponentiation public key distribution system as above, some implementation difficulties can be overcome by considering the exponentiation system in the extension fields GF($2^n$).

First consider the exponentiation over GF(q) from a system design point of view. Initially one needs a suitable set of routines to generate and test for large prime numbers. The tests mentioned in Section 10.2 can be used for this purpose. Further it is required that the routine must generate a prime q such that q-1 has no small factors. Blakely [71 ] refers to such primes of the form q=2p+1 where p is a prime as 'safe' primes. Then one must choose a primitive element 'a' in GF(q). The number of primitive elements (mod q) is given by the Euler totient function $\phi(q-1)$ and hence the probability that an arbitrarily chosen element 'a' is primitive is given by $\phi(q-1)/q-1$. If q is of the form 4p+1 where p is a prime then $2^n$(mod q) is a primitive element (mod q) for any n relatively prime to q-1 and hence the problem of determining a primitive element is simplified. Otherwise finding a primitive root of an arbitrary prime q may be more cumbersome requiring the knowledge of the factors of q-1. In addition, multiple precision arithmetic is required which is cumbersome and slows down the response time.

To overcome some of these problems associated with the GF(q) implementation, Berkovits [72] proposed to perform the exponentiation in the extension field $GF(2^n)$,instead of GF(q). As this is an extension field of GF(2)all operations are based on modulo 2 arithmetic which is easily implementable using digital logic systems. Addition and subtraction are performed with exclusive-or operation alone, while operations in GF(q) require carry and borrow propagation. Further advantages arise from the choice of n to be a prime so that $2^n-1$ is a Mersenne prime. Since there are no subgroups within the multiplicative group of such a field, the logarithmic attack reduces to an exhaustive search. Furthermore, since the order of the multiplicative group is a prime, every element except one is primitive. Thus the selection of 'a' becomes arbitrary. Finally,since all the resulting y's ($y\equiv a^x$ in $GF(2^p)$) are also primitive, the number of possible keys is maximized.

As seen in Section 10.5, the elements of $GF(p^n)$ can be represented using polynomials of degree less than n whose coefficients are in GF(p). In this system p=2 and hence the coefficients are all either 0 or 1. Using the earlier notation, the field $Z/2Z[x]/f(x)$, where f(x) is an irreducible polynomial of degree n over Z/2Z, is isomorphic to $GF(2^n)$. Multiplication of two elements in $GF(2^n)$ are programmed as multiplication of two polynomials and the terms in the product with exponent n or higher are reduced modulo the generating irreducible polynomial f(x). Let f(x) be a monic polynomial and is equal to

$$f(x) = x^n + \sum_{i=0}^{n-1} b_i x^i \qquad \text{where } b_i \varepsilon \ \{0,1\}$$

Then

$$x^n \equiv \sum_{i=0}^{n-1} b_i x^i \qquad (\text{mod } f(x))$$

Then the reduction process of a polynomial P(x) modulo f(x) is performed as follows: (cf Section 10.5)

Let

$$P(x) \equiv Q(x) + R(x)\, x^n$$

where the degree of Q(x) is less than n.

Then

$$P(x) \equiv Q(x) + R(x) \sum_{i=0}^{n-1} b_i x^i$$

$$\equiv Q(x) + \sum_{b_i=1} R(x)\, x^i \pmod{f(x)}$$

This result may still have terms with exponents equal to n or greater and the process is repeated. The reduction process continues until the result is of degree less than n.

As $2^n-1$ is prime, any number 'a' less than $2^n-1$ (other than 1) can be used as the base (primitive element in $GF(2^n)$) of the PKD system. The base 'a' can be represented as a vector whose value as a binary number is equal to 'a'. That is, let

$$a = \sum_{i=0}^{n-1} a_i 2^i \qquad \text{where } a_i \in Z/2Z$$

Let the secret keys of users A and B be $X_A$ and $X_B$ which are integers less than $2^n-1$ (except 1). Then,

$$y_A(x) = \left( \sum_{i=0}^{n-1} a_i x^i \right)^{X_A} \pmod{f(x)}$$

where $y_A(x)$ is a polynomial of degree less than n with coefficients over GF(2). $y_A(x)$ can be represented as an integer $Y_A$ in the range $0<Y_A<2^n-1$ by calculating the value of the polynomial $y_A(x)$ in binary. Similarly the user B calculates $Y_B$. Thus the public key and secret key pairs of users A and B are given by $(Y_A,X_A)$ and $(Y_B,X_B)$ respectively. The common key $K_{AB}$ is derived by users A and B using

$$K_{AB} \equiv (y_A(x))^{X_B} \equiv (y_B(x))^{X_A} \pmod{f(x)}$$

The choice of the extension field $GF(2^n)$ depends on the required difficulty of computing logarithms over the field. The particular choice of the Mersenne prime $2^{127}-1$, that is, n=127 is very attractive from implementation point of view. In $GF(2^{127})$, manipulation of 127-bit blocks are conveniently performed in most computers which have 8,16,32 or 64-bit architectures. A further attraction is that there exists a particularly simple irreducible polynomial, a trinomial, over GF(2) namely $f(x) = x^{127} + x + 1$ [73] which can be used to generate all the elements of $GF(2^{127})$. But with the advent of the subexponential algorithm for computing logarithm over finite fields by Adleman, it is necessary to work in higher extension fields to offer a similar amount of work factor as the DES to break the system. The next Mersenne prime occurs when n=521 and $2^{521}-1$ is a prime. Thus one can work in GF($2^{521}$) to overcome the attack using Adleman's subexponential logarithm algorithm. In this field,the corresponding irreducible trinomial over GF(2) is given by $f(x) = x^{521} + x^{32} + 1$ [73]. Tore Herlestam [74] has proposed a hauristic method for computing logarithms over GF($2^p$) where p is a prime. When $2^p-1$ is a Mersenne prime less than or equal to $2^{31}-1$, the method is reported to work in very short running times on a general purpose computer [74]. Although the numerical results obtained so far and the complexity of the problem does not allow to assess the security of the corresponding public key distribution systems with p=127, 521 and larger, this may induce doubts that such systems could be considered secure.

Even so,this system, referred to as the Mitre system [75], is probably the most practical of the public key algorithms that have been proposed so far. However there is another attack called the short cycling attack which can be used with any public key system discussed so far. This attack may enable 'backdoor' penetrations, for example, in the Mitre system, a penetrator could use his knowledge of the system parameters namely the system base 'a' and the modulus polynomial f(x) to superencipher intercepted cipher until a cycle occurs. The effectiveness of such an attack on the Mitre system is now considered. This attack is of the same type as the one used by Simmons and Norris [76] against the RSA system.

### 13.4 Short Cycling Attack

Suppose g is a function of a set S into itself. Then given an x in S, the sequence defined by $x_0=x$ and $x_i=g(x_{i-1})$ is called the path of the element x under the action of g. If S is a finite set, then the path of each element must eventually repeat itself. When g is a one to one map, then repetition begins with $x_k=x_0=x$ for some minimal k. Under these circumstances, the path of x cycles around the same k elements. This circular path is called the orbit of x under g. The orbit of any element not in the orbit of x is completely disjoint from the orbit of x. Thus the set S is partitioned into disjoint orbits and hence the sum of the numbers of elements in the distinct orbits is the cardinality of S which is equal to the size of the largest orbit possible.

Now consider the cycling in the PKD system based on exponentiation in GF($2^n$). Let

$$P = \{x : x \text{ an integer between 1 and } 2^n-1\}$$

and $$C = \{y : y \text{ a non-zero element of } GF(2^n)\}$$

Let P denote the set of plaintexts (secret keys) and C the set of ciphertexts (public keys) under the encrypting function E. Let us define another function DEC from C to P as follows: If y is an element of C, it is a polynomial of degree less than n. Writing y as the n-tuple of its coefficients and evaluating that n-tuple as the binary expansion of an integer, the result DEC(y) is obtained. Since both E and DEC are one to one functions, their composition F is a one to one map of the finite set P onto itself. The set P is partitioned into disjoint orbits under the F-map. These orbits are the cycles. A schematic diagram of the mappings is shown in Figure 13.1.

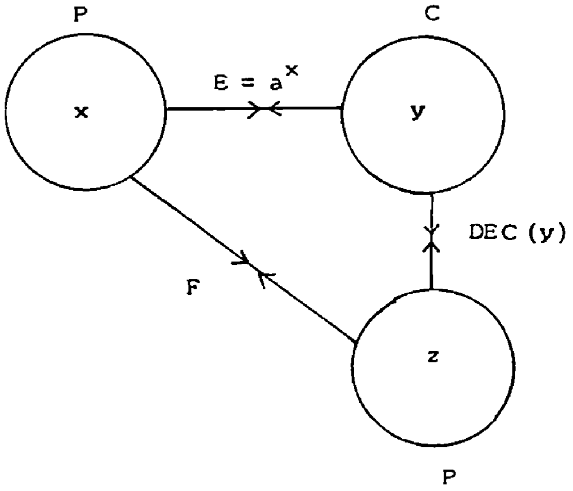


Fig 13.1

The threat of short cycling arises as follows:

Let

$$y_0 \equiv a^{x_0} \pmod{f(x)}$$

Then

$$x_1 \equiv \mathrm{DEC}(y_0)$$

$$y_1 \equiv a^{x_1} \pmod{f(x)}$$

$$x_2 \equiv \mathrm{DEC}(y_1)$$

$$\begin{matrix} \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{matrix}$$

$$y_i \equiv a^{x_i} \pmod{f(x)}$$

$$x_{i+1} \equiv \mathrm{DEC}(y_i)$$

Hence at some point r, as the set is finite

$$y_r \equiv a^{x_r} \pmod{f(x)}$$
$$x_{r+1} \equiv \mathrm{DEC}(y_r)$$
$$\equiv x_0$$

and

$$y_{r+1} \equiv a^{x_0} \pmod{f(x)}$$
$$x_1 \equiv \mathrm{DEC}(y_{r+1})$$

When this point is reached, the opponent realizes that the penultimate number of the sequence gives the original secret key $x_0$.

This type of short cycling analysis has been carried out in small extension fields using irreducible polynomials of degrees 3 and 7. The primitive polynomials used are $f(x) = x^3+x+1$ in $GF(2^3)$ and $f(x) = x^7+x+1$ in $GF(2^7)$ respectively. The system base 'a' is allowed to vary from 2 to 7 and 2 to 127 respectively. Then the cycle lengths are determined for various values of the secret exponent x, using the program CYCLE.FTN given in Appendix 15. The cycle lengths obtained in $GF(2^7)$ for several values of x are given in Appendix 17 (Section A17.1). The complete set of results shows that in the case of $GF(2^7)$ with $f(x) =x^7+x+1$, the base a=38 (evaluated using x=2 in $x^5+x^2+x$, an element of $GF(2^7)$) gives the maximum cycle length of 127. [In the case of $GF(2^3)$ with $f(x)=x^3+x+1$, a=5 ($x^2+1$ in $GF(2^3)$) gave the maximum cycle length of 7]. For all the other values of the base, the cycle lengths are less than 127. This can be explained by the reasoning that the choice of the system base 'a' partitions the set P into disjoint orbits; in the case of a=38 with the primitive polynomial $x^7+x+1$, this has created a partition consisting only of the entire set P. Thus every element of P is in the same orbit and hence every value x gives the maximum cycle length with a=38. The other choices of 'a' partitioned P into several disjoint orbits with different cycle lengths. Two different exponent values of x having the same cycle length (for instance a=9, $x_1$=15, $x_2$=32, cycle length =116) may be due to two or more orbits having the same cardinality or it may be that $x_1$ and $x_2$ lie in the same orbit.

These results may indicate that certain bases and certain generating polynomials are superior to others thus giving rise to maximum cycle lengths irrespective of the exponent x. If so, the system parameters should not be chosen at random, for instance, the system base 'a' should not be chosen randomly among all primitive elements. To determine whether an optimum set of parameters is possible, further analysis is carried out in the extension field GF($2^7$).

Considering the mappings shown in Figure 13.1, it is seen that an opponent is not restricted to choosing the function DEC to get back from C to P. Any one to one map of C to P will serve his purpose and the system designer cannot guard against all possible choices of the opponent. Consider for instance, the function G(y) where

$$G(y) = DEC\ (b\cdot y)$$

Let b take all non-zero values in the set C. That is, b can be any one of the 127 polynomials of degree less than 7 in GF($2^7$). Consider the following algorithm.

1. Choose a system base 'a'.
2. Choose a particular value for 'b' in G(y) = DEC(b·y)
3. Vary the exponent values x (secret keys in PKD) from 2 to 127 and in each case, calculate the corresponding cycle length.
4. Do steps 2 and 3 for 127 values of 'b'.

This algorithm has been implemented using the program RANDCYCLE.FTN given in Appendix 16. Several values for the system base 'a' have been tried; only the results for a=38 are given in Appendix 17, Section A17.2. The results show the expected cycle lengths obtained as the secret exponent x varies from 1 to 127 for different values of the polynomial b (evaluated as a binary vector). From the results, it is seen that by varying b in G(y) = DEC(b.y) the expected cycle length can be changed for a fixed base 'a'. Hence the opponent can obtain a shorter cycle length than the maximum, by appropriately choosing the value of b in G(y) for any system base 'a'. Section A17.2 shows that with a=38, even though the expected cycle length is equal to the maximum 127, when b=1, with b=2, the expected cycle length ≑ 38.87, with b=13, expected cycle length ≑ 27.18, with b=125, expected cycle length ≑ 32.64 and so on. Hence even if the system designer had chosen the 'best' system base, a=38, in his PKD system in GF($2^7$), if the opponent chooses b=13 in his G(y) = DEC(b.y) then he only needs to superencipher on average 28 times before obtaining the secret exponent x. The system designer has no control over the opponent choice of 'b'. Thus it seems that there is no best choice for the system base 'a'.

Section A17.3 of Appendix 17 gives the average expected cycle lengths for several values of the system base 'a'. That is, having selected a system base 'a', the expected cycle lengths are calculated for the 127 non-zero specialized values of the polynomial b. Then the average of these expected cycle lengths is determined. From the results it seems that the average expected cycle length for a chosen system base is around 63.5. Recalling that in the algorithm given above, one has considered only 127 functions of a specialized form for the polynomial 'b', it appears that the average expected cycle length will approach 63.5 if one has averaged over all the 127! possible functions from the set C to the set P. Dr R Odoni has in fact explained using a heuristic argument that the cycle length will be n/2 when all the n! functions are taken into account.

Thus once the opponent chooses the function from C to P, that is, the value of b in G(y), then he has actually chosen the order in which he will try elements of P to search for the one that encrypts into cipher y. That is, he has effectively decided on the elements

$$G(y_1) = DEC\ (b.y_1)$$
$$G(y_2) = DEC\ (b.y_2)$$
$$\vdots \qquad \vdots$$
$$G(y_n) = DEC\ (b.y_n)$$

The opponent tries each of the elements in the above sequence in turn in the equation $a^{G(y_i)} = y_{i+1}$ to find the one which matches with the given $y_1$. For different values of b, the order of the sequence of elements changes and hence the number of elements to be searched to find the match with $y_1$ changes. The average number of such elements to be tried is about 63.5. This implies that for a randomly chosen system base, the expected cycle length of an arbitrary cycle is about half the number of non-zero field elements. (That is, $(2^{127}-1)/2$ and $(2^{521}-1)/2$ for $GF(2^{127})$ and $GF(2^{521})$ respectively). Thus although the cycling attack will eventually find a solution, the work required appears to be equivalent to a "random" exhaustive key search and hence confirms [77].

In the next section, the implementation of this PKD system in conjunction with the conventional DES system. using Apple microcomputers is considered.

## 13.5 DES/PKD Hybrid System

A hybrid DES/PKD demonstration system has been developed using the DES interface unit (Chapter 4) and the exponentiation system over GF($2^{127}$). The system has been experimented using two Apple microcomputers forming a link. Each Apple microcomputer is assumed to be shared by n users (here n = 5 ). Each of these n users can compose and send a message to any other user at the other end in a secure way. This system is primarily intended as a testbed for investigating the problems of developing an application system incorporating DES/PKD techniques on a microprocessor based system.

The DES interface card is used to encrypt messages under one of the three modes namely the ECB, CFB or CBC (Chapter 5). The PKD function performs the basic GF($2^{127}$)* exponentiation system using $f(x) = x^{127} + x + 1$ as the modulus irreducible polynomial. This is essentially used to transfer the session keys securely between the two Apple microcomputers over a public telephone network. With 6502 microprocessor running at 1 MHZ, an average time of 4 seconds with a worst case of 6 seconds is required to perform the exponentiation in GF($2^{127}$). The PKD program size is approximately 400 bytes and the listing of the program is given in Appendix 18 (see Section 13.6).

During system operation, the user can communicate with an user at the other end under DES or DES/PKD modes or generate a new secret/public key pair for the PKD system. The system base 'a' and the modulus polynomial f(x) are input once to the program and they are assumed to be fixed. The public key generated is stored in some preallocated memory location depending upon the user ID, n. That is, in this simple system, each terminal stores its own list of public keys. This is sometimes referred to as the local Public Key File (PKF) mode.

A secure connection is established between users i and j through a simple connection protocol sequence. This sequence allows to establish a DES session key and initialization vector

*As mentioned in Section 13.2, this can be changed to GF($2^{521}$) with $f(x) = x^{521} + x^{32} + 1$ to overcome Adleman's algorithm.

required for secure communication between the users i and j. It also enables each user to authenticate the other's identity. A new session key and a new initialization vector, derived using the common key $K_{ij}$, are established for each session between i and j. This sequence is used with the local PKF mode mentioned above.

When user i wishes to establish a secure connection with user j, i sends his user ID(I) along with the ID of user j(J) to the other end. That is,

$$i \rightarrow j \; : \; I, J$$

The user j is informed at the receiving end that a communication has been requested by user i and the system asks the user j to input his secret key via the keyboard of the terminal. As soon as user j has entered his 16 character (127-bit) secret key, the system requests user i at the sending end to input his secret key. At this point, users i and j fetch the other user's locally stored public key value from their memories and compute the common key $K_{ij}$ using their own secret keys independently at their respective ends. Then the sending end generates a pseudo-random number, R, and encrypts this number under the ECB mode of DES using the first 64 bits of $K_{ij}$ as the DES key. ($K_{ij1}$). This cipher is then transmitted to the receiving end. That is,

$$i \rightarrow j \; : \; (R)_{K_{ij1}}$$

The receiving end decrypts the cipher using $K_{ji1}$ as the DES key in ECB mode to obtain the pseudo-random number R. Then he modifies the number R by adding 1 to it and encrypts (R+1) under the ECB mode of DES with $K_{ji2}$ and transmits to user i.
That is,

$$j \rightarrow i \; : \; (R+1)_{K_{ji2}}$$

This operation is done by user j to prove his identity to user i which requires the knowledge of the common key $K_{ij} = K_{ji}$. The user i at the sending end decrypts the information and tests to see if the message is equal to R+1. If this is the case, then the user at the receiving end must have computed the key $K_{ji} = K_{ij}$ and hence it must be user j as he is the only one who knows the user j's secret key. Hence user i can start a conversation with user j at the other end. If they do not match, then the user at the receiving end is informed that the conversation cannot begin. This may either be due to some error during the communication or more seriously due to the false identity of user j at the receiving end.

In this simple procedure, the dynamically generated pseudo-random number is used as the session key to be used in the DES algorithm for encrypting data. Note that the number R is never actually transferred over the link between users i and j in its plain form. If the DES algorithm is used in its ECB mode for encrypting data messages no more initialization procedure is required. The users i and j can communicate with each other in a secure manner using the DES system with R as the secret DES key. On the other hand, if the DES is used in either CFB or CBC modes, then the initialization vector needs to be transferred from the sender to the receiver. This is done using the normal procedure explained in Section 5.3.2, by generating another pseudo-random number, encrypting it under the ECB mode using the session key and transmitting it to the receiver.

To authenticate the identity of user i and establish an initialization vector for transfers from j to i, user j generates the pseudo-random number which he then sends to i in encrypted from. Establishment of authentication of user i proceeds in a similar fashion as described above by the modification of the random number by user i. Note that user authentication in such a public key distribution system is based on the possession of the secret key which the user employs to compute the common key for any other user. It is the calculation of the common key that results in user authentication and DES protection.

A set of 10 pairs of public/secret keys used in this demonstration system is given in Figure 13.2. The public keys although they do not have to be kept secret, they still have to be authentic. Otherwise an imposter could manufacture his own public key, claim it belonged to another user and then employ it to impersonate him. Thus public keys have to be public in the broadest sense - not only non-secret, but guaranteed,'accessible to or shared by all users of the network' [78]. Three possible methods by which the public keys can be distributed in this hybrid DES/PKD system are considered [75].

System Base a = 1 2 3 4 5 6 7 8 1 2 3 4 5 6 7 8

System Polynomial, f(x) : $x^{127} + x + 1$

| User No. | Secret key (Bytes) | Public key (Hexadecimal code) |
|---|---|---|
| 1 | ASDFGHJKLMNBVCXZ | FEDB15A175Ø2AF4E13C71678863ØCD41 |
| 2 | 8765432187654321 | 9F6AC7C3DD227686F3974Ø9CFA53DE96 |
| 3 | QWERTYUIOPLKJHGF | C82C8F4781B7CC85DED399F7Ø7DED4DØ |
| 4 | ØPL,MKO98IJNBHU7 | 9Ø52E6CØE3DCAEØ326F82DE99FC9EAØ9 |
| 5 | 1QAZXSW23EDCVFR4 | 228Ø521679EBCE31F29227A3194BA2F2 |
| 6 | 5TGB6YHN7UJM8ZA1 | 18EDF7ØF67CF4F2158EE37BE2F8Ø725 |
| 7 | !"#$&'()Ø*=@+<> | 91314E65CF6D5231D258A8BAØ6D3E3AØ |
| 8 | 1!2"3#4$5%6&7'8( | 1287266652187B15Ø338581F6E3B83ØC |
| 9 | AZSXDCFVGBHNJMKL | 619Ø5Ø8319F9Ø5A1CØAD2ØA7FF92EBCA |
| 10 | W28DF6&N^MTG!RAM | 4877538FFEØ3934F7B8DE1Ø924CØFE94 |

Fig. 13.2 - Public & Secret Key Pairs in GF($2^{127}$)

### 13.5.1 Central Public Key File

In this approach, as in Section 9.6, the existence of a Public Key Distribution Centre (PKDC) is envisaged which controls the formation of user connections. The connection protocol sequence may be described as follows. Each user registers his public key with the PKDC and each user knows the public key of the Centre. When a user i wishes to communicate with user j, he first enters his secret key on the terminal. This key can then be combined with the PKDC's public key, which is assumed to be locally stored on the terminal, to form a common key between the user and the PKDC. Then the user can connect with the PKDC, using the common key in his DES based system, to transmit an encrypted request to the PKDC for user j's public key (or the entire list of public keys). The PKDC decrypts the request and encrypts and transmits the desired public key(s) to the user i. Hence the connection to the PKDC is authentic and private so that an opponent cannot modify the public keys transmitted or impersonate the PKDC without detection. This approach insures that the PKDC controls the access of users to the system and centralizes the dissemination of public keys. Note that a connection to the PKDC is required to obtain a list of public keys but once the user has stored the public keys, the PKDC is no longer needed unless the user wishes to update his public key list.

### 13.5.2 Local Public Key File

This method has been used in the simple demonstration system described earlier and is very attractive when only a small number of users are involved. Each user possesses an authentic list of public keys. The distribution of this local directory can be accomplished manually through hardcopy or electronic storage media such as programmable memory PROM. Precautions must be taken to protect the public keys from modification or substitution but not from privacy.

### 13.5.3 No Public Key File

The third approach assumes a more benign environment in which the opponent is content to passively eavesdrop. If this applies, keys do not need to be stored. To establish a connection, each user generates a new secret key, computes a new public key, exchanges it and then calculates the secret session key. An active opponent however can interpose between the two users, can mirror each half of the scenario and establish an imposter connection with each. This method is mentioned here for completeness sake and it is recommended that it should be avoided in practice.

Thus this arrangement shows that the implementation of the PKD algorithm and its use in a hybrid system using the DES is entirely feasible. The value of combining the protection provided by the conventional cryptosystem with the user authentication attributes of a public key system is most advantageous. The integration of these two methods gives the designer of secure systems flexibility in these areas:

1. The distribution and management of keys; while most conventional cryptosystems require centralized key management and connection establishment, integration of public keys systems with conventional cryptosystems promises centralized control of key management functions with distributed connection establishment.
2. Decentralized user authentication; since a user's identity can be confirmed using public parameters and a single secret parameter known only to the user.
3. Document or file protection; the conventional cryptosystem

can be used to protect against information release by encryption using document or file key while access to the file key is restricted by the public key system.

## 13.6 Exponentiation in GF($2^n$)

In this section two different methods of performing exponentiation in GF($2^n$) are considered. These indicate that fast exponentiation in GF($2^n$) is possible using dedicated hardware and make the DES/PKD hybrid system described above a practical way of providing both security and authentication.

Both the methods, use the well known 'square and multiply' technique to perform exponentiation in GF($2^n$). This technique has been used throughout Chapters 10, 11 and 12 and the flowchart is given in Figure 10.2. But in this case, the operations are performed modulo 2 which makes the implementation easy and reduces the running time of the algorithm. The two methods differ in the way they perform multiplication and squaring in GF($2^n$).

### 13.6.1 Method 1

#### 13.6.1.1 Squaring

Squaring operation in Galois field can be performed very efficiently if the irreducible generator polynomial f(x) is fixed as shown below.

Let the irreducible polynomial be a trinomial, $f(x) = x^n+x+1$. (In the demonstration hybrid DES/PKD system described in Section 13.5, $f(x) = x^{127}+x+1$). Let 'a' represent an element of the field GF($2^n$) (for example, the system base) and representing 'a' as a binary polynomial

$$a = a_0 + a_1x + \dots + a_{n-1}x^{n-1} \qquad a_i \in GF(2)$$

As GF($2^n$) is a field with characteristic 2, the property that

$$(a+b)^2 = a^2 + b^2$$

holds true for the operations in the field. Thus the representation of $a^2$ is given by

$$a^2 = a_0 + a_1x^2 + a_2x^4 + \dots + a_{n-1}x^{2(n-1)}$$

Now reducing the powers greater than n using the recursive function $x^{n+j} = x^{j+1} + x^j$ for $j \geq 0$, gives

$$a^2 = a_0 + a_{n/2}x + (a_1+a_{n/2})x^2 + \dots + a_{n-1}x^{n-2} + (a_{n/2} + a_{n-1})x^{n-1}$$

where n/2 denotes $\lceil n/2 \rceil$, the least integer greater than n/2.

As the characteristic of the field is 2, one can represent $a^2$ using exclusive-or operator $\oplus$

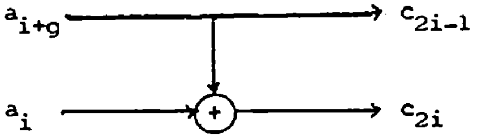
$$a^2 = a_0 + a_{n/2}\ x + (a_1 \oplus a_{n/2})x^2 + \ldots + a_{n-1}x^{n-2} + (a_{n/2} \oplus a_{n-1})\ x^{n-1}.$$

This expression can be rewritten as

$$a^2 = a_0 + \sum_{i=1}^{g} \{a_{i+g}x^{2i-1} + (a_i \oplus a_{i+g})\ x^{2i}\} \qquad (13.2)$$

where $g = n/2 - 1$.

Implementation of the equation (13.2) from the hardware point of view gives



where

$$(c_0,c_1,\ldots,\ c_{n-1}) = (a_0,a_1,\ldots,\ a_{n-1})^2$$

Thus squaring can be very efficiently accomplished using exclusive-or gates alone. Such an implementation in GF($2^7$) using 7-bit vector is shown in Figure 13.3.

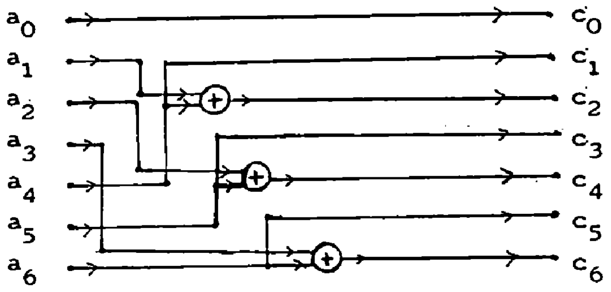


Fig 13.3-Squaring in GF($2^7$)

### 13.6.1.2 Multiplication

The multiplication in GF($2^n$) can be performed using the standard canonical basis representation of elements. That is, the elements of GF($2^n$) are expressed in terms of a canonical basis for GF($2^n$) over GF(2) and the multiplication rule is derived as follows:

Suppose that U and V in GF($2^n$) have the representation $(u_0,u_1,\ \ldots\ u_{n-1})$ and $(v_0,v_1,\ldots,\ v_{n-1})$ respectively in terms of the canonical basis $(1,\alpha\ ,\ldots,\alpha^{n-1})$ where $\alpha$ is a root of an irreducible polynomial of degree n over GF(2). This means that

$$U = [u_0,u_1,u_2,\ \ldots,\ u_{n-1}] \begin{bmatrix} 1 \\ \alpha \\ \vdots \\ \alpha^{n-1} \end{bmatrix}$$

and

$$V = [v_0, v_1, \ldots, v_{n-1}] \begin{bmatrix} 1 \\ \alpha \\ \cdot \\ \cdot \\ \alpha^{n-1} \end{bmatrix} = [1, \alpha, \ldots, \alpha^{n-1}] \begin{bmatrix} v_0 \\ v_1 \\ \cdot \\ \cdot \\ \cdot \\ v_{n-1} \end{bmatrix}$$

Hence $Z = U.V$

$$= [u_0, u_1, \ldots, u_{n-1}] \begin{bmatrix} 1 \\ \alpha \\ \cdot \\ \cdot \\ \cdot \\ \alpha^{n-1} \end{bmatrix} [1, \alpha, \ldots, \alpha^{n-1}] \begin{bmatrix} v_0 \\ v_1 \\ \cdot \\ \cdot \\ \cdot \\ v_{n-1} \end{bmatrix}$$

$$Z = [u_0, u_1 \ldots, u_{n-1}] \begin{bmatrix} 1 & \cdots\cdots & \alpha^{n-1} \\ \alpha & & \cdot \\ \vdots & & \vdots \\ \alpha^{n-1} & \cdots\cdots & \alpha^{2n-2} \end{bmatrix} \begin{bmatrix} v_0 \\ \cdot \\ \cdot \\ v_{n-1} \end{bmatrix} \qquad (13.3)$$

Now expanding the nxn matrix in (13.3) as

$$\begin{bmatrix} 1 & \cdots\cdots & \alpha^{n-1} \\ \alpha & & \cdot \\ \vdots & & \vdots \\ \alpha^{n-1} & \cdots & \alpha^{2n-2} \end{bmatrix} = M_0 + M_1\alpha + M_2\alpha^2 + \ldots + M_{n-1}\alpha^{n-1}$$

where $M_k$ is the GF(2) matrix whose entry in row i and column j is the coefficient of $\alpha^k$ when $\alpha^{i+j-2}$ is expanded in the canonical basis $(1, \alpha, \ldots, \alpha^{n-1})$. Hence if $(z_0, \ldots, z_{n-1})$ is the representation of Z we have

$$z_0 + z_1 \alpha + \ldots + z_{n-1} \alpha^{n-1} = \underset{\sim}{u}^t M_0 \underset{\sim}{v} + \underset{\sim}{u}^t M_1 \underset{\sim}{v} \alpha + \ldots + \underset{\sim}{u}^t M_{n-1} \underset{\sim}{v}\alpha^{n-1}$$

where

$\underset{\sim}{u} = (u_0, u_1, \ldots, u_{n-1})$ and $\underset{\sim}{v} = (v_0, v_1 \ldots, v_{n-1})$.

It now follows from the uniqueness of the representation in terms of a fixed basis that

$$z_k = \underset{\sim}{u}^t M_k \underset{\sim}{v} \qquad \text{for } k = 0, 1, \ldots, n-1 \qquad (13.4)$$

The right hand side of (13.4) is sometimes referred to as the bilinear form in the vectors $\underset{\sim}{u}$ and $\underset{\sim}{v}$.

To avoid the need to store the matrices $M_k$ in the calculation of the product of two elements, the algorithm given by Berlekamp in [56] has been used.

Let the multiplicand be U and the multiplier be V and assume that they are stored in two n-bit registers U and V respectively. Let the partial product register be called Z. To perform multiplication, the Z register is initially set to zero. Depending on the lowest bit of the multiplicand $v_0$, U is either added or not added into Z (modulo 2 addition). If $v_0 = 1$, then U is added into Z; if $v_0 = 0$, Z is left unchanged. The V register is then shifted right, the U register is multiplied by $\alpha$ and the process is repeated. Multiplication by $\alpha$ is done according to the recursive equation $\alpha^n = \alpha + 1$. At the mth step, U contains $\alpha^m$ times the original multiplicand, $v_0$ contains $v_m$, the mth bit of the original multiplier and Z contains $\sum_{i=0}^{m} V_i(U\alpha^i)$. After n such steps, the multiplication is complete. Z contains the product of the original U and V registers. The V register has been cycled completely around to its original position but the U register now contains U $\alpha^{n-1}$. A schematic diagram using feedback shift registers is shown in Figure 13.4 for the field GF($2^7$) with $f(x) = x^7+x+1$ as the generating irreducible polynomial.

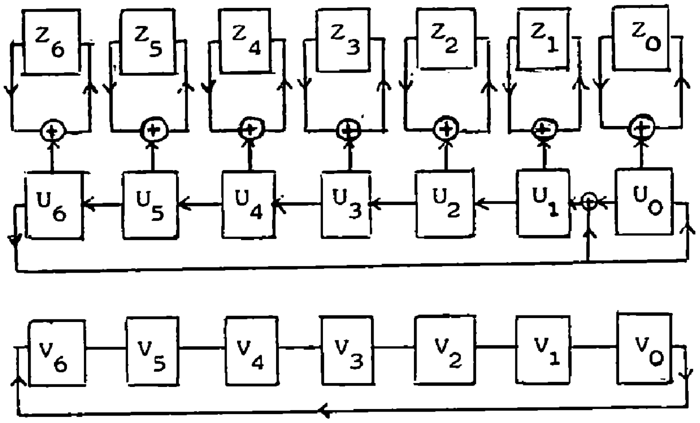


Fig 13.4 - Multiplication in GF($2^7$)

This method of multiplying and squaring elements in GF($2^n$) has been used in the software implementation of the PKD algorithm in the hybrid system. With dedicated hardware, the exponentiation algorithm would take approximately 16256 (=127 x 128) clock cycles in the worst case. Hence using 1MHZ clock, this gives a worst running time of less than 20 milliseconds compared to 6 seconds when carried out in software using 6502 machine code programming.

### 13.6.2 Method 2

Here exponentiation is performed in GF($2^n$) using a novel technique based on normal basis representation of elements [79]. This section contains some of the notes communicated by Prof J Massey, ETH, Zurich, in private correspondence. The method has been used to design a hardware exponentiation system in GF($2^7$).

13.6.2.1 Multiplication

The multiplication rule given in method 1 can be used with any basis not only with the canonical basis.

Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be any basis for GF($2^n$) over GF(2) and let $\underset{\sim}{u} = (u_1, \ldots, u_n)$, $\underset{\sim}{v} = (v_1, v_2, \ldots, v_n)$ and $Z = (z_0, z_1, \ldots, z_n)$ be the representations of U, V and Z in terms of this basis. It follows that if Z = UV, then

$$Z = [u_1, \ldots, u_n] \begin{bmatrix} \alpha_1^2 & \ldots & \alpha_1\alpha_n \\ \alpha_2\alpha_1 & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \alpha_n\alpha_1 & \ldots & \alpha_n^2 \end{bmatrix} \begin{bmatrix} v_1 \\ \cdot \\ \cdot \\ \cdot \\ v_n \end{bmatrix} \qquad (13.5)$$

The matrix in (13.5) can be expanded as

$$\begin{bmatrix} \alpha_1^2 & \ldots & \alpha_1 \alpha_n \\ \cdot & & \cdot \\ \vdots & & \vdots \\ \alpha_n \alpha_1 & \ldots & \alpha_n^2 \end{bmatrix} = M_1\alpha_1 + M_2\alpha_2 + \ldots + M_n \alpha_n$$

where $M_k$ is the GF(2) matrix whose entry in row i and column j is the coefficient of $\alpha_k$ when $\alpha_i\alpha_j$ is expanded in the basis $\alpha_1, \alpha_2, \ldots, \alpha_n$. It follows that

$$z_k = \underset{\sim}{u}^t \; M_k \; \underset{\sim}{v} \qquad \text{for } k = 1,2,\ldots, n.$$

In particular let us now consider a special basis called the normal basis.

Suppose that E is an extension of the field F and E is a vector space of dimension n over F. Then $\alpha_1, \alpha_2, \ldots, \alpha_n$ is said to be a normal basis for E over F if $\alpha_1, \ldots, \alpha_n$ are a basis for E over F and are also roots of the same irreducible polynomial in F[x]. For F = GF(2) and E = GF($2^n$), this is equivalent to saying that for some element $\alpha$ of GF($2^m$), $\alpha_i = \alpha^{2^{i-1}}$ for i = 1,2,...,n are linearly independent over GF(2). Now letting $[u_0, u_1, \ldots, u_{n-1}]$ be the representation of U ε GF($2^n$) in terms of the normal basis $\alpha, \alpha^2, \ldots \alpha^{2^{n-1}}$, then

$U = u_0\alpha + u_1\alpha^2 + \dots + u_{n-2}\alpha^{2^{n-2}} + u_{n-1}\alpha^{2^{n-1}}$

so that

$U^2 = u_0\alpha^2 + u_1\alpha^4 + \dots + u_{n-2}\alpha^{2^{n-1}} + u_{n-1}\alpha$

where the identity $u^2 \equiv u$ for all $u \in GF(2)$ and the fact that cross-term products vanish when raising to the power 2 in GF(2) have been used. Thus it is seen that squaring an element U in $GF(2^n)$ merely consists of a right cyclic shift of its representation in terms of a normal basis for $GF(2^n)$ over GF(2). In a general extension field $GF(q^n)$ where q is a prime, raising to the qth power an element of $GF(q^n)$ corresponds to a right cyclical shift of its representation in terms of a normal basis for $GF(q^n)$ over GF(q). Thus the implementation of squaring in'multiply and square' technique exponentiation is made very simple using normal basis.

Now letting $\underset{\sim}{u} = (u_0, \dots, u_{n-1})$, $\underset{\sim}{v} = (v_0, v_1, \dots, v_{n-1})$ and $\underset{\sim}{z} = (z_0, z_1, \dots, z_{n-1})$ be the normal basis representation of U, V and Z respectively and Z = U.V, then

$$z_{n-1} = (u_0, u_1, \dots, u_{n-1})^t \; M \; (v_0, v_1, \dots, v_{n-1}) \qquad (13.6)$$

where M is the GF(2) matrix whose entry in row i and column j is the coefficient of $\alpha^{2^{n-1}}$ when $\alpha^{2^{i-1}} \alpha^{2^{j-1}} = \alpha^{2^{i+j-2}}$ is expanded in the normal basis $\alpha, \alpha^2, \dots, \alpha^{2^{n-1}}$. But $Z^2 = U^2V^2$ and the elements $U^2, V^2$ and $Z^2$ have the normal basis representations $(u_{n-1}, u_0, \dots, u_{n-2})$, $(v_{n-1}, v_0, \dots, v_{n-2})$ and $(z_{n-1}, z_0, \dots, z_{n-2})$ respectively. Thus it follows that for the same matrix M as in (13.6)

$$z_{n-2} = (u_{n-1}, u_0, \dots, u_{n-2})^t \; M \; (v_{n-1}, v_0, \dots v_{n-2})$$

and in general that, for this same GF(2) matrix M

$$z_k = (u_{k+1}, \dots, u_{n-1}, u_0, \dots, u_k) \; M \; (v_{k+1}, \dots v_{n-1}, v_0, \dots v_k)$$

for $0 \leq k \leq n-1$.

Thus when a normal basis representation of $GF(2^n)$ over GF(2) is employed, each digit in the product is given by the same bilinear form with appropriate cyclic shifting of the representations of the factors. This pröperty is the key to construction of simple multipliers for finite fields.

In the PKD system, it is necessary to implement the equation (13.6) to perform multiplication in $GF(2^n)$. The expression (13.6) can be written as

$$z_{n-1} = u_{n-2}\, v_{n-2} + (u_0, \dots, u_{n-1})^t \Lambda \, (v_0, v_1, \dots, v_{n-1}) \qquad (13.7)$$

where the binary matrix $\Lambda$ is symmetric with an all zero main diagonal.

This can be seen as follows:

Writing $m_{ij}$ for the entries in M,

$$z_{n-1} = \sum_i \sum_j u_i v_j m_{ij}$$

Suppose $u_k = v_k = 1$ and $u_s = v_s = 0$ for $s \neq k$. Then $z_{n-1} = m_{kk}$. But for this case $U = V = \alpha^{2^k}$ which implies that $Z = UV = \alpha^{2^{k+1}}$. Thus $z_{n-1} = 1$ if and only if $k = n-2$. Hence $m_{kk} = 1$ if and only if $k = n-2$, so that the next to last entry on the main diagonal of M is the only one which is non-zero.

Because of its symmetry and zero diagonal, the nxn matrix $\Lambda$ can be written as

$$\Lambda = \begin{bmatrix} 0 & & \\ \vdots & T & \\ 0 & \cdots & 0 \end{bmatrix} + \begin{bmatrix} 0 & \cdots & 0 \\ & T^t & \vdots \\ & & 0 \end{bmatrix}$$

where T is the binary upper triangular (n-1) x (n-1) matrix.

$$T = \begin{bmatrix} t_{01} & t_{02} & \cdots & t_{0,n-1} \\ 0 & t_{12} & & t_{1,n-1} \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & & t_{n-2,n-1} \end{bmatrix}$$

Thus,

$$\begin{aligned} z_{n-1} = u_{n-2}v_{n-2} &+ (u_0,\ldots, u_{n-2})^t \; T \; (v_1,\ldots, v_{n-1}) \\ &+ [T(u_1,\ldots, u_{n-1})]^t \; (v_0,v_1 \ldots v_{n-2}) \end{aligned} \qquad (13.8)$$

Using (13.8) one particular implementation of the multiplication function is shown in Figure 13.5. The boxes labelled T contain Exclusive-or gates only. Each component of the output is that sum of input bits corresponding to the locations of the '1's in the corresponding row of the matrix T. The outputs of this box are then And-ed with the appropriate components of the other factor. The outputs of these n-1 And gates are then summed by a tree of n-2 Exclusive-or gates. The outputs of the two Exclusive-or trees are summed and added to the term $u_{n-2}v_{n-2}$. Alternatively, the two Exclusive-or trees and the Exclusive-or gate that combines their outputs can be replaced by a single Exclusive-or tree with 2n-2 gates having 2n-1 inputs. The circuit shown in Figure 13.5 computes $z_{n-1}$. If U and V are shifted as shown in the diagram, the same circuit will compute $z_{n-2}$, $z_{n-3}$, $\ldots, z_1, z_0$ during the

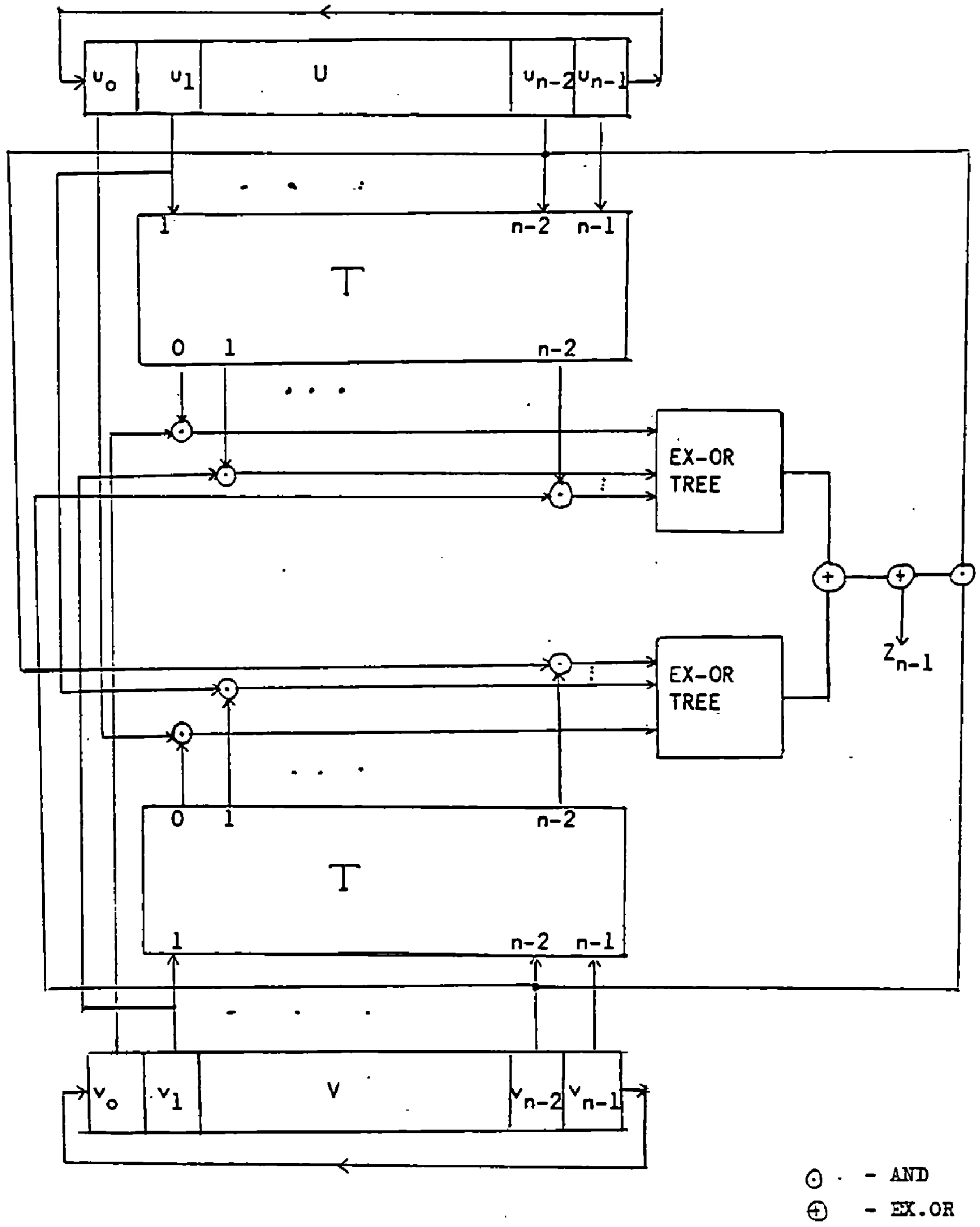


Fig. 13.5 - Multiplier Configuration using T-matrix

next n-1 subsequent shifts.

A different implementation of the multiplication function is based on the alternating property of $\Lambda$.

Let B denote any nxn matrix in any field F and let $\underset{\sim}{u}$ and $\underset{\sim}{v}$ be any vectors in $F^n$. Then the bilinear form $\underset{\sim}{u}^t B \underset{\sim}{v}$ defined by B is said to be alternating if it vanishes whenever $\underset{\sim}{u} = \underset{\sim}{v}$, that is, if $\underset{\sim}{u}^t B \underset{\sim}{u} = 0$ for all $\underset{\sim}{u}$ in $F^n$. Two nxn matrices over F, B and D, are said to be congruent if there is an invertible matrix P such that

$$B = P^t \; D \; P$$

Then

$$\underset{\sim}{u}^t \; B \; \underset{\sim}{v} = (P\underset{\sim}{u})^t \; D \; (P\underset{\sim}{v})$$

Thus the bilinear form $\underset{\sim}{u}^t$ B $\underset{\sim}{v}$ can be evaluated using the matrix D if the basis is changed by the transformation, $P^{-1}$. When D has only a small number of non-zero entries, the evaluation of $(P\underset{\sim}{u})^t$ D $(P\underset{\sim}{v})$ requires only a small number of multiplications and additions beyond those needed to form $P\underset{\sim}{u}$ and $P\underset{\sim}{v}$.

In [48], a binary matrix B which is symmetric and has an all zero main diagonal is shown to be alternating. Thus the matrix $\Lambda$ which defines the normal basis multiplier is alternating. The rank of the matrix $\Lambda$ is n when n is even and is n-1 when n is odd. The matrix $\Lambda$ can be reduced to a diagonal matrix D using the elementary divisor method given in Section 12.2.4. Thus for the nxn matrix $\Lambda$, there exists elementary matrices $E_1$, $E_2$,..., $E_m$ (for some m) such that

$$E_m \ldots E_1 \Lambda E_1^t \ldots E_m^t = D$$

where D is a diagonal matrix with n non-zero entries when n is even or n-1 non-zero entries when n is odd. Thus

$$P^t = E_m \ldots E_1$$

The number of '1's present in the P-matrix is indicative of the number of Exclusive-or gates required to implement this matrix. Hence using this P-box, one can implement the multiplier function as

$$z_{n-1} = u_{n-2} \, v_{n-2} + (P\underset{\sim}{u})^t \; D \; (P\underset{\sim}{v})$$

This configuration is shown in Figure 13.6. The connections in the P-box are determined by the P-matrix and the connections to the And gates are determined by the D-matrix. Subsequent shifts of $\underset{\sim}{u}$ and $\underset{\sim}{v}$ vectors will produce $z_{n-2}, \ldots, z_0$.

As a final method of implementing the multiplier function $z_{n-1} = \underset{\sim}{u}^t \; M \; \underset{\sim}{v}$, the 'brute-force' approach is considered. This approach consists of first forming the vector $M\underset{\sim}{v}$ and then And-ing

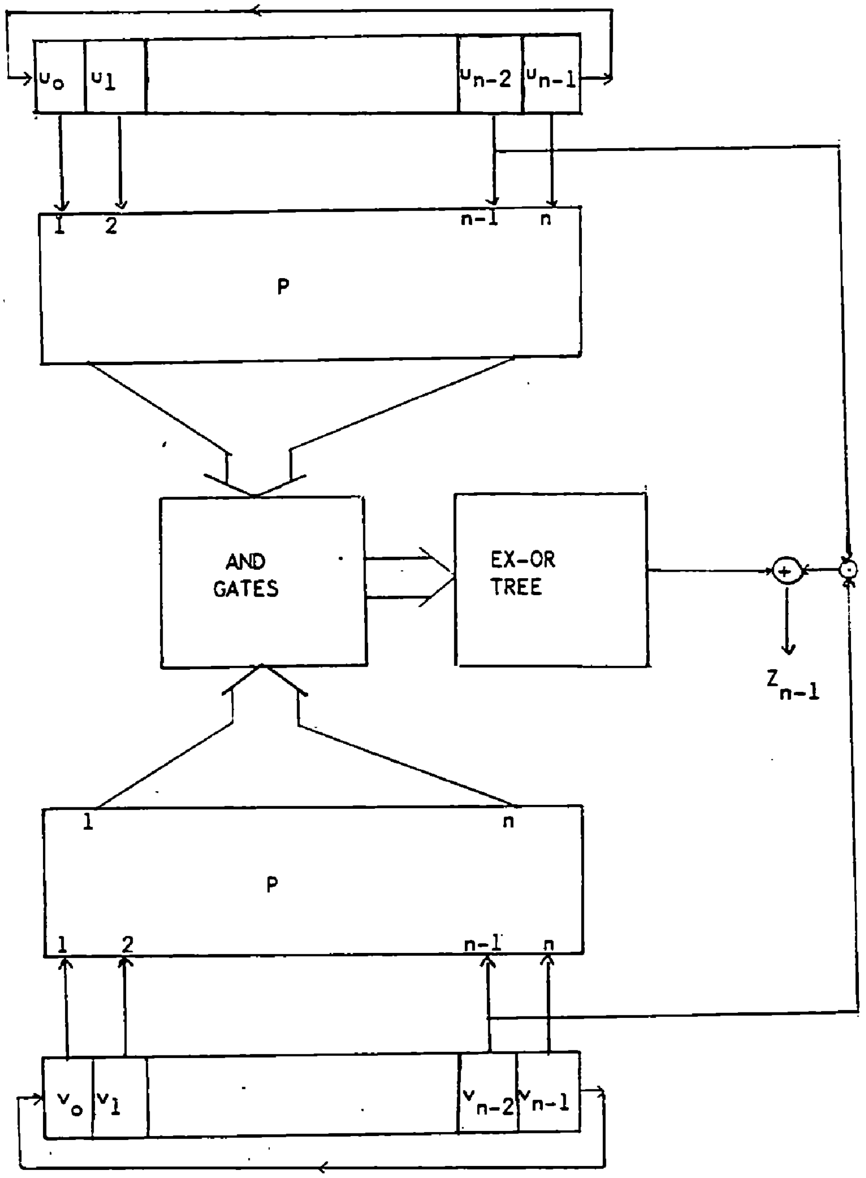


Fig. 13.6 - Multiplier Configuration using P matrix

each of its components with the corresponding component of $\underset{\sim}{u}$, then finally summing the outputs of the n And gates. This approach requires n And gates and n-1 Exclusive-or gates in addition to the Exclusive-or gates needed to form $M\underset{\sim}{v}$. Compared to the previous two methods, this one lacks the 'modularity' - several sub-circuits of the same kind.

## 13.7 Hardware Design of An Exponentiator in GF($2^7$)

A hardware exponentiation system in GF($2^7$) has been designed using the second method which involves the normal basis representation. To begin with, a program is written to search for a generator of a normal basis in GF($2^7$). This consists of finding seven conjugates in GF($2^7$) which are linearly independent over GF(2). A complete listing of the program is given in Appendix 19. The steps involved in this algorithm are given as follows:

1. Choose an element $\alpha \in$ GF($2^n$)
2. Compute $\alpha_i = \alpha^{2^{i-1}}$ for i=1,..., n in GF($2^n$) using the irreducible polynomial $f(x) = x^n + x + 1$.
3. Form a nxn matrix using the n-bit vectors $\alpha_1,\dots,\alpha_n$.
4. Calculate the determinant of the matrix over GF(2).
5. If determinant $\equiv 0$ in GF(2), Go to step 1.
6. If determinant $\not\equiv 0$ in GF(2), then $\alpha$ generates a normal basis.

In this case n= 7. Initially, it is decided to look at the elements $\alpha$ of the form $\alpha = x^a + 1$ for $1 \leq a \leq n-1$. One such generator $\alpha$ is found to be $\alpha_1 = x^3 + 1$. Then the powers $\alpha_i = \alpha^{2^{i-1}}$ for i = 1,2,..., 7 modulo $(x^7 + x + 1)$ are given by:
$\alpha_1 = x^3 + 1, \alpha_2 = x^6+1, \alpha_3 = x^6 + x^5 +1, \alpha_4 = x^6 + x^5 + x^4 + x^3 + 1,$
$\alpha_5 = x^5 + x^4 + x^3 + x^2 + x + 1, \alpha_6 = x^6 + x^3 + x + 1$ and $\alpha_7 = x^5 +$
$x^2 + 1$.

The symmetric matrix of the bilinear form is then given by

$$\begin{array}{ccccccc}
x^6+1 & x^6+x^2+1 & x^6+x^5+x+1 & x^5+x^4+x^3 & x^6+x & x^4+x^3+x^2+x+1 & x^3+x+1 \\
* & x^6+x^5+1 & x^6+x^5+x^4+1 & x^6+x^5+x^4+x^3+x^2+1 & x^6+x^4+x^3+x^2+x & x^6+x^5+x^2 & x^6+x^4+x+1 \\
* & * & x^6+x^5+x^4+x^3+1 & x^6+x^5+x^4+x^3+x^2+x+1 & x^5+x^3+x^2+x+1 & x^5+x^4+x & x^6+x^5+x^3 \\
* & * & * & x^5+x^4+x^3+x^2+x+1 & x^5+x^3+x+1 & x^6+x^3+x^2+1 & x^4+x^3+x \\
* & * & * & * & x^6+x^3+x+1 & x^6+x^4+x^3+x^2+1 & x^5+x^4+1 \\
* & * & * & * & * & x^5+x^2+1 & x^5+x^4+x^2+x+1 \\
* & * & * & * & * & * & x^3+1
\end{array}$$

* - indicates symmetry

Rewriting the above matrix in terms of the normal basis, gives

$$\begin{matrix}
\alpha_2 & \alpha_4+\alpha_5+\alpha_6+\alpha_1+\alpha_2 & \alpha_5+\alpha_7+\alpha_4 & \alpha_4+\alpha_2 & \alpha_1+\alpha_6 & \alpha_2+\alpha_3+\alpha_5 & \alpha_1+\alpha_3+\alpha_4+\alpha_5+\alpha_7 \\
* & \alpha_3 & \alpha_2+\alpha_3+\alpha_5+\alpha_6+\alpha_7 & \alpha_1+\alpha_6+\alpha_5 & \alpha_5+\alpha_3 & \alpha_2+\alpha_7 & \alpha_3+\alpha_4+\alpha_6 \\
* & * & \alpha_4 & \alpha_1+\alpha_3+\alpha_4+\alpha_6+\alpha_7 & \alpha_2+\alpha_6+\alpha_7 & \alpha_4+\alpha_6 & \alpha_1+\alpha_3 \\
* & * & * & \alpha_5 & \alpha_1+\alpha_2+\alpha_4+\alpha_5+\alpha_7 & \alpha_1+\alpha_3+\alpha_7 & \alpha_5+\alpha_7 \\
* & * & * & * & \alpha_6 & \alpha_1+\alpha_2+\alpha_3+\alpha_5+\alpha_6 & \alpha_1+\alpha_2+\alpha_4 \\
* & * & * & * & * & \alpha_7 & \alpha_2+\alpha_6+\alpha_3+\alpha_4+\alpha_7 \\
* & * & * & * & * & * & \alpha_1
\end{matrix}$$

* - indicates symmetry

Expanding the above matrix in terms of $\alpha_1, \alpha_2, \ldots, \alpha_7$, yields

$$M = M_1\ \alpha_1 + M_2\ \alpha_2 + M_3\ \alpha_3 + M_4\ \alpha_4 + M_5\ \alpha_5 + M_6\ \alpha_6 + M_7\ \alpha_7$$

where

$$M_1 = \begin{bmatrix}
0&1&0&0&1&0&1 \\
1&0&0&1&0&0&0 \\
0&0&0&1&0&0&1 \\
0&1&1&0&1&1&0 \\
1&0&0&1&0&1&1 \\
0&0&0&1&1&0&0 \\
1&0&1&0&1&0&1
\end{bmatrix}$$

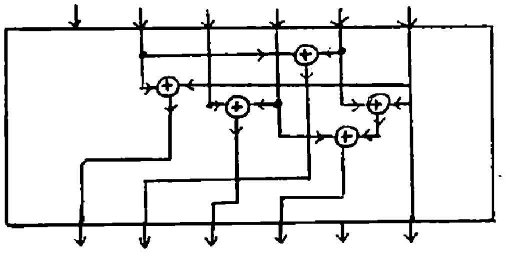
and

$M_2$ is obtained by rotating the rows of $M_1$ downward by 1 position. and then rotating the columns right by 1 position. Similar operations on $M_2$ yield $M_3$ and so on. In the implementation of the multiplier function

$M = M_7$ is used. The T-matrix is therefore given by

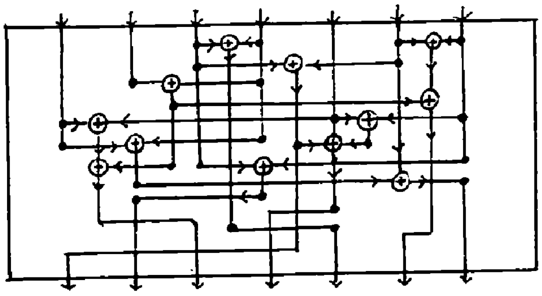
$$T = \begin{bmatrix} 0&1&0&0&0&1 \\ 0&1&0&0&1&0 \\ 0&0&1&1&0&0 \\ 0&0&0&1&1&1 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&1 \end{bmatrix}$$

The T-box can therefore be realized as



Note that one of the six outputs is not used and is identically equal to 0. The And gate is therefore unnecessary that is fed by this 0 output. Thus the complete circuit requires 2 T-boxes with 5 Exclusive-or gates each, 2 sets of 5 And gates operating on their outputs, 2 Exclusive-or trees with 4 Exclusive-or gates each and 2 additional Exclusive-or gates and 1 And gate for producing the final output. The total gate count for the multiplier function is equal to 20 Exclusive-or gates and 11 And gates.

Using the brute force method the multiplier function can be implemented as follows:



The number of gates required using this approach is equal to 18 Exclusive-or gates and 7 And gates. It appears to give the minimum number of gates of the three realizations; but the realization using the T-matrix or the P-matrix is preferrable for large extension fields because then the circuit is composed of two identical subcircuits, that is, a modular design is achieved.

The circuit diagram of the designed complete exponentiation system in GF($2^7$) is given in Appendix 20. Here the multiplier function has been implemented using the 'brute-force' approach.

As the number of '1's present in the M-matrix give an indication of the number of Exclusive-or gates required to implement the multiplier function, to minimise the number of gates, one needs to reduce the number of '1's in the M-matrix to a minimum. In an effort to obtain such an optimum M-matrix, various normal basis generators $\alpha$ with four different irreducible polynomials are tried in GF($2^7$) and the number of '1's in their corresponding M-matrices are calculated. The results are given in Figure 13.7. From the results it is seen that

a) the difference between the minimum and the maximum number of '1's for the tried cases is not large, that is, the variance does not appear to be high.

b) The average number of '1's in the M-matrix is approximately equal to 23 which is roughly equal to $n^2/2 = \frac{49}{2} \doteq 24$.

This may imply that for a randomly chosen normal basis generator and an irreducible polynomial, the number of '1's is approximately equal to half the entries in the M-matrix. It appears that the random choice of normal basis generators and different irreducible polynomials does not seem to yield any substantial reduction in the number of '1's in the M-matrix. The above claim should be read with caution as this is based on a small number of trials in a small extension field GF($2^7$).

## 13.8 Normal Basis Generators in GF($2^{127}$)

From cryptography point of view, one is interested in large extension fields namely GF($2^{127}$) or GF($2^{521}$). The next step is therefore to determine the generators of normal basis in these extension fields which can then be used in the PKD exponentiation system. Albert [48] proved that if F is a subfield of E and E is normal over F, then E has a normal basis over F. Thus, GF($2^n$) has a normal basis over GF(2) for all n. Although the theorem by Albert establishes the existence of a normal basis for any field GF($2^n$), it does not give any help to determine the generators of normal basis in practice. As no systematic method for finding these generators was apparently evident, it is decided to resort to the trial and error procedure using the algorithm given in Section 13.7. However, let us first consider the probability of finding a normal basis using this random search procedure.

| Irreducible polynomial of degree 7 over GF(2) | Normal Basis Generator | No. of '1's in the M-matrix | |
|---|---|---|---|
| $X^7+X+1$ | $X^3+1$ | 21 | (1)† |
| $X^7+x+1$ | $X^5+1$ | 27 | (2) |
| $X^7+X+1$ | $X^3+1$ | 25 | (3) |
| $X^7+X+1$ | $X^5+X+1$ | 27 | (4) |
| $X^7+X+1$ | $X^6+X+1$ | 19 | (5) |
| $X^7+X+1$ | $X^6+X^2+1$ | 25 | (6) |
| $X^7+X^6+X^5+X^6+X^2+X+1$ | $X^4+X+1$ | 21 | (7) |
| $X^7+X+1$ | $X^6+1$ | 21 | (1) |
| $X^7+X+1$ | $X^5X^2+1$ | 21 | (1) |
| $X^7+X+1$ | $X^3+X^2+1$ | 21 | (7) |
| $X^7+X^6+X^5+X^4+X^2+X+1$ | $X^2+X+1$ | 19 | (5) |
| | $X^6+X+1$ | 27 | (4) |
| | $X$ | 27 | (2) |
| | $X^2$ | 27 | (2) |
| | $X^4$ | 27 | (2) |
| | $X^3$ | 19 | (5) |
| | $X^6$ | 19 | (5) |
| $X^7+X^3+X^2+X+1$ | $X^5$ | 27 | (4) |
| | $X+1$ | 27 | (2) |
| | $X^2+1$ | 27 | (2) |
| | $X^3+1$ | 25 | (3) |
| | $X^4+1$ | 27 | (2) |
| | $X^6+1$ | 25 | (3) |
| | $X^2+X+1$ | 19 | (5) |
| | $X^3+X+1$ | 25 | (3) |
| | $X^4+X+1$ | 21 | (7) |
| | $X^6+X+1$ | 19 | (5) |

† Same numbers in brackets indicate same M-matrix

Fig. 13.7 - Table showing some normal basis generators in GF($2^7$) and the number of '1's in the corresponding M-matrices

From [56, Theorem 11.39], the number of elements in GF($q^m$) which have m linearly independent conjugates over GF(q) is given by

$$q^m \prod_k (1 - q^{-d_k})$$

where the $d_k$ are the degrees of the distinct irreducible factors of $x^m-1$ over GF(q).

For the extension field GF($2^{127}$)

$$x^{127}-1 = \prod_{d|127} Q^{(d)}(x)$$

where $Q^{(d)}(x)$ denotes the cyclotomic polynomial.

That is,

$$x^{127}-1 = Q^{(1)}(x) \quad Q^{(127)}(x)$$

$Q^{(1)}(x)$ is an irreducible polynomial of degree 1. The degrees of the irreducible factors of $Q^{(127)}(x)$ are determined using the following result [56] :

Since every element of order n has the same number of conjugates with respect to GF(2), every irreducible factor of the cyclotomic polynomial $Q^{(n)}(x)$ has the same degree over GF(2). This degree is the multiplicative order of 2 modulo n.

For the case n = 127, the multiplicative order of 2 modulo 127 is 7 as $2^7 \equiv 1 \pmod{127}$. Thus every irreducible factor of $Q^{(127)}(x)$ has degree 7 over GF(2). Hence the probability of finding a normal basis in GF($2^{127}$) = $(1-\frac{1}{2})(1-\frac{1}{2^7})^{18} \doteq 0.434$.

The program FINDNORBAS.F77 in Appendix 19 is used to determine a normal basis generator in GF($2^{127}$) using the random search algorithm. Initially it is again decided to consider the elements of the form $\alpha = x^a+1$ for $1 \leq a \leq 126$. The operations are performed modulo the irreducible polynomial $x^{127}+x+1$. The first such generator of normal basis found in GF($2^{127}$) is $\alpha = x^{119}+1$. In GF($2^{521}$), with irreducible polynomial $x^{521}+x^{32}+1$, the first generator of the form $x^a+1$ is found to be $\alpha = x+1$.

The M-matrix and the T-matrix required for the implementation of the multiplier function are also determined, using programs M-MATRIX.F77 and T-MATRIX.F77. The listing of these two programs are given in Appendices 21 and 22. The number of Exclusive-or gates needed to implement the multiplier function using the T-matrix approach is calculated using the program EXORNO.F77 (Appendix 23). Without employing any optimization techniques, the T-matrix required 3794 Exclusive-or gates. Thus a rough estimate of the total number of

Exclusive-or and And gates required to implement the multiplier and hence the basic exponentiation function in GF($2^{127}$) is about 2x3794+2 (125)+2 = 7840 Exclusive-or gates and 2(126)+1 = 253 And gates. This can be conveniently manufactured using very large scale integration (VLSI) techniques.

## 13.9 Extension of Diffie-Hellman System to Matrix Rings

This chapter on public key distribution system is concluded by presenting an extension of the Diffie-Hellman system to matrix rings.

As the ring of all nxn matrices over a finite field contains nilpotent elements when n>1 (Section 10.4), again the group formed by only the non-singular matrices of order n, $M_n$, is considered. In particular, the group of non-singular matrices over Z/pZ where p is a prime is considered. To form a public key distribution system, it is required to choose an element $A \in M_n(Z/pZ)$ where p is a very large prime such that

$$A^r \equiv I \pmod{p}$$

where r is the order of A, the base matrix.

The base matrix A, the prime p and the order r are to be made public. Each user chooses a secret random number $x_i$ less than r and generates a public matrix $C_i$ where

$$C_i \equiv A^{x_i} \pmod{p}$$

Two users can arrive at the common key in the same way as in the Diffie-Hellman system. For instance, if user 1 wishes to initiate an interchange of secret information with user 2, he extracts the public matrix $C_2$ of user 2 and computes $C_2^{x_1} \pmod{p}$. Similarly user 2 computes $C_1^{x_2} \pmod{p}$ and the process yields the common key K where

$$K = K_{12} = K_{21} \equiv C_1^{x_2} \pmod{p} \equiv C_2^{x_1} \pmod{p} \equiv A^{x_1 x_2} (\text{mod } p)$$

With the Diffie-Hellman system operated in Z, the maximum number of secret keys possible is limited to p-1 whereas with this extended system it depends on the order of the base matrix, r. The larger the value of r, greater the number of users that the system can support. Again the security of this system is dependent on the difficulty of computing logarithms modulo p.

### 13.9.1 Design of Base Matrix

The system designer needs to construct a base matrix A in $M_n$

$(Z/pZ)$ and determine its order $r$. One method of construction of A with a given order is outlined below.

Consider an irreducible polynomial $f(x)$ of degree $m$ for which $\lambda$ is a root where $\lambda \; \varepsilon \; F_q$ and $q = p^m$.

$$f(x) = a_0 + a_1 x + \ldots + a_{m-1} x^{m-1} \qquad a_i \; \varepsilon \; F_p$$

Regarding $F_q$ as a m-dimensional vector space over $F_p$ with basis $(1, \lambda, \lambda^2 \ldots, \lambda^{m-1})$, let T represent the following linear transformation on $F_q$.

$$T : x \rightarrow \lambda x.$$

Under T, then $1 \longrightarrow \lambda, \lambda \longrightarrow \lambda^2, \ldots, \lambda^m \longrightarrow -a_{m-1} \lambda^{m-1} \ldots -a_0$.

Hence the matrix representation of the linear transformation T relative to the basis $(1, \lambda, \lambda^2, \ldots, \lambda^{m-1})$ is given by the companion matrix

$$B = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ -a_{m-1} & -a_{m-2} & -a_{m-3} & \ldots & -a_0 \end{bmatrix}$$

Linear independence of $I, T, T^2, \ldots, T^{m-1}$ implies that $I, B, B^2, \ldots, B^{m-1}$ are linearly independent. Since $f(x) = 0$, we have $f(B) = 0$. But $f(x)$ has degree $m$ and so the linear independence implies that $f(x)$ is the minimum function of B.

Hence the order of the matrix B is equal to $p^m - 1$ and

$$B^{p^m - 1} \equiv I \pmod p$$

Thus the system designer can choose irreducible polynomials of degrees $m_1, m_2, \ldots, m_s$ in $Z/pZ$ (see Section 10.5.2.1) and form the composite matrix B as shown below:

$$B = \begin{bmatrix} B_1 & & & \\ & B_2 & & \\ & & \ddots & \\ & & & B_s \end{bmatrix}$$

where the order of $B_i$ is equal to $p^{m_i} - 1$ for $1 \leq i \leq s$. The order of the matrix B is then given by the expression

$$\ell cm \left\{ (p^{m_1} - 1), (p^{m_2} - 1), \ldots, (p^{m_s} - 1) \right\}$$

The matrix A to be used in the public key distribution can then be obtained by conjugating B with an arbitrary non-singular matrix Y belonging to $M_n(Z/pZ)$. That is,

$$A = Y B Y^{-1}$$

The order of A is the same as that of B and they are of dimension n where n is given by:

$$n = \sum_{i=1}^{s} m_i$$

This above method has been implemented on the Prime computer system and the program used is given in Appendix 25. A small example of such a public key distribution system is considered below:

13.9.2 Example

Let $p = 5$

Let $f_1(x) = x^2+x+1$ where $f_1(x)$ is irreducible over $Z/5Z$.

The matrix $B_1$ is therefore given by

$$B_1 = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \equiv \begin{bmatrix} 0 & 1 \\ 4 & 4 \end{bmatrix} \pmod 5$$

and

$$\begin{bmatrix} 0 & 1 \\ 4 & 4 \end{bmatrix}^{5^2-1} \equiv I \pmod 5$$

Let $f_2(x) = x^3+3x^2+x+2$ which is irreducible over $Z/5Z$.

Hence the matrix $B_2$ is given by

$$B_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -1 & -3 \end{bmatrix} \equiv \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 3 & 4 & 2 \end{bmatrix} \pmod 5$$

and

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 3 & 4 & 2 \end{bmatrix}^{5^3-1} \equiv I \pmod 5$$

Now it is necessary to choose Y and $Y^{-1}$ such that

$$Y \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} Y^{-1} = A \text{ where } A \in M_n(Z/5Z)$$

Let Y be an arbitrary 5x5 non-singular matrix given below:

$$Y = \begin{bmatrix} 2 & 3 & 1 & 0 & 1 \\ 1 & 2 & 2 & 3 & 1 \\ 1 & 0 & 3 & 2 & 4 \\ 1 & 2 & 1 & 4 & 3 \\ 2 & 4 & 0 & 1 & 1 \end{bmatrix}$$

The determinant of Y is calculated using program DETMOD.F77 (Appendix 12) and is equal to 4 (mod 5), hence Y is non-singular. The inverse of Y is calculated using program INVMOD.F77 (Appendix 24) and is given by

$$Y^{-1} = \begin{bmatrix} 0 & 2 & 1 & 3 & 0 \\ 0 & 1 & 2 & 2 & 0 \\ 3 & 3 & 1 & 4 & 3 \\ 2 & 2 & 4 & 2 & 4 \\ 3 & 0 & 1 & 4 & 2 \end{bmatrix}$$

and hence

$$A = \begin{bmatrix} 0 & 2 & 0 & 4 & 3 \\ 1 & 1 & 3 & 1 & 3 \\ 4 & 0 & 0 & 3 & 2 \\ 3 & 3 & 2 & 4 & 4 \\ 1 & 2 & 4 & 1 & 1 \end{bmatrix}$$

The order of A is equal to $\ell cm \ \{(5^2-1)(5^3-1)\} = 744$ and $A^{744} \equiv I \pmod 5$, verified using program MATEXP.FTN (Appendix 13). Hence the key space is $2 \leq x \leq 743$ where $A^x \equiv C \pmod 5$ compared to the key space $2 \leq x \leq 4$ in the Diffie-Hellman system with p = 5.

### 13.9.3 Use of Upper Triangular Matrices Over Z/pZ

On the other hand, the ring of upper triangular matrices over Z/pZ (p prime) can also be used. If the base matrix A is chosen from this ring, then the maximum order of such a matrix is equal to p(p-1) which can be obtained by having non-zero elements along the main super-diagonal. This can be shown as follows.

Partitioning A into a diagonal matrix D and an upper triangular nilpotent matrix U, that is, A = U+D, then it is seen that D.U, U.D and $U^2$ are also upper triangular nilpotent matrices. In Section 10.4.4, it is shown that

$(D + U)^{\emptyset(p)} \equiv (I + U_{\emptyset}) \pmod p$ where $\emptyset$ is the Euler totient function and $U_{\emptyset}$ is some upper triangular nilpotent matrix.

and $(I + U_{\emptyset})^{p^t} \equiv I$ for some t

If p is assumed to be greater than n-1 (which is valid in a PKD system as the prime p is very large), then t = 1. Thus the order of A is $p\emptyset(p) = p(p-1)$. Hence in this case the key space for x is $2 \leq x \leq p(p-1)$.

# CHAPTER 14

## PERMUTATION POLYNOMIALS IN THE DESIGN OF PUBLIC KEY SYSTEMS

### 14.1 General

The fact that permutation polynomials can be used in the construction of cryptographic systems of a general mathematical nature should not be surprising since they determine the permutation of elements of the set, which is the essential basis of a cipher system. The RSA permutation polynomials and some others which may be used in the design of public key systems are investigated.

A polynomial $f(x)$ with coefficients in a finite field $F_q$ is called a permutation polynomial if the numbers $f(a)$ where $a \varepsilon F_q$ are a permutation of the a's. An equivalent statement is that the equation

$$f(x) = a$$

is solvable in $F_q$ for every 'a' in $F_q$ and that the equation has a unique solution in $F_q$ for each $a \varepsilon F_q$ [80].

### 14.2 Polynomial $y \equiv x^n \pmod m$)

First consider the 'famous' power polynomial $y = x^n$ which has been used in the RSA public key cryptosystem and the Diffie-Hellman public key distribution system.

Lemma 1

The polynomial

$y \equiv x^n$ in $Z/pZ$ where p is a prime

represents a permutation if and only if n is prime to p-1.

Let d be the greatest common divisor of n and p-1. Then letting $y \equiv x^n \pmod p$ and raising y to the power $(p-1)/d$, gives,

$$(x^{n/d})^{p-1} \equiv 1 \pmod p$$

ie $$y^{(p-1)/d} \equiv 1 \pmod p$$

The above congruence has $(p-1)/d$ roots in $Z/pZ$ and each root is a nth power in $Z/pZ$. If $d = 1$, that is, n is relatively prime to p-1, then there exists only one nth root of each element in $Z/pZ$ and hence $y \equiv x^n$ is a permutation modulo p.

Now the above argument can be extended to the case where $y = x^n$ is a permutation in $Z/mZ$ where $m = \prod_{i=1}^{r} p_i$ and the $p_i$, $1 \leqslant i \leqslant r$, are distinct primes and n is a positive integer such that $\gcd (n,(p_1-1)$

$(p_2-1)\ldots(p_r-1))= 1$.

To prove that $y \equiv x^n$ (mod m) is a permutation one needs to show that $x^n \equiv y^n$ (mod m) implies that $x \equiv y$ (mod m). Following [81], the proof is considered in three steps.

(i) x and y are relatively prime to m.

(ii) $x = ap_i$ and $y = bp_i$ for $1 \leqslant a,b \leqslant m/p_i$

(iii) $x = ap_i$ and $y = bp_j$ for $i \neq j$, $1 \leqslant a \leqslant m/p_i$, $1 \leqslant b \leqslant m/p_j$

(i) The proof is by induction on r. From Lemma 1, if r=1, then $y \equiv x^n$ is a permutation as gcd $(n,p_1-1) = 1$. Assume it holds for $r \leqslant k-1$. Let $\{a_1,\ldots,a_{\phi(m)}\}$ be a complete reduced residue system mod m, that is, gcd $(a_i,m) = 1$. If $a_i^n \equiv a_j^n$ (mod m), then $(a_i/a_j)^n \equiv x^n \equiv 1$ (mod m) and hence $x^n \equiv 1$ (mod $p_i$), $1 \leqslant i \leqslant r$. The gcd $(n,p_i-1) = 1$ and Lemma 1 imply that $x \equiv 1$ (mod $p_i$). But then $x \equiv 1$ (mod m). Thus $\{a_1^n, \ldots, a^n_{\phi(m)}\}$ are distinct mod m.

(ii) Now suppose $a^n p_i^n \equiv b^n p_i^n$ (mod m) where $1 \leqslant a,b \leqslant m/p_i$. Then $a^n p_i^{n-1} \equiv b^n p_i^{n-1}$ (mod $m/p_i$) and so $a^n \equiv b^n$ (mod $m/p_i$) since $p_i$ is invertible mod $m/p_i$. By induction hypothesis, a = b. Consequently, $\{a^n p_i^n\}$, $1 \leqslant a \leqslant m/p_i$ are distinct mod m. Moreover for any i, the sets $\{a^n p_i^n\}$, $1 \leqslant a \leqslant m/p_i$ and $\{a_1^n,\ldots a^n_{\phi(m)}\}$ are distinct since $a_j^n \not\equiv 0$ (mod $p_i$), $1 \leqslant j \leqslant \phi(m)$.

(iii) If $a^n p_i^n \equiv b^n p_j^n$ (mod m) where $1 \leqslant a \leqslant m/p_i$, $1 \leqslant b \leqslant m/p_j$, $i \neq j$, then $p_i | b$, $p_j | a$. Let $a = p_j \bar{a}$ and $b = p_i \bar{b}$, then

$$\bar{a}^n (p_i p_j)^{n-1} \equiv \bar{b}^n (p_i p_j)^{n-1} \left(\text{mod } \frac{m}{p_i p_j}\right)$$

and

$$p_i p_j \text{ invertible mod } \frac{m}{p_i p_j} \text{ implies that } \bar{a}^n \equiv \bar{b}^n \left(\text{mod } \frac{m}{p_i p_j}\right)$$

By the induction hypothesis and by $1 \leqslant \bar{a},\bar{b} \leqslant m/p_i p_j$, $\bar{a} = \bar{b}$ and so $ap_i = bp_j$. The only intersection of the sets $\{ap_k^n\}$ $1 \leqslant a \leqslant m/p_k$, k = i,j have then, is at the common multiples of $p_i p_j$.

Suppose now that $x^n \equiv y^n$ (mod m) with $x \not\equiv y$ (mod m). From the first case, neither x nor y can be relatively prime to m. But then $x = ap_i$, $y = bp_j$, for some i,j with $1 \leqslant i,j \leqslant r$. Again by the above,this implies $x \equiv y$ (mod m). Thus $\{x^n\}$ $1 \leqslant x \leqslant m$ are distinct (mod m). This completes the induction.

Conversely, if the proposition is not true, then there is at least one i such that gcd $(n,p_i-1) \neq 1$ and hence $x^n$ does not yield a permutation (mod $p_i$). If $x \not\equiv y$ (mod $p_i$) and $x^n \equiv y^n$ (mod $p_i$), then for $z = m/p_i$ $xz \not\equiv yz \pmod{m}$ and $(xz)^n \equiv (yz)^n \pmod{m}$. So $x^n$ is not a permutation (mod m).

The above paragraphs thus show why in the RSA system which uses the polynomials $x^e$ and $x^d$ (mod m), one needs to choose the coding exponents e and d relatively prime to $(p_1-1)(p_2-1)$ where, $m = p_1p_2$.

### 14.3 Polynomial $y \equiv ax + b \pmod{m}$

Consider the linear permutation polynomial f(x) given by

$$f(x) \equiv ax + b \pmod{m} \qquad (14.1)$$

where a and b are elements in Z/mZ and gcd (a,m) = 1 and m is a square free integer. Assume that the message is x, $1 \leq x < m$ and the encryption procedure consists of evaluating f(x). The decryption procedure is given by the inverse polynomial, $f^{-1}(x)$. Rewriting (14.1) as

$$y \equiv ax + b \pmod{m}$$

ie,

$$ax \equiv y - b \pmod{m} \qquad (14.2)$$

Letting $y - b = y'$, gives

$$ax \equiv y' \pmod{m} \qquad (14.3)$$

The congruence (14.3) has as solution

$$x \equiv y' \, a^{\phi(m)-1} \pmod{m}$$

where $\phi(m)$ is the Euler totient function. But the opponent does not actually need to calculate $\phi(m)$ to find x given y. The congruence (14.3) can be easily solved using Euclid's algorithm without the knowledge of $\phi(m)$ as follows.

One can find $a^{-1}$ using Euclid's algorithm where

$$a \, a^{-1} \equiv 1 \pmod{m}$$

Hence $\quad x \equiv y' \, a^{-1} \pmod{m}$

Thus as expected, the linear congruence (14.1), does not provide a secure public key system since the opponent can easily recover the message x without factoring the modulus m to its prime factors. On the other hand, this polynomial can be used to form a conventional cryptosystem where the parameters a, b and m are kept secret. Further the parameters a and b can be varied in some prearranged manner resulting in a variable substitution such as

$$a_i = g_1(a_{i-1}, \ldots, a_1)$$

and

$$b_i = g_2(b_{i-1}, \ldots, b_1)$$

## 14.4 Linear Fractional Substitution

Now consider the linear fractional substitution function

$$f(x) \equiv \frac{ax+b}{cx+d} \pmod p$$

From Section 10.4.5, the above function permutes the elements in Z/pZ if ad - bc $\not\equiv 0 \pmod p$. In Section 10.4.5, the elements a,b,c and d were used to form the message. Here the element x is considered to be the message. For designing a public key system which is to be based on the difficulty of factoring a large integer, consider

$$y \equiv \frac{ax+b}{cx+d} \pmod m$$

where say $m = p_1p_2$ and $p_1, p_2$ are distinct primes. y is the cipher. A symbol $\infty$ is adjoined to Z/mZ where $\infty = 1/0$, $0 = 1/\infty$ and for $x \in Z/mZ$, $\infty + x = \infty$, $\infty\, x \equiv \infty$ for $x \not\equiv 0$. The legitimate receiver can decrypt the cipher by finding the inverse operations modulo $p_1$ and modulo $p_2$ separately and then use the Chinese Remainder Theorem to obtain the message $x(\text{mod } p_1p_2)$. Actually this system has no security at all because the cryptanalyst can also find the message very easily with the knowledge of the parameters a,b,c,d and m. That is, he does not need to factorize m into its prime factors. This is because the inverse of the matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \pmod m$ can be found even without the knowledge of the prime factors of m. This can be done using a process similar to the Gauss-Jordan elimination process over the real numbers except in this case, the elementary operations are chosen to ensure that the determinant of the resulting matrix is relatively prime to m (assuming determinant of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is relatively prime to m). This type of algorithm would require a similar number of operations as its counterpart over the reals, that is, something like $O(n^3)$ operations for a nxn matrix [45]. Thus the opponent can find the A,B,C,D such that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \pmod m$$

If the greatest common divisor of cipher y (mod m) and m is greater than 1, then this gives one of the prime factors of m. The probability of this occurring must be small for large m,as the factorization of a large m is known to be hard. Therefore neglecting this case and considering gcd(y,m) = 1, the opponent can choose two elements w and z mod m such that

$$zy \equiv w \pmod m$$

Once such a pair (w,z) is found, he can easily solve for the message x as

$$\begin{pmatrix} x \\ 1 \end{pmatrix} \equiv \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} w \\ z \end{pmatrix}$$

and

$$x \equiv Aw + Bz \pmod{m} \qquad (14.4)$$

Thus the message x can be recovered without having to factorize m. An example illustrating the above method is given below:

Let $m = p_1 p_2 = 3.7 = 21$.

Let

$$y \equiv \frac{3x + 5}{x + 2} \pmod{21} \qquad \det \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix} \equiv 1 \pmod{21}$$

Let the message to be encrypted is, x = 6. Then the cipher y is given by

$$y \equiv \frac{3.6 + 5}{6 + 2} \equiv 16 \pmod{21}$$

Using the modified Gauss-Jordan method, the opponent calculates the inverse to be

$$\begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}^{-1} \equiv \begin{pmatrix} 2 & 16 \\ 20 & 3 \end{pmatrix} \pmod{21}$$

Choosing $w \equiv 2 \pmod{21}$, gives

$$16z \equiv 2 \pmod{21}$$

Using Euclid's algorithm, $z \equiv 8 \pmod{21}$. Now using equation (14.4), x is calculated to be

$$\begin{aligned} x &\equiv 2.2 + 16.8 \pmod{21} \\ &\equiv 6 \pmod{21} \end{aligned}$$

Again such a system can be used in the design of a conventional symmetric cryptosystem. One can vary the parameters $a_i, b_i, c_i, d_i$ such that $a_i d_i - b_i c_i \not\equiv 0 \pmod{m}$, by some prearranged manner. For instance, one can initially select two matrices $M_1$ and $M_2$ such that $\det(M_1) \not\equiv 0 \pmod{m}$ and $\det(M_2) \not\equiv 0 \pmod{m}$ and $M_{i+2} = M_{i+1} M_i$, where $M_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$, for subsequent i's.

## 14.5 Rédei Rational Functions

Now consider a general rational function $f(x) = \frac{g(x)}{h(x)}$, a quotient of polynomials over Z where g(x) and h(x) are relatively prime in Z[x]. Then f(x) is a permutation function modulo m if h(a) (mod m) is a prime residue class (mod m) for any $a \in Z$ and the mapping $\theta$: $Z/mZ \rightarrow Z/mZ$, $\theta(a) \equiv h(a)^{-1} \cdot g(a)$ is a permutation [80]. If $m = p_1 p_2$, then f(x) is a permutation function (mod m) if and only if it is a permutation function (mod $p_1$) and (mod $p_2$). Again a symbol $\infty$ is adjoined to Z/mZ where $\infty = 1/0$ and $0 = 1/\infty$ and for $a \in Z/mZ, \infty + a = \infty$, $a\infty = \infty$ for $a \not\equiv 0$. If the quantities $f(a)$ are distinct for all $a \in Z/mZ \cup \{\infty\}$, then $f(x)$ is a permutation function over $Z/mZ \cup \{\infty\}$.

Rédei [82] considered certain particular functions $f_n(x)$ which could be used in the design of public key cryptosystems as shown below:

Let $\alpha$ be a residue modulo p (p, a prime $\neq 2$) such that $(\frac{\alpha}{p}) = -1$ (ie, $\alpha$ is a fixed non-square). Rédei then proved that the function $f_n(x)$ given by

$$\left(\frac{x+\sqrt{\alpha}}{x-\sqrt{\alpha}}\right)^n = \frac{f_n(x)+\sqrt{\alpha}}{f_n(x)-\sqrt{\alpha}}$$

ie,

$$f_n(x) = \sqrt{\alpha}\ \frac{(x+\sqrt{\alpha})^n + (x-\sqrt{\alpha})^n}{(x+\sqrt{\alpha})^n - (x-\sqrt{\alpha})^n}$$

is a permutation (mod p) if n is odd and gcd $(n, p+1) = 1$ and $p \nmid n$. Further he showed that

$$f_{ed}(x) = f_e(f_d(x)) \qquad \text{over } Z/pZ$$

as

$$\frac{f_{ed}(x)+\sqrt{\alpha}}{f_{ed}(x)-\sqrt{\alpha}} = \frac{f_e(f_d(x))+\sqrt{\alpha}}{f_e(f_d(x))-\sqrt{\alpha}}$$

Thus $f_d$ is the inverse permutation of $f_e$ if d is chosen such that

$$ed \equiv 1 \pmod{p+1}$$

and

$$f_e(f_d(x)) = f_d(f_e(x)) = x$$

The proof can be found in [82].

One can use such a function f in the design of public key systems as follows.

If the ring $Z/mZ$ is considered where say $m = p_1p_2$ ($p_1, p_2$ are primes $\neq 2$) then the encrypting exponent e can be chosen such that $\gcd(e, p_1+1) = 1$, $\gcd(e, p_2+1) = 1$ and $e \nmid p_1$ and $e \nmid p_2$, then $f_e(x)$ is a permutation function over $Z/mZ$. This is a consequence of the Chinese Remainder Therorem. Now one can determine the decoding exponent d such that

$$ed \equiv 1 \pmod{(p_1+1)(p_2+1)} \qquad (14.5)$$

The encryption procedure transforms the message $x \in Z/mZ$ using the function $y = f_e(x)$ and the decryption procedure recovers the message by evaluating $f_d(y)$ in $Z/mZ$. The parameter $\alpha$ in the function f is chosen such that $(\frac{\alpha}{p_1}) = -1$ and $(\frac{\alpha}{p_2}) = -1$. The public key is therefore equal to $(e, f_e, m, \alpha)$ and the secret key is $(d, f_d, m, \alpha)$.

From (14.5), it is seen that to calculate the decrypting exponent d, the opponent needs to factorize m into its prime factors. Thus this function seems to be suitable for a public key cryptosystem whose security again depends on the difficulty of factorizing a large integer m.

## 14.6 Dickson Polynomial Based Public Key System

Let us now consider a special class of polynomials called the Dickson polynomials [53]. As personally suggested by Prof S D Cohen of Glasgow University, such Dickson polynomials are used in the design of public key cryptosystems.

For $\alpha \varepsilon F_p$ and any odd positive integer k, the Dickson polynomial $g_k(x, \alpha)$ is given by

$$g_k(x,\alpha) = \sum_{r=0}^{(k-1)/2} \frac{k}{k-r} \binom{k-r}{r} (-\alpha)^r \quad x^{k-2r} = \left(\frac{x+\sqrt{x^2-4\alpha}}{2}\right)^k + \left(\frac{x-\sqrt{x^2-4\alpha}}{2}\right)^k$$

For $\alpha = 0$, $g_k(x,0)$ is equal to the familiar power function $x^k$ used by the RSA system.

From [53], it is known that if $\gcd(k,p^2-1) = 1$, then $g_k(x,\alpha)$ is a permutation polynomial in $F_p$. Further

$$g_{ed}(x,\alpha) = g_e(g_d(x,\alpha), \alpha^d) = g_d(g_e(x,\alpha), \alpha^e)$$

Thus the inverse permutation of $g_e(x,\alpha)$, with $\gcd(e,p^2-1) = 1$ can be found using

$$ed \equiv 1 \pmod{p^2-1}$$

and

$$g_e^{-1}(x,\alpha) = g_d(x,\alpha^e) \qquad \text{over } Z/pZ$$

Now consider the polynomial $g_e(x,\alpha)$ over Z/mZ where say $m=p_1p_2$, a product of distinct primes. Then $g_e(x,\alpha)$ represents a permutation if and only if

$$\gcd(e,(p_1^2-1)(p_2^2-1)) = 1$$

Thus one can design a public key system using the Dickson polynomial $g_k(x,\alpha)$ as follows:

1. Choose large random primes $p_1$ and $p_2$ and let $m = p_1p_2$.
2. Choose the encryption exponent e such that $\gcd(e,(p_1^2-1$ $(p_2^2-1)) = 1$.
3. Calculate the decrypting exponent d such that
   $ed \equiv 1 \pmod{(p_1^2-1)(p_2^2-1)}$ (14.6)

4. Choose an arbitrary element $\alpha$ in $Z/mZ$.

The encryption procedure then consists of transforming the message $x \in Z/mZ$ using $g_e(x,\alpha)$ and the decryption procedure recovers the message using $g_d(x,\alpha^e)$. The public key is therefore given by $(e,m,g_e(x,\alpha),\alpha)$and the secret key consists of $(d,m, g_d(x,\alpha^e),\alpha)$. The security of this system again lies in the difficulty of factoring the modulus m to its prime factors to be able to calculate d using (14.6).

Thus Dickson polynomials can be used to design secure public key systems like the RSA system. It has been simulated on the Prime Computer system using the program DPOLY.F77 given in Appendix 26. An example showing the various parameters is given below.

Example

Let $p_1 = 5$, $p_2 = 7$ and $m = p_1p_2 = 35$. Let $\alpha = 1$

Choosing the coding exponents e and d such that

$$ed \equiv 1 \pmod{(5^2-1)(7^2-1)}$$

$$ed \equiv 1 \pmod{1152}$$

Therefore e = 11 and d = 419 are suitable.

Thus the encryption Dickson polynomial function is given by $g_{11}(x,1)$:

```
polynomial degree = 11
polynomial coefficients =
          x^11                                   x^0
         (1  0  24  0  9  0  28  0  20  0  24  0  )
```

The decryption Dickson polynomial function is given by $g_{419}(x,1)$:

```
polynomial degree = 419
polynomial coefficients =
          x^419
        ( 1   0   1   0   2   0   5   0  14   0
         28   0  13   0  26   0   5   0  31   0
         25   0   0   0   0   0  15   0   0   0
          0   0   0   0   0   0   0   0  14   0
         21   0  21   0   7   0   0   0   0   0
          0   0   0   0  20   0  15   0  15   0
         30   0   5   0   0   0   0   0  30   0
         20   0  20   0   5   0  30   0   0   0
          0   0   0   0   0   0   0   0  14   0
         21   0  21   0   7   0   0   0   0   0
         21   0  21   0   7   0   0   0  14   0
         28   0  28   0  21   0   0   0  21   0
          0   0   0   0   0   0   0   0   0   0
```

$$\begin{array}{cccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7 & 0 \\
28 & 0 & 28 & 0 & 21 & 0 & 0 & 0 & 0 & 0 \\
14 & 0 & 34 & 0 & 8 & 0 & 15 & 0 & 16 & 0 \\
12 & 0 & 7 & 0 & 14 & 0 & 30 & 0 & 34 & 0 \\
20 & 0 & 5 & 0 & 30 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 15 & 0 & 15 & 0 \\
30 & 0 & 5 & 0 & 0 & 0 & 0 & 0 & 20 & 0 \\
5 & 0 & 5 & 0 & 10 & 0 & 25 & 0 & 0 & 0 \\
0 & 0 & 15 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
18 & 0 & 3 & 0 & 31 & 0 & 20 & 0 & 22 & 0 \\
14 & 0 & 14 & 0 & 13 & 0 & 25 & 0 & 18 & 0 \\
15 & 0 & 20 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7 & 0 \\
28 & 0 & 28 & 0 & 11 & 0 & 25 & 0 & 15 & 0 \\
20 & 0 & 0 & 0 & 0 & 0 & 10 & 0 & 20 & 0 \\
20 & 0 & 5 & 0 & 30 & 0 & 0 & 0 & 0 & 0 \\
25 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 21 & 0 \\
14 & 0 & 14 & 0 & 28 & 0 & 0 & 0 & 25 & 0 \\
24 & 0 & 24 & 0 & 13 & 0 & 15 & 0 & 21 & 0 \\
7 & 0 & 27 & 0 & 4 & 0 & 25 & 0 & 29 & 0 \\
20 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 28 & 0 \\
7 & 0 & 22 & 0 & 29 & 0 & 30 & 0 & 5 & 0 \\
21 & 0 & 21 & 0 & 27 & 0 & 5 & 0 & 19 & 0 \\
3 & 0 & 18 & 0 & 21 & 0 & 0 & 0 & 1 & 0)
\end{array}$$

$$x^{0}$$

Message $x \equiv 10 \pmod{35}$

Encryption : cipher $y = g_{11}(x,1) \equiv 25 \pmod{35}$

Decryption : $g_{419}(y,1) \equiv 10 \pmod{35}$ = Message

## 14.7 Discussion

Some permutation polynomials for which the inverse permutations are easy to construct have been considered. The linear polynomial ax+b and the linear fractional substitution function (ax+b)/(cx+d) are found to be insecure when used as public key systems. The Rèdei function and the Dickson polynomials seem to offer a similar level of security as the prototype RSA system. In general, it is not easy to find the inverse permutation for an arbitrary permutation polynomial. One way of finding the inverse permutation of a polynomial g(x) in Z/pZ consists of raising the function g(x) successively to powers 2,3,4,... and combining an appropriate set of them to give the inverse function. Using this method, both the system designer and the cryptanalyst have the same work factor. On the other hand, if a permutation polynomial g(x) can be found whose inverses over $Z/p_iZ$ for i=1,..., r are easy to calculate but whose inverse over Z/mZ where $m = \prod_{i=1}^{r} p_i$ is difficult to calculate without knowing the prime factors $p_i$, then one can design a public key system based on the factorization trapdoor as follows:

1. Choose large primes $p_1, \ldots, p_r$ and let $m = \prod_{i=1}^{r} p_i$.
2. The permutation polynomial g(x) is made public along with m.
3. The encryption procedure consists of transforming a message $x_0 \in Z/mZ \to g(x_0) \in Z/mZ$.
4. The decryption procedure finds $g^{-1}(x_0)$ in $Z/p_iZ$ for i=1,...r and then uses the Chinese Remainder Theorem to recover the message $x_0$ in Z/mZ.

Note that the decryption procedure should require the knowledge of prime factors of m to be able to provide security.

Public key systems can also be designed using permutation polynomials based on the law of composition. One can combine the permutation polynomials, for which inverses can be found if some 'extra' information is known, under the law of composition to construct public key systems which can be more secure than the individual polynomial based systems. Consider for instance, the encryption procedure given by the composite permutation function g where

$$g = y_1 \, O \, y_2 \, O \, y_3 \, O \ldots O \, y_n$$

and

$y_i$, 1≤i≤n, are permutation functions.

(O - denotes composition)

The decryption procedure is then obtained by the composition of the inverses of the composition factors of g in the opposite order given by

$$h = y_n^{-1} \, O \, y_{n-1}^{-1} \, O \ldots O \, y_1^{-1}$$

Hence g O h = h O g = Identity, I.

The system designer can easily obtain the inverse composite permutation function h as he knows the factors of g. But the opponent only knows g and its inverse is very difficult to calculate since it is very difficult to find the decomposition of g with respect to composition.

As an illustration of this scheme, consider the following simple example given below:

Let $y_1 \equiv x^{e_1}$ (mod m) where $m = p_1p_2$ and $e_1d_1 \equiv 1$ (mod $\phi(m)$)
Let $y_2 \equiv a_1y_1+b_1$ (mod m) where $\gcd(a_1,m) = 1$ and $b_1 \in Z/mZ$
Let $y_3 \equiv y_2^{e_2}$ (mod m) where $e_2d_2 \equiv 1$ (mod $\phi(m)$)
Let $y_4 \equiv a_2y_3+b_2$ (mod m) where $\gcd(a_2,m) = 1$ and $b_2 \in Z/mZ$

Then the composite permutation g is given by

$$g(x) = a_2(a_1x^{e_1} + b_1)^{e_2} + b_2$$

Letting $e_1 = 3$ and $e_2 = 5$, gives

$$g(x) = \sum_{i=0}^{15} c_ix^i \pmod{m} \text{ where } c_i \in Z/mZ.$$

The system designer would make the function g(x) and m public and keep the prime factors $p_1$ and $p_2$ and the factors $y_1,y_2,y_3$ and $y_4$ of the composition secret. If someone only knows g(x) and m, then it is very difficult to find the inverse permutation. As seen earlier, one way is to try g, $g^2$, ... until the inverse permutation can be constructed by trial and error procedure. On the other hand, the legal receiver can find the inverse permutation very easily by calculating $y_4^{-1}$, followed by $y_3^{-1}$, followed by $y_2^{-1}$, followed by $y_1^{-1}$ to recover the message.

The complexity of the composite system can be dramatically increased using such an approach and there are numerous ways of constructing such composite systems. For instance, one could combine the RSA system and the knapsack system. The message can be encrypted using the RSA system or one of its extensions and the encrypted message can be interpreted as a string of numbers in say the binary system,which can be encrypted using knapsack system. This gives rise to a number of possibilities to increase the security.

# CHAPTER 15

## CHAINING TECHNIQUES AND BROADCASTING WITH PUBLIC KEY SYSTEMS

### 15.1 General

In the following section, some of the chaining techniques are applied to the generalized RSA matrix system discussed in Section 10.4. It may be recalled that some of these techniques have been used with the symmetric DES system (see Chapter 5). Such techniques can also be applied to the RSA polynomial system described in Section 10.5. By these techniques, the undesirable effects of redundancy and structure present in the plaintext data are eliminated.

Finally, a precaution which should be taken when using the RSA system or any one of its extensions in a broadcasting situation is mentioned, where a single message is encrypted under several public keys to be sent to several users.

### 15.2 Chaining Techniques

Let E and D denote the encryption and decryption of a message M under the generalized RSA system. Initially, consider the intersymbol dependence in an individual block using the RSA matrix system.

First consider the messages to be non-singular matrices over Z/mZ, where $m = \prod_{i=1}^{r} p_i^{a_i}$, under the normal encryption mode $C \equiv M^e$ (mod m). If there is a single error in cipher matrix C then the recovered plain matrix will be completely in error. An example illustrating this is shown in Figure 15.1. This indicates that there is a strong intersymbol dependence within an individual ciphertext block.

On the other hand, with upper triangular matrix messages (including the diagonal elements), this is not the case. Consider a message M, a 3x3 matrix over Z/mZ, given by

$$M = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} \quad \text{where } \gcd(a_{ii}, m) = 1,\ 1 \leqslant i \leqslant 3$$

and the corresponding cipher matrix C

$$C \equiv M^e \pmod{m}$$

Let $m = 5.3^2 = 45$

Let $e = d = 5$

$$M \equiv \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix} \equiv \begin{pmatrix} 8 & 13 \\ 3 & 4 \end{pmatrix} \quad (\text{mod } 45)$$

$$C \equiv M^5 \equiv \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} \equiv \begin{pmatrix} 29 & 7 \\ 12 & 13 \end{pmatrix} \quad (\text{mod } 45)$$

An error in $c_1$ affects decryption of all elements $m_1, m_2, m_3$ and $m_4$.

Denoting this as

$(c \rightarrow c_1^*) \longrightarrow (m_1^*, m_2^*, m_3^*, m_4^*)$

$(29 \rightarrow 30) \longrightarrow (39, 40, 30, 19)$

Similarly,

$(c_2 \rightarrow c_2^*) \longrightarrow (m_1^*, m_2^*, m_3^*, m_4^*)$

$(7 \rightarrow 8) \longrightarrow (11, 5, 30, 1)$

$(c_3 \rightarrow c_3^*) \longrightarrow (m_1^*, m_2^*, m_3^*, m_4^*)$

$(12 \rightarrow 13) \longrightarrow (11, 10, 25, 1)$

$(c_4 \rightarrow c_4^*) \longrightarrow (m_1^*, m_2^*, m_3^*, m_4^*)$

$(13 \rightarrow 14) \longrightarrow (29, 2, 42, 44)$

Fig 15.1 - Intersymbol Dependence in Non-singular Matrix Message Space.

$$C = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ 0 & c_{22} & c_{23} \\ 0 & 0 & c_{33} \end{pmatrix}$$

If there is an error in $c_{11}$, denoted by $c^*_{11}$ , then after decryption it is found that

$$\begin{pmatrix} c^*_{11} & c_{12} & c_{13} \\ 0 & c_{22} & c_{23} \\ 0 & 0 & c_{33} \end{pmatrix}^d \xrightarrow{D} \begin{pmatrix} a^*_{11} & a^*_{12} & a^*_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} = M^*$$

It is seen that only the elements $a_{11}$, $a_{12}$ and $a_{13}$ are affected due to an error in $c_{11}$. That is, there is not a strong intersymbol dependence between the elements as in the case of the non-singular matrix messages. More generally, it is possible to work out which elements are being affected by errors. Considering again 3x3 matrices, let the message be

$$M = \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}$$

Then

$$M^{n+1} = \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}^{n+1}$$

$$= \begin{pmatrix} a_n & b_n & c_n \\ 0 & d_n & e_n \\ 0 & 0 & f_n \end{pmatrix} \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}$$

Thus

$$a_{n+1} = a^{n+1} \qquad (15.1)$$

$$d_{n+1} = d^{n+1} \qquad (15.2)$$

$$f_{n+1} = f^{n+1} \qquad (15.3)$$

and

$$b_{n+1} = a\,b_n + b\,d^n$$

$$e_{n+1} = d\,e_n + e\,f^n$$

$$c_{n+1} = a\,c_n + b\,e_n + e\,f^n$$

Hence

$$b_n = \frac{b}{a-d} \quad (a^n - d^n)$$

$$e_n = \frac{e}{d-f} \quad (d^n - f^n)$$

$$c_n = \frac{be}{d-f} \quad \frac{(a^n - d^n)}{(a-d)} + \left(c - \frac{be}{d-f}\right) \left(\frac{a^n - f^n}{a-f}\right)$$

Now if there is an error in 'a', that is, a*, this results in error in $a_n$, $b_n$ and $c_n$. Denoting this as,

$$a^* \longrightarrow a_n^*, b_n^*, c_n^*$$

then

$$b^* \longrightarrow b_n^*, c_n^*$$

$$c^* \longrightarrow c_n^*$$

$$d^* \longrightarrow d_n^*, b_n^*, e_n^*, c_n^*$$

$$e^* \longrightarrow e_n^*, c_n^*$$

$$f^* \longrightarrow f_n^*, e_n^*, c_n^*$$

Thus from the point of view of intersymbol dependence within a block, it is preferrable to use the set of non-singular matrices as messages.

Now consider some chaining techniques using this RSA matrix system. Suppose an user A wishes to communicate to an user B a total of r messages $M_1, \ldots, M_r$. Let the corresponding ciphers be $C_1, \ldots, C_r$.

One way of chaining would be to modify the first cipher $C_1$ using some function $f_1$ before transmitting to user B. The function $f_1$ can be communicated to user B under some secure means. For the subsequent ciphers $C_2, \ldots, C_r$, both users can derive the functions $f_2, \ldots, f_r$ using some publicly known function g,

$$f_i = g(f_{i-1}, \ldots, f_1, M_{i-1}, \ldots, M_1, C_{i-1}, \ldots, C_1),\ i > 1$$

One could also transmit the function $f_1$ to begin with using a public key distribution system such as the extended matrix version of the Diffie-Hellman public key distribution system discussed in Chapter 13.

Now consider the case where the function $f_1$ is transmitted to the user B via the first message $M_1$. Two such possible methods are looked at.

### 15.2.1 Method 1

The operation of this method follows the pattern given below:

User A

$$\begin{tikzcd}
M_1 \arrow[r, "E"] & C_1 \\
M_2+C_1 \arrow[r, "E"] & C_2 \\
M_3+C_2 \arrow[r, "E"] & C_3 \\
\vdots &
\end{tikzcd}$$

User B

$$\begin{tikzcd}
C_1 \arrow[r, "D"] & M_1 & \\
C_2 \arrow[r, "D"] & M_2+C_1 \arrow[r, "-C_1"] & M_2 \\
C_3 \arrow[r, "D"] & M_3+C_2 \arrow[r, "-C_2"] & M_3 \\
\vdots & &
\end{tikzcd}$$

In this method, the previous cipher matrix is simply added with the next message matrix before encryption. Here a simple modulo m addition has been suggested. More generally, one could use a publicly known function g to form $g(M_i, C_{i-1})$. With this method, an error in $C_i$ will cause an error in the decryption of $M_i$ and $M_{i+1}$ and then the system will synchronize. That is,

$$\begin{tikzcd}
M_i + C_{i-1} \arrow[r, "E"] & C_i \\
M_{i+1} + C_i \arrow[r, "E"] & C_{i+1}
\end{tikzcd}
\qquad
\begin{tikzcd}
C_i^* \arrow[r, "D"] & M_i^* + C_{i-1}^* \arrow[r, "-C_{i-1}"] & M_i^{**} \\
C_{i+1} \arrow[r, "D"] & M_{i+1} + C_i \arrow[r, "-C_i^*"] & M_{i+1}^*
\end{tikzcd}$$

This system has the same error characteristic as the stream cipher feedback (CFB) or the cipher block chaining (CBC) modes of DES. Note that this method does not provide authentication as anyone can encrypt a message.

### 15.2.2 Method 2

The operation of this method follows the pattern given below:

User A

$$\begin{tikzcd}
M_1 \arrow[r, "E"] & C_1 \\
M_1+C_1+M_2 \arrow[r, "E"] & C_2 \\
\vdots &
\end{tikzcd}$$

User B

$$\begin{tikzcd}
C_1 \arrow[r, "D"] & M_1 & \\
C_2 \arrow[r, "D"] & M_1+C_1+M_2 \arrow[r, "-M_1-C_1"] & M_2 \\
\vdots & &
\end{tikzcd}$$

In this method, the previous cipher matrix as well as the previous message matrix are added modulo m to the next message matrix prior to encryption. More generally, again a publicly known function g can be used instead of modulo m addition where the arguments of g are $M_i$, $M_{i-1}$ and $C_{i-1}$. With this method, an error in $C_i$, will cause error in the decryption of all subsequent message matrices.

$$\begin{array}{ll}
\underline{\text{User A}} & \underline{\text{User B}} \\
M_{i-1}+C_{i-1}+M_i \xrightarrow{E} C_i & C_i^* \xrightarrow{D} M_i^*+M_{i-1}^* \ +C_{i-1}^* \xrightarrow{-C_{i-1}\ \ -M_{i-1}} M_i^{**} \\
M_i+C_i+M_{i+1} \xrightarrow{E} C_{i+1} & C_{i+1} \xrightarrow{D} M_i+C_i+M_{i+1} \xrightarrow{-C_i^*\ -M_i^{**}} M_{i+1}^* \\
\vdots & \vdots
\end{array}$$

This system has the same error characteristic as the CBCP or the CFBV modes of DES discussed in Chapter 5. It possesses the error propagation property.

Note that in both these methods, the first message-cipher pair $(M_1,C_1)$ is essentially used to set up some form of 'initialization vector' (Section 2.3.2) and it is a direct block encryption.

## 15.3 Broadcasting of Messages

In many applications, it is necessary to transmit the same set of messages to a group of users in the network. Simmons has shown that [83]in such broadcasting situations, special precautions should be taken to avoid the message being recovered by a cryptanalyst without having to 'break' the underlying cryptosystem. His argument given below applies to the prototype RSA system (over Z) or any of its extensions proposed earlier. Thus the messages M and ciphers C in the argument can be rational integers, polynomials, matrices or algebraic integers.

Let the modulus of the system be m $(=p_1p_2)$ and two public encryption exponents be $e_1$ and $e_2$. Let a message M be encrypted with each of these exponents to form $C_1$ and $C_2$. That is,

$$\begin{array}{ll} C_1 \equiv M^{e_1} \pmod{m} & \\ & \text{where } M,\ C_1,\ C_2\ \varepsilon\ Z/mZ \\ C_2 \equiv M^{e_2} \pmod{m} & \end{array}$$

Now if it is assumed that $\gcd(e_1,e_2) = 1$, then there exists a and b such that

$$ae_1 + be_2 = 1$$

The values of a and b can be found using the Euclid's algorithm. One of the coefficients is positive and the other is negative. If a is negative, then using Euclid's algorithm, the multiplicative inverse of $C_1$ (mod m) is given by

$$C_1 C_1^{-1} \equiv 1 (\bmod m)$$

If gcd($C_1$,m) = 1, then $C_1^{-1}$ exists and

$$\begin{aligned}(C_1^{-1})^{|a|} (C_2)^b &= (M^{e_1})^{-|a|} (M^{e_2})^b \\ &\equiv M^{ae_1+be_2} (\bmod m) \\ &\equiv M (\bmod m)\end{aligned}$$

Thus the message can be recovered even without factoring m into $p_1$ and $p_2$. The only information needed are the integer values of m, $e_1$, $e_2$, $C_1$and $C_2$ all of which are assumed to be publicly available. Thus precautions must be taken to avoid such situations.

An example illustrating such a situation is given below:

Let m = 13.23 = 299

Let $e_1 = 5$ $d_1 = 53$

Let $e_2 = 7$ $d_2 = 151$

Let the message M = 4. Then,

$$C_1 \equiv (4)^5 \equiv 127 (\bmod 299)$$

$$C_2 \equiv (4)^7 \equiv 238 (\bmod 299)$$

Now using Euclid's algorithm

$$5a + 7b = 1$$

$$\Rightarrow a = 3, b = -2$$

As b is negative, we form

$$C_2 C_2^{-1} \equiv 1 (\bmod 299)$$

$$(238) C_2^{-1} \equiv 1 (\bmod 299)$$

Using Euclid's algorithm $C_2^{-1} \equiv 49 (\bmod 299)$

Hence

$$\begin{aligned}(49)^2 (127)^3 &= 4918167583 \\ &\equiv 4 (\bmod 299) \\ &\equiv M\end{aligned}$$

# CHAPTER 16

## CONCLUSIONS

Some cryptographic techniques for secure data communication over an insecure channel have been investigated in this thesis. The first part has been primarily concerned with conventional cryptosystems, in particular, the Data Encryption Standard (DES) whereas the second part focussed on public key cryptosystems.. The main results and conclusions are summarized below [87-90].

### Part 1

A software implementation of the DES algorithm has been carried out using an Apple microcomputer which allowed a study of some of the properties of the DES such as the complementary property and the avalanche effect. It has been found however that such a software implementation is too slow for many real time applications and since it is necessary to store the secret key within the computer system, there is a possibility of its recovery by an unauthorized user. These problems can be overcome by a hardware LSI implementation of the DES and this has been used in the design and construction of a microprocessor based data encryption interface unit.

The use of the interface in a point to point communication system allowed secure data transfer between Apple microcomputers in either plain or encrypted format. The interface has been satisfactorily tested over the public switched telephone network with data rates up to 1200 bits per second. If required, the security can be increased by performing multiple encryption with independent keys or using chaining techniques. Several chaining techniques have been investigated using the developed system namely Cipher Block Chaining (CBC), Stream Cipher Feedback (CFB), Cipher Block Chaining with Plaintext Feedback (CBCP) and stream Cipher Feedback with Vector Feedback (CFBV). Each of these schemes gave rise to a cryptographic system with different error characteristics, speed of operation and level of security and hence is suitable for different applications. The error propagation property of CBCP and CFBV schemes made them particularly suitable for message authentication purposes but unsuitable for use in communication links prone to noise. The self synchronizing property of CFB and CBC made them more useful for links prone to small amounts of noise. The four chaining modes are found to be less susceptible to attacks of replay, insertion,deletion and code book analysis in comparison to the standard Electronic Code Book (ECB) mode.Chaining also helped to eliminate the undesirable effects of data redundancy and structure. A statistical analysis of the randomness of the output sequences produced under these different chaining modes confirmed the good pseudo-random generator property of the DES.

The use of the developed encryption interface has been extended to provide off-line file security using the Apple disk system. A self synchronizing mode is found to be more suitable for file encryption as in this case, recovery from an error must be effected with ciphertext alone. If a ciphering procedure with error propagation property is used for file security, subsequent inability to read a portion of the ciphertext because of damage to the physical medium or the recorded bits, may prevent all the following ciphertext being deciphered. With communication security, it is possible to recover from an error by retransmitting the original message.

Further the developed system can be used on the Prestel public network allowing storage and retrieval of completely and partly encrypted frames of information on the Prestel database. A 6-bit cipher feedback technique has been found to be suitable for such an application. This technique prevented the occurrence of control characters in the ciphertext which are not acceptable to the Prestel control unit.

The use of such a DES based encryption system in a communication network requires the keys be distributed to the users over a separate secure channel. Several methods of key distribution using Key Centres and public key systems have been discussed.

Part 2

A generalization of the RSA system in the ring of matrices over Z/mZ, where m is a composite integer, is proposed. It is shown that a factorization of the modulus m is needed to compute the exponent of the group formed by either non-singular matrix messages or upper triangular matrices with invertible diagonal elements over Z/mZ, thus offering a similar level of security as the prototype RSA system. This system allows the use of a non-square free modulus which is not possible with the RSA system over the integers. This scheme is as suitable for both privacy and authentication as its predecessor. The use of chaining techniques in this generalized system to overcome the difficulties of data redundancy has also been demonstrated.

An extension of the RSA system to polynomial rings has been considered. It is found that the difficulty of factorization of a polynomial into its irreducible factors over a finite field does not in itself provide a secure public key cryptosystem. However, if the difficulty of factorizing an integer is compounded with the difficulty of factorizing a polynomial, then a secure RSA type system in the ring of polynomials is seen to be possible. For cryptographic application, both the modulus polynomial and the modulus integer need to be square free to enable proper decryption.

The design of public key systems in some quadratic algebraic number fields using the factorization trapdoor concept has been presented. The security of such systems is found to be dependent on the difficulty of factorizing the norm of the modulus. Thus a similar level of security as the prototype RSA system can be achieved if the norm is made to be sufficiently large. One method of message representation in such systems involves the use of the elementary divisor theory to choose a standard set of representatives.

The investigation of such extensions indicate that rings other than the ring of rational integers,can be used to construct public key systems with the factorization trapdoor property. From a practical point of view, it seems that the complexity of such systems may favour the implementation of the factorization trapdoor in the ring of rational integers.

The Diffie-Hellman public key distribution has been implemented in the Galois extension field GF($2^n$) with n = 127,where the computations required for exponentiation can be easily performed using digital logic. To withstand the most recently published Adleman's subexponential algorithm to compute logarithms, it is necessary to work in a higher extension field of GF($2^{521}$) to maintain a work factor equivalent to that of DES key exhaustion.

Short cycling attacks consisting of repeated encipherings have been carried out against the exponentiation system in GF($2^n$) with n = 3 and 7. It is found that if the primitive element is chosen at random, then the expected cycle length of an arbitrary cycle is very close to half the number of non-zero elements in the field. Thus this type of attack appears to be very much analogous to a random search procedure.

The exponentiation system in GF($2^{127}$) has been used in conjunction with the DES encryption to form a hybrid system which combines the protection provided by the conventional cryptosystem with the user authentication attributes of a public key system. Such a hybrid arrangement is found to be feasible in practice.

A dedicated hardware exponentiation system in GF($2^7$) has been designed and constructed. The use of normal basis representation in the design allows a modular construction which is very useful in large scale integrated circuit design.

An extension of the Diffie-Hellman public key distribution system to matrix rings is proposed. Using rings of non-singular matrices over Z/pZ (p prime) and upper triangular matrices with invertible elements along the diagonal over Z/pZ, it is shown that the number of possible secret keys is much greater for a given prime p compared to the original system.

The role of permutation polynomials in the design of public key systems has been investigated and it is shown that the class of Dickson permutation polynomials and certain Rèdei rational functions can be used to construct public systems with a similar level of security as the prototype RSA system. Further, a method of designing public key systems using permutation polynomials under the law of composition has been presented. The complexity of such composite systems can be dramatically increased to provide high security.

## REFERENCES

1. Kahn, D., The Codebreakers, MacMillan, New York 1972.

2. Diffie, W. and Hellman, M. E., 'Privacy and Authentication: An introduction to cryptography', Proc. IEEE, Vol.67, No.3, 1979.

3. Meyer, C. and Matyas, S., Cryptography: A new dimension in Computer Data Security, John Wiley, 1982

4. Denning, D. E. R., Cryptography and Data Security, Addison-Wesley, 1982.

5. Simmons, G. J., 'Symmetric and Asymmetric Encryption', Computing Surveys, Vol.11, No.4, 1979.

6. Konheim, A. G., Cryptography: A Primer, John Wiley, 1981.

7. Shannon, C., 'Communication Theory of Secrecy Systems', Bell System Tech.Journal, Vol.28, 1949, pp 656-715.

8. Garey, M. R. and Johnson, D. S., Computers and Intractability, A Guide to the Theory of NP-Completeness, W. H. Freeman and Co., 1979.

9. Brassard, G., 'A note on the complexity of cryptography', IEEE Trans. on Information Theory, Vol.IT-25, No.2, 1979.

10. Shamir, A., 'A polynomial Time Algorithm for breaking the basic Merkle-Hellman Cryptosystem', Proc. 23rd Annual Symp. Found. Computers Sci, 1982, pp 145-152.

11. Data Encryption Standard (DES), FIPS publication 46, NBS, US Dept. of Commerce, 1977.

12. Rivest, R. L., Shamir, A. and Adleman, L., 'A method for obtaining Digital Signatures and Public Key Cryptosystems', Commun. ACM, Vol.21, No.2, 1978, pp 120-126.

13. Merkle, R. C. and Hellman, M. E., 'Hiding Information and Signatures in Trapdoor Knapsacks', Vol.IT-24, No.5, 1978.

14. Beker, H. and Piper, F. C., Cipher Systems: The Protection of Communications, Northwood Pub., 1982.

15. Rivest, R., 'The Impact of Technology on Cryptography', Proc. IEEE Int. Comm. Conf., Toronto, 1978.

16. Vernam, G., 'Cipher Printing Telegraph Systems for Secret Wire and Radio Telegraphic Communications', Journal AIEE, Vol.45, 1926, pp 109-115.

17. Smith, J. L., 'The Design of Lucifer, a Cryptographic Device for Data Communications', IBM, Watson Research Centre, Yorktown Heights, New York, 1971.

18. Meyer, C. H., 'Ciphertext/Plaintext and Ciphertext/Key Dependence vs. number of rounds for the DES', AFIPS Conference Proc., 47, 1978, pp 1119-1126.

19. Hellman, M. E., Merkle, R., Schroeppel, R., Washington, L., Diffie, W., Pohlig, S. and Schweitzer, P., 'Results of an initial attempt to cryptanalyze the NBS Data Encryption Standard', Elec. Engg. Dept., Stanford University, CA, SEL 76-042, 1976.

20. Branstad, D. K., Gait, J. and Katzke, S., 'Report of the Workshop on Cryptography in support of Computer Security', NBS, USA, IR 77-291, 1976.

21. NBS, 'Report of the 1976 Workshop on Estimation of Significant Advances in Computer Technology', NBS, USA, 1976.

22. Diffie, W. and Hellman, M. E., 'A Critique of the proposed Data Encryption Standard', Comm. ACM, Vol.19, No.3, 1976, pp 164-165.

23. Hellman, M. E., 'A cryptanalytic Time-Memory Trade off', IEEE Trans. on Information Theory, Vol.IT-26, No.4, 1980, pp 401-406.

24. LEXAR Corporation, 'An Evaluation of the NBS Data Encryption Standard', Los Angeles, USA, 1976.

25. Bichl, I., Biermeier, J., Gollmann, D. and Pichler, F., 'Cryptanalysis of the Data Encryption Standard by Formal Coding', Proc. IEEE International Symposium on Information Theory, USA, 1981.

26. DES modes of operation FIPS Pub. 81, NBS, USA, 1980.

27. Buckley, D. C., 'Hershey Characters', Plymouth Polytechnic Computer Centre Internal Handout No.U5. 8-14, 1983.

28. Chatfield, C., Statistics for Technology, Chapman and Hall, 1978.

29. Good, I. J., 'On the Serial Test for random sequences', Ann. Math. Statist., 1957, pp 262-264.

30. Gibbons, J. D., Non-Parametric Statistical Inference, McGraw Hill, New York, 1971.

31. Prestel Viewdata System Manuals, British Telecom, 1979.

32. Price, W. L. and Davies, D. W., 'Issues in the design of a Key Distribution Centre', NPL Report, DNACS 43/81, 1981.

33. Denning, D. E. and Sacco, G. M., 'Timestamps in Key Distribution Protocols', Comm. ACM, Vol.24, No.8, 1981, pp 533-536.

34. Denning, D. E. and Schneider, F. B., 'Master keys for group sharing', Information Processing letters, Vol.12, No.1, 1981, pp 23-25.

35. Diffie, W. and Hellman, M. E., 'New Directions in cryptography', IEEE Transactions on Information Theory, Vol. IT-22, 1976, pp 644-654.

36. Keyes, R. W., 'Physical limits in digital electronics', Proc. IEEE, Vol.63, 1975, pp 740-767.

37. Landauer, R. W., 'Irreversibility and heat generation in the computing process', IBM Journal of Research and Development, Vol.5, 1961, pp 183-191.

38. Hardy, G. H. and Wright, E. M., An Introduction to the Theory of Numbers, Fifth edition, Oxford, 1983.

39. Ayoub, F. N., 'Some aspects of the design of secure encryption algorithms', Council for National Academic Awards, Hatfield Polytechnic, Thesis, March 1983.

40. Miller, G. L., 'Riemann's Hypothesis and Tests for Primality', Proc. of the Seventh Annual ACM Symposium on the Theory of Computing, pp 234-239.

41. Ingemarsson, I., Tang, D. T. and Wong, C. K., 'A Conference Key Distribution System', IEEE Trans. on Information Theory, Vol.IT-28, No.5, 1982, pp 714-720.

42. Rabin, M. O., 'Probabilistic algorithms', in Symposium on New Directions and Recent Results in Algorithms and Complexity' Academic Press, Ed. J. F. Traub, 1976.

43. Bond, D. J. 'Practical Primality Testing', International Conference on Secure Communication Systems, IEE, London, 1984.

44. Solovay, R. and Strassen, V., 'A Fast Monte-Carlo test for Primality', SIAM Journal of Computing, Vol.6, 1977, pp 84-85.

45. Knuth, D. E., The Art of Computer Programming, Vol.2: Seminumerical Algorithms, 2nd Edition, Addison-Wesley, 1981.

46. Smith, D. E. and Palmer, J. T., 'Universal fixed messages and the RSA Cryptosystem', Mathematika, Vol.26, 1979, pp 44–52.

47. Blakely, G. R. and Borosh, I., 'RSA public key cryptosystems do not always conceal messages', Computers and Maths with Appls., Vol.5, pp 169-178.

48. Albert, A. A., Fundamental Concepts of Higher Algebra, The University of Chicago Press, 1956.

49. Burton, D. M., Rings and Ideals Addison-Wesley Pub., 1968.

50. Van der Waerden, B. L., Modern Algebra, Vol.2, 1949.

51. MacWilliams, J., 'Orthogonal matrices over finite fields', American Mathematical Monthly, Feb 1969.

52. Turnbull, H. W. and Aitken, A. C., An Introduction to the Theory of Canonical Matrices, Blackie and Son Pub., 1932.

53. Dickson, L. E., Linear Groups with an Exposition of The Galois Field Theory, Dover Pub., 1958.

54. Postnikov, M. M., Foundations of Galois Theory, Translated by A. Swinfen and P. J. Hilton, Pergamon Press, International Series of Monographs on Pure and Applied Mathematics, Vol.29, 1962.

55. Kravitz, D. W. and Reed, I. S., 'Extension of RSA Cryptostructure: A Galois Approach', IEE Electronic letters, Vol.18, No.6, 1982, pp 255–256.

56. Berlekamp, E. R., Algebraic Coding Theory, McGraw Hill, New York, 1968.

57. Berlekamp, E. R., 'Factoring polynomials over large finite fields', Mathematics of Computation, Vol.24, No.111, 1970, pp 713–735.

58. Rabin, M. O., 'Probabilistic Algorithms in Finite Fields', Siam Journal of Computing, Vol.9, No.2, 1980, pp 273-280.

59. Gait, J., 'Short cycling in the Kravitz-Reed Public Key Encryption System', Electronic letters, Vol.18, No.16, 1982, pp 706-707.

60. Northcott, D. G., Ideal Theory, Cambridge Tracts in Mathematics and Mathematical Physics, Cambridge Univ. Press, 1953.

61. Hancock, H., Foundations of the Theory of Algebraic Numbers, Vol.1, Dover 1931.

62. McCoy, N. H., Theory of Rings, MacMillan Co., New York, 1964.

63. Pollard, H., The Theory of Algebraic Numbers, The Carus Mathematical Monographs, No.9, Pub. by Mathematical Assoc. of America, John Wiley, 1950.

64. Rosen, M. and Ireland, K., A Classical Introduction to Modern Number Theory, Springer-Verlag, 1980.

65. Cohn, H., A Second Course in Number Theory, John Wiley, 1962.

66. Andrews, G. E., Number Theory, W. B. Saunders Co., 1971.

67. Hartley, B. and Hawkes, T. O., Rings, Modules and Linear Algebra, Chapman and Hall, 1980 (Reprint).

68. Weyl, H., Algebraic Theory of Numbers, Princeton University Press, 1940.

69. Pohlig, S. C. and Hellman, M. E., 'An Improved Algorithm for Computing Logarithms over GF(p) and its Cryptographic Significance', IEEE Trans. On Information Theory, Vol. IT-24, No.1, 1978, pp 106-110.

70. Adleman, L., 'A Subexponential Algorithm for the Discrete Logarithm Problem with Applications to Cryptography', MIT, Dept. of Maths. and Lab. of Computer Science Report, 1980.

71. Blakely, B. and Blakley, G. R., 'Security of Number Theoretic Public Key Cryptosystems against Random Attack I', Cryptologia, Vol.2, No.4, 1978, pp 305-321.

72. Berkovits, S., Kowalchuk, J., and Schanning, B., 'Implementing Public Key Scheme', IEEE Communications Magazine, Vol.17, No.3, 1979, pp 2-3.

73. Zierler, N., 'Primitive Trinomials Whose Degree is a Mersenne Exponent', Information and Control, Vol.15, 1969, pp 67-69.

74. Herlestam, T., and Johanneson, R., 'On Computing Logarithms Over GF($2^p$),' BIT, Vol.21, 1981, pp 326-334.

75. Schanning, B. P., 'Data Encryption with Public Key Distribution, IEEE Conference EASCON, 1979.

76. Simmons, G. and Norris, M. J., 'Preliminary Comments on the MIT Public Key Cryptosystem', Cryptologia, Vol.1, 1977, pp 406-414.

77. Berkovits, S., 'Cycling is just a Random Search', IEEE Communication Magazine Vol.17, Nov. 1979, pp 2-3.

78. Webster's New Collegiate Dictionary, G & C Merriam Co., 1975.

79. Massey, J. L. and Omwia, J. K., 'A new multiplication method for GF($2^m$) and its application in Public Key Cryptography', Presented at Eurocrypt '83, Udine, Italy, March 1983.

80. Carlitz, L., 'Some theorems on permutation polynomials', Bulletin Amer. Math. Soc., Vol.68, 1962, pp 120-122.

81. Cordes, C. M., 'Permutations mod m in the form $x^n$', American Mathematical Monthly, Jan. 1976.

82. Rédei, L., 'Über eindeutig umkehrpare Polynome in endlichen Körpern', Acta. Sci. Math., Vol.11, 1946-48, pp 85-92.

83. Simmons, G. J., 'A Weak Privacy Protocol Using the RSA Cryptoalgorithm,'Vol.7, No.2, 1983, pp 180-182.

84. Diffie, W., 'Cryptographic Technology: Fifteen Years Forecast', BNR Inc., Mountain View, California, USA, 1981.

85. Rédei, L., Algebra, Volume 1, International Series of Monographs in Pure and Applied Mathematics, Pergamon Press, 1967.

86. Davenport, H., The Higher Arithmetic, An Introduction to The Theory of Numbers, Hutchinson and Co. Pub. Ltd., 1968.

87. Sanders, P. W. and Varadharajan, V., 'Secure Communications between Microcomputer Systems', Computer Communications, Vol.6, No.5, Oct. 1983.

88. Odoni, R. W. K., Varadharajan, V. and Sanders, P. W., 'Public Key Distribution in Matrix Rings', IEE Electronic Letters, Vol.20, No.9, Apr. 1984.

89. Varadharajan, V. and Odoni, R. W. K., 'Extension of RSA System to Matrix Rings', submitted to Cryptologia for Publication.

90. Sanders, P. W. and Varadharajan, V. 'DES can keep your data safe', Apple User (Windfall), Vol.2, No.12, June 1983.

# APPENDICES

## Appendix 1

## Data Encryption Standard : A Software Design

A description of the Data Encryption Standard program together with a partial listing is given in this Appendix.

The program is written as a subroutine which can be called at hexadecimal address 1EØØ. In order to use the routine, three things must be supplied: mode, plaintext and key. The user selects the encryption or decryption mode by inputting E or D. In the case of encryption, the input plaintext can be provided either from the keyboard or from a file stored in memory whereas in the case of decryption, the input ciphertext is expected to be present in the memory. The program is almost the same for both encryption and decryption. In the case of decryption, the left shifts are replaced by the right shifts and the shift schedule operation is carried out in the reverse order to that of encryption. Hence only the encryption program is considered here.

### A 1.1 Main Program : Encryption

```
RND CNT   :   Iteration count
 IEØØ     :   JSR   (INIT - 1)
              JSR   (E1)
              JSR   (E2)
              LDA @   00
              STA  RND  CNT
L3        :   LDA  RND  CNT
              CMP @ 10
              BEQ   L1
              CMP @ 08
              BEQ   L2
              CMP @ 0F
              BEQ   L2
              CMP @ 00
              BEQ   L2
              CMP @ 01
              BEQ   L2
```

@ : Hexadecimal data

```
L2          JSR      (LEFT SHIFT)
            JSR      (LEFT SHIFT)
            JSR      (PC - 2)
            JSR      (E4)
            INC      RND  CNT
            JSR      (INIT - 2)
            JMP      L3

L1          JSR      L5
            HALT
```

### A 1.2 Subroutines : Encryption

1. INIT.1, INIT.2 : The subroutine INIT-1 initializes the key register, the data register and all the temporary registers to Zero whereas the routine INIT.2 only clears the temporary registers.

2. E1 : This subroutine takes 64 bits of key as input block and carries out the permutation PC-1. The output is stored in memory locations 9000 to 9007 (Key Register).

3. E2 : This subroutine takes 64 bits of data as input block and carries out the Initial Permutation (IP). The result is stored in memory locations 7000 to 7007. (Data Register).

4. LEFT SHIFT : The input to this routine is found in locations 9000 to 9007. It performs a single circular left shift on locations 9000 to 9003 and on 9004 to 9007. The valid output bits in locations 9000 to 9007 are the seven most significant bits, i.e.,

   9000 to 9007 : | V A L I D | ▨ |

   ←—7—→

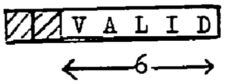
5. PC-2 : This routine performs the permutation PC-2 on the memory locations 9000 to 9007. (Only the seven most significant bits are used). The result is stored in 4000 to 4007 where the six middle bits are the valid bits i.e.,

   4000 to 4007 : | ▨ | V A L I D | ▨ |

   ←—6—→

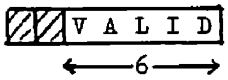
6. E4 : This routine carried out a number of tasks. First, the right half of the original data containing 32 bits into 48 bits using permutation E. Input to this operation is found in locations 7004 to 7007 and the output is stored in 3500 to 3507. In each location, the valid bits are the six least significant ones, i.e.,

3500 to 3507 :



The next part of the routine combines this expanded version with the key bits after PC-2 (in 4000 to 4007) using exclusive-or operation. The result is stored in locations 4500 to 4507. In each location, the valid bits are the six least significant ones, i.e.,

4500 to 4507 :



Then the routine used the 6-bit blocks in 4500 to 4507 as input to the S-boxes and yields a 4-bit blocks as outputs. The first and sixth bits in each location are combined to represent a number, i, in the range Ø to 3 in base 2. The middle four bits are used to represent a number, j, in the range 0 to 15 in base 2. The output is the entry (i, j) in the corresponding S-box. These S-box outputs are stored in locations 4600 to 4607. Then, the Permutation P is carried out using the values in 4600 to 4607 as its input. The result is stored in 5700 to 5703 where all 8 bits in each location are valid. They are exclusive-ored with the left half of the original data in 7000 to 7003 to form the right half of the next iteration. Finally the right half of the original data is transferred to the left half.

7. E5 : This routine carried out the Inverse Initial Permutation (IP$^{-1}$) on 7000 to 7007. The output of this routine is stored in 6100 to 6107 which is the final ciphertext output.

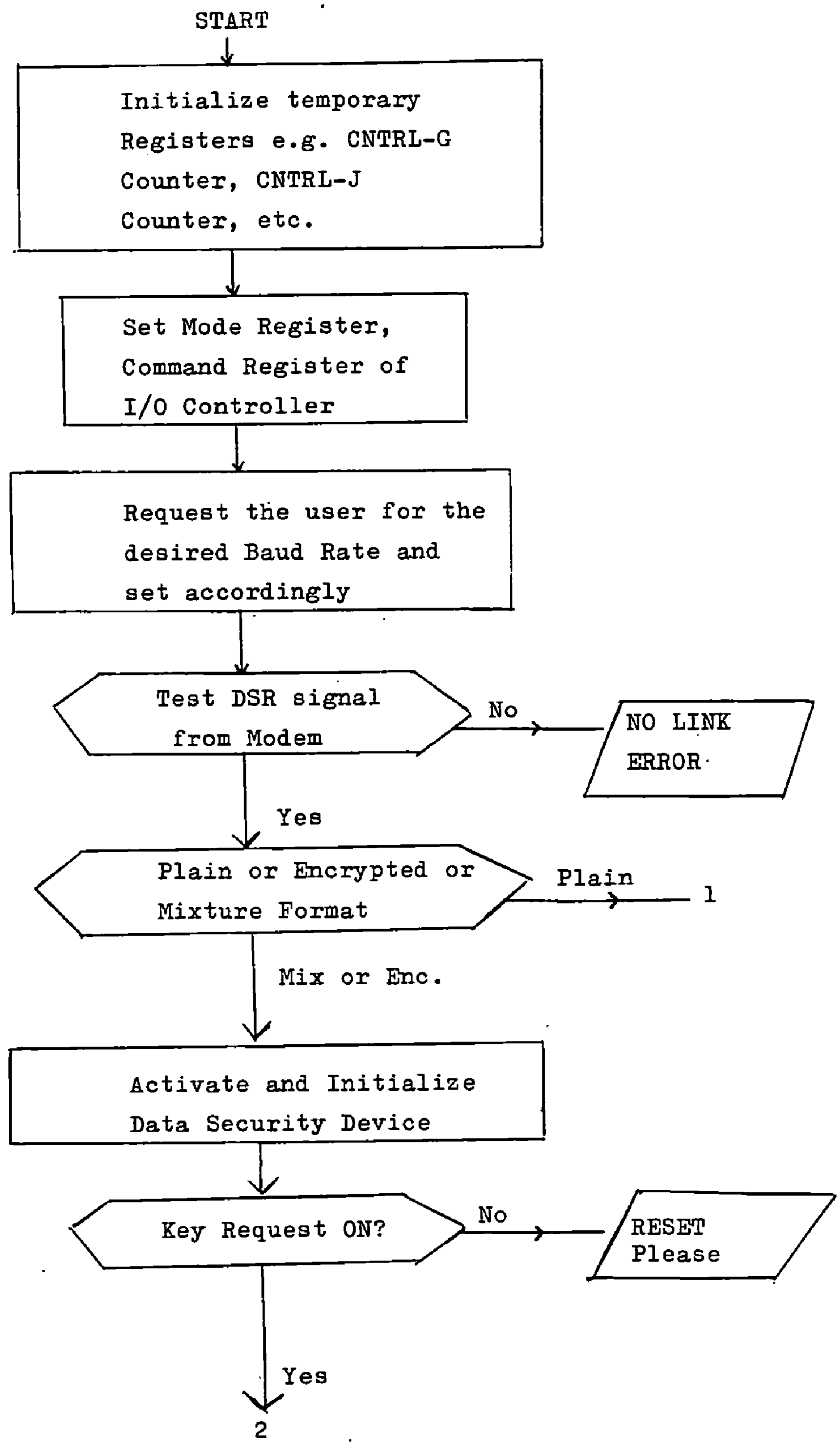
### A 1.3 Use of the DES Program

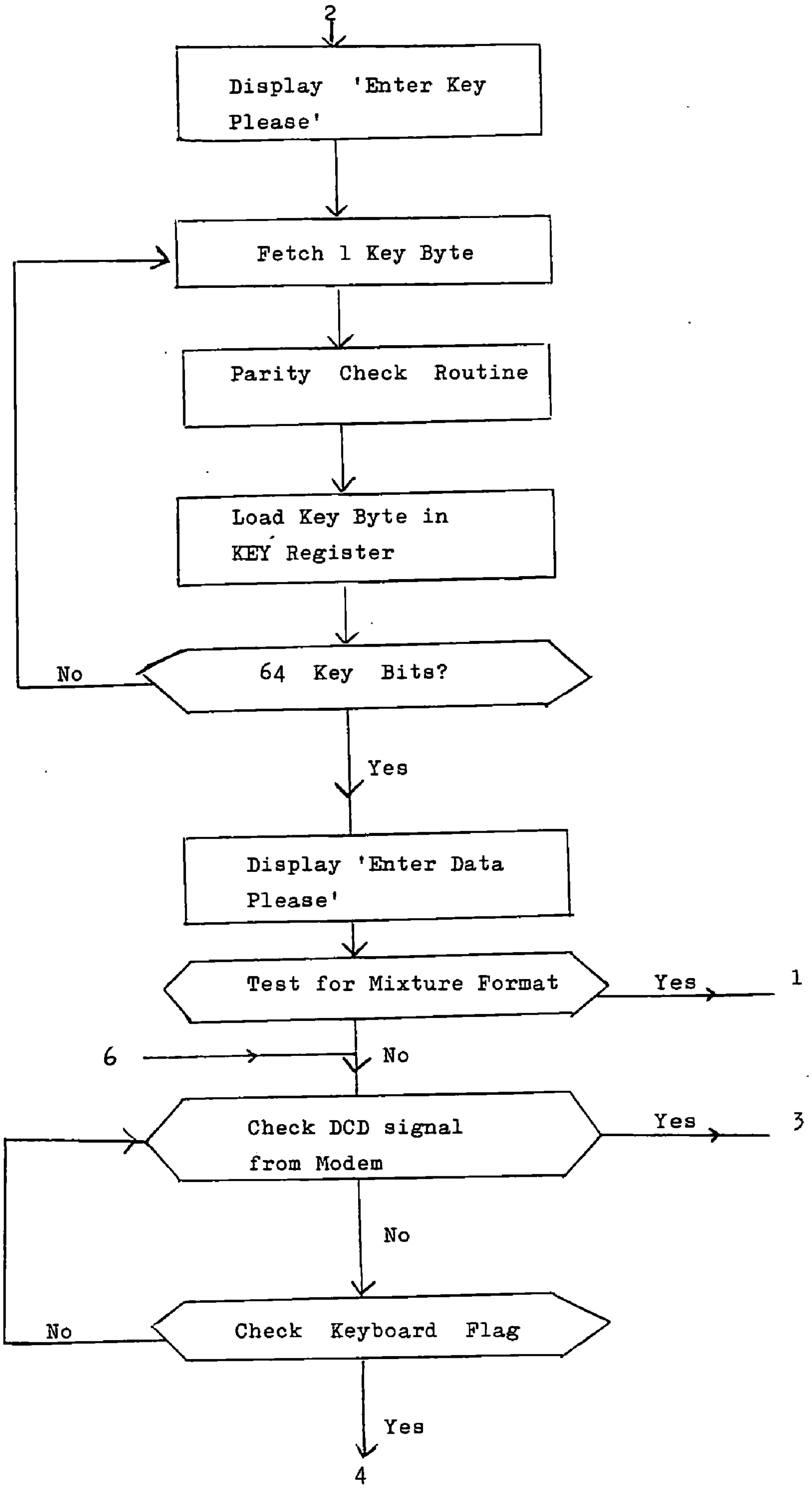
```
*    1EØØ G

*    ENTER ENCRYPT (E) or DECRYPT (D)

     E

*    ENTER KEY PLEASE

*    ENTER DATA BLOCK PLEASE

*    ENCRYPTED DATA BLOCK IS FOUND IN (6100 - 6107)

                    - - - - - -

*    1EØØ  G

*    ENTER ENCRYPT (E) or DECRYPT (D)

     D

*    DECRYPTED DATA BLOCK IS FOUND IN (3800 - 3807)

                    - - - - - -
```

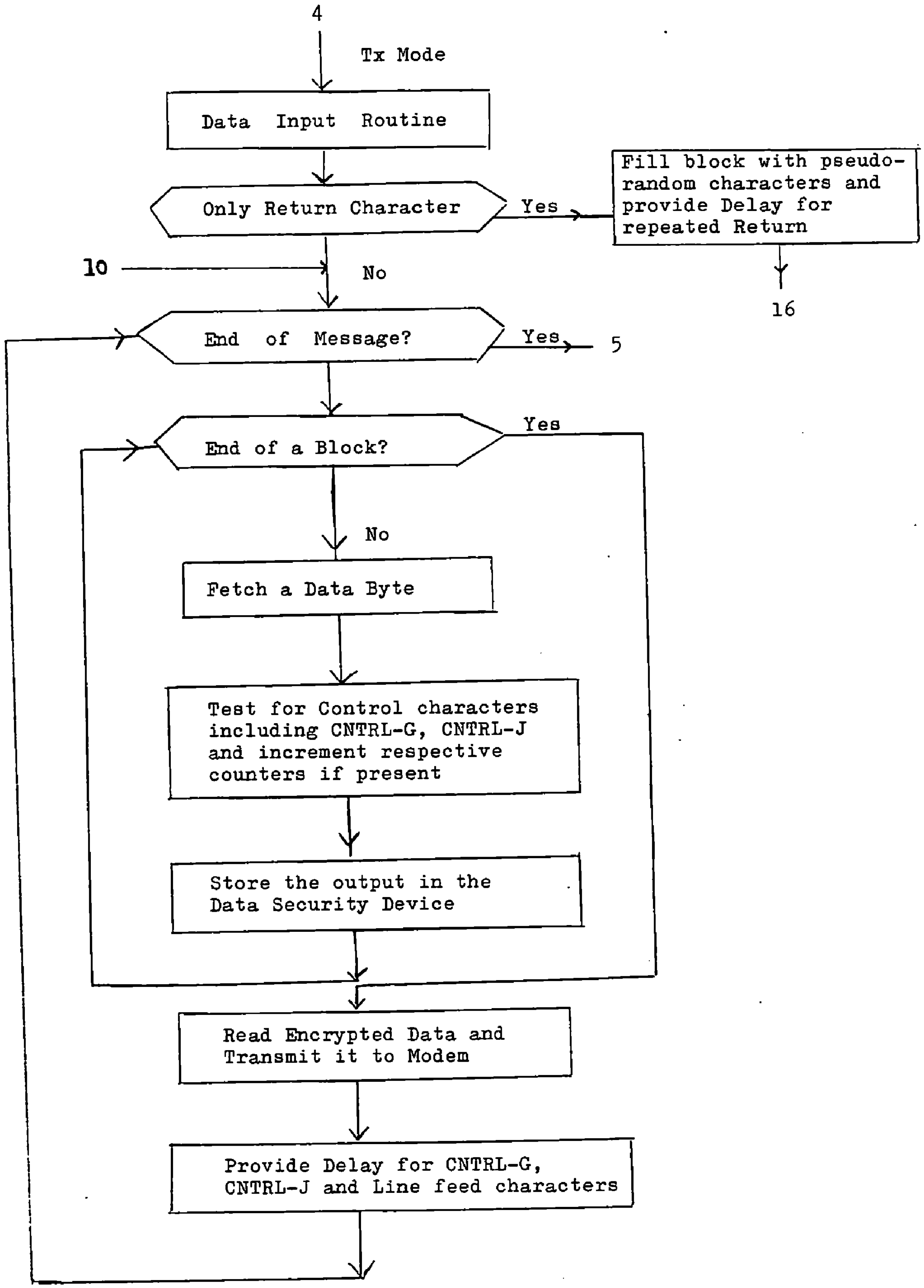
## Appendix 2

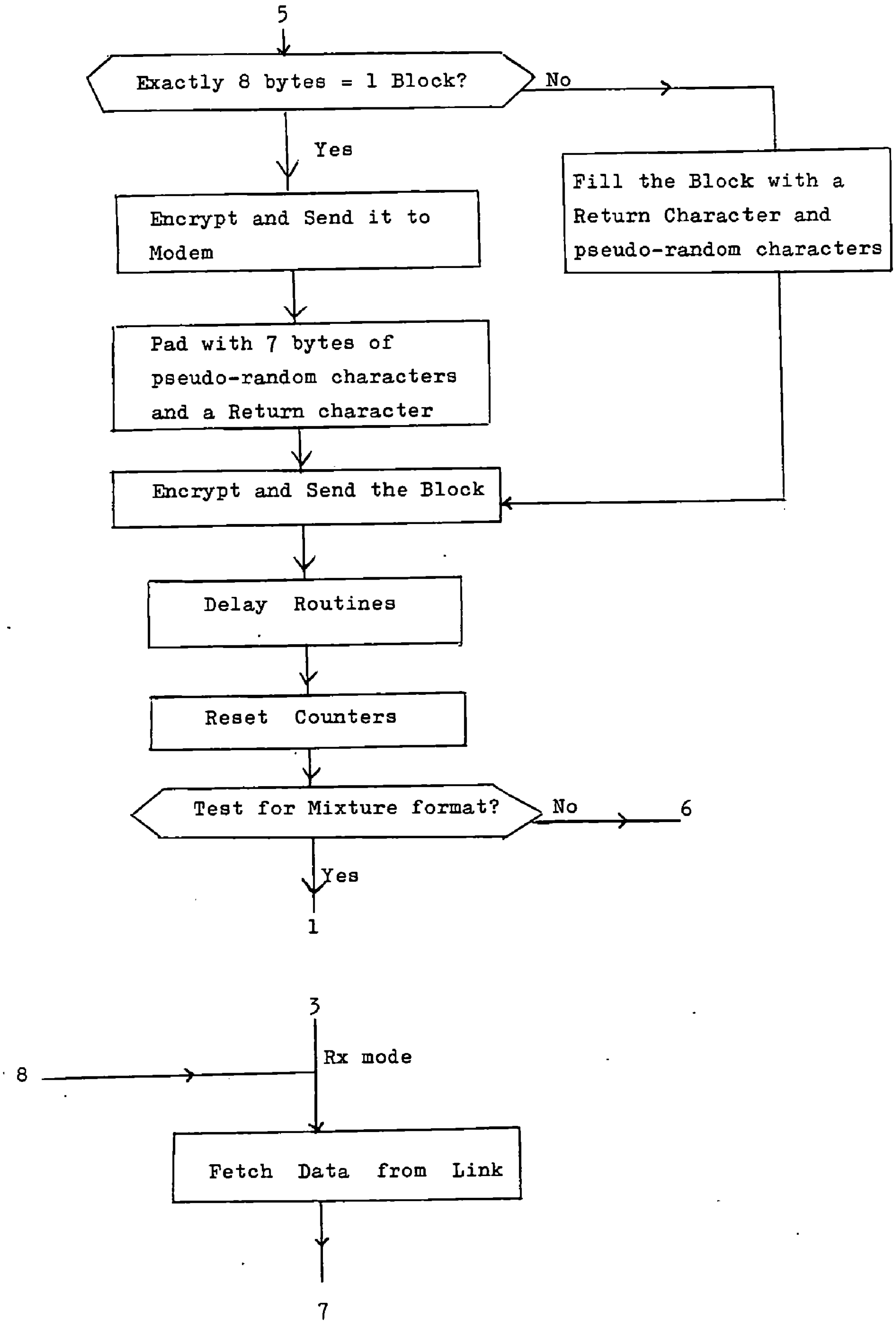
## Point-to-point Communication: Block Encryption Program (ECB)

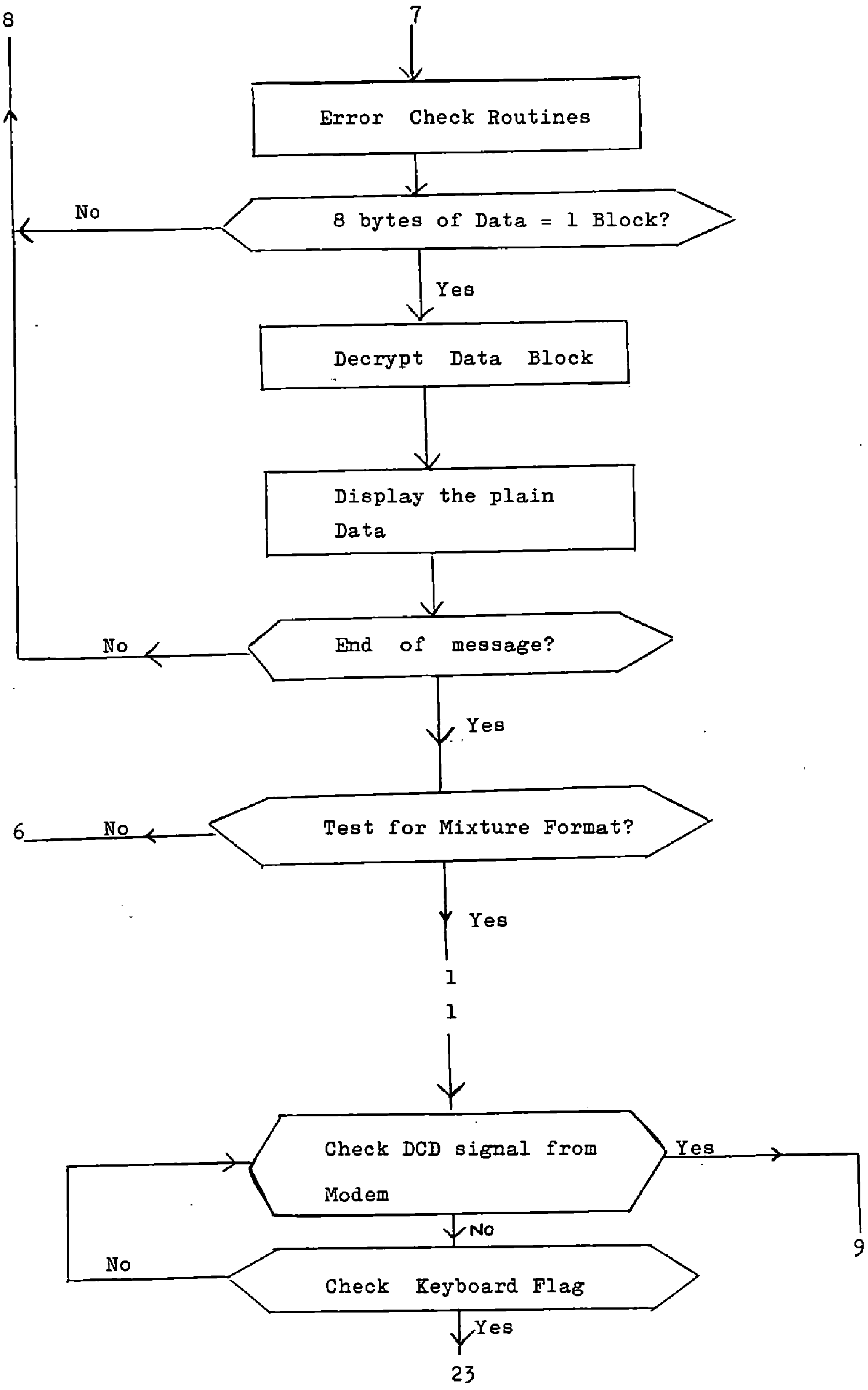
### A 2.1 Basic Flowchart

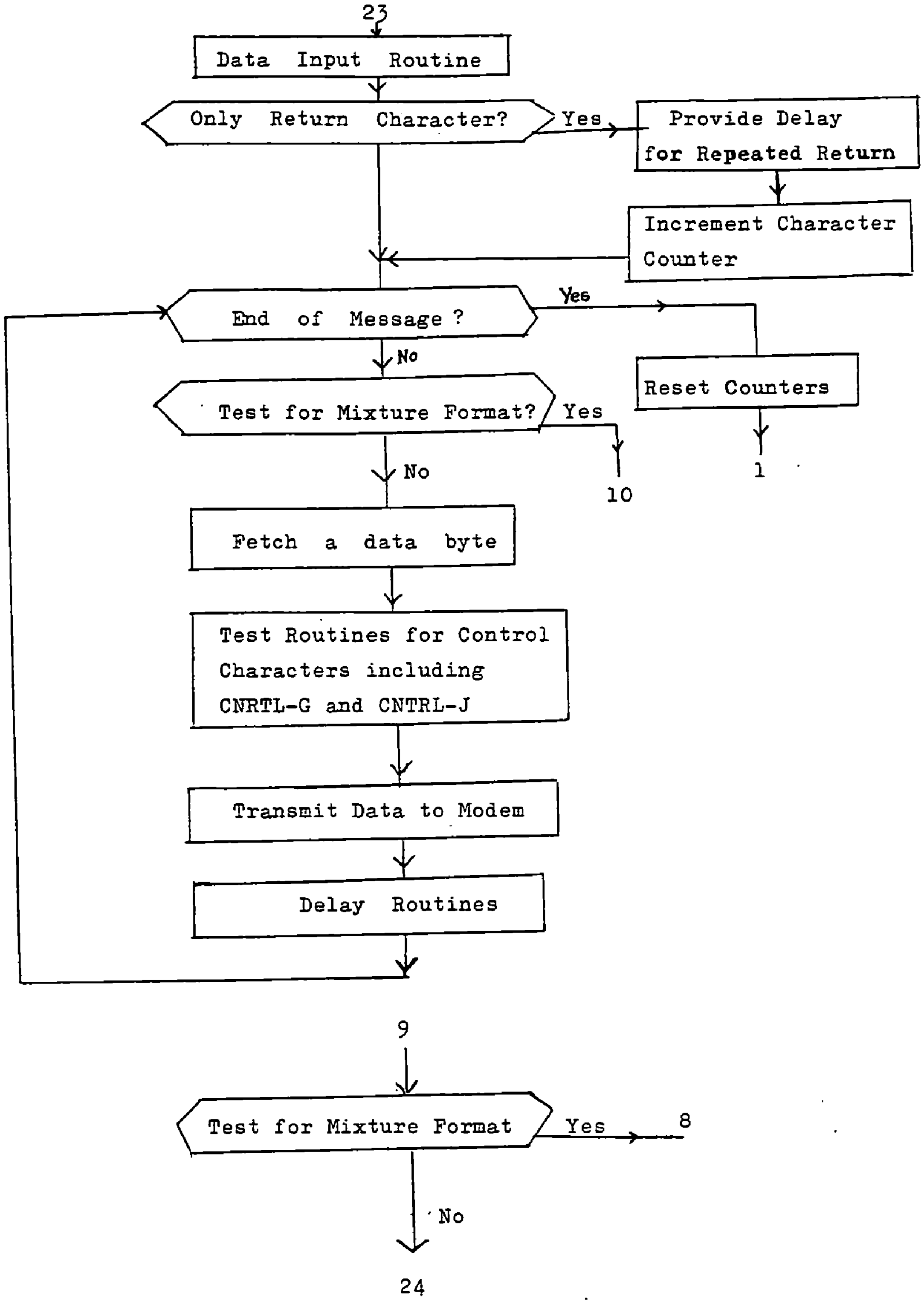

START
Initialize temporary Registers e.g. CNTRL-G Counter, CNTRL-J Counter, etc.
Set Mode Register, Command Register of I/O Controller
Request the user for the desired Baud Rate and set accordingly
Test DSR signal from Modem
No
NO LINK ERROR·
Yes
Plain or Encrypted or Mixture Format
Plain
1
Mix or Enc.
Activate and Initialize Data Security Device
Key Request ON?
No
RESET Please
Yes
2


2
Display 'Enter Key Please'
Fetch 1 Key Byte
Parity Check Routine
Load Key Byte in KEY Register
64 Key Bits?
No
Yes
Display 'Enter Data Please'
Test for Mixture Format
Yes
1
6
No
Check DCD signal from Modem
Yes
3
No
Check Keyboard Flag
No
Yes
4

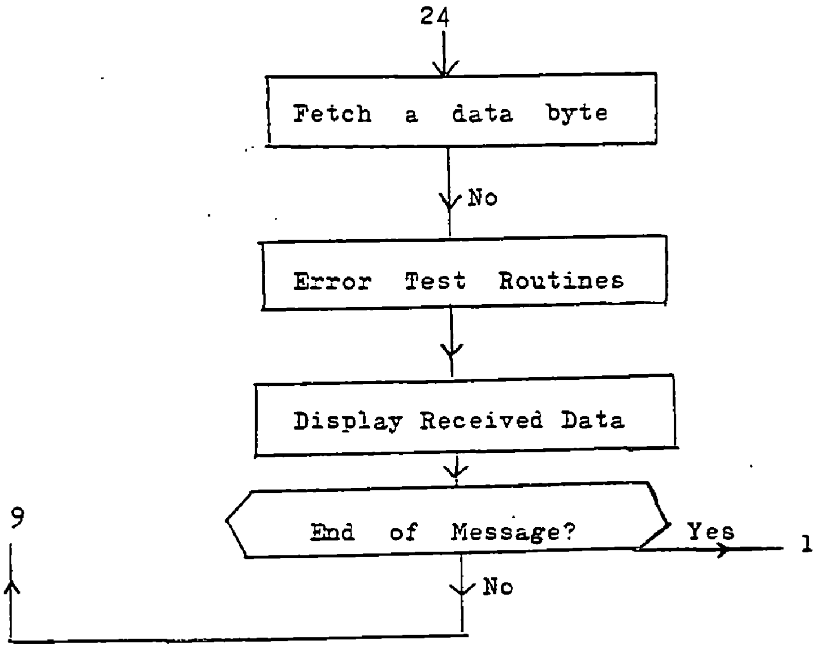

4
Tx Mode
Data Input Routine
Only Return Character
Yes
Fill block with pseudo-random characters and provide Delay for repeated Return
16
10
No
End of Message?
Yes
5
End of a Block?
Yes
No
Fetch a Data Byte
Test for Control characters including CNTRL-G, CNTRL-J and increment respective counters if present
Store the output in the Data Security Device
Read Encrypted Data and Transmit it to Modem
Provide Delay for CNTRL-G, CNTRL-J and Line feed characters


5
Exactly 8 bytes = 1 Block?
No
Yes
Fill the Block with a Return Character and pseudo-random characters
Encrypt and Send it to Modem
Pad with 7 bytes of pseudo-random characters and a Return character
Encrypt and Send the Block
Delay Routines
Reset Counters
Test for Mixture format?
No
6
Yes
1
3
Rx mode
8
Fetch Data from Link
7


8
7
Error Check Routines
No
8 bytes of Data = 1 Block?
Yes
Decrypt Data Block
Display the plain Data
No
End of message?
Yes
6
No
Test for Mixture Format?
Yes
1
1
Check DCD signal from Modem
Yes
No
9
No
Check Keyboard Flag
Yes
23


23
Data Input Routine
Only Return Character?
Yes
Provide Delay for Repeated Return
Increment Character Counter
End of Message ?
Yes
No
Reset Counters
1
Test for Mixture Format?
Yes
10
No
Fetch a data byte
Test Routines for Control Characters including CNRTL-G and CNTRL-J
Transmit Data to Modem
Delay Routines
9
Test for Mixture Format
Yes
8
No
24

A-9a


24
Fetch a data byte
No
Error Test Routines
Display Received Data
End of Message?
Yes
1
No
9

A-9b

### A 2.2 Main Program

```
                                Comments

30AE  :   LDA  C051   Set Text Mode
          STA  C0A8   Reset I/O Controller WD 2123
          LDX @ 00
          LDY @ 00
          LDA @ 00
          STA    1B   Counter:  No. of CNTRL-G within a block.
          STA    F9   Counter:  No. of CNTRL-J within a block.
          STA    06   Initialize memory location (06) to be
                      used in determining end of a line of
                      display on the VDU.
          LDA @ 01
          STA    08   Counter:  No. of characters in a line.
          LDA @ 10
          STA    07   Counter used in delay routine to avoid
                      noise spikes created during relay change-
                      over in the modem.
          LDA  C0A6
          AND @ 01    Check DSR = 1 ?
          BEQ    L1
          JMP NOLINK  Display No Link Error Message.

L1        LDA @ 6E    Set MODE Register of I/O Controller
          STA  C0AD   WD 2123.
          LDA @ 12    Set COMMAND Register of I/O Controller
                      WD 2123.
```

```
        STA   COAD
        LDA @  14      Set SELCLK = 1
        STA   COA4
        JSR BAUD       Subroutine to set appropriate Baud Rate.
        JSR PLAENC     Subroutine displaying the message 'Plain
                       or Encrypt'.
        JSR   FD6A     Subroutine to input data into the 6502
                       Apple System from keyboard.
        LDA   0200
        CMP @  DO
        BNE     L2     Encryption format requested.
        JSR    DATA    Display Message - 'Enter Data Please'.
        JSR     R7
        JMP  PLAIN     Jump to plain data communication routines.
L2      JSR    DSD     Subroutine which handles the secret key.
                       Fetches key from keyboard, checks for
                       correct parity and stores in KEY Register
                       in Security Device WD 2001.
        JSR   DATA
        JSR     R7


L4      LDA   COA6     Check keyboard and the communication link
        AND @   03     for the presence of valid data in an
        NOP            alternating manner.
        BNE     L3
        JMP     RX
L3      LDA   C000
        BPL     L4
        LDA @  33     Set COMMAND Register of I/O Controller
        STA  COAD     WD 2123 to Transmit Mode.
        LDA @  06     Set Encryption Mode in Security Device
        STA  COA1     WD 2001.

        JSR  FD6A     Data Input Subroutine.
        CPX @  00
        BEQ    L5
        JMP    L6

L5      LDA @  FF
        JSR  FCA8
        JSR   END     Subroutine to encrypt and send a block
                      consisting of a Return character and
                      seven random   characters.
        JSR SWITCH    Switch the I/O device to Receive Mode.
        JMP    L4

L6      TXA
        LDY @  00
        STA    1D

L10     LDX @  00

L9      LDA @  00
        CMP    1D     Check end of message.
        BEQ    L7
        CPX    08     Check completion of a block of 64 bits.
        BEQ    L8
        JSR   DIR     Check Data Input Request.
        LDA 0200,y
        JSR TEST      Test for CNTRL-G or CNTRL-J character.
        DEC    1D
        INY
        INX
        JMP    L9

L8      JSR READSEND  Subroutine to send an encrypted block.
        JSR DELAY LINE Subroutine to provide delay for some
                      special characters.
        JMP   L10
```

```
L7      CPX @  08
        BEQ    L11
        JMP    L12

L11     JSR    READSEND
        JSR    DELAYLINE
        LDX @  00

L12     JSR    END
        JSR    DELAYLINE
        JSR    SWITCH
        JMP     L4

RX      LDA @  02      )
        JSR   FCA8     )
        DEC     07     )
        LDA     07     )    Check DCD repeatedly to avoid
        CMP @  00      )    noise spikes.
        BEQ     Z1     )
        JMP     L4     )

Z1      LDA @  10
        STA     07
        LDA   COAA
        LDA @  16          Set Receive Mode in the I/O Controller.
        STA   COAD
        LDA @  0E          Set Security Device to Decryption Mode.
        STA   COA1

L18     LDA @  00          Counter: To determine whether 8 bytes
                           have been received.
        STA     FC

L17     LDA   COAB
        JSR   DELAY        Test TxRDY of I/O Controller.
        AND @  02
        BEQ    L17
        JSR    DIR
        INC     FC
        LDA   COAB
        AND @  10
        BNE    L23
        LDA   COAB           Check for parity,
        AND @  08            framing and overrun errors.
        BNE    L23
        LDA   COAB
        AND    20
        BNE    L23
        JMP    B1
L23     JSR    ERR
        JMP    L4

B1      LDA   COAA
        STA   COA0           Input the received cipher block
                             to the Security Device.
        LDA    FC
        CMP @  08
        BNE    L17
        JSR READPRINT        Decrypt the cipher and display the
                             message on the VDU screen.
                             If end of message, discard any random
                             bytes in the last block.
        CMP @  8D
        BNE    L18
        LDX @  00
L21     CPX @  08
        BEQ    L19
        LDA 9500, x
        CMP @  87
        BEQ    L20
        CMP @  8A
        BEQ    L20
        INx
        JMP    L21
```

```
L19     JSR    END1
        JMP      L4

L20     LDA  @  FF
        JSR    FCA8
        JMP     L19
```

### A 2.3 Subroutines

Some of the important subroutines called by the main program are now briefly described.

#### A 2.3.1 READSEND

This subroutine first checks the TxRDy and the CTS status bits to see if they are in the correct state. Then it checks the DOR status bit and reads data bytes from the Security Device one by one until a whole block is formed. Then the block is sent down the line.

```
        LDX @ 00

L25     CPX @ 08
        BEQ  L22

L23     LDA COAB
        BPL  L23
        LDA COAB
        JSR DELAY
        AND @ 01
        BEQ  L23
        INX
        LDA COA2
        STA COAC
        JMP  L25

L22     RTS
```

#### A 2.3.2 READPRINT

This subroutine initially checks the DOR status bit of the Security Device and then reads the data bytes. The decrypted data block is stored in 9500 to 9507 for later use and is displayed on the VDU screen.

```
        LDX @ 00

L27     CPX @ 08
        BEQ  L24
```

```
L26     LDA   COA3
        BPL    L26
        INX
        LDA   COA2
        STA   9500,x
        JSR   FDFO
        JMP    L27
L24     RTS.
```

### A 2.3.3 _ END

This subroutine is concerned with the last block of the message in the Transmission Mode. First it checks whether the number of characters present in the last block which is not completely full is equal to seven. If so, it adds the Return character at the end and encrypts this block and sends the cipher over the link using READSEND. If however the block contains less than seven characters, it fills the block with a Return and random characters, encrypts the block and transmits the cipher over the link. In both cases, the subroutine SUB1 is used to increment the appropriate CNTRL-G, CNTRL-J character counters. The routine SUB1 is part of the TEST subroutine described in Section A 2.3.4.

```
        JSR  DIR
        LDA @ 8D
        STA COAO
        JSR SUB1
        INX
M12     CPX @ 08
        BEQ  M14
        LDA 02F8,x
        STA COAO
        JSR SUB1
        INX
        JMP  M12
M14     JSR  READSEND
        RTS.
```

### A 2.3.4 _ TEST

This subroutine checks whether the input character is a CNTRL-G or a CNTRL-J or any other control character and increments the appropriate counters accordingly.

```
             STA    COAO
             CMP  @  8A
             BEQ     R11
             CMP  @  87
             BEQ     R12
             CMP  @  AO
             BCS     R13
             RTS

R11          INC      F9
             RTS

R12          INC      1B
             RTS

R13          JSR    SUB1
             RTS

SUB1         INC     ·08
             LDA      08
             CMP   @  29
             BEQ     R14
             RTS

R14          LDA   @  FF
             STA      06
             LDA   @  01
             STA      08
             RTS
```

### A 2.3.5 _ DELAYLINE

This subroutine provides the necessary delay required after sending a CNTRL-G, CNTRL-J or a linefeed (end of line) character.

```
             LDA    06
             CMP  @ FF
             BEQ   R16
             JMP   R17
```

```
R16      LDA @ F0
         JSR FCA8
         LDA @ 00
         STA   06
R17      LDA   1B
         CMP @ 00
         BEQ  R18
         LDA @ FF
         JSR FCA8
         DEC   1B
         JMP  R17
R18      LDA   F9
         CMP @ 00
         BEQ  R19
         LDA @ 80
         JSR FCA8
         DEC   F9
         JMP  R18
R19      RTS.
```

### A 2.3.6 _ BAUD

This subroutine takes a number between 0 to 8 from the keyboard as its input and sets the Baud Rate in the I/O controller accordingly in the range 50 to 1200 bits per second (bauds).

```
         LDA  0200
         AND @  0F
         STA  COAE
         ASL
         CMP @  06
         BCS   R20
         CLC
         ADC @  04
R20      STA    09
         SEC
         LDA @  11
         SBC    09
         STA    09
         RTS.
```

### A 2.3.7 _ ERR

This subroutine prints an ERROR MESSAGE and helps the system to resynchronize after an error has occurred. This is done by resetting DIR and DOR status flags and initializing the character counter (08) and the memory location (06) to zero.

```
          JSR  FF2D
R21       LDA  COA3
          AND  @ 40
          BNE   R23

R22       LDA  COA3
          BPL   R22
          LDA  COA2
          JMP   R22

R23       STA  COAO
          JMP   R21

R24       LDA  @  01
          STA     08
          LDA  @  00
          STA     06
          LDX  @  08

R25       CPX  @  00
          BEQ   R26
          LDA  COA2
          DEX
          JMP   R25

R26       RTS.
```

### A 2.3.8 _ PLAIN

This subroutine allows plain data communication over the link.

```
L29       LDA  COA6
          AND  @  03
          BNE   L28
          JMP   RX1
```

```
L28       LDA  C000
          BPL   L29
          LDA @  33
          STA  COAD
          JSR  FD6A
          CPX @  00
          BEQ   L30

L32       INX
          JMP   L31

L30       LDA @  FF
          JSR  FCA8
          JMP   L32

L31       TXA
          STA    ID
          LDY @  00

L33       CPX @  00
          BEQ   L34

L35       LDA  COAB
          BPL   L35
          LDA  COAB
          JSR DELAY
          AND @  01
          BEQ   L35
          LDA 0200,Y
          STA  COAC
          JSR  TEST
          JSR DELAYLINE
          INY
          DEX
          JMP   L33

L34       JSR SWITCH
          JMP   L29

RX1       LDA @  02
          JSR  FCA8
          DEC    07
          LDA    07
          CMP @  00
          BEQ    Z3
          JMP   L29
```

```
Z3      LDA  @ 10
        STA    07
        LDA  COAA
        LDA  @ 16
        STA  COAD
        LDA  @ 00
        STA  6500

L40     LDA  COAB
        AND  @ 02
        BEQ   L40
        LDA  COAB
        AND  @ 20
        BNE   L23
        LDA  COAB
        AND  @ 10
        BNE   L23
        LDA  COAB
        AND  @ 08
        BNE   L23
        JSR  FF2D
        JMP   L29

L23     LDA  COAA
        STA  6501
        JSR  FDFO
        LDA  6501
        CMP  @ 8D
        BEQ   L41
        LDA  6501
        STA  6500
        JMP   L40

L41     LDA  @ 12
        STA  COAD
        LDA  @ FF
        JSR  FCA8
        LDA  6500
        CMP  @ 87
        BEQ   L42
        CMP  @ 8A
        BEQ   L42
        JMP   L29
```

```
L42         LDA @  F0
            JSR  FCA8
            JMP    L29
```

### A 2.3.9 _ DSD

This subroutine initializes the Data Security Device, WD 2001, and enters the DES secret key into the KEY Register of the device after checking/correcting its parity. This routine also generates a 64-bit pseudo-random number.

```
            LDA  0200
            CMP @  C5
            BEQ    S1
            LDA  0200
            CMP @  C4
            BEQ    S2
            JMP  FF2D

S2          LDA @  01
            STA    FC
            LDA @  0E
            JMP    S3

S1          LDA @  00
            STA    FC
            LDA @  06

S3          STA  C0A1
            LDA  C0A3
            AND @  10
            BNE    S4
            JMP RESET

S4          JSR ENTER KEY PLEASE        Display 'Enter Key Please'.
            LDA @  00
            STA    FA
            LDA @  F8
            STA    1D
            LDA @  02
            STA    1E

S6          JSR  FD0C
            JSR  FDF0
            JSR  PARITYCHECK    Parity check subroutine called.
```

```
                    LDA      FB
                    STA    COAO
                    LDA    COA3
                    AND @    20
                    BEQ      S5
                    JMP    FF2D

S5                  LDY @    00
                    LDA      4E
                    STA (1D),Y
                    INC      1D
                    INC      FA
                    LDA      FA
                    CMP @    08
                    BNE      S6
                    LDY @    08

S8                  CPY @    00
                    BEQ      S7
                    LDA @    88
                    JSR    FDFO
                    DEY
                    JMP      S8

S7                  JSR    FD8B
                    RTS

PARITYCHECK:        STA      FB
                    LDX @    00
                    LDY @    00

S10                 ROL
                    BCC      S9
                    INY

S9                  INX
                    CPX @    08
                    BNE     S10
                    TYA
                    AND @    01
                    ROR
                    BCS     S11
                    LDA      FB
                    CLC
                    ROL
                    STA      FB

S11                 RTS
```

## Appendix 3

### Extended Character Set

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | A | B | C | D | E | F | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | A | B | C | D | E | F | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 00 | ∞ | a | b | c | d | e | f | g | h | i | j | k | l | m | n | o | p | q | r | s | t | u | v | w | x | y | z | ≤ | 0 | ≥ | ▲ | ☆ | ˘ | α | β | γ | δ | ε | ζ | η |
| 28 | ϑ | ι | κ | λ | μ | ν | ξ | φ | π | ρ | σ | τ | ∂ | φ | χ | ψ | ω | θ | 0 | 8 | ± | ≠ | ∓ | ⊕ | ≡ | Λ | B | C | D | E | F | G | H | I | J | K | L | M | N | Γ |
| 50 | P | Q | R | n | T | U | V | W | X | Y | Σ | ∩ | ∝ | ∪ | ^ | --- | ˙ | ! | " | # | $ | \| | & | ' | ( | ) | * | + | , | - | . | / | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 78 | 8 | 9 | ! | ; | ← | = | ▪ | □ | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| A0 | | ! | " | # | $ | % | & | ' | ( | ) | ● | + | , | − | . | / | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | : | ; | < | = | > | ? | @ | A | B | C | D | E | F | G |
| C8 | H | I | J | K | L | M | N | O | P | Q | R | S | T | U | V | W | X | Y | Z | [ | \ | ] | ⌢ | — | ' | a | b | c | d | e | f | g | h | i | j | k | l | m | n | o |
| F0 | p | q | r | s | t | u | v | w | x | y | z | } | \| | { | ~ | ● | | | | | | | | | | | | | | | | | | | | | | | | |

## Appendix 4

### Graphics_Display_Program_ (RESTRY._F77)

```
C  x x x x x x x x x x x x x x x x x x x x x x x x x x x x x x
C  This program prints the plaintext and ciphertext examples
C  using Hershey character set.  The file VVV contains the 96
C  possible ASCII characters (32 out of 128 are control
C  characters and hence not printed).  The program used GINO
C  library subroutines for drawing characters and it also
C  invokes special Hershey subroutines.
C  x  x  x  x  x  x  x  x  x  x  x  x  x  x  x  x  x  x  x  x

        INTEGER TABLE (256), N, TOTAL1 (20)
C  Select the graphics printer as the output device.
                   CALL CC906
C  Select a new paper for a new set of axis
                   CALL PICCLE
C  Set the window size, horizontal and vertical axes scale.
                   CALL UNITS (2.94)
                   CALL WINDOW (2)
                   CALL CHASIZ (1.0, 1.0).
C  Read the character set from file VVV.
                   OPEN (7, FILE = 'VVV')
                   READ (7,*)  (TABLE (I), I=1,256)
                   CLOSE (7).
C  Input the example number and mode of encryption number.
                   K = 0
                   READ (1,*) IP1, IP2
                   IPP1 = 1200 + IP1
                   IPP2 = 1200 + IP2
                   N = 0
                   Y = 61.2
C  Draw the characters using GINO and HERSHEY subroutines.
                   CALL MOVTO2 (1.0, Y)
                   CALL HERCEN (IPP1)
                   CALL MOVBY2 (1.0, 0.0)
                   CALL HERCEN (IPP2)
                   Y = Y-1.6
```

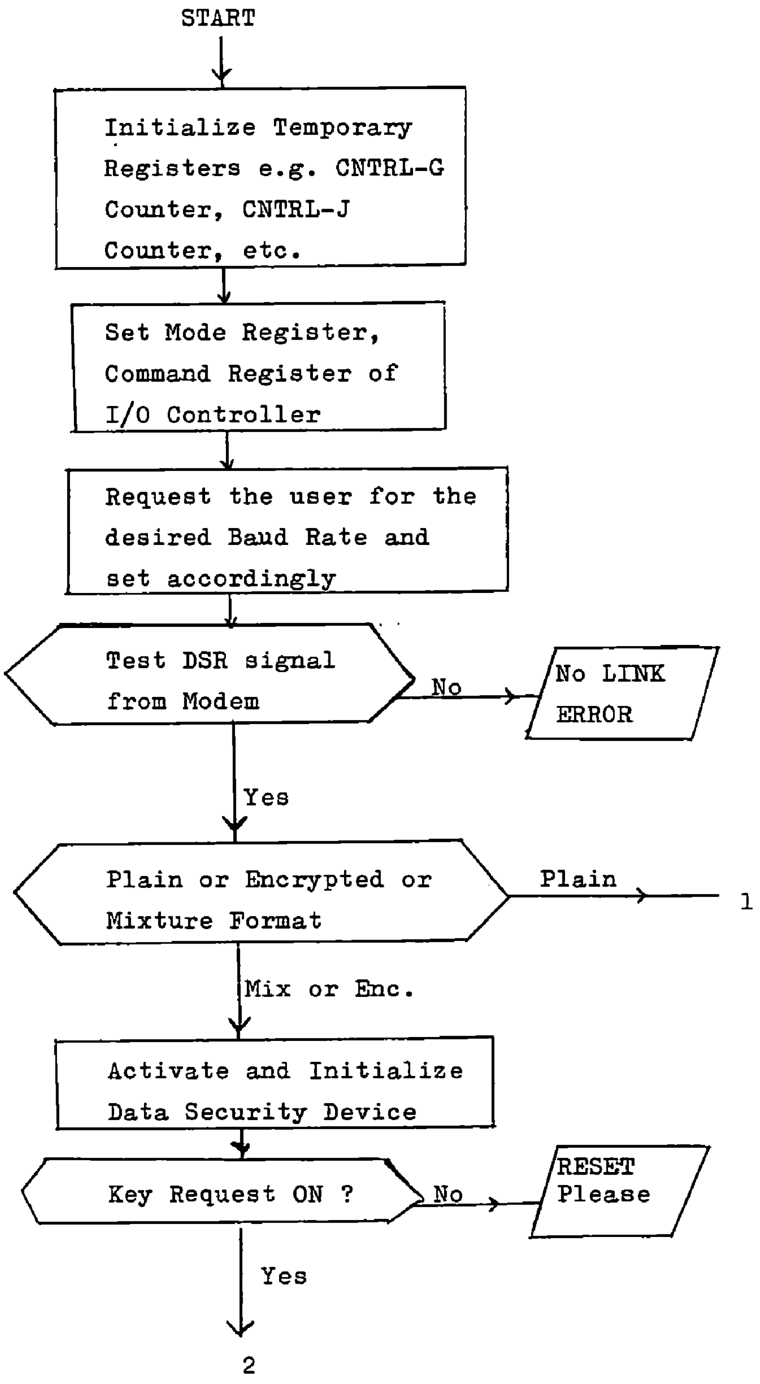
```
            CALL MOVTO2 (1.0,Y)
40          CALL CHOSE (TOTAL1, L,IK)
            lF (IK  .EQ. 1)  Go to 50
            DO 101  JJ = 1, L
            TOTAL1 (JJ) = TOTAL1(JJ) + 1
101         CONTINUE
            II = 1
            L1 = L + 1
30          lF (II.EQ.L1) Go to 10
            lF (N.EQ.40) Go to 20
            NO = TOTAL1 (II)
            NOO = TABLE (NO)
            lF (NOO.EQ.0) NOO = 161
            CALL HERCEN (NOO)
            CALL MOVBY2 (2.1,0.0)
            N = N + 1
            II = II + 1
            Go to 30
10          Go to 40
20          Y = Y - 1.6
            IF (Y. EQ.0) Go to 50
            CALL MOVTO2 (1.0, Y)
            N = 0
            Go to 30
50          CALL DEVEND
            STOP
            END
c  The subroutine CHOSE transforms a character into its
c  corresponding Hershey character.  Hence this allows a
c  possible representation of all256 codes using the
c  extended character set (Appendix 3).  It calls another
c  subroutine CONV2.
            SUBROUTINE CHOSE  (TOTAL1, L, IK)
            CHARACTER * 1  A(38)
            INTEGER I, NUM1, NUM2, NUM, TOTAL, TOTAL1 (20),
         *          LENGT, L
            READ (5,1) (A(I), I = 1, 38)
1           FORMAT (38 A1)
```

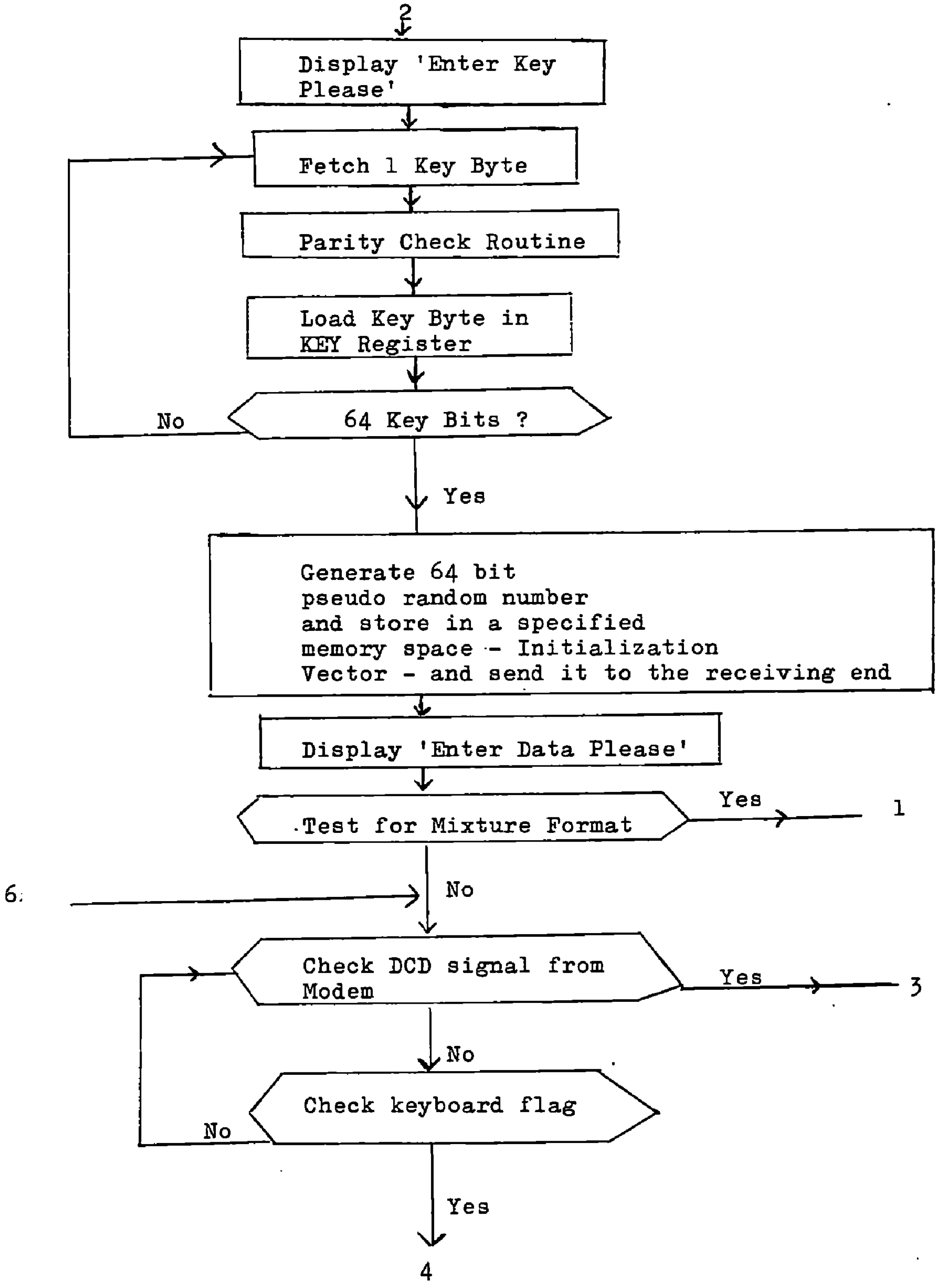
```
         IK = 0
         I  = 1
300      IF (A(I). EQ. ' ' ) Go To 200
         I = I + 1
         Go to 300
200      LENGT = I
         1F (LENGT.Lt.5) Go To 211
         J = 6
         L = 0
201      L = L + 1
         CALL CONV2 (A(J), NUM1)
         J = J + 1
         CALL CONV2 (A(J), NUM2)
         TOTAL = 16 * NUM1 + NUM2
         TOTAL1 (L) = TOTAL
         J = J + 1
         IF (J.NE. LENGT) Go to 201
         RETURN
211      IK = 1
         RETURN
         END
         SUBROUTINE CONV2 (CHAR, NUM)
         CHARACTER * 1 CHAR
         INTEGER NUM
         IF (CHAR. EQ. '0') NUM = 0
         IF (CHAR. EQ. 'A') NUM = 10
         IF (CHAR. EQ. 'B') NUM = 11
         IF (CHAR. EQ. 'C') NUM = 12
         IF (CHAR. EQ. 'D') NUM = 13
         IF (CHAR. EQ. 'E') NUM = 14
         IF (CHAR. EQ. 'F') NUM = 15
         RETURN
         END
```

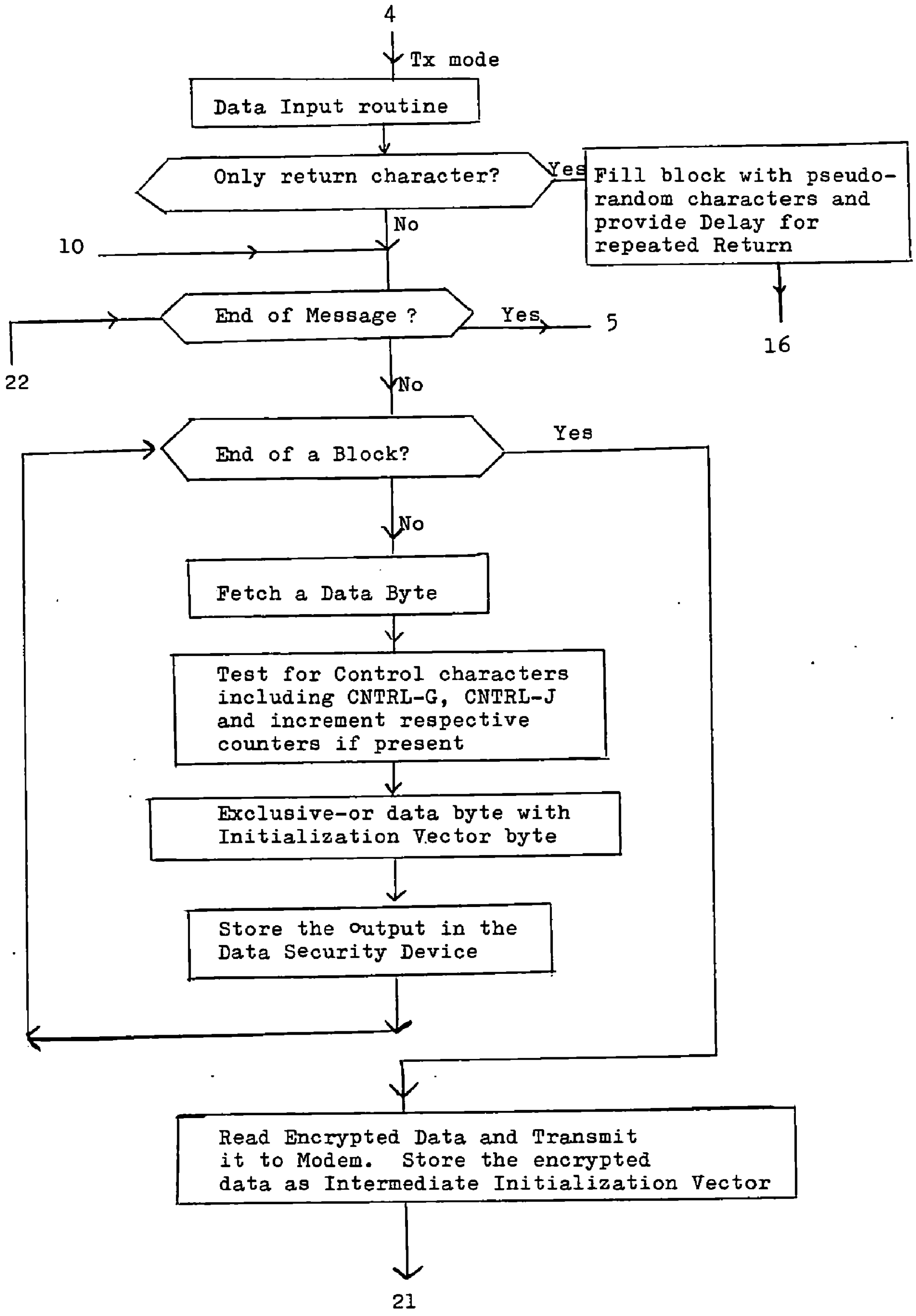
Appendix 5

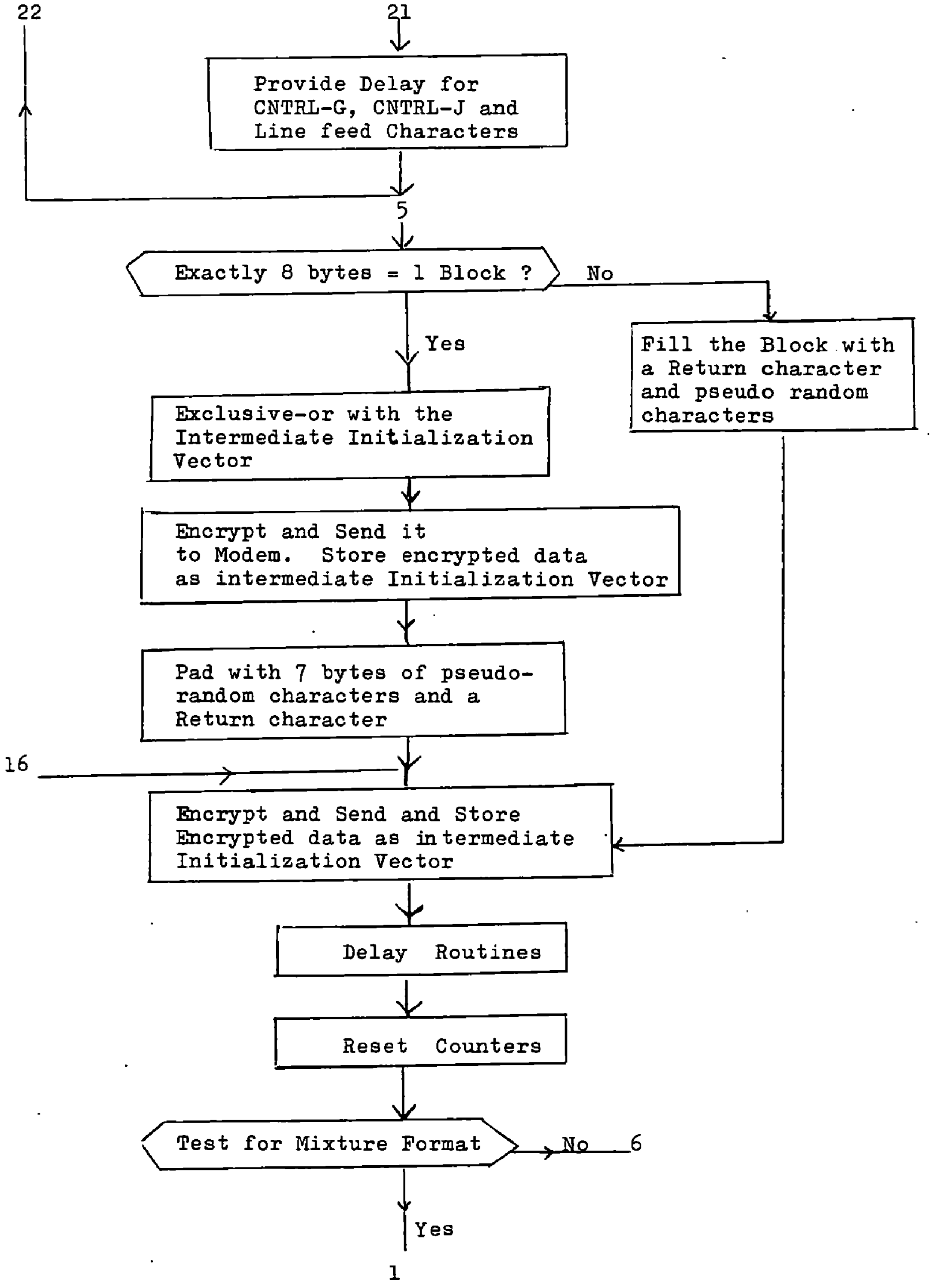
## Point to Point Communication : Cipher Block Chaining Program (CBC)

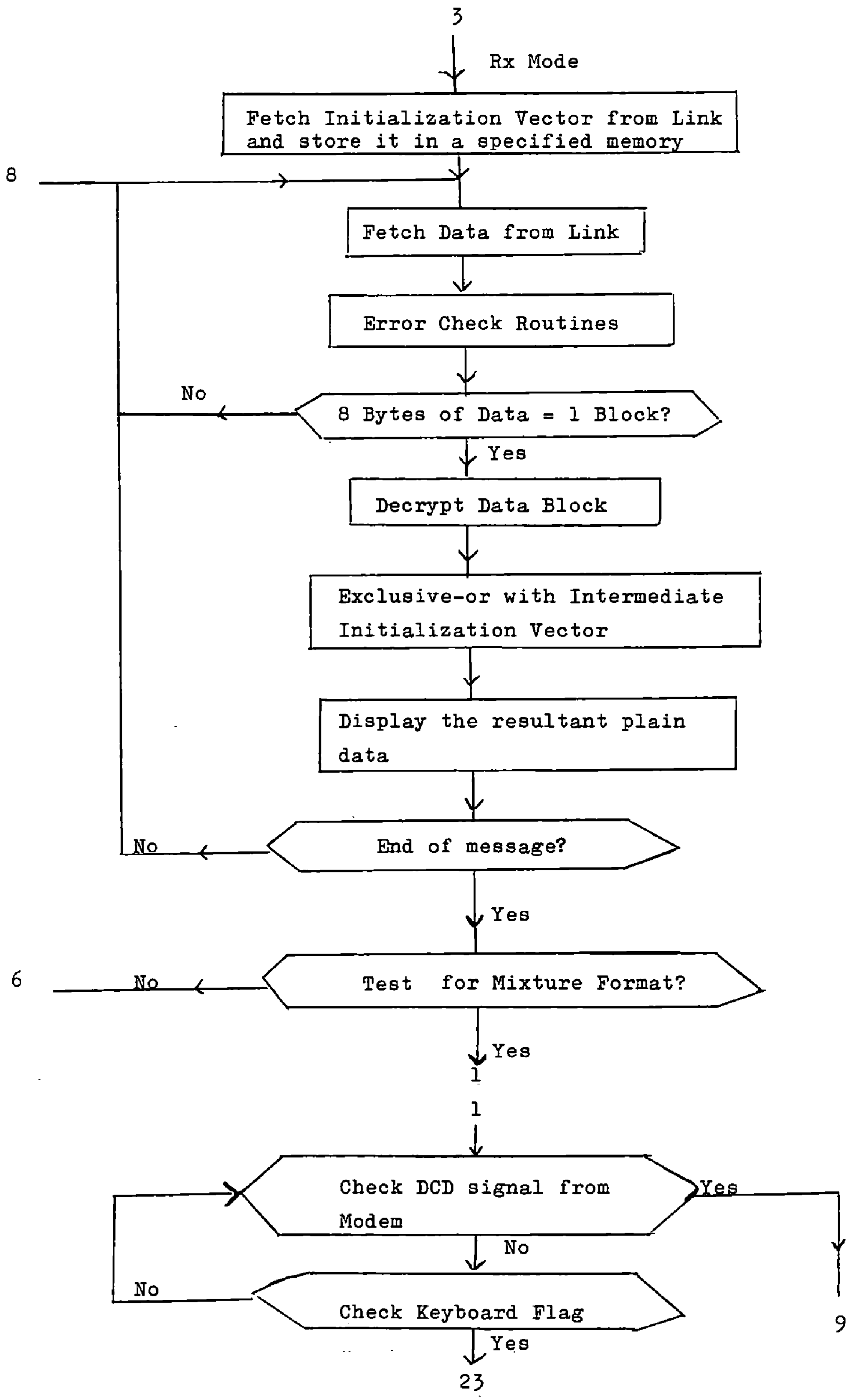
### A 5.1 Basic Flowchart

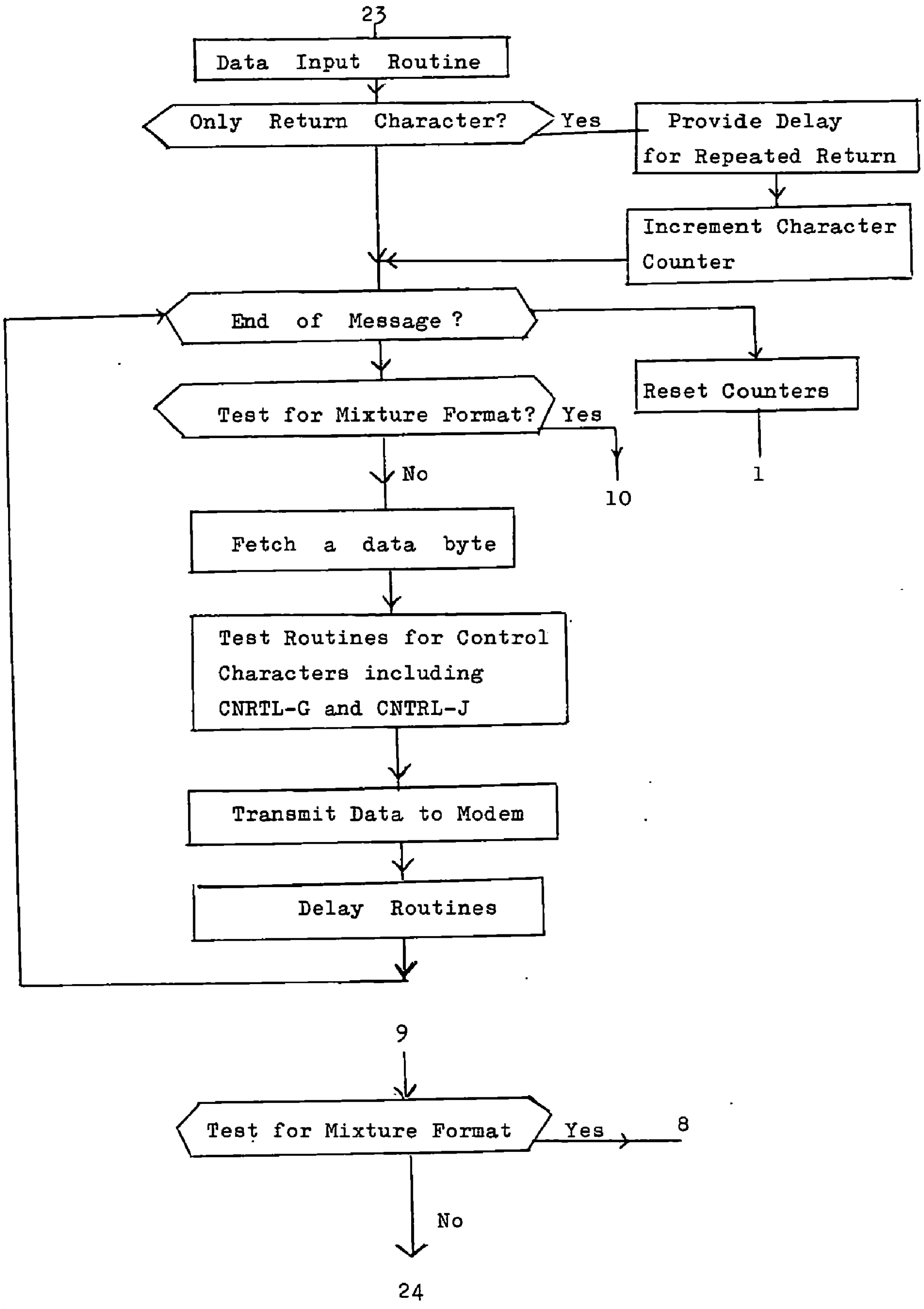

START
Initialize Temporary Registers e.g. CNTRL-G Counter, CNTRL-J Counter, etc.
Set Mode Register, Command Register of I/O Controller
Request the user for the desired Baud Rate and set accordingly
Test DSR signal from Modem
No
No LINK ERROR
Yes
Plain or Encrypted or Mixture Format
Plain
1
Mix or Enc.
Activate and Initialize Data Security Device
Key Request ON ?
No
RESET Please
Yes
2

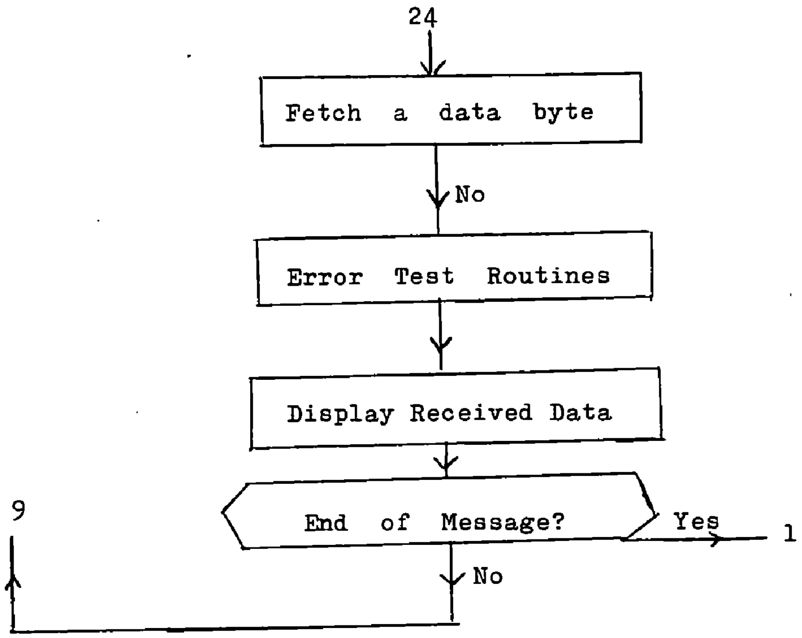

2
Display 'Enter Key Please'
Fetch 1 Key Byte
Parity Check Routine
Load Key Byte in KEY Register
No
64 Key Bits ?
Yes
Generate 64 bit pseudo random number and store in a specified memory space - Initialization Vector - and send it to the receiving end
Display 'Enter Data Please'
Test for Mixture Format
Yes
1
6.
No
Check DCD signal from Modem
Yes
3
No
Check keyboard flag
No
Yes
4


4
Tx mode
Data Input routine
Only return character?
Yes
Fill block with pseudo-random characters and provide Delay for repeated Return
16
No
10
End of Message ?
Yes
5
22
No
End of a Block?
Yes
No
Fetch a Data Byte
Test for Control characters including CNTRL-G, CNTRL-J and increment respective counters if present
Exclusive-or data byte with Initialization Vector byte
Store the output in the Data Security Device
Read Encrypted Data and Transmit it to Modem. Store the encrypted data as Intermediate Initialization Vector
21


22
21
Provide Delay for CNTRL-G, CNTRL-J and Line feed Characters
5
Exactly 8 bytes = 1 Block ?
No
Yes
Fill the Block with a Return character and pseudo random characters
Exclusive-or with the Intermediate Initialization Vector
Encrypt and Send it to Modem. Store encrypted data as intermediate Initialization Vector
Pad with 7 bytes of pseudo-random characters and a Return character
16
Encrypt and Send and Store Encrypted data as intermediate Initialization Vector
Delay Routines
Reset Counters
Test for Mixture Format
No
6
Yes
1


3
Rx Mode
Fetch Initialization Vector from Link and store it in a specified memory
8
Fetch Data from Link
Error Check Routines
No
8 Bytes of Data = 1 Block?
Yes
Decrypt Data Block
Exclusive-or with Intermediate Initialization Vector
Display the resultant plain data
No
End of message?
Yes
6
No
Test for Mixture Format?
Yes
1
1
Check DCD signal from Modem
Yes
No
No
Check Keyboard Flag
Yes
9
23


23
Data Input Routine
Only Return Character?
Yes
Provide Delay for Repeated Return
Increment Character Counter
End of Message ?
Reset Counters
1
Test for Mixture Format?
Yes
10
No
Fetch a data byte
Test Routines for Control Characters including CNRTL-G and CNTRL-J
Transmit Data to Modem
Delay Routines
9
Test for Mixture Format
Yes
8
No
24


24
Fetch a data byte
No
Error Test Routines
Display Received Data
9
End of Message?
Yes
1
No

## A 5.2 Program Listing

### A 5.2.1 Main Program

Most comments are similar to the ones given in the Block Encryption Program listing in Appendix 2.

```
30AE       :  LDA  C051
              STA  COA8
              LDX  @ 00
              LDY  @ 00
              LDA  @ 00
              STA    1B
              STA    F9
              STA    06
              LDA  @ 01
              STA    08
              LDA  @ 10
              STA    07
              LDA  COA6
              AND  @ 01
              BEQ    L1
              JMP NOLINK

L1            LDA  @ 6E
              STA  COAD
              LDA  @ 12
              STA  COAD
              LDA  @ 14
              STA  COA4
              JSR  BAUD
              LDA  COAA
              JSR PLAENC
              JSR  FD6A
              LDA  0200
              CMP  @ DO
              BNE    L2
              JSR  DATA
              JSR    R7
              JMP PLAIN
```

```
L2          JSR   DSD
P3          LDA  C0A6
            AND @  03
            BNE    P1
            JMP    P2
P1          LDA  C000
            BPL    P3
            LDA @  33
            STA  C0AD
            LDA @  06
            STA  C0A1
            JSR  FD6A
            LDY @  00
P6          CPY @  08
            BEQ    P4
P5          LDA  C0AB
            BPL    P5
            LDA 02F8,Y         Pseudo random numbers are
            STA 3000,Y         stored in 3000 to 3007 which
            STA  C0AC          act as the Initialization
            INY                Vector (IV).  IV is
            JMP    P6          transmitted to the other end.
P4          JMP    P7
L4          LDA  C0A6
            AND @  03
            BNE    L3
            JMP    RX
            LDA  C000
            BPL    L4
            LDA @  33
            STA  C0AD
            LDA @  06
            STA  C0A1
            JSR  FD6A
P7          CPX @  00
            BEQ    L5
            JMP    L6
```

```
L5          LDA @  FF
            JSR  FCA8
            JSR   END
            JSR SWITCH
            JMP    L4
L6          TXA
            LDY @  00
            STA    1D

L10         LDX @  00

L9          LDA @  00
            CMP    1D
            BEQ    L7
            CPX @  08
            BEQ    L8
            JSR   DIR
            LDA 0200,Y
            EOR 3000,X
            JSR  TEST
            DEC    1D
            INY
            INX
            JMP    L9

L8          JSR READSEND MOD
            JSR DELAYLINE
            JMP   L10

L7          CPX @  08
            BEQ   L11
            JMP   L12

L11         JSR READSEND MOD
            JSR DELAYLINE
            LDX @  00

L12         JSR ENDMOD
            JSR DELAYLINE
            JSR SWITCH
            JMP    L4
```

```
RX          LDA @  02
            JSR  FCA8
            DEC    07
            LDA    07
            CMP @  00
            BEQ    Z1
            JMP    L4
Z1          LDA @  10
            STA    07
            LDA  COAA
            LDA @  16
            STA  COAD
            LDA @  OE
            STA  COA1

L18         LDY @  00

L17         LDA  COAB
            JSR  DELAY
            AND @  02
            BEQ   L17
            JSR   DIR
            INY
            LDA  COAB
            AND @  10
            BNE   L23
            LDA  COAB
            AND @  08
            BNE   L23
            LDA  COAB
            AND    20
            BNE   L23
            JMP    B1

L23         JSR   ERR
            JMP    L4

B1          LDA  COAA
            STA  COAO
            STA 3010,Y
            CPY @  08
            BNE   L17
```

```
             JSR READPRINTMOD
             CMP  @  8D
             BNE    L18
             LDX  @  00
L21          CPX  @  08
             BEQ    L19
             LDA  9500,X
             CMP  @  87
             BEQ    L20
             CMP  @  8A
             BEQ    L20
             INX
             JMP    L21
L19          JSR   END 1
             JMP     L4
L20          LDA  @  FF
             JSR   FCA8
             JMP    L19
```

## A 5.2.2 Subroutines

Most of the subroutines used in the program are the same as the ones used in the Block Encryption Program given in Appendix 2. The subroutine READSENDMOD is a slightly modified version of READSEND given in Appendix 2 (A 2.3.1). The modified section in READSENDMOD is given below.

```
              .
              .
              .
              LDA   C0A2
              STA   C0AC
              STA   3000, X
              JMP    L25
L22           RTS
```

The subroutine READPRINTMOD is a slightly modified version of READPRINT given in Appendix 2 (A 2.3.2). The modified section in READPRINTMOD is given below.

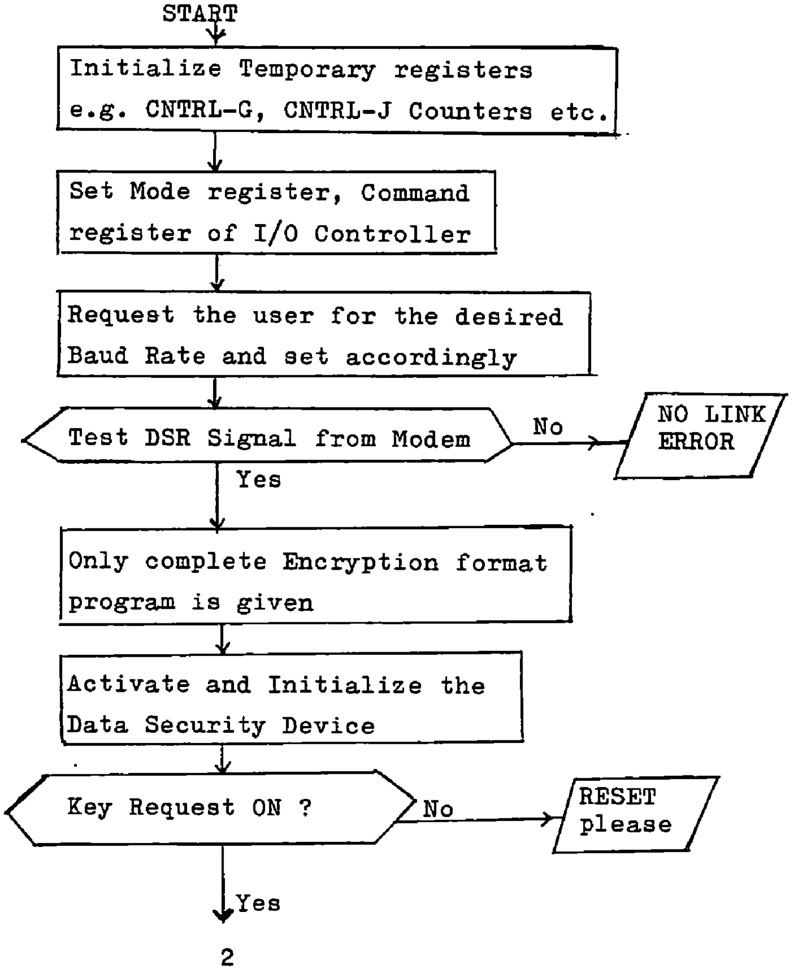
```
               .
               .
               .
               LDA  COA2
               EOR  3000,X
               STA  9500,X
               JSR  FDFO
               LDA  3010,X
               STA  3000,X
               JMP   L27
L24            RTS
```

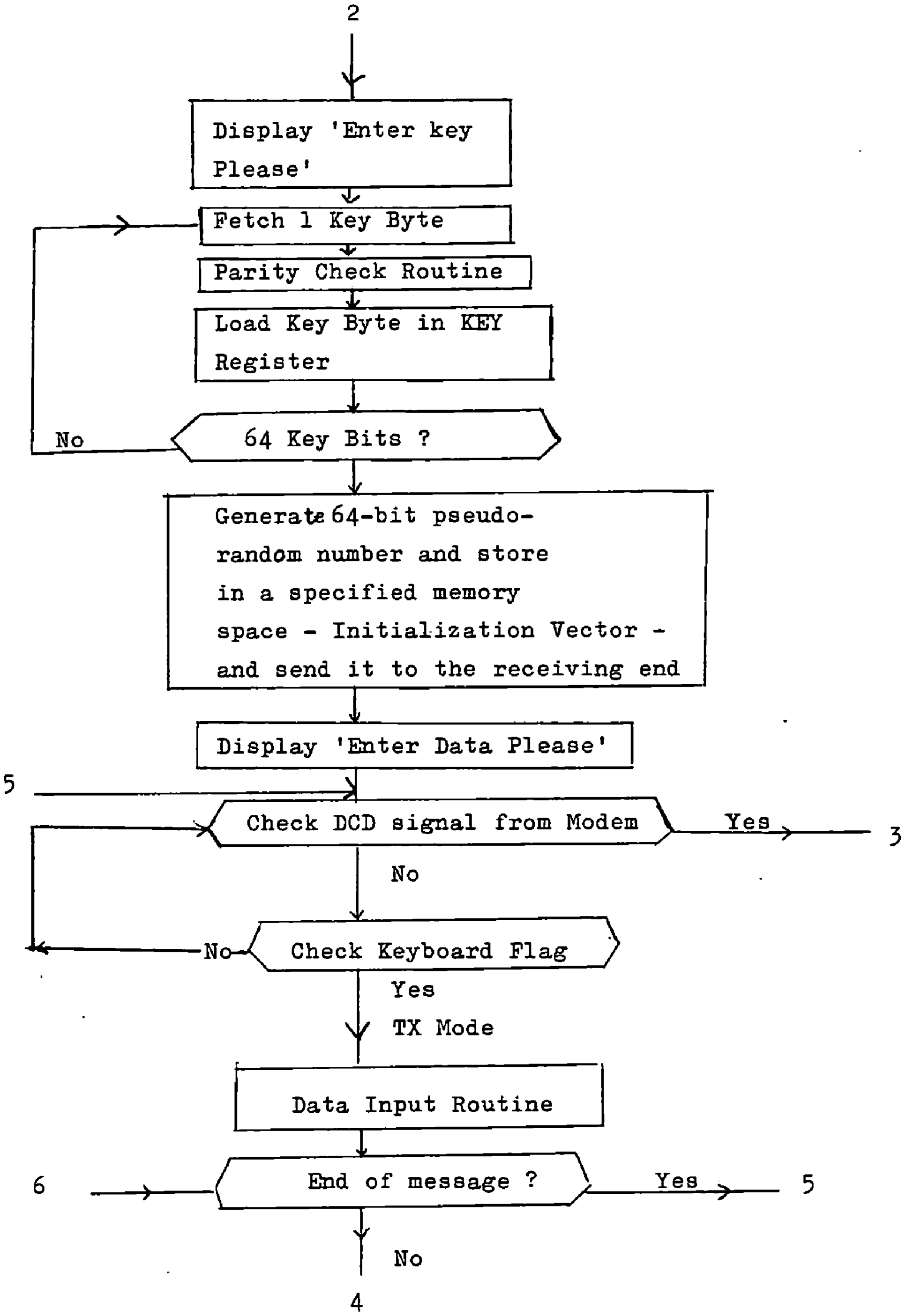
The subroutine ENDMOD is the same as the subroutine END of Appendix 2 (A 2.3.3) except that ENDMOD calls the subroutine READSENDMOD instead of READSEND.

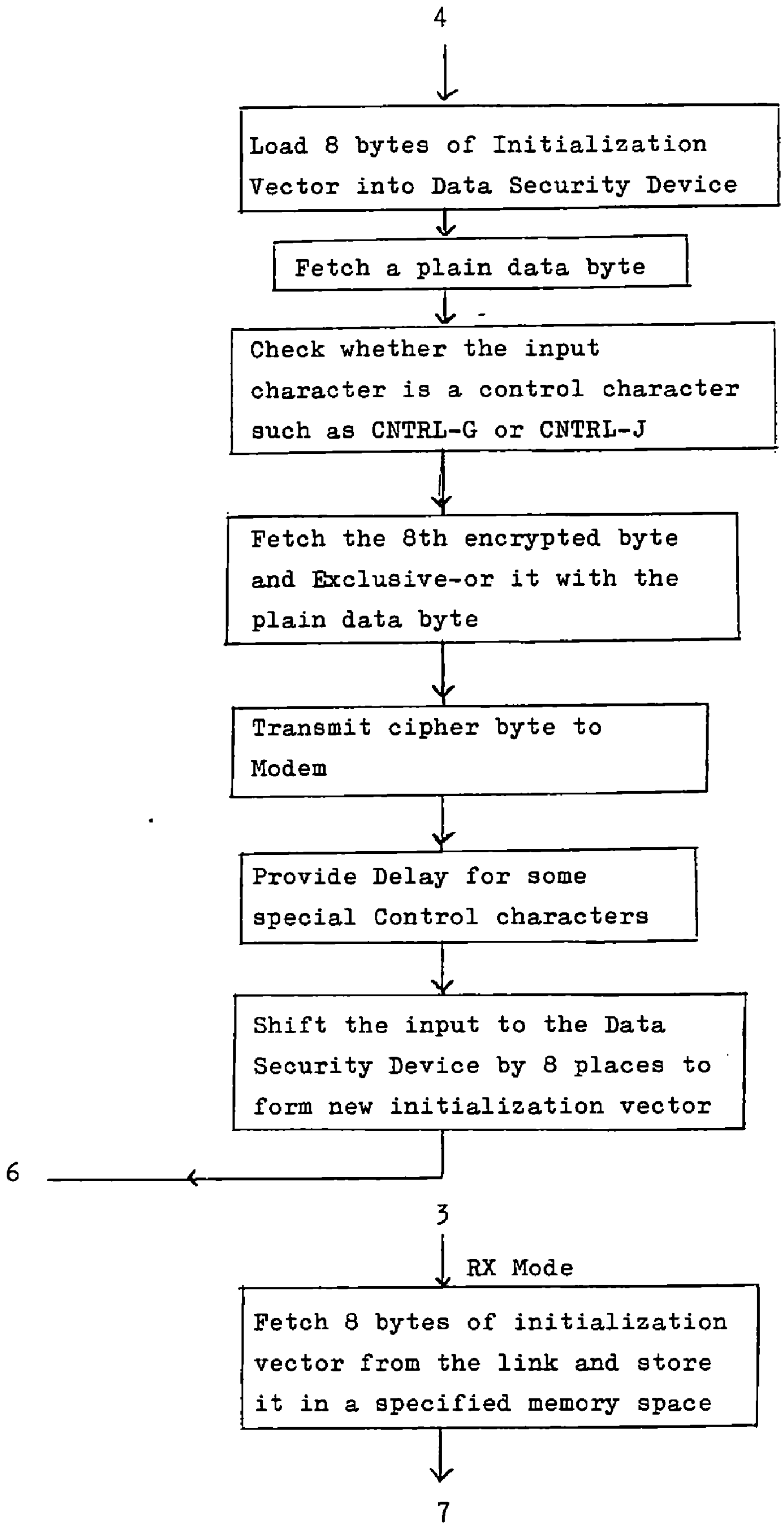
Appendix 6

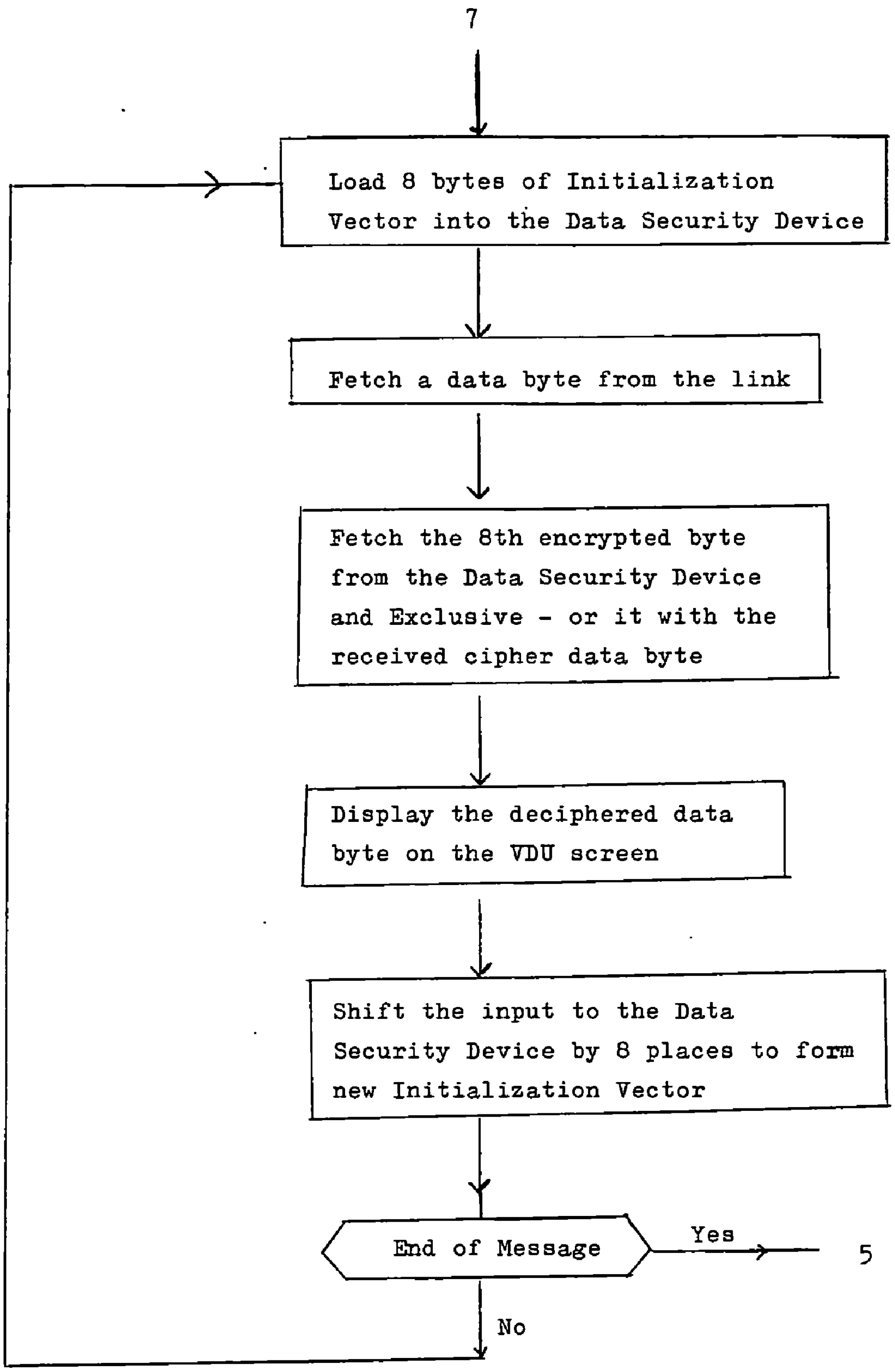
Point to Point Communication : Stream Cipher Feedback Program (CFB)

## A 6.1 Basic Flowchart


START
Initialize Temporary registers e.g. CNTRL-G, CNTRL-J Counters etc.
Set Mode register, Command register of I/O Controller
Request the user for the desired Baud Rate and set accordingly
Test DSR Signal from Modem
No
NO LINK ERROR
Yes
Only complete Encryption format program is given
Activate and Initialize the Data Security Device
Key Request ON ?
No
RESET please
Yes
2


2
Display 'Enter key Please'
Fetch 1 Key Byte
Parity Check Routine
Load Key Byte in KEY Register
No
64 Key Bits ?
Generate 64-bit pseudo-random number and store in a specified memory space - Initialization Vector - and send it to the receiving end
Display 'Enter Data Please'
5
Check DCD signal from Modem
Yes
3
No
No
Check Keyboard Flag
Yes
TX Mode
Data Input Routine
6
End of message ?
Yes
5
No
4


4
Load 8 bytes of Initialization Vector into Data Security Device
Fetch a plain data byte
Check whether the input character is a control character such as CNTRL-G or CNTRL-J
Fetch the 8th encrypted byte and Exclusive-or it with the plain data byte
Transmit cipher byte to Modem
Provide Delay for some special Control characters
Shift the input to the Data Security Device by 8 places to form new initialization vector
6
3
RX Mode
Fetch 8 bytes of initialization vector from the link and store it in a specified memory space
7


7
Load 8 bytes of Initialization Vector into the Data Security Device
Fetch a data byte from the link
Fetch the 8th encrypted byte from the Data Security Device and Exclusive - or it with the received cipher data byte
Display the deciphered data byte on the VDU screen
Shift the input to the Data Security Device by 8 places to form new Initialization Vector
End of Message
Yes
5
No

## A 6.2 _ Program Listing

### A 6.2.1 _ Main Program

Most comments are similar to the ones given in the Block Encryption Program listing in Appendix 2.

```
30AE       :  LDA  C051
              STA  COA8
              LDX @ 00
              LDY @ 00
              LDA @ 00
              STA   1B
              STA   F9
              STA   06
              LDA @ 01
              STA   08
              LDA @ 10
              STA   07
              LDA  COA6
              AND @ 01
              BEQ   L1
              JMP  NOLINK

L1            LDA @ 6E
              STA  COAD
              LDA @ 12
              STA  COAD
              LDA @ 14
              STA  COA4
              JSR  BAUD
              LDA  COAA
              JSR  PLAENC
              JSR  FD6A
              LDA  0200
              CMP @ DO
              BNE   L2
              JSR  DATA
              JSR   R7
              JMP  PLAIN
```

```
L2          JSR  DSD
P3          LDA COA6
            AND @ 03
            BNE   P1
            JMP   P2
P1          LDA C000
            BPL   P3
            LDA @ 33
            STA COAD
            LDA @ 06
            STA COA1
            JSR FD6A
            LDY @ 00
P6          CPY @ 08
            BEQ   P4
P5          LDA COAB
            BPL   P5
            LDA COAB
            JSR DELAY
            AND @ 01
            BEQ   P5
            LDA 02F8,Y
            STA 3000,Y
            STA COAC
            INY
            JMP   P6
P4          JMP   P7
L4          LDA COA6
            AND @ 03
            BNE   L3
            JMP   RX
            LDA C000
            BPL   L4
            LDA @ 33
            STA COAD
            LDA @ 06
            STA COA1
            JSR FD6A
```

```
P7          CPX @ 00
            BEQ   L5
            JMP   L6
            LDA @ FF
            JSR  FCA8
            INX
            TXA
            LDY @ 00
            STA   1D
L9          LDX @ 00
            LDA @ 00
            CMP   1D
            BEQ   L7
            JSR SUB3
            LDA 0200,Y
            STA   EC
            JSR  TEST
            JSR  SUB4
L8          LDA  COAB
            BPL   L8
            LDA  COAB
            JSR DELAY
            AND @ 01
            BEQ   L8
            LDA   EC
            STA  COAC
            JSR  DELAYLINE
            JSR  SHIFT 1
            DEC   ID
            INY
            JMP   L9
L7          JSR  SWITCH
            JMP   L4
P2          LDA @ 02
            JSR  FCA8
            DEC   07
            LDA   07
            CMP @ 00
            BEQ   P11
            JMP    P3
P11         LDA @ 10
            STA    07
            LDA  COAA
            LDA @ 16
            STA  COAD
            LDA    06
            STA  COA1
            LDY @ 00
P14         CPY    08
            BEQ   P12
P13         LDA  COAB
            JSR  DELAY
            AND @ ´02
            BEQ   P13
            LDA  COAA
            STA  3000,Y
            INY
            JMP   P14
P12         JMP   P15

RX          LDA    02
            JSR  FCA8
            DEC    07
            LDA    07
            CMP @ 00
            BEQ    Z1
            JMP    L4
Z1          LDA  COAA
            LDA    16
            STA  COAD
            LDA @ 06
            STA  COA1
P15         LDA  COAB
            JSR DELAY
            AND @ 02
            BEQ   P15
            LDA  COAB
            AND @ 10
            BEQ   L23
            LDA   COAB
            AND @  08
            BEQ    L23
            LDA   COAB
            AND @  20
            BEQ    L23
            JMP    L24
L23         JMP    ERR
L24         JSR   SUB3
            LDA   COAA
            STA     EE
            STA     EC
            JSR   SUB4
            JSR   FDFO
            JSR   SHIFT2
            LDA     EC
            CMP @  8D
            BEQ    L25
            JMP    P15
L25         JSR    END
            JMP     L4
```

### A 6.2.2 Subroutines

Most of the subroutines are the same as the ones given in the Block Encryption Program in Appendix 2. Some additional subroutines called by this program are now briefly described.

SUB3 : This subroutine inputs the data stored in the memory locations 3000 to 3007 (Initialization Vector) into the Data Security device after testing the DIR status flag each time.

```
      K3     CPX @  08
             BEQ     K1
             LDA   COA3
             AND @  40
             BNE     K2
             JMP   FF2D
      K2     LDA   3000,X
             STA   COAO
```

```
          INX
          JMP    K3
     K1   RTS
```

SUB4 : This subroutine reads the encrypted Initialization Vector from the Data Security Device and exclusive-ores the 8th encrypted data byte with the plain data byte (the cipher data byte) to produce the cipher data byte (the plain data byte).

```
          LDX  @  00
     K6   CPX  @  07
          BEQ     K4
     K5   LDA   COA3
          BPL     K5
          LDA   COA2
          INX
          JMP     K6
          LDA   COA2
          EOR     EC
          STA     EC
          RTS
```

SHIFT 1, SHIFT 2 : These two subroutines are essentially the same. They shift the memory locations 3000 to 3007 (Initialization Vector) sequentially as follows: 3007 ⟶ 3006, 3006 ⟶ 3005, - - -, 3001 ⟶ 3000 and (cipher data byte) ⟶ 3007.

```
          LDX  @  01
          LDA  @  00
          STA     ED
     K7   LDA   3000,X
          DEX
          STA   3000,X
          INX
          INX
          INC     ED
          LDA  @  07
          CMP     ED
```

```
BNE    K7
LDA    EC
DEX
STA  3000,X
LDX @  00
RTS
```

## Appendix 7

## Results of some Statistical tests on DES Output Sequences

### A 7.1 _ Input Samples and Keys used in Statistical Tests

### A 7.1.1 _ Input Samples

| | Input sample period in digits | Input sample (binary/hex) | Abbreviation |
|---|---|---|---|
| 1. | 5-digits | 01110 (binary) | 1.1 |
| | | 10010 " | 1.2 |
| | | 01011 " | 1.3 |
| | | 00011 " | 1.4 |
| | | 10111 " | 1.5 |
| 2. | 8-digits | 00000001 (binary) | 2.1 |
| | | 10101010 " | 2.2 |
| | | 11111111 " | 2.3 |
| | | 01000110 " | 2.4 |
| | | 11000011 " | 2.5 |
| 3. | 10-digits | 1011111111 (binary) | 3.1 |
| | | 0100010000 " | 3.2 |
| | | 0000011111 " | 3.3 |
| | | 0110000111 " | 3.4 |
| | | 1100111000 " | 3.5 |
| 4. | 20-digits | FFFFC (hex) | 4.1 |
| | | 003FF " | 4.2 |
| | | E3C3F " | 4.3 |
| | | 68D47 " | 4.4 |
| | | BBBOF " | 4.5 |
| 5. | 40-digits | FFFFFFFFFC (hex) | 5.1 |
| | | CE3C3E0FC0 " | 5.2 |
| | | 00000FFFFF " | 5.3 |
| | | 999EEE1116 " | 5.4 |
| | | 62B5A45197 " | 5.5 |
| 6. | 64-digits | FFFFFFFFFFFFFFF0 (hex) | 6.1 |
| | | 00000000FFFFFFFF " | 6.2 |
| | | C4C4C433CCC4C4C4 " | 6.3 |
| | | B1B2B3B4B5B6B7B8 " | 6.4 |
| | | 3141592654138547 " | 6.5 |

A 7.1.2 Keys

| | | | |
|---|---|---|---|
| I. | 1F1F1F1F0E0E0E0E | - | 'weak' |
| II. | 1FE01FE00EF10EF1 | - | 'semiweak' |
| III. | 039649C539317965 | - | 'random' |
| IV. | 85CDCB1C9BD0851A | - | 'random' |
| V. | 3131313131313131 | - | 'non-random' |

A 7.2 Results of Frequency, Serial and Runs tests on ciphertext sequences: using different modes under a fixed key

| | | | |
|---|---|---|---|
| 5% Significant levels | : | Frequency (F) test | : 3.84 |
| | | Serial (S) test | : 5.99 |
| | | Runs (R) test | : 1.96 |

Key : 3131313131313131 (hex)

| Input Sample | Mode | F | S | R |
|---|---|---|---|---|
| 1.1 | ECB | 0.19 | 3.65 | 1.19 |
| | CFB | 0.77 | 2.90 | -0.29 |
| | CBC | 0.66 | 2.61 | -0.04 |
| | CBCP | 2.25 | 4.17 | -0.06 |
| | CFBV | 0.39 | 7.36 (**) | -2.24 (**) |
| 1.2 | ECP | 4.00 (**) | 7.69 (**) | -1.26 |
| | CFB | 0.25 | 3.71 | -1.24 |
| | CBC | 0.39 | 4.16 | 1.33 |
| | CBCP | 2.06 | 4.98 | 0.94 |
| | CFBV | 0.77 | 3.22 | -0.73 |
| 1.3 | ECB | 20.81 (**) | 23.09 (**) | 1.37 |
| | CFB | 0.004 | 2.52 | -0.75 |
| | CBC | 0.14 | 2.38 | 0.44 |
| | CBCP | 0.32 | 2.40 | 0.20 |
| | CFBV | 0.04 | 2.66 | -0.81 |

| Input Sample | Mode | F | S | R |
|---|---|---|---|---|
| 1.4 | ECB | 8.63 (**) | 15.22 (**) | -1.87 (*) |
| | CFB | 0.47 | 3.37 | 0.95 |
| | CBC | 0.25 | 2.89 | 0.76 |
| | CBCP | 0.06 | 2.33 | -0.56 |
| | CFBV | 0.02 | 2.72 | 0.81 |
| 1.5 | ECB | 4.25 (**) | 6.18 (**) | -0.12 |
| | CFB | 2.64 | 4.87 | 0.40 |
| | CBC | 7.22 (**) | 9.69 (**) | -0.34 |
| | CBCP | 0.04 | 4.79 | 1.63 (*) |
| | CFBV | 2.44 | 4.35 | 0.01 |
| 2.1 | ECB | 9.00 (**) | 10.83 (**) | 0.28 |
| | CFB | 2.25 | 4.35 | 0.07 |
| | CBC | 3.52 (*) | 5.63 (*) | 0.55 |
| | CBCP | 0.32 | 5.75 (*) | -1.87 (*) |
| | CFBV | 0.88 | 5.48 | -1.60 (*) |
| 2.2 | ECB | 1.00 | 3.06 | -0.03 |
| | CFB | 0.77 | 3.43 | 0.78 |
| | CBC | 1.56 | 4.00 | 0.74 |
| | CBCP | 1.41 | 4.68 | -1.08 |
| | CFBV | 4.52 (**) | 6.61 (**) | -0.36 |
| 2.3 | ECB | 25.00 (**) | 30.62 (**) | -1.25 |
| | CFB | 0.47 | 5.61 (*) | -1.80 (*) |
| | CBC | 0.02 | 2.72 | -0.88 |
| | CBCP | 0.10 | 2.09 | -0.12 |
| | CFBV | 0.004 | 2.95 | -1.00 |
| 2.4 | ECB | 9.00 (**) | 14.96 (**) | 2.30 (**) |
| | CFB | 4.79 (**) | 7.27 (**) | -0.67 |
| | CBC | 2.85 | 4.83 | 0.34 |
| | CBCP | 0.25 | 3.77 | -1.24 |
| | CFBV | 0.14 | 2.33 | -0.43 |
| 2.5 | ECB | 0 | 18.27 (**) | 4.00 (**) |
| | CFB | 2.44 | 6.33 (**) | 1.46 |
| | CBC | 1.72 | 8.03 (**) | 2.12 (**) |
| | CBCP | 0.39 | 2.45 | -0.18 |
| | CFBV | 1.27 | 3.50 | 0.42 |

| Input Sample | Mode | F | S | R |
|---|---|---|---|---|
| 3.1 | ECB | 0.19 | 5.18 | 1.70 (*) |
| | CFB | 3.06 | 7.50 (**) | -1.53 (*) |
| | CBC | 1 | 3.10 | 0.41 |
| | CBCP | 1.27 | 3.28 | 0.29 |
| | CFBV | 2.25 | 4.16 | 0.07 |
| 3.2 | ECB | 2.64 | 5.88 (*) | -1.11 |
| | CFB | 0.47 | 2.65 | 0.45 |
| | CBC | 4.00 (**) | 7.33 (**) | 1.19 |
| | CBCP | 0.02 | 2.52 | 0.69 |
| | CFBV | 0.32 | 4.33 | 1.39 |
| 3.3 | ECB | 1.72 | 4.58 | 0.99 |
| | CFB | 0.004 | 2.52 | 0.69 |
| | CBC | 1 | 3.22 | 0.53 |
| | CBCP | 0.004 | 3.99 | 1.38 |
| | CFBV | 0.035 | 2.48 | -0.69 |
| 3.4 | ECB | 2.07 | 19.25 (**) | -3.88 (**) |
| | CFB | 0.02 | 2.17 | -0.44 |
| | CBC | 3.29 | 5.34 | -0.34 |
| | CBCP | 0.77 | 2.79 | -0.29 |
| | CFBV | 1.89 | 3.82 | -0.07 |
| 3.5 | ECB | 4.52 (**) | 6.82 (**) | -0.77 |
| | CFB | 0.66 | 2.93 | 0.46 |
| | CBC | 2.44 | 5.17 | -0.86 |
| | CBCP | 0.88 | 6.95 (**) | 2.03 (**) |
| | CFBV | 0.04 | 2.96 | -1.00 |
| 4.1 | ECB | 0.004 | 2.62 | 0.75 |
| | CFB | 1.89 | 7.45 (**) | 1.94 (*) |
| | CBC | 0.56 | 2.68 | -0.42 |
| | CBCP | 1.13 | 3.71 | 0.724 |
| | CFBV | 0.66 | 2.88 | -0.42 |
| 4.2 | ECB | 0.04 | 10.14 (**) | 2.81 (**) |
| | CFB | 0.02 | 2.10 | 0.25 |
| | CBC | 0.04 | 2.27 | 0.44 |
| | CBCP | 0.53 | 7.92 (**) | -0.71 |
| | CFBV | 0.004 | 2.13 | 0.31 |

| Input Sample | Mode | F | S | R |
|---|---|---|---|---|
| 4.3 | ECB | 0.00 | 2.01 | 0.06 |
| | CFB | 1.41 | 5.50 (*) | 1.48 |
| | CBC | 1.56 | 5.65 (*) | -1.45 |
| | CBCP | 0.32 | 2.80 | -0.74 |
| | CFBV | 1.27 | 6.98 (**) | -1.90 (*) |
| 4.4 | ECB | 2.64 | 5.06 | 0.77 |
| | CFB | 0.19 | 2.38 | 0.38 |
| | CBC | 2.44 | 5.37 | 0.95 |
| | CBCP | 0.14 | 2.28 | 0.38 |
| | CFBV | 0.004 | 2.94 | -1.00 |
| 4.5 | ECB | 3.50 (*) | 5.43 | 0.24 |
| | CFB | 0.04 | 3.09 | -1.06 |
| | CBC | 0.10 | 2.47 | 0.57 |
| | CBCP | 0.00 | 2.00 | 0.00 |
| | CFBV | 0.00 | 2.94 | 0.94 |
| 5.1 | ECB | 0.02 | 2.03 | 0.06 |
| | CFB | 0.32 | 2.63 | 0.51 |
| | CBC | 1.00 | 4.26 | -1.10 |
| | CBCP | 2.07 | 4.15 | -0.37 |
| | CFBV | 0.14 | 3.92 | 1.32 |
| 5.2 | ECB | 0.56 | 5.69 (*) | -1.80 (*) |
| | CFB | 0.02 | 2.00 | -0.06 |
| | CBC | 0.88 | 4.80 | -1.41 |
| | CBCP | 0.02 | 3.65 | 1.25 |
| | CFBV | 0.47 | 2.63 | 0.328 |
| 5.3 | ECB | 0.39 | 3.07 | 0.825 |
| | CFB | 0.19 | 2.24 | 0.131 |
| | CBC | 0.04 | 2.76 | -0.87 |
| | CBCP | 0.25 | 3.28 | -1.06 |
| | CFBV | 1.72 | 4.16 | 0.74 |
| 5.4 | ECB | 2.64 | 25.3 (**) | -4.49 (**) |
| | CFB | 1.12 | 3.21 | -0.10 |
| | CBC | 0.14 | 2.33 | -0.43 |
| | CBCP | 0.85 | 5.40 | -1.76 (*) |
| | CFBV | 0.04 | 2.31 | 0.50 |

| Input Sample | Mode | F | S | R |
|---|---|---|---|---|
| 5.5 | ECB | 0.04 | 4.57 | -1.63 (*) |
| | CFB | 0.06 | 3.53 | -1.25 |
| | CBC | 0.04 | 2.05 | 1.10 |
| | CBCP | 0.56 | 2.80 | 0.52 |
| | CFBV | 0.00 | 2.00 | 0.06 |
| 6.1 | ECB | 36.00 (**) | 37.70 (**) | 1.17 |
| | CFB | 0.25 | 2.80 | 0.70 |
| | CBC | 0.32 | 2.28 | 0.01 |
| | CBCP | 2.44 | 4.98 | 0.70 |
| | CFBV | 0.39 | 4.59 | 1.45 |
| 6.2 | ECB | 1.00 | 3.07 | 0.03 |
| | CFB | 0.77 | 3.76 | 0.96 |
| | CBC | 0.06 | 2.16 | 0.25 |
| | CBCP | 0.10 | 2.30 | 0.44 |
| | CFBV | 0.47 | 2.68 | -0.42 |
| 6.3 | ECB | 4.00 (**) | 105.60 (**) | 10.11 (**) |
| | CFB | 0.56 | 2.74 | 0.46 |
| | CBC | 1.56 | 3.92 | 0.68 |
| | CBCP | 0.56 | 8.11 (**) | -2.36 (**) |
| | CFBV | 0.19 | 4.38 | -1.50 (*) |
| 6.4 | ECB | 4.00 (**) | 9.76 (**) | -1.88 (*) |
| | CFB | 2.44 | 4.63 | -0.49 |
| | CBC | 4.52 (**) | 8.00 (**) | -1.05 |
| | CBCP | 0.04 | 3.09 | -1.00 |
| | CFBV | 0.02 | 3.36 | -1.19 |
| 6.5 | ECB | 36.00 (**) | 53.46 (**) | -2.98 (**) |
| | CFB | 1.56 | 3.85 | -0.08 |
| | CBC | 2.25 | 5.17 | -0.87 |
| | CBCP | 0.39 | 2.48 | 0.20 |
| | CFBV | 4.25 (**) | 7.00 (**) | -0.68 |

### A 7.2.1 Number of sequences classified as non-random by each of the three tests

Let the number of sequences classified as non-random by Frequency test be nF

Let the number of sequences classified as non-random by Serial test be nS

Let the number of sequences classified as non-random by Runs test be nR

| Mode | nF | nS | nR |
|---|---|---|---|
| ECB | 12 | 16 | 7 |
| CFB | 1 | 4 | 0 |
| CBC | 3 | 4 | 1 |
| CBCP | 0 | 3 | 2 |
| CFBV | 2 | 4 | 1 |

### A 7.2.2 Number of sequences classified as non-random by more than one test

| No. | Mode | Input Sample | Tests which indicate non-randomness |
|---|---|---|---|
| 1 | ECB | 1.2 | F, S |
| 2 | | 1.3 | F, S |
| 3 | | 1.4 | F, S |
| 4 | | 1.5 | F, S |
| 5 | | 2.1 | F, S |
| 6 | | 2.3 | F, S |
| 7 | | 2.4 | F, S, R |
| 8 | | 2.5 | S, R |
| 9 | | 3.4 | S, R |
| 10 | | 3.5 | F, S |
| 11 | | 4.2 | S, R |
| 12 | | 5.4 | S, R |
| 13 | | 6.1 | F, S |
| 14 | | 6.3 | F, S, R |
| 15 | | 6.4 | F, S |
| 16 | | 6.5 | F, S, R |
| 17 | CBC | 1.5 | F, S |
| 18 | | 2.5 | S, R |

| No. | Mode | Input Sample | Tests which indicate non-randomness |
|---|---|---|---|
| 19 | | 3.2 | F, S |
| 20 | | 6.4 | F, S |
| 21 | CBCP | 3.5 | S, R |
| 22 | | 6.3 | S, R |
| 23 | CFBV | 1.1 | S, R |
| 24 | | 2.2 | F, S |
| 25 | | 6.5 | F, S |

## A 7.3 Results of Frequency, Serial and Runs tests on Ciphertext sequences; using different keys

5% Significant levels : Frequency (F) test : 3.84

Serial (S) test : 5.99

Runs (R) test : 1.96

| Sample No. | Input Sample |
|---|---|
| (i) (1.5) | 10111 |
| (ii) (2.1) | 00000001 |
| (iii) (3.3) | 0000011111 |
| (iv) (4.1) | FFFFC (hex) |
| (v) (5.2) | CE3C3E0FC0 (hex) |
| (vi) (6.5) | 3141592654138547 (hex) |

| Input Sample No. | Mode | Key | F | S | R |
|---|---|---|---|---|---|
| (i) | ECB | I | 0.004 | 3.07 | 1.00 |
| | | II | 1.56 | 4.26 | -0.71 |
| | | III | 7.22 (**) | 12.03 (**) | 1.93 (*) |
| | | IV | 0.25 | 8.32 (**) | 2.45 (**) |
| | | V | 4.25 | 6.18 (**) | -0.12 |
| (i) | CFB | I | 1.13 | 3.28 | 0.29 |
| | | II | 4.00 (**) | 6.95 (**) | -0.82 |
| | | III | 0.004 | 2.23 | 0.44 |
| | | IV | 4.25 (**) | 6.56 (**) | -0.56 |
| | | V | 2.64 | 4.87 | 0.40 |

| Input<br>Sample No. | Mode | Key | F | S | R |
|---|---|---|---|---|---|
| (i) | CBC | I | 2.25 | 5.10 | 1.01 |
| | | II | 0.06 | 3.53 | -1.25 |
| | | III | 1.89 | 4.10 | -0.32 |
| | | IV | 2.25 | 4.35 | 0.07 |
| | | V | 7.22 (**) | 9.69 (**) | -0.34 |
| (i) | CBCP | I | 0.06 | 2.90 | -0.94 |
| | | II | 1.13 | 3.68 | 0.79 |
| | | III | 0.56 | 2.78 | 0.39 |
| | | IV | 1.27 | 3.50 | 0.42 |
| | | V | 0.04 | 4.79 | 1.63 (*) |
| (i) | CFBV | I | 0.77 | 2.93 | 0.46 |
| | | II | 0.88 | 8.73 (**) | -2.41 (**) |
| | | III | 1.72 | 4.00 | 0.62 |
| | | IV | 0.47 | 3.50 | -1.05 |
| | | V | 2.44 | 4.35 | 0.01 |
| (ii) | ECB | I | 25.00 (**) | 4.30 | -3.37 (**) |
| | | II | 1.00 | 39.35 (**) | 6.04 (**) |
| | | III | 4.00 (**) | 22.15 (**) | -3.96 (**) |
| | | IV | 2.07 | 9.19 (**) | -2.25 (**) |
| | | V | 9.00 (**) | 10.83 (**) | 0.28 |
| (ii) | CFB | I | 1.27 | 4.54 | -1.15 |
| | | II | 1.27 | 3.91 | -0.84 |
| | | III | 0.66 | 4.05 | -1.17 |
| | | IV | 0.10 | 2.24 | 0.32 |
| | | V | 2.25 | 4.35 | 0.07 |
| (ii) | CBC | I | 1.27 | 3.28 | -0.27 |
| | | II | 1.00 | 5.41 | 1.53 (*) |
| | | III | 0.56 | 5.36 | -1.67 (*) |
| | | IV | 0.47 | 4.41 | -1.42 |
| | | V | 3.52 (*) | 5.63 (*) | 0.55 |
| (ii) | CBCP | I | 4.00 (**) | 15.33 (**) | 3.14 (**) |
| | | II | 1.13 | 3.50 | -0.65 |
| | | III | 1.41 | 5.29 | 1.36 |
| | | IV | 0.19 | 3.81 | 1.26 |
| | | V | 0.32 | 5.75 (*) | -1.87 (*) |

| Input Sample No. | Mode | Key | F | S | R |
|---|---|---|---|---|---|
| (ii) | CFBV | I | 0.004 | 2.61 | 0.75 |
| | | II | 0.39 | 2.57 | 0.45 |
| | | III | 0.10 | 3.46 | -1.19 |
| | | IV | 3.06 | 4.98 | 0.22 |
| | | V | 0.88 | 5.48 | -1.60 (*) |
| (iii) | ECB | I | 1.27 | 3.20 | -0.02 |
| | | II | 3.29 | 10.33 (**) | -2.16 (**) |
| | | III | 6.25 (**) | 24.37 (**) | 4.22 (**) |
| | | IV | 2.25 | 4.35 | 0.07 |
| | | V | 1.72 | 4.58 | 0.99 |
| (iii) | CFB | I | 0.004 | 2.03 | 0.13 |
| | | II | 1.27 | 3.21 | -0.09 |
| | | III | 0.04 | 3.85 | 1.31 |
| | | IV | 1.41 | 8.00 (**) | -2.15 (**) |
| | | V | 0.004 | 2.52 | 0.69 |
| (iii) | CBC | I | 1.27 | 5.01 | -1.34 |
| | | II | 1.72 | 3.89 | 0.30 |
| | | III | 0.004 | 2.44 | -0.69 |
| | | IV | 1.41 | 3.92 | -0.65 |
| | | V | 1.00 | 3.22 | 0.53 |
| (iii) | CBCP | I | 1.27 | 3.32 | -0.34 |
| | | II | 0.32 | 2.31 | 0.14 |
| | | III | 0.39 | 2.40 | -0.24 |
| | | IV | 0.06 | 3.69 | -1.31 |
| | | V | 0.004 | 3.99 | 1.38 |
| (iii) | CFBV | I | 2.44 | 4.47 | 0.39 |
| | | II | 1.27 | 4.83 | -1.21 |
| | | III | 0.14 | 2.14 | -0.18 |
| | | IV | 2.25 | 4.63 | -0.49 |
| | | V | 0.04 | 2.48 | -0.69 |
| (iv) | ECB | I | 0.004 | 2.05 | 0.19 |
| | | II | 5.94 (**) | 11.06 (**) | -1.57 (*) |
| | | III | 0.04 | 7.52 (**) | 2.32 (**) |
| | | IV | 0.56 | 3.68 | 1.02 |
| | | V | 0.004 | 2.62 | 0.75 |

| Input<br>Sample No. | Mode | Key | F | S | R |
|---|---|---|---|---|---|
| (iv) | CFB | I | 1.41 | 4.10 | 0.80 |
| | | II | 1.89 | 6.94 (**) | 1.75 (*) |
| | | III | 1.13 | 3.14 | 0.29 |
| | | IV | 0.56 | 2.61 | -0.05 |
| | | V | 1.89 | 7.45 (**) | 1.94 (*) |
| (iv) | CBC | I | 1.41 | 4.83 | -1.21 |
| | | II | 0.04 | 4.59 | -1.63 (*) |
| | | III | 0.32 | 3.07 | -0.87 |
| | | IV | 2.07 | 4.26 | -0.50 |
| | | V | 0.56 | 2.68 | -0.42 |
| (iv) | CBCP | I | 0.47 | 2.44 | 0.08 |
| | | II | 0.19 | 2.22 | 0.006 |
| | | III | 0.06 | 2.90 | -0.94 |
| | | IV | 0.77 | 4.02 | 1.09 |
| | | V | 1.13 | 3.71 | 0.72 |
| (iv) | CFBV | I | 4.00 (**) | 5.89 (*) | 0.19 |
| | | II | 0.56 | 2.68 | 0.39 |
| | | III | 0.10 | 2.14 | -0.18 |
| | | IV | 0.56 | 3.95 | 1.14 |
| | | V | 0.66 | 2.88 | -0.42 |
| (v) | ECB | I | 1.56 | 4.43 | 1.00 |
| | | II | 2.07 | 4.87 | 0.88 |
| | | III | 4.52 (**) | 6.91 (**) | -0.61 |
| | | IV | 2.07 | 9.19 (**) | -2.25 (**) |
| | | V | 0.56 | 5.69 (*) | -1.80 (*) |
| (v) | CFB | I | 5.35 (**) | 9.02 (**) | 1.49 |
| | | II | 0.06 | 2.90 | 0.88 |
| | | III | 0.39 | 3.37 | 0.95 |
| | | IV | 0.10 | 3.57 | -1.25 |
| | | V | 0.02 | 2.00 | -0.06 |

| Input<br>Sample No. | Mode | Key | F | S | R |
|---|---|---|---|---|---|
| (v) | CBC | I | 1.13 | 3.25 | -0.22 |
| | | II | 2.64 | 6.73 (**) | 1.46 |
| | | III | 0.77 | 3.07 | 0.59 |
| | | IV | 1.00 | 3.14 | 0.28 |
| | | V | 0.88 | 4.80 | -1.41 |
| (v) | CBCP | I | 4.25 (**) | 6.74 (**) | 0.70 |
| | | II | 4.52 (**) | 6.66 (**) | 0.14 |
| | | III | 0.66 | 2.66 | -0.23 |
| | | IV | 1.89 | 5.18 | 1.12 |
| | | V | 0.02 | 3.65 | 1.25 |
| (v) | CFBV | I | 2.64 | 4.76 | 0.52 |
| | | II | 0.04 | 2.07 | 0.19 |
| | | III | 0.88 | 6.58 (**) | -1.91 (*) |
| | | IV | 0.14 | 2.45 | 0.51 |
| | | V | 0.47 | 2.63 | 0.328 |
| (vi) | ECB | I | 9.00 (**) | 14.96 (**) | 2.30 (**) |
| | | II | 1.00 | 7.07 (**) | 2.03 (**) |
| | | III | 16.00 (**) | 53.94 (**) | -5.59 (**) |
| | | IV | 64.00 (**) | 65.63 (**) | 2.07 (**) |
| | | V | 36.00 (**) | 53.46 (**) | -2.98 (**) |
| (vi) | CFB | I | 0.14 | 2.33 | -0.43 |
| | | II | 0.06 | 2.36 | -0.56 |
| | | III | 0.10 | 2.14 | 0.13 |
| | | IV | 0.47 | 2.71 | -0.55 |
| | | V | 1.56 | 3.85 | -0.08 |
| (vi) | CBC | I | 5.94 (**) | 11.28 (**) | 1.95 (*) |
| | | II | 1.89 | 4.26 | 0.56 |
| | | III | 1.89 | 3.47 | -0.38 |
| | | IV | 1.00 | 2.96 | -0.16 |
| | | V | 2.25 | 5.17 | -0.87 |
| (vi) | CBCP | I | 30.25 (**) | 81.45 (**) | -6.31 (**) |
| | | II | 0.77 | 6.70 (**) | -1.98 (**) |
| | | III | 3.16 | 2.36 | -0.30 |
| | | IV | 0.04 | 3.39 | -1.19 |
| | | V | 0.39 | 2.48 | 0.20 |

| Input Sample No. | Mode | Key | F | S | R |
|---|---|---|---|---|---|
| (vi) | CFBV | I | 0.77 | 2.74 | 0.15 |
| | | II | 0.04 | 2.03 | 0.06 |
| | | III | 0.14 | 2.28 | 0.32 |
| | | IV | 0.04 | 3.83 | -1.38 |
| | | V | 4.25 (**) | 7.00 (**) | -0.68 |

### A 7.3.1 Number of sequences classified as non-random by each of the three tests

Let the number of sequences classified as non-random by Frequency test be nF

Let the number of sequences classified as non-random by Serial test be nS

Let the number of sequences classified as non-random by Runs test be nR

| Key | nF | nS | nR |
|---|---|---|---|
| I | 8 | 6 | 4 |
| II | 3 | 10 | 5 |
| III | 5 | 7 | 4 |
| IV | 2 | 6 | 5 |
| V | 4 | 6 | 1 |

### A 7.3.2 Number of sequences classified as non-random by more than one test

| Key | Input Sample | Mode | Tests which indicate non-randomness |
|---|---|---|---|
| I | (ii) | ECB | F,R |
| | (ii) | CBCP | F, S, R |
| | (v) | CFB | F, S |
| | (v) | CBCP | F, S |
| | (vi) | ECB | F, S, R |
| | (vi) | CBC | F, S |
| | (vi) | CBCP | F, S, R |

| Key | Input Sample | Mode | Tests which indicate non-randomness |
|---|---|---|---|
| II | (i) | CFB | F, S |
| | (ii) | ECB | S, R |
| | (iii) | ECB | S, R |
| | (iv) | ECB | F, S |
| | (v) | CBCP | F, S |
| | (vi) | ECB | S, R |
| | (vi) | CBCP | S, R |
| III | (i) | ECB | F, S |
| | (ii) | ECB | F, S, R |
| | (iii) | ECB | F, S, R |
| | (iv) | ECB | S, R |
| | (v) | ECB | F, S |
| | (vi) | ECB | F, S, R |
| IV | (i) | ECB | S, R |
| | (i) | CFB | F, S |
| | (ii) | ECB | S, R |
| | (iii) | CFB | S, R |
| | (v) | ECB | S, R |
| | (vi) | ECB | F, S, R |
| V | (i) | CBC | F, S |
| | (ii) | ECB | F, S |
| | (vi) | ECB | F, S, R |
| | (vi) | CFBV | F, S |

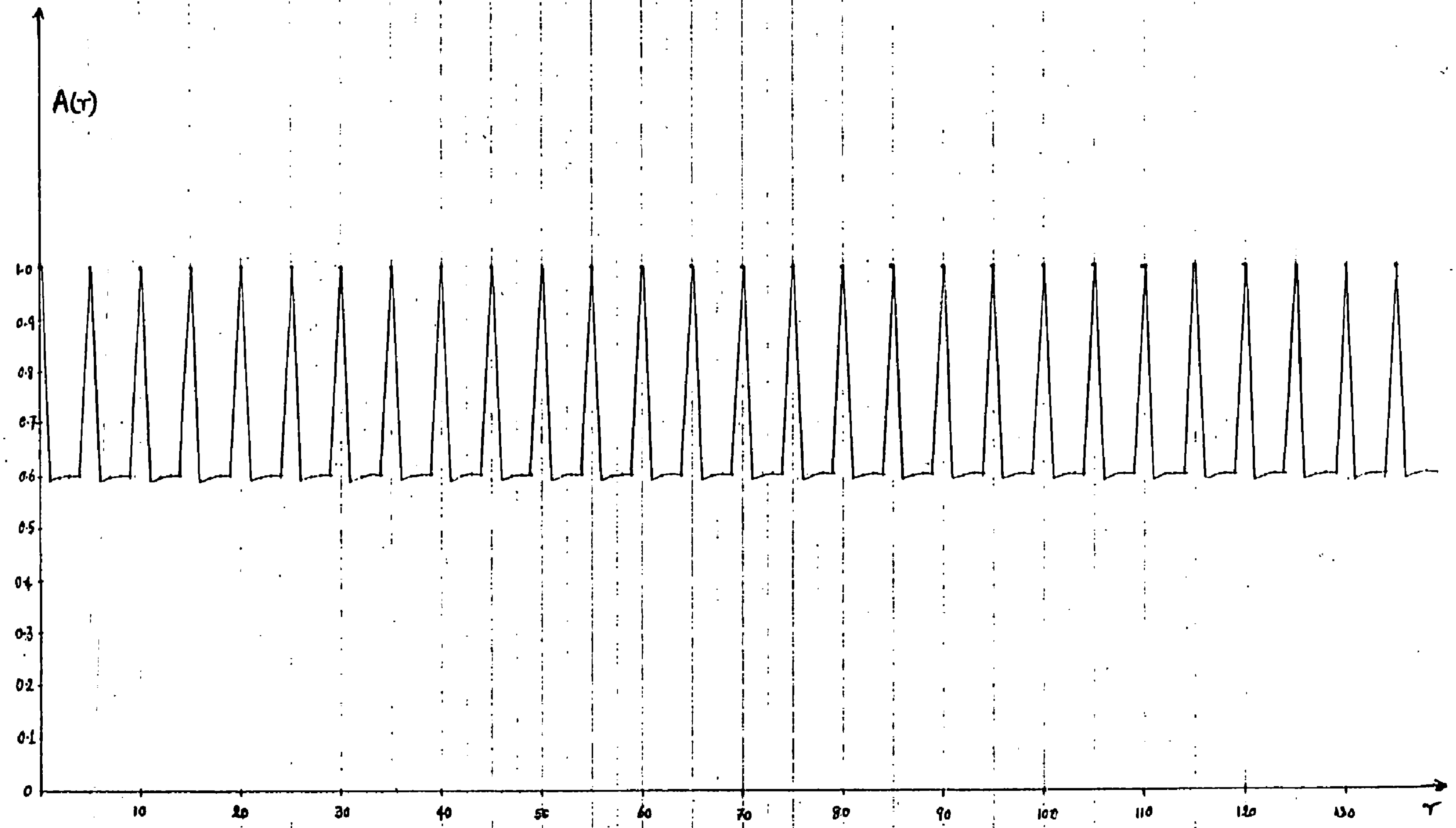
## A 7.4 Autocorrelation graphs (See pages A-65a to A-65f)

### A 7.4.1 Autocorrelation test : a measure of randomness

Let M be equal to the number of points which lie beyond the 5% significance level calculated using normal approximation (section 6.2.4) Total number of points = 400.

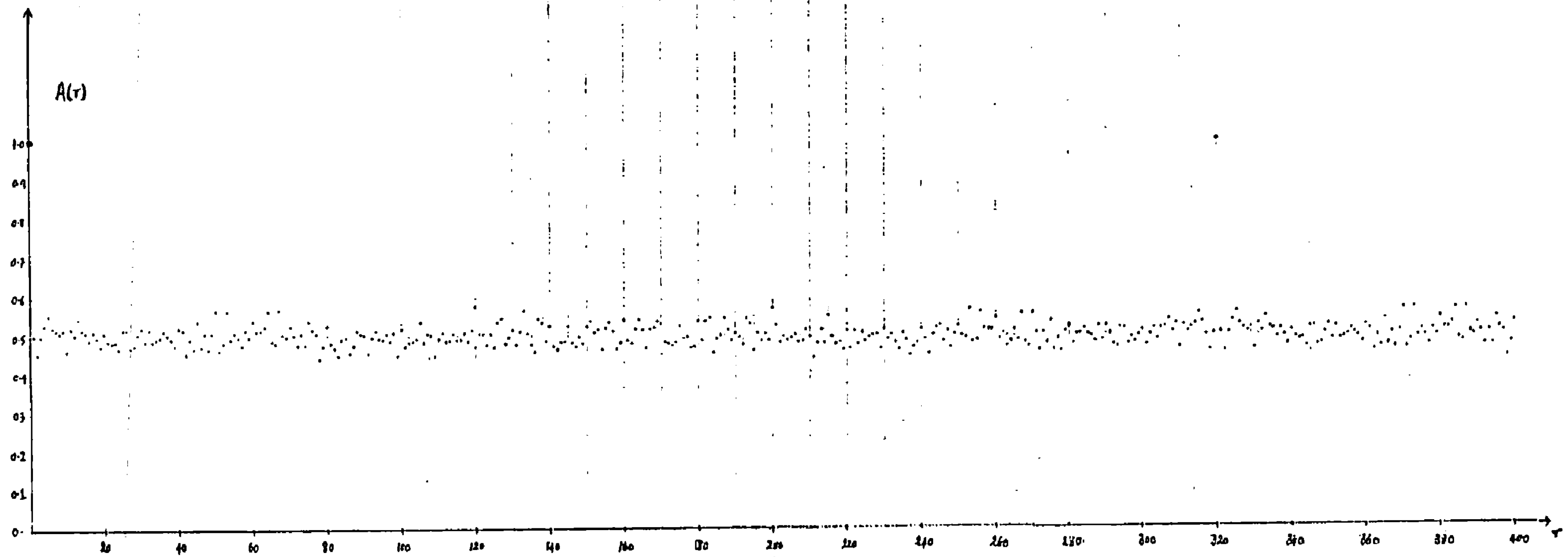
| Key | Input Sample | Mode | M/400 |
|---|---|---|---|
| I | (i) | ECB | 76/400 |
| | | CFB | 14/400 |
| | | CBC | 21/400 |
| | | CBCP | 18/400 |
| | | CFBV | 15/400 |

(Continued on page A-66)



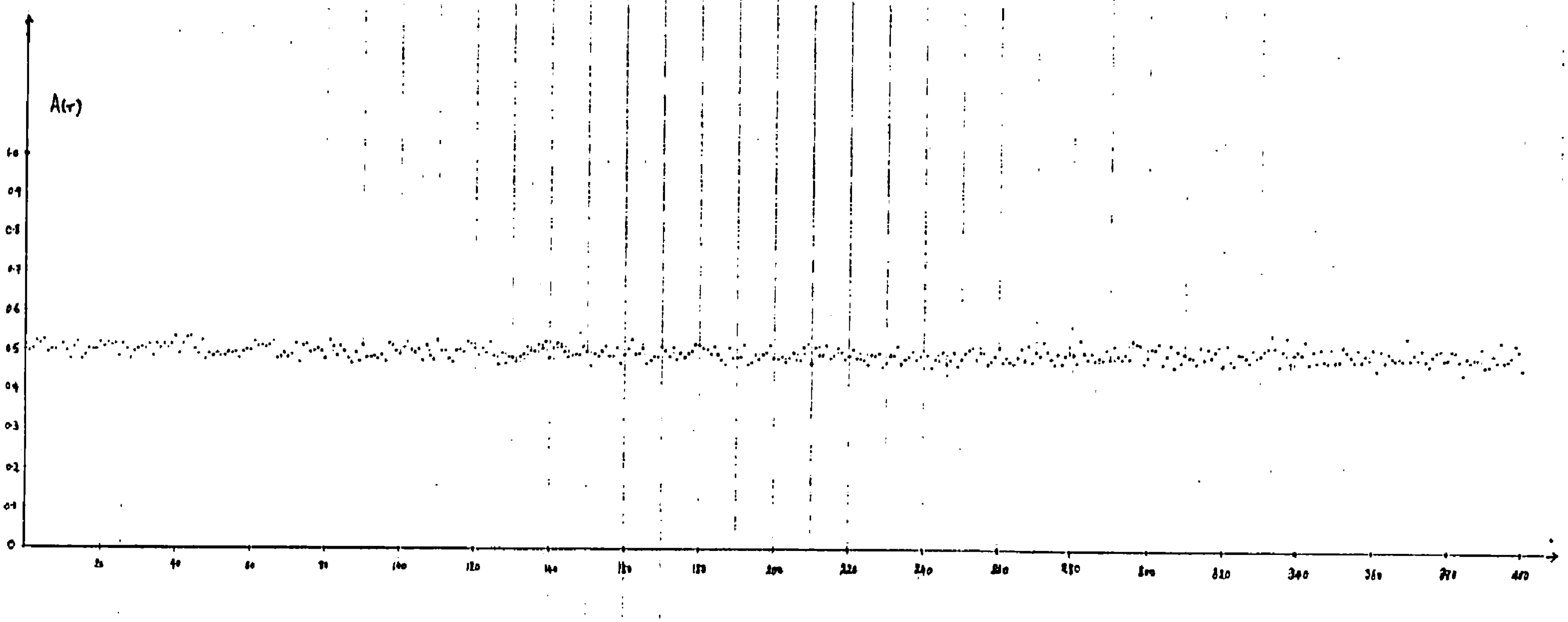
A 7.4 (a) Autocorrelation function of input sample (i)

A-65a



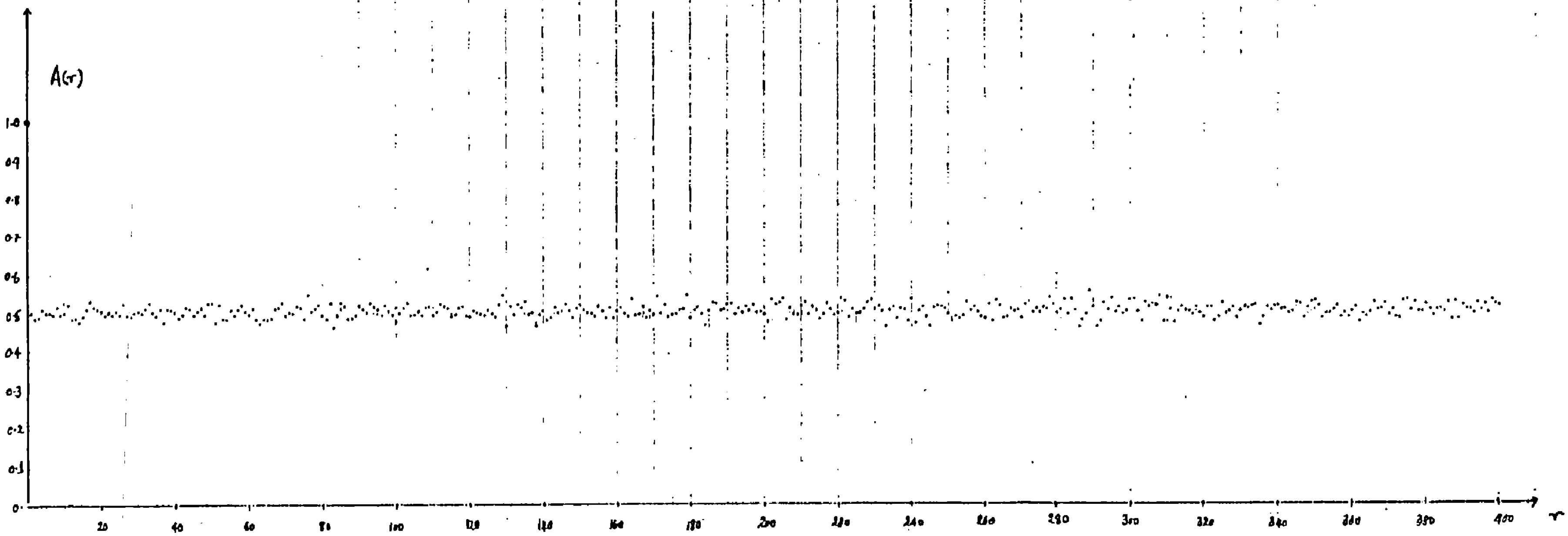
A 7.4 (b) Autocorrelation of ECB output sequence of input sample (1)

A-65b



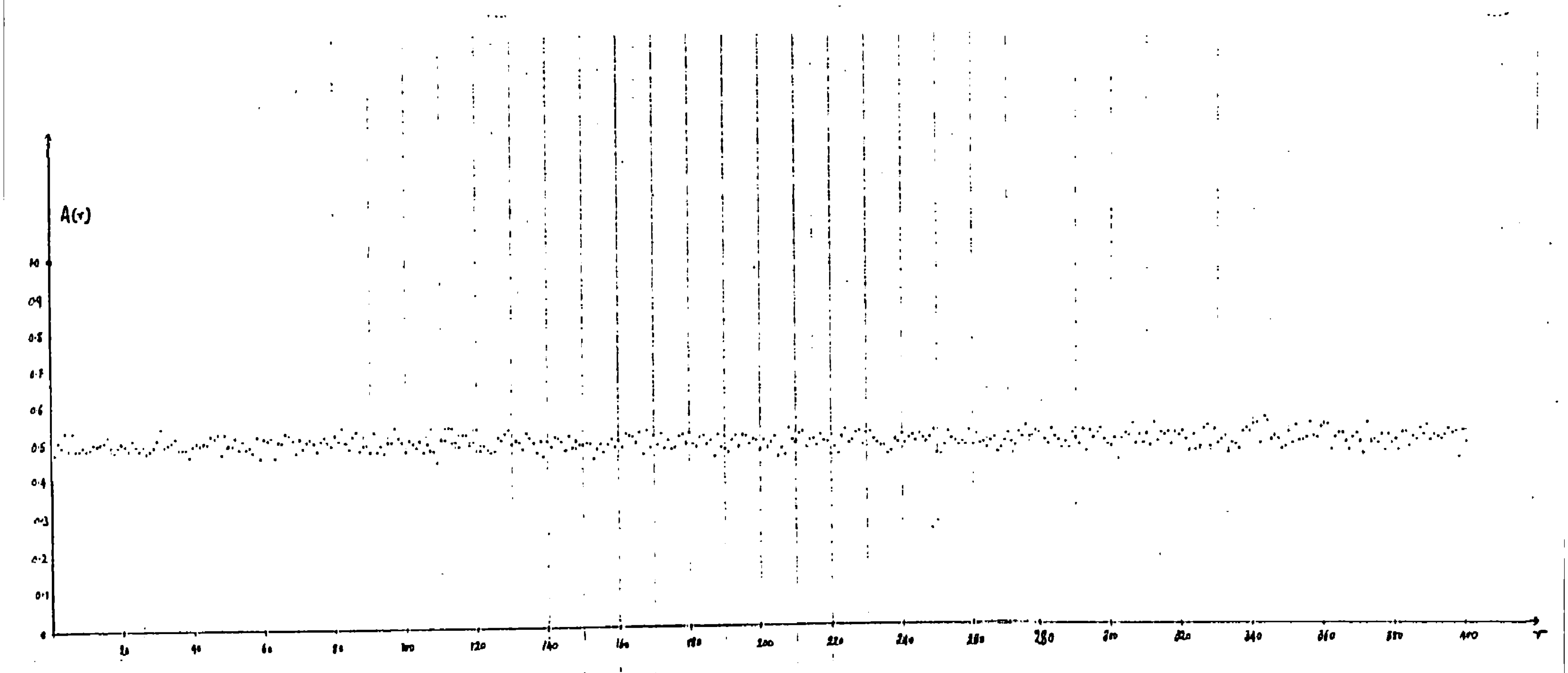
A 7.4 (c) Autocorrelation function of CFB output sequence of input sample (i)

A-65c



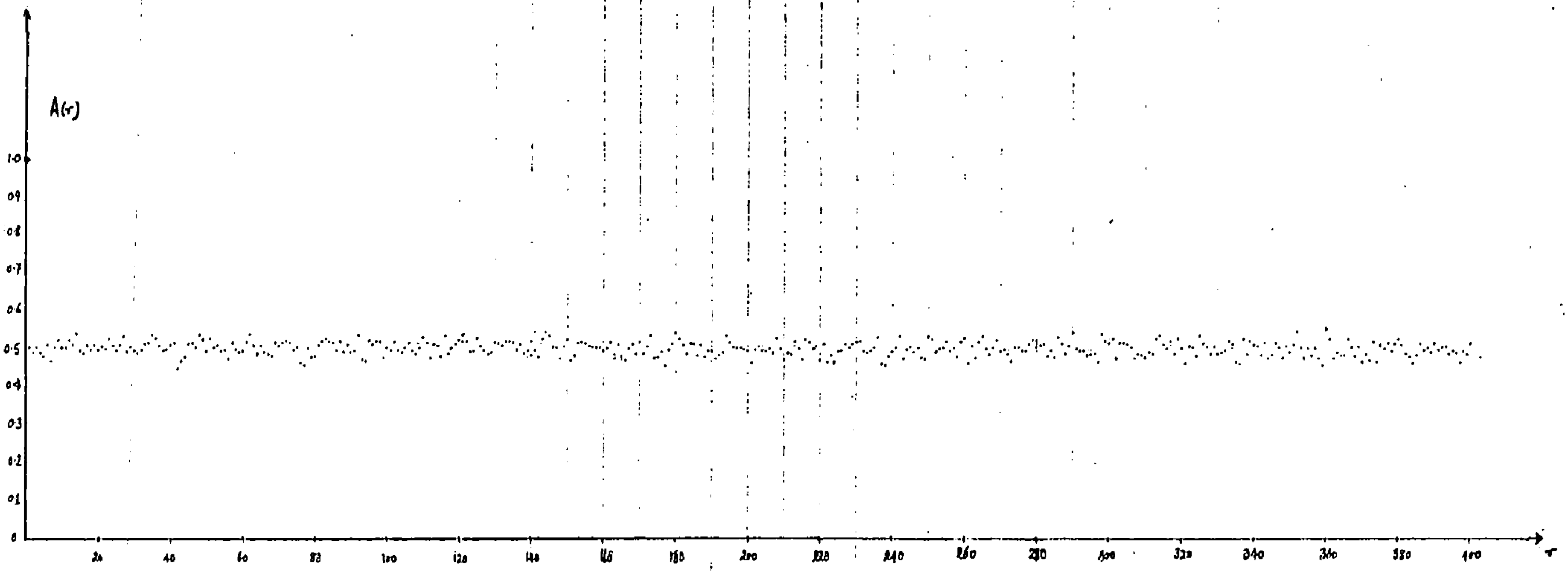
A 7.4 (d) Autocorrolation function of CBC output sequence of input sample (i)

A-65d



A 7.4 (c) Autocorrelation function of CBCP output sequence of input sample (1)

A-650



A 7.4 (f) Autocorrelation function of CFBV output sequence of input sample (1)

A-65f

| Key | Input Sample | Mode | M/400 |
|---|---|---|---|
| I | (iii) | ECB | 65/400 |
| | | CFB | 17/400 |
| | | CBC | 13/400 |
| | | CBCP | 10/400 |
| | | CFBV | 21/400 |
| V | (i) | ECB | 76/400 |
| | | CFB | 10/400 |
| | | CBC | 18/400 |
| | | CBCP | 37/400 |
| | | CFBV | 29/400 |
| | (iii) | ECB | 77/400 |
| | | CFB | 17/400 |
| | | CBC | 20/400 |
| | | CBCP | 22/400 |
| | | CFBV | 21/400 |

### A 7.5 Results of Frequency, Serial and Runs tests on intermediate DES output sequences

| 5% Significant levels : | Frequency | (F) test | : 3.84 |
|---|---|---|---|
| | Serial | (S) test | : 5.99 |
| | Runs | (R) test | : 1.96 |

(a) Plaintext : FFFFFFFFFFFFFFFF (hex)
Key : 1F1F1F1F0E0E0E0E (hex) - ('Weak')

| Round No. | F | S | R |
|---|---|---|---|
| 1 | 14.06 | 30.80 | -2.91 |
| 2 | 0.06 | 2.40 | 0.51 |
| 3 | 3.06 | 6.01 | 0.94 |
| 4 | 2.25 | 4.72 | 0.29 |
| 5 | 0.25 | 2.40 | 0.03 |
| 6 | 0.06 | 2.40 | 0.51 |
| 7 | 0.25 | 2.15 | -0.22 |
| 8 | 0.06 | 3.41 | -1.25 |
| 9 | 0.00 | 2.78 | -1.01 |
| 10 | 1.56 | 4.08 | -0.31 |

| Round No. | F | S | R |
|---|---|---|---|
| 11 | 1.00 | 3.31 | 0.13 |
| 12 | 0.25 | 3.16 | 0.79 |
| 13 | 0.00 | 2.02 | 0.00 |
| 14 | 3.06 | 5.99 | -0.39 |
| 15 | 2.25 | 4.71 | -0.75 |
| 16 | 0.06 | 2.02 | 0.01 |
| Ciphertext | 0.06 | 2.15 | 0.01 |

(b) Plaintext : FFFFFFFFFFFFFFFF (hex)
Key : 85CDCB1C9BD0461A (hex)

| Round No. | F | S | R |
|---|---|---|---|
| 1 | 18.06 | 46.26 | -4.23 |
| 2 | 0.00 | 2.02 | -0.25 |
| 3 | 1.56 | 7.88 | 2.27 |
| 4 | 2.25 | 4.72 | 1.08 |
| 5 | 1.00 | 3.31 | 0.13 |
| 6 | 0.25 | 2.27 | -0.47 |
| 7 | 0.56 | 2.92 | -0.44 |
| 8 | 0.56 | 2.78 | -0.18 |
| 9 | 0.25 | 2.40 | 0.03 |
| 10 | 0.06 | 2.40 | 0.51 |
| 11 | 0.25 | 3.16 | 0.79 |
| 12 | 0.06 | 2.02 | 0.01 |
| 13 | 0.06 | 2.15 | 0.01 |
| 14 | 0.25 | 3.42 | 1.04 |
| 15 | 0.25 | 3.42 | 1.04 |
| 16 | 0.00 | 2.02 | 0.00 |
| Ciphertext | 0.25 | 6.72 | 2.06 |

(c) Plaintext : FFFFFFFFFFFFFFFE (hex)
Key : 3131313131313131 (hex)

| Round No. | F | S | R |
|---|---|---|---|
| 1 | 10.56 | 27.31 | -3.25 |
| 2 | 0.25 | 2.27 | -0.47 |
| 3 | 0.56 | 3.67 | 1.09 |
| 4 | 1.56 | 4.70 | 0.98 |
| 5 | 1.56 | 10.93 | -2.64 |

| Round No. | F | S | R |
|---|---|---|---|
| 6 | 0.06 | 4.69 | -1.76 |
| 7 | 0.56 | 3.17 | 0.83 |
| 8 | 4.00 | 5.63 | 0.54 |
| 9 | 5.06 | 7.95 | -0.68 |
| 10 | 3.06 | 4.74 | 0.14 |
| 11 | 1.56 | 4.07 | 0.46 |
| 12 | 2.25 | 5.98 | -1.01 |
| 13 | 1.56 | 5.21 | -1.35 |
| 14 | 0.00 | 6.59 | -2.27 |
| 15 | 1.56 | 6.61 | -1.61 |
| 16 | 2.25 | 4.72 | 0.29 |
| Ciphertext | 2.25 | 4.34 | -0.49 |

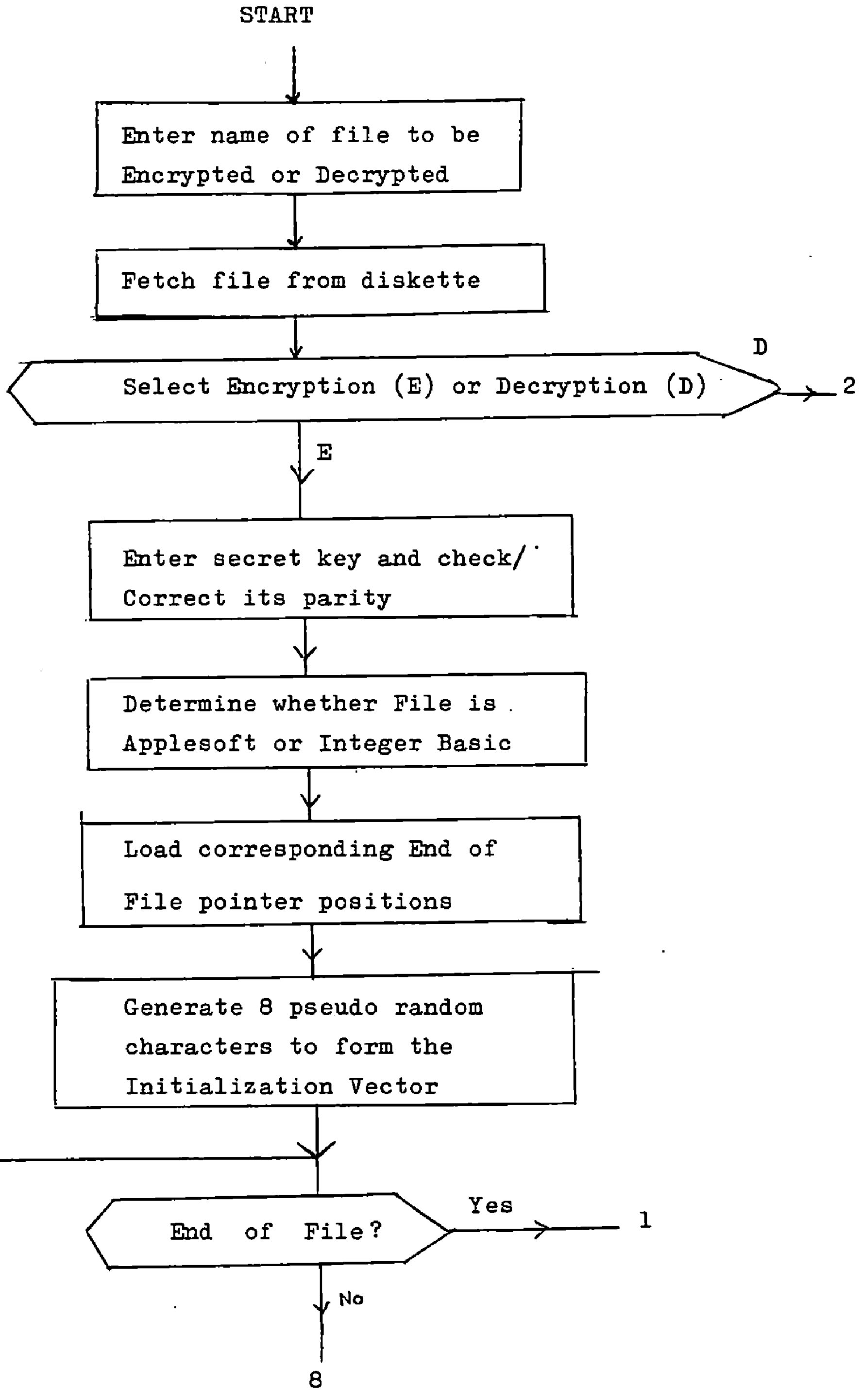
(d) Plaintext : 016319E1C3C04780 (hex)
Key : 3131313131313131 (hex)

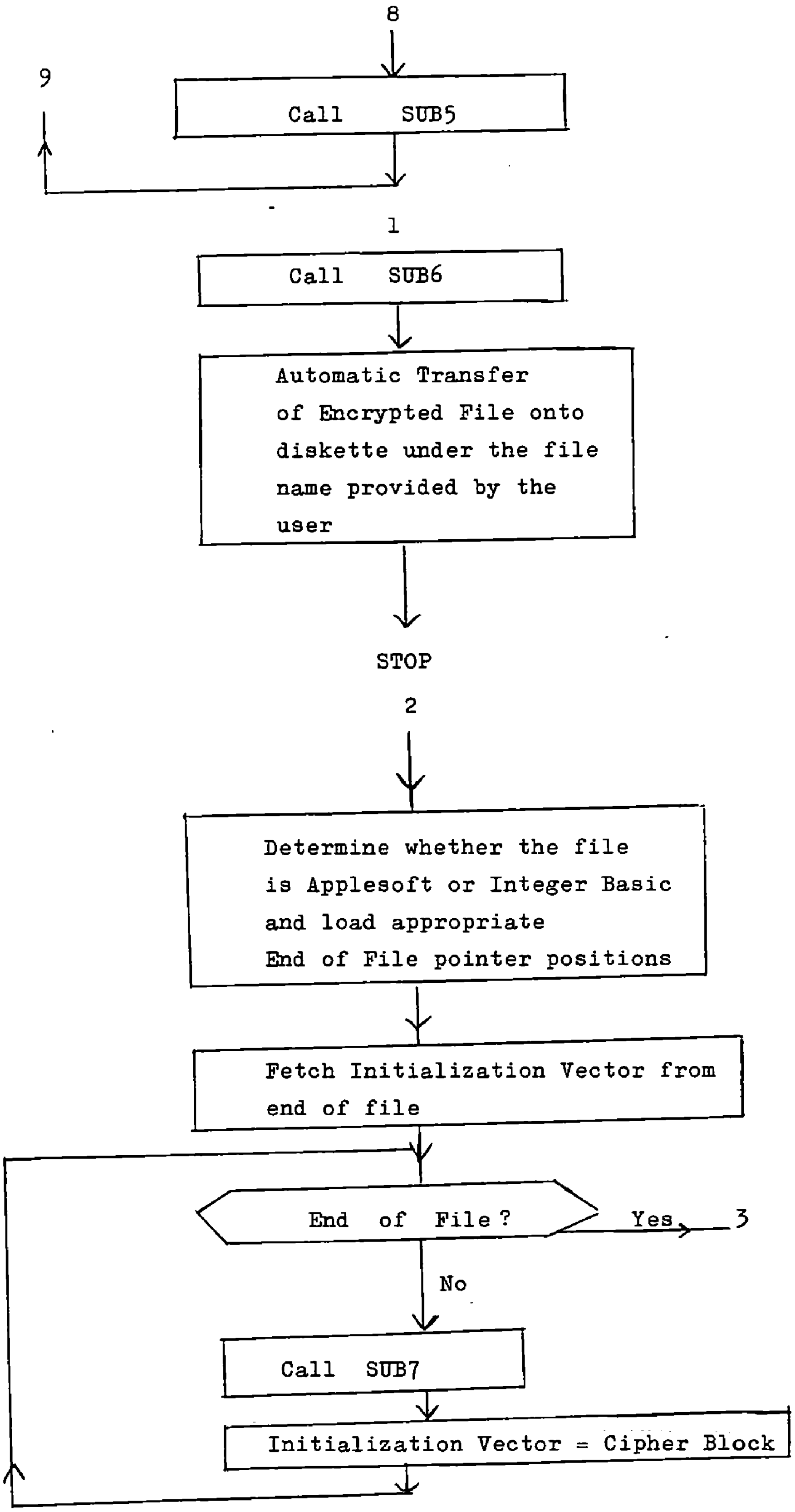
| Round No. | F | S | R |
|---|---|---|---|
| 1 | 1.56 | 4.32 | -0.58 |
| 2 | 0.25 | 2.27 | -0.47 |
| 3 | 0.25 | 2.53 | 0.54 |
| 4 | 2.25 | 5.09 | -0.49 |
| 5 | 1.00 | 2.79 | 0.13 |
| 6 | 1.00 | 4.06 | 1.15 |
| 7 | 0.56 | 3.55 | 0.83 |
| 8 | 0.06 | 2.02 | 7.88 |
| 9 | 0.56 | 2.78 | -0.18 |
| 10 | 2.25 | 4.34 | -0.49 |
| 11 | 0.25 | 2.78 | -0.73 |
| 12 | 3.06 | 4.72 | 0.41 |
| 13 | 0.25 | 3.42 | -1.23 |
| 14 | 0.25 | 2.27 | -0.47 |
| 15 | 0.00 | 3.29 | 1.01 |
| 16 | 1.56 | 3.43 | -0.31 |
| Ciphertext | 1.56 | 4.07 | 0.46 |

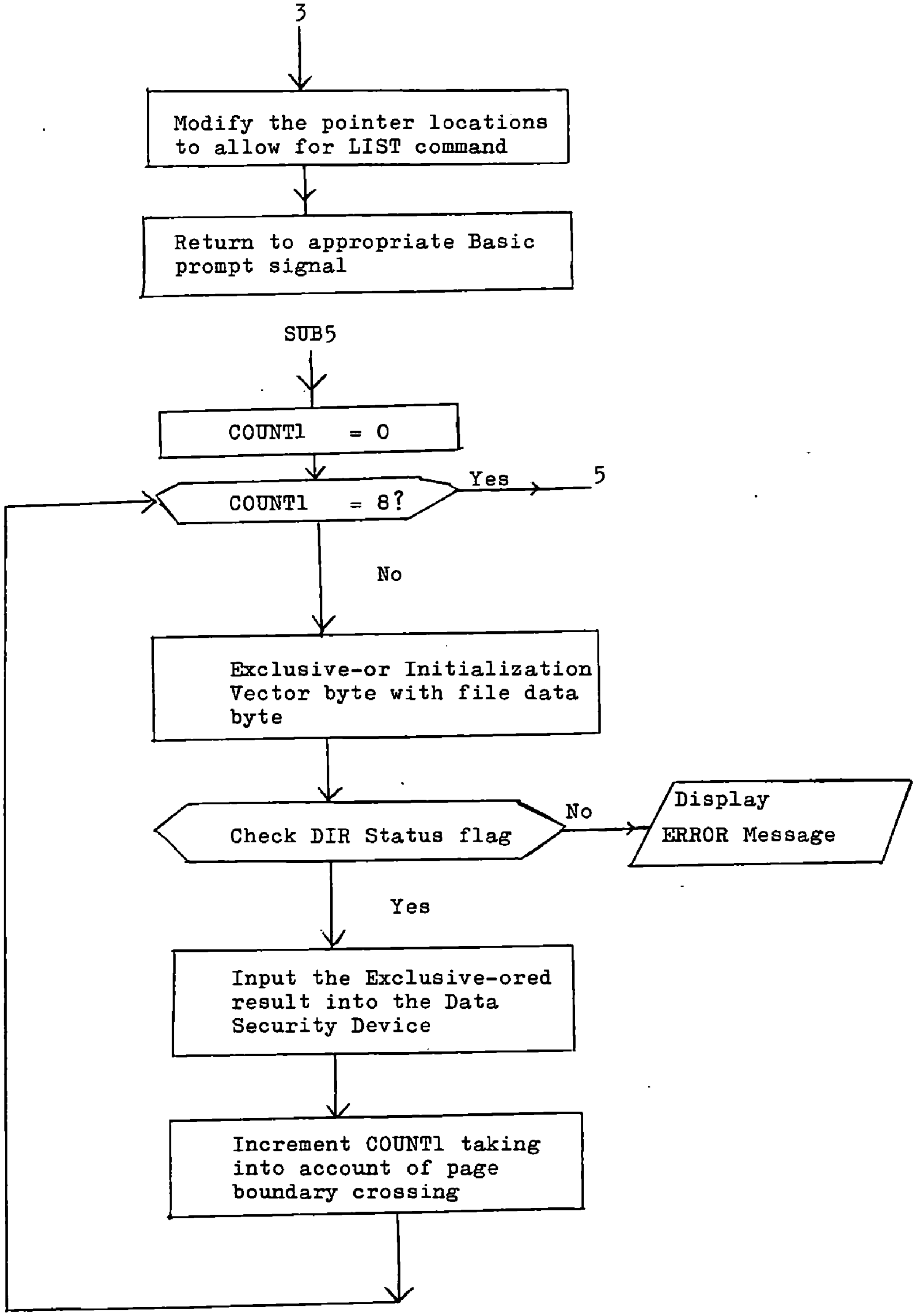
## Appendix 8

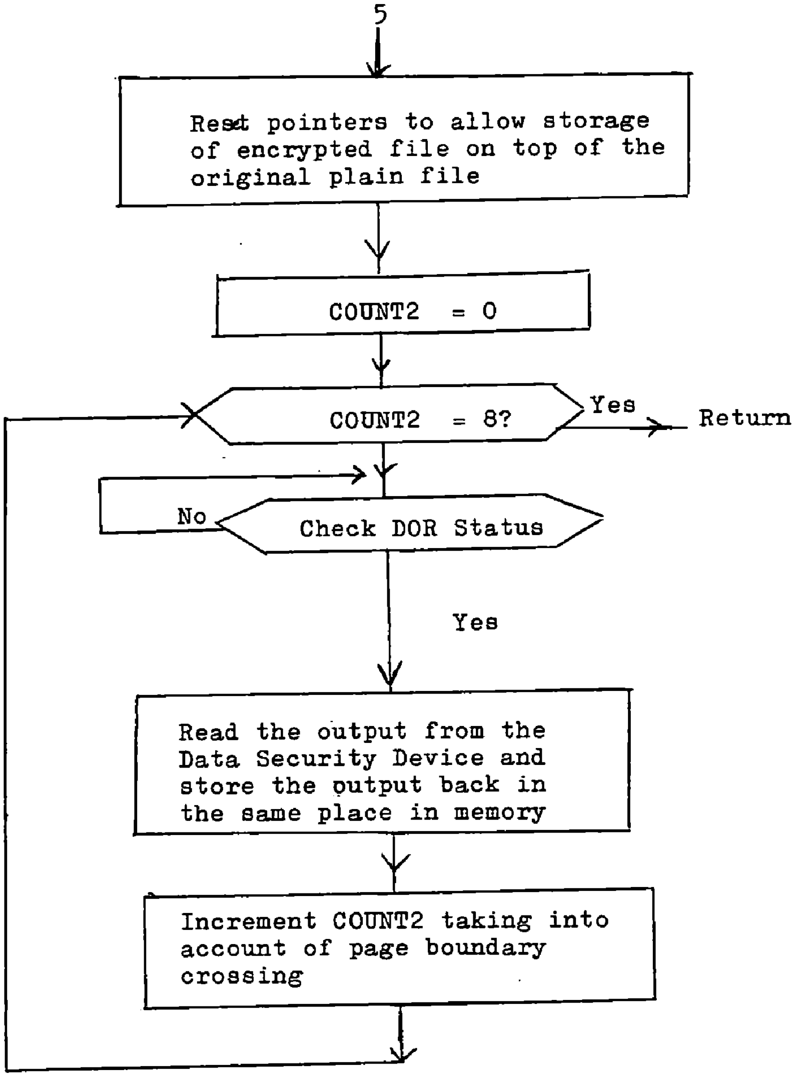
## File Security : Cipher Block Chaining Program (CBC)

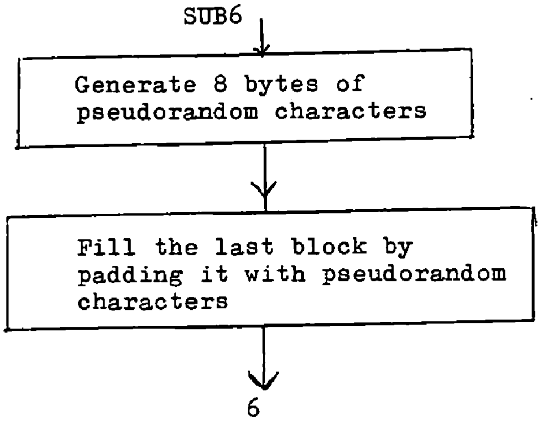
### A 8.1 Basic Flowchart

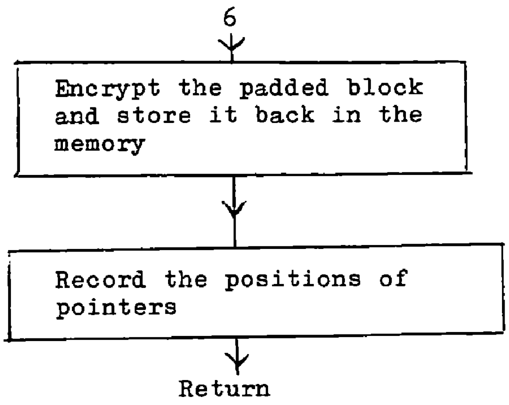

START
Enter name of file to be Encrypted or Decrypted
Fetch file from diskette
Select Encryption (E) or Decryption (D)
D
2
E
Enter secret key and check/ Correct its parity
Determine whether File is Applesoft or Integer Basic
Load corresponding End of File pointer positions
Generate 8 pseudo random characters to form the Initialization Vector
End of File?
Yes
1
No
9
8

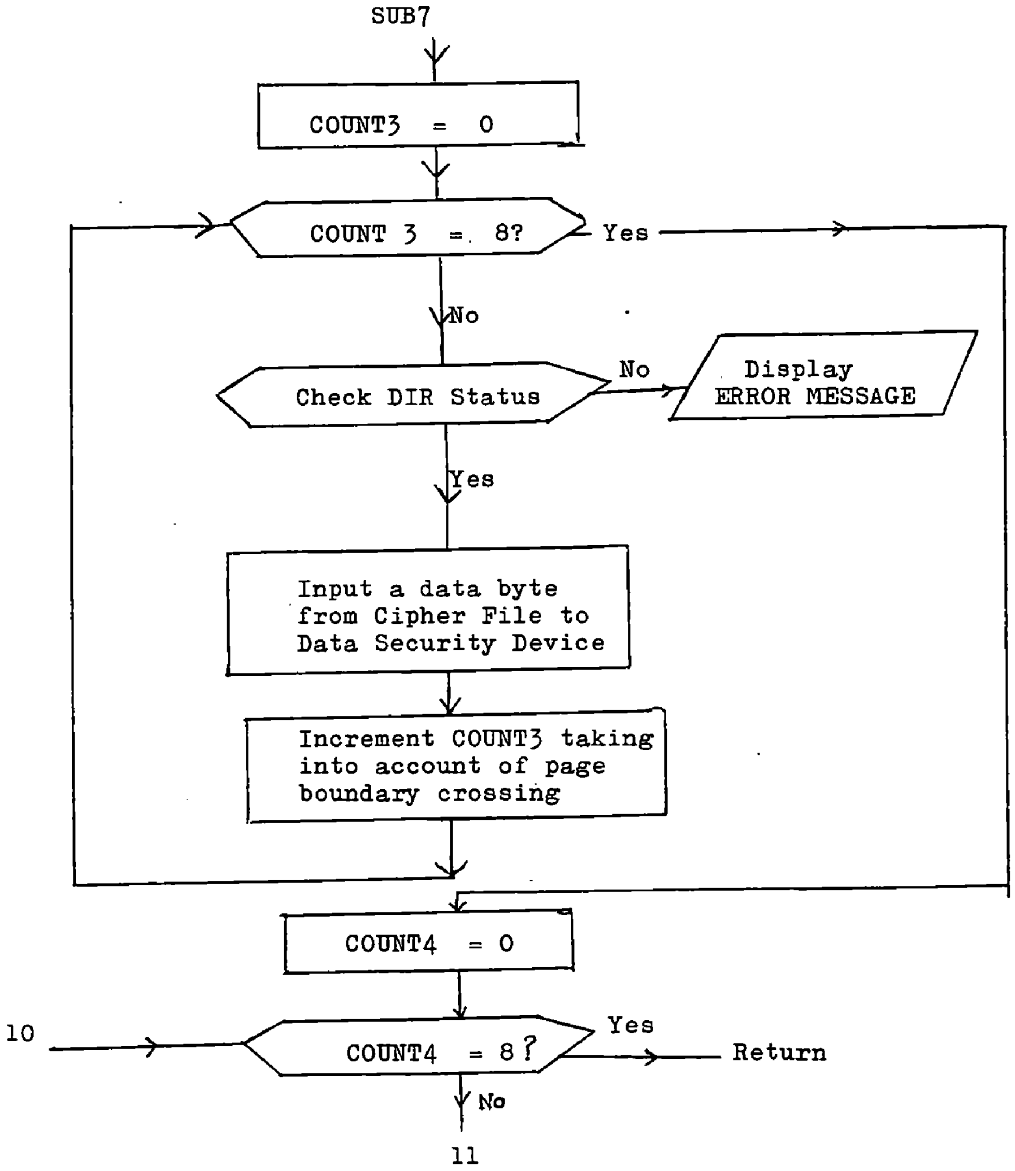

8
9
Call SUB5
1
Call SUB6
Automatic Transfer of Encrypted File onto diskette under the file name provided by the user
STOP
2
Determine whether the file is Applesoft or Integer Basic and load appropriate End of File pointer positions
Fetch Initialization Vector from end of file
End of File ?
Yes
3
No
Call SUB7
Initialization Vector = Cipher Block

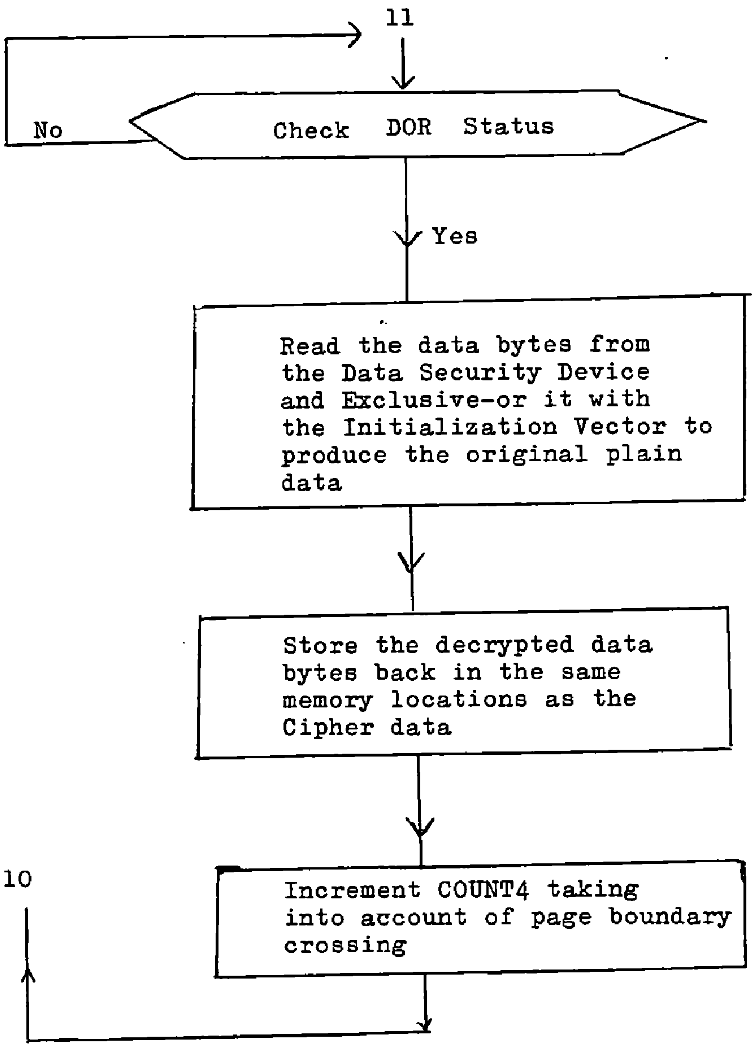

3
Modify the pointer locations to allow for LIST command
Return to appropriate Basic prompt signal
SUB5
COUNT1 = 0
COUNT1 = 8?
Yes
5
No
Exclusive-or Initialization Vector byte with file data byte
Check DIR Status flag
No
Display ERROR Message
Yes
Input the Exclusive-ored result into the Data Security Device
Increment COUNT1 taking into account of page boundary crossing


5
Reset pointers to allow storage of encrypted file on top of the original plain file
COUNT2 = 0
COUNT2 = 8?
Yes
Return
No
Check DOR Status
Yes
Read the output from the Data Security Device and store the output back in the same place in memory
Increment COUNT2 taking into account of page boundary crossing


SUB6
Generate 8 bytes of pseudorandom characters
Fill the last block by padding it with pseudorandom characters
6


6
Encrypt the padded block and store it back in the memory
Record the positions of pointers
Return


SUB7
COUNT3 = 0
COUNT 3 =. 8?
Yes
No
Check DIR Status
No
Display ERROR MESSAGE
Yes
Input a data byte from Cipher File to Data Security Device
Increment COUNT3 taking into account of page boundary crossing
COUNT4 = 0
10
COUNT4 = 8?
Yes
Return
No
11


11
No
Check DOR Status
Yes
Read the data bytes from the Data Security Device and Exclusive-or it with the Initialization Vector to produce the original plain data
Store the decrypted data bytes back in the same memory locations as the Cipher data
10
Increment COUNT4 taking into account of page boundary crossing

## A 8.2 _ _Program_Listing

### A 8.2.1 _ Main_Program

```
                                        Comments

5000   :   JSR  FILESECR                Displays 'File Security System'.
           LDA @ 00
           STA   06                     Locations 6000 to 6007 contain
           LDA @ 60                     the Initialization Vector.
           STA   07
           JSR  FILENAME                Displays 'Enter Filename Please'.
           JSR  LOAD                    Loads file into system memory
           JSR  ENCDEC                  Displays 'Encrypt (E) or Decrypt (D)'.
           JSR  DSD                     Initializes  the Data Security
                                        Device and requests and stores the
                                        secret key in the KEY Register
                                        of the device.
                                        Same subroutine as the one given
                                        in Appendix 2.
           LDA   FC
           CMP @ 00
           BEQ   L1                     L1 - Encryption
           JMP   L2                     L2 - Decryption
  L1       LDA AAB6
           BEQ   L3
           LDA   69
           STA   1B
           LDA   6A
           STA   1C
           JMP   L4
  L3       LDA   CA
           STA   1B
           LDA   CB
           STA   1C
  L4       JSR SUB5
           JSR SUB6
           JSR FILETRANSFER

  L2       LDA AAB6
           BEQ   L5
           LDA 07FE
           STA   1B
```

```
          STA    69
          LDA  07FF
          STA    1C
          STA    6A
          JMP    L6
L5        LDA  9600
          CMP    FF
          BNE    L7
          LDA    00
          STA    1B
          LDA  9601
          TAX
          INX
          STX    1C
          JMP    L6
L7        TAX
          INX
          STX    1B
          LDA  9601
          STA    1C

L6        JSR   SUB7
          JSR   PROMPT
```

### A 8.2.2 Subroutines

Some of the subroutines used by this program are the same as the ones used by the Block Encryption Program in Appendix 2. (for instance, subroutine DSD). Three additional subroutines namely a merged version of subroutines SUB5 and SUB7, SUB6 and FILETRANSFER are given in this section.

```
Merged SUB5 and SUB7  :    LDY @  00
              S22          LDX @  00
              S6           LDA    1B
                           STA    FC
                           LDA    19
                           CMP    FC
                           BNE    S1
                           LDA    1C
```

```
                    STA    FC
                    LDA    1A
                    CMP    FC
                    BNE    S1
                    RTS
S1                  CPX @  08
                    BEQ   S19
                    LDA  C0A3
                    AND @  40
                    BNE    S2
                    JMP  FF2D
S2                  LDA    08
                    CMP @  00
                    BEQ    S3
                    JMP    S4
S3                  LDA  (19),Y
                    EOR  6000,X
                    JMP    S5
S4                  LDA  (19),Y
                    STA  6010,X
S5                  STA  C0A0
                    INX
                    JSR   S10
                    JMP    S6
S19                 JSR   S18
                    JMP   S22
S10                 LDA  AAB6
                    BEQ    S7
                    LDA    19
                    CMP @  FF
                    BEQ    S8
                    INC    19
                    RTS
```

```
S8          LDA  @ 00
            STA    19
            INC    1A
            RTS

S7          LDA    19
            CMP  @ 00
            BEQ    S9
            DEC    19
            RTS

S9          LDA  @ FF
            STA    19
            DEC    1A
            RTS

S18         CPX  @ 00
            BEQ   S12
            LDA  AAB6
            BEQ   S13
            LDA    19
            CMP  @ 00
            BEQ   S14
            DEC    19
            JMP   S15

S14         LDA  @ FF
            STA    19
            DEC    1A
            JMP   S15

S13         LDA    19
            CMP  @ FF
            BEQ   S16
            INC    19
            JMP   S15

S16         LDA  @ 00
            STA    19
            INC    1A

S15         DEX
            JMP   S18
```

```
          S12              LDX  @ 00

          S21              CPX  @ 08
                           BNE   S23
                           RTS

          S23              LDA  COA3
                           BPL   S23
                           LDA    08
                           CMP  @ 00
                           BEQ   S24
                           LDA  COA2
                           EOR  6000,X
                           STA  (19),Y
                           LDA  6010,X
                           STA  6000,X
                           JMP   S20

          S24              LDA  COA2
                           STA  (19),Y
                           STA  6000,X

          S20              INX
                           JSR   S10
                           JMP   S21


SUB6      :                LDA @  00
                           STA     FC
                           CPX @  00
                           BEQ   S23

          S26              CPX @  08
                           BEQ   S24
                           LDA  (ID),Y
                           EOR  6000,X
                           STA  COAO
                           LDA  AAB6
                           BNE   S25
                           INC   FC

          S25              JSR   S10
                           INX
                           DEC    1D
                           JMP   S26
```

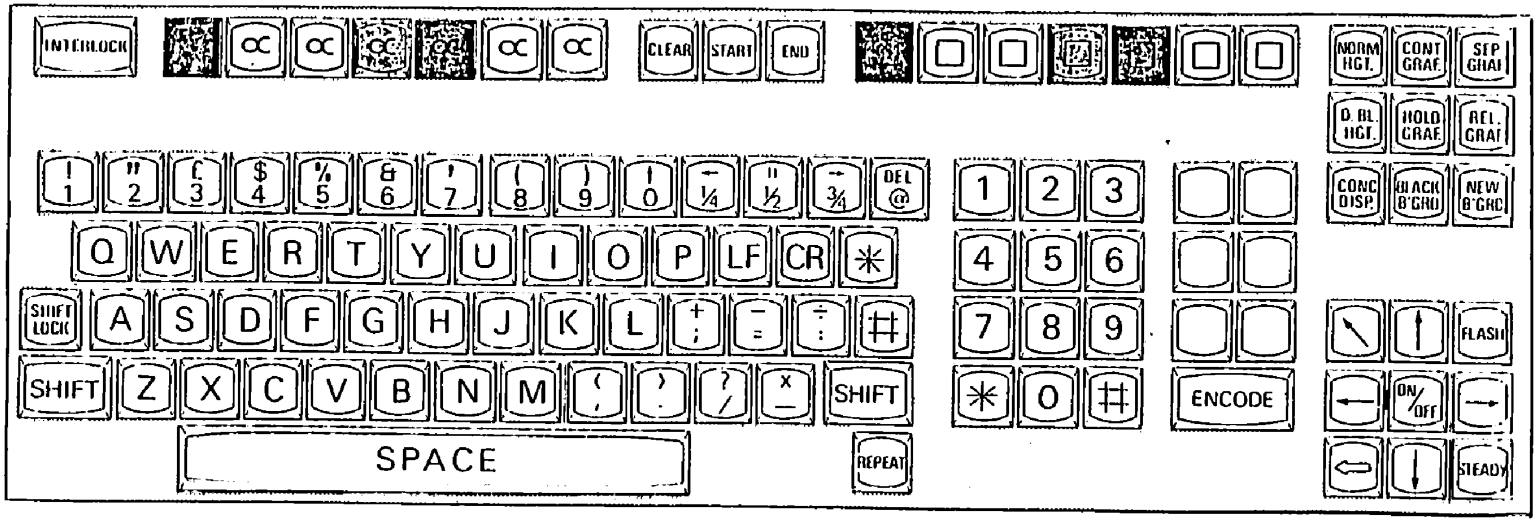
```
        S24              JSR   S18

        S23              LDA  AAB6
                         BEQ   S27
                         LDA    19
                         STA  07FE
                         LDA    1A
                         STA  07FF
                         JMP   S28

        S27              LDA @  00
                         CMP    19
                         BNE   S29
                         LDA @  FF
                         STA    19
                         DEC    1A
                         JMP   S30

        S29              DEC    19

        S30              LDA    FC
                         STA  (19),Y
                         LDA  19
                         STA  9600
                         LDA    1A
                         STA  9601

        S28              RTS

FILETRANSFER   :         JSR  FE93
                         JSR  FE89
                         CLD
                         LDA @  30
                         STA  AA5F
                         LDA    03
                         STA  AA64
                         LDA @  09
                         STA  AA65
                         LDA  AAB6
                         BEQ   S31
                         LDA @  FE
                         STA  AA72
                         LDA @  07
```

```
                    STA  AA73
                    SEC
                    LDA    19
                    SBC    FE
                    STA    FC
                    LDA    1A
                    SBC  @ 07
S32                 STA  AA6D
                    LDA    FC
                    STA  AA6C
                    JSR  FD8E
                    LDA    C6
                    JSR  FDF0
                    LDA  @ C9
                    JSR  FDF0
                    LDA  @ CC
                    JSR  FDF0
                    LDA  @ C5
                    JSR  FDF0
                    LDA  @ FF
                    JSR  FDF0
                    JSR  FD8E
                    JSR  FD6A
                    JSR   S33
                    JSR  A331
                    JSR  FD8E
                    JSR  03D0

S31                 LDA    1A
                    STA  AA73
                    LDA    19
                    STA  AA72
                    SEC
                    LDA  @ 02
                    SBC    19
                    STA    FC
                    LDA  @ 96
                    SBC    1A
                    JMP   S32
```

```
S33             LDY @  00
                TXA
                STA    06
                LDX @  00

S34             LDA @  00
                CMP    06
                BEQ   L35
                LDA  0200,Y
                STA  AA75,X
                INX
                INY
                DEC    06
                JMP   S34

S35             CPY @  15
                BEQ   S36
                LDA @  A0
                STA  AA75,Y
                INY
                JMP   S35

S36             RTS
```

## Appendix 9

PRESTEL Editing Keyboard



## Appendix 10

### Chinese Remainder Theorem

The Chinese Remainder Theorem can be stated as follows [38]:

Let $m_1$, $m_2, \cdots m_r$ denote r positive integers that are relatively prime in pairs and let $a_1, a_2, \cdots, a_r$ denote any r integers. Then the congruences $x = a_i \pmod{m_i}$ $i = 1, 2, \cdots, r$, have common solutions. Any two solutions are congruent modulo $m_1 \cdot m_2 \cdots m_r$.

Let us now consider an algorithm to implement the Chinese Remainder Theorem to compute a (mod m) where $m = m_1 \cdots m_r$

### Algorithm

1. For $i = 1$ to r compute $M_i = m/m_i$
2. For $i = 1$ to r compute $M_i^{-1} \pmod{m_i}$
3. Compute $a = \sum_{i=1}^{r} a_i \; M_i \; M_i^{-1}$

$M_i^{-1} \pmod{m_i}$, $1 \leq i \leq r$, can be calculated by r applications of the Euclid's algorithm (Appendix 11). Thus step 2 requires at most $O(r \log n)$ operations where $m \ll n$. Computation of 'a' in step 3 requires at most $O(r)$ operations. As $M_i < n$, the amount of storage required for $M_i$ and $M_i^{-1}$, $1 \leq i \leq r$, is at most $O(r \log n)$.

## Appendix 11

### Euclid's Algorithm

Euclid's algorithm is a rapid method of testing relative primeness and it can be used to compute inverses modulo m. In this thesis, the Euclid's algorithm has been frequently assumed to have been used in determining inverses modulo m. Here the basic algorithm is given. For more detailed treatment refer to [45]

Given positive integers a and b calculate the greatest common divisor, gcd (a,b) = d and x and y such that $ax + by = d$.

This is carried out using the Division Theorem which can be stated as

Given integers $b > o$ and $a \geq o$, there exist unique integers $q \geq o$ and r, $o \leq r < b$ such that $a = b q + r$
Now applying the Division Theorem successeively, it is seen that

$$\begin{array}{llll} a & = & b\, q + r_o & \text{(dividing b into a)} \\ b & = & r_o\, q_o + r_1 & \text{(dividing } r_o \text{ into b)} \\ r_o & = & r_1\, q_1 + r_2 & \text{(dividing } r_1 \text{ into } r_o\text{)} \\ r_1 & = & r_2\, q_2 + r_3 & \text{(etc.)} \\ \cdot & & \cdot & \\ \cdot & & \cdot & \\ \cdot & & \cdot & \\ & & & \\ r_{n-2} & = & q_{n-1} \cdot r_{n-1} + r_n & \\ r_{n-1} & = & q_n \cdot r_n + 0 & \end{array}$$

where $r_n < r_{n-1} < r_{n-2} \ldots < r_o < b$

and $r_n$ is the greatest common divisor of a and b. That is, $d = r_n = \gcd (a,b)$

The numbers $x$ and $y$ can be calculated as follows. To begin with, let $x = o$, $u = 1$ and $v = o$. Then at each division step, form

$$\begin{aligned} s &\longleftarrow x \\ t &\longleftarrow y \\ x &\longleftarrow u - q_{i-1}\, x \\ y &\longleftarrow v - q_{i-1}\, y \\ u &\longleftarrow s \\ v &\longleftarrow t \end{aligned}$$

When the repeated division algorithm ends, $d = r_n$ and $r_n = ax + by$.

Let us now sketch a more formal proof.

Theorem : If $r_n$ is the last non-zero remainder in Euclid's algorithm for $a$ and $b$, then $r_n$ is the gcd $(a,b)$ and $r_n = ax + by$ for some $x$ and $y$.

Proof : If $r_n$ is the last non-zero ramainder in Euclid's algorithm for $a$ and $b$, then the number of steps in the algorithm is $n+1$. If $n = o$, then $b$ divides $a$ and the theorem is trivial. If $n = 1$, then the Euclid's algorithm for $a$ and $b$ has the form

$$\begin{aligned} a &= bq_1 + r_1 \\ b &= r_1\, q_2 + 0 \end{aligned}$$

$r_1$ is the gcd $(a,b)$ and $r_1 = a.1 + b \cdot (-q_1)$

Using induction, assume that the theorem is true for $N = n-1$. That is, the theorem is true for any two numbers whose Euclid's algorithm takes $n$ steps. Suppose the Euclid's algorithm takes $n + 1$ steps for $a$ and $b$. If $a = bq_1 + r_1$ then the rest of the algorithm for $a$ and $b$ is the Euclid's algorithm for $r_1$ and $b$. So the Euclid's algorithm takes $n$ steps for $r_1$ and $b$. If $r_n$ is the last non-zero remainder of Euclid's algorithm applied to a and b, it is the last non-zero remainder for Euclid's algorithm applied to $r_1$ and b. By induction $r_n$ is the gcd $(b, r_1)$ and $r_n$ = $bu + r_1 v$. Now $a = bq_1 + r_1$ and hence $r_n$, the gcd $(b,r_1)$ is also the gcd (a, b). Substituting for $r_1$ in the expression $r_n = b u + r_1 v$ gives $r_n = bu + v (a-bq_1)$ ie, $r_n = b (u - vq_1) + a v$. Thus the theorem is true by induction.

To compute inverse of an integer a (mod m) where gcd (a,m) = 1, let b = m. The repeated division algorithm can then be used to find x and y such that $ax + my = d$. Hence $ax \equiv d \pmod{m}$. As d = 1, x is the inverse of a (mod m). Knuth [45] shows that inverses (mod m) can be computed in O (log m) operations. Furthermore, note that the factorization of m into primes is not required to determine the relative primeness and calculate the inverse.

## Appendix 12

## Determinant of a Matrix over GF(2) Program (DETMOD.F77)

```
C x  x  x  x  x  x  x  x  x  x  x  x  x  x  x  x  x  x  x  x  x  x
C  This program calculates the determinant of a given NxN matrix
C  over G F (2) by reducing the matrix to upper triangular form using
C  elementary operations.
C x  x  x  x  x  x  x  x  x  x  x  x  x  x  x  x  x  x  x  x  x  x

          INTEGER  B(127, 127),  TEMP(127),  XM1, XM,
        +          X, XM2, XM3, XM4, XM5, SIGN
           N = 127
           READ (1,* ) ((B(I,J), J = 1,N), I = 1, N)
           L = 1
           SIGN = 1
           K  =  2
           XM1  = 1
70         L1 = L
           XM = B (L,L)
           XM = ABS (XM)
           XM = MOD (XM,2)
           IF (XM. EQ. 0) GO TO 215
214        DO 95  J  = L, N
           XM2  =  B (L, J)
           XM2 = ABS (XM2)
           XM2 = MOD (XM2, 2)
           XM2 = XM2/XM
95         CONTINUE
           DO 141  I = K, N
           X = B (I, L)
           X = ABS (X)
           X = MOD (X, 2)
           IF (X. EQ. 0) GO TO 141
           DO  140 J = L, N
           XM3 = B (I, J)
           XM4 = B (L, J)
           XM3 = ABS (XM3)
           XM4 = ABS (XM4)
           XM3 = MOD (XM3, 2)
           XM4 = MOD (XM4, 2)
```

```
            B (I, J)  =  XM3 - XM4 * X
            IF (B(I,J). EQ. -1) B (I,J) = 1
140         CONTINUE
141         CONTINUE
            L = L + 1
            K = K + 1
            XM1 = XM1 * XM
            IF (L - N) 70, 190, 190
190         XM5 = B (N,N)
            XM5 = ABS (XM5)
            XM5 = MOD (XM5, 2)
            XM1 = XM1 * XM5
            XM1 = XM1 * SIGN

211         WRITE (1, 210) XM1
210         FORMAT (1X, 'DETERMINANT =  ' , I10 )
            STOP

215         L1 = L1 + 1
            IF (L1. EQ. 128) GO TO 212
            XM = B (L1, L)
            XM = ABS (XM)
            XM = MOD (XM, 2)
            IF (XM.EQ.0) GO TO 215
            DO 96 J = L, N
            TEMP (J) = B(L,J)
            B (L,J) = B(L1, J)
            B (L1,J) = TEMP (J)
96          CONTINUE
            SIGN = (-1) * SIGN
            GO TO 214
212         XM1 = 0
            WRITE (1,813)
813         FORMAT (1X, 'MATRIX IS SINGULAR')
            GO TO 211

             END
```

## Appendix 13

### Matrix Exponentiation Program (MATEXP. FTN)

```
c x x x x x x x x x x x x x x x x x x x x x x
c This program is used to encrypt and decrypt matrix messages
c using the extended RSA matrix system.  Exponentiation of the
c matrix message is performed using the square and multiply
c technique.
c x x x x x x x x x x x x x x x x x x x x x x

            INTEGER  M(3,3), C(3,3), Z(3,3), U(3,3), V(3,3), X(100), EXP
            REAL Q

            N = 3
            WRITE (1,1)
1           FORMAT (1 H, 'INPUT THE MESSAGE/CIPHER MATRIX PLEASE')
            READ (1,*) ((M(I,J), J = 1, N), I = 1, N)
            WRITE (1,2)
2           FORMAT (1 H, 'ENTER THE ENCRYPTING/DECRYPTING
                           EXPONENT PLEASE')
            READ (1,*) EXP
c       Initialize the C matrix to identity matrix
            DO 3  I = 1, N
            DO 3  J = 1, N
            C (I,J) = 0
3           C (I,I) = 1
c       Convert EXP to a k-bit binary vector (K≤100)
            DO 4 L1 = 1, 100
            Q = EXP/2
            IR = EXP - INT (Q) * 2
            L2 = 101 - L1
            X (L2) = IR
            EXP = INT (Q)
4           CONTINUE
c        Raise the matrix M to the power EXP
            II = 100
9           IF (X(II) .EQ.0) Go to 5
            CALL MULT (M, C, Z)
            DO 6 I = 1, N
            DO 6 J = 1, N
```

```
6             C(I,J)  =  Z (I,J)
5             IF (II.EQ.1) GO TO 7
              CALL MULT (M, M, Z)
              DO  8  I = 1, N
              DO  8  J = 1, N
8             M (I,J)  = Z (I,J)
              II = II - 1
              GO TO 9

7             WRITE (1, 10)
10            FORMAT (1H, 'THE CIPHER/MESSAGE MATRIX IS')
              WRITE (1, 11) ((M(I,J), J = 1, N), I = 1, N)
11            FORMAT (1H,  3 (I6, 2X))
              STOP
              END

c  The subroutine MULT multiplies the matrix U by matrix V
c  Over Z/mZ where  m  is the modulus of the public key
c  system.  In this case  m = 299.
              SUBROUTINE MULT (U, V, Z)
              INTEGER U (3,3), V(3,3), Z(3,3)

              DO  12  I1 = 1, N
              DO  12  J1 = 1, N
              Z (I1, J1) = 0
              DO  12  K1 = 1, N
              IP = Z (I1,J1) + U (I1, K1) * V (K1, J1)
              Z (I1, J1) = MOD (IP, 299)
12            CONTINUE
              RETURN
              END.
```

## Appendix 14

### Polynomial Exponentiation Program (POLYEXT. F 77)

```
c  x  x  x  x  x  x  x  x  x  x  x  x  x  x  x  x  x  x  x  x  x
c  This program is used to encrypt and decrypt polynomial
c  messages using the extended RSA polynomial system.
c  Exponentiation of the polynomial message is performed using
c  the square and multiply technique.
c  x  x  x  x  x  x  x  x  x  x  x  x  x  x  x  x  x  x  x  x  x


               INTEGER  M(4), V(4), Z(4), EXP, X(100)
               REAL Q
               N = 4
               WRITE (1,1)
1              FORMAT (1H, 'INPUT THE MESSAGE/CIPHER
                       POLYNOMIAL PLEASE')
               READ (1,*) (M(J), J = 1, N)
               WRITE (1,2)
2              FORMAT (1 H, 'ENTER THE ENCRYPTING/
                            DECRYPTING EXPONENT PLEASE')
               READ (1,*) EXP
c          Initialize the V polynomial to identity vector.
               DO 3  II = 1, N
3              V (II) = 0
               V (1) = 1

c          Convert EXP to a K-bit binary vector (K≤100)
               DO  4  L1 = 1, 100
               Q = EXP/2
               IR = EXP - INT(Q) * 2
               L2  = 101 - L1
               X(L2) = IR
               EXP = INT(Q)
4              CONTINUE
c          Raise the polynomial M to the power EXP
               II = 100
9              IF (X (II). EQ.0) GO TO 5
               CALL MULT (M, V, Z)
               DO  6 I = 1, N
```

```
6            V (I) = Z (I)
5            1F (II.EQ.1) GO TO 7
             CALL MULT (M, M, Z)
             DO  8 I = 1, N
8            M (I) = Z (I)
             II = II - 1
             GO TO 9
7            WRITE (1, 10)
10           FORMAT (1 H, 'THE CIPHER/MESSAGE POLYNOMIAL 1S')
             WRITE (1, 11)  (M (J), J = 1, N)
11           FORMAT (1H, 4 (I6, 2X))
             STOP
             END
C  The subroutine multiplies the polynomial U by polynomial V
C  over Z/mZ where m is the modulus of the public key
C  system.  The product polynomial is reduced modulo an
C  irreducible polynomial.  In this subroutine, the irreducible
C  polynomial is f (x) = x * * 4  + x * * 2  + 1  and
C  m is equal to 35
             SUBROUTINE MULT (U, V, Z)
             INTEGER U (4), V(4), Z(4)
   Z(1) = U(1) * V(1) - U(4) * V(2) - U(3) * V(3)-U(2)*V(4)+U(4)*V(4)
   Z(2) = U(2)*V(1)+U(0)*V(2)-U(4)*V(3) - U(3)*V(4)
   Z(3) = U(3)*V(1)+U(2)*V(2)+U(1)*V(3)-U(4)*V(2)-U(3)*V(3)-U(2)*V(4)
   Z(4) = U(4)*V(1)+U(3)*V(2)+U(2)*V(3)+U(1)*V(4)-U(4)*V(3)-U(3)*V(4)
             DO 12  I = 1, N
             IP = Z(I)
             Z(I) = MOD (IP, 35)
12           CONTINUE
             STOP
             END
```

## Appendix 15

### Cycle Length Calculation Program in GF (2 * * 7) (CYCLE.F77)

```
C x  x  x  x  x  x  x  x  x  x  x  x  x  x  x  x  x  x  x  x  x
C  This program is used to calculate the cycle lengths in
C  GF (2**7) for a given secret exponent key when the system
C  base polynomial, evaluated  as a binary vector, is varied
C  from 2 to 127   .  The generator irreducible polynomial used in
C  the system is x * * 7 + x + 1.
C  x  x  x  x  x  x  x  x  x  x  x  x  x  x  x  x  x  x  x  x  x

     INTEGER  M(7,7), S(7,7), U(7), V(7), Z1(7), X(7),
              TEMP(7)
     REAL     Q, Q1

C  Read the multiplication matrix (M) and square matrix (S)
C  from the data file MITDATA
     READ (5, *) ((S(I,J), J = 1,7), I = 1,7)
     READ (5, *) ((M(I,J), J = 1,7), I = 1,7)

     WRITE (1,1)
1    FORMAT (1H, 'INPUT THE BASE AND SECRET KEY PLEASE')
     READ (1,*) BASE, KEY
501  BASE 1 = BASE
C  Convert BASE and KEY to 7-bit binary vectors.
     DO  2  L1 = 1, 7
     Q  = BASE/2
     Q1 = KEY/2
     IR = BASE - INT (Q) * 2
     IR1 = KEY - INT (Q1) * 2
     L2  = 8 - L1
     U (L2) = IR
     X (L2) = IR1
     BASE = INT (Q)
     KEY = INT (Q1)
2    CONTINUE

C  Store the BASE in temporary vector TEMP
     DO  3  I = 1, 7
     TEMP (I) = U (I)
3    CONTINUE
```

```
C  Initialize the V - vector to 1.
        DO  4  I = 1, 6
        V (I) = 0
4       V (7) = 1
C  Compute the cycle length for a given base.
        ICL = 1
        I = 7
9       1F (X (I).EQ.0) GO TO 5
        CALL MULT (U, V, M)
        DO  6 II = 1, 7
        U (II) = TEMP (II)
6       CONTINUE
5       1F (I.EQ.1) GO TO 7
        CALL SQUARE (U, S)
        DO  8 II = 1, 7
        TEMP  (II) = U (II)
8       CONTINUE
        I = I -1
        GO TO 9
        DO 51 JJ = 1, 7
        CHECK (IC, JJ) = V(JJ)
51      CONTINUE
        ICL = ICL -1
        1F (ICL .EQ. 0) GO TO 56
        DO  521  II = 1, IC
        DO  52  IJ = 1, 7
        1F (V(JJ) .EQ. CHECK (II, JJ)) FLAG = 1
        1F (V(JJ) .NE. CHECK (II, JJ)) FLAG = 0
52      CONTINUE
        1F (FLAG.EQ.1) GO TO 53
521     CONTINUE
56      DO  54 L = 1, 7
        X (L) = V(L)
        U (L) = TEMP1 (L)
54      CONTINUE
        ICL = ICL + 2
        1F (ICL.EQ. 350) GO TO 55
        GO TO 50
```

```
53      ICCL = ICL + 1
        WRITE (6,400) ICCL
400     FORMAT (1H, 'THE BASE IS =', 2 x, 7 (I4, 1X))
        GO TO 502
55      ICL = ICL -1
        WRITE (6,402) IC, (U(L), L = 1,7)
402     FORMAT (1H, 'MAXIMUM CYCLE', I4, 2X, 7 (I4,1X))
502     BASE = BASE1 + 1
        BASE1 = BASE
        IF (BASE .EQ.128) GO TO 503
        WRITE (6,504)
504     FORMAT (' * * * * * * * * * * * * * *  ')
        GO TO 501
503     STOP
        END
C   The subroutine MULT multiplies the polynomial U by the
C   polynomial V over GF (2 * * 7) using the irreducible
C   trinomial x * * 7 + x + 1
        SUBROUTINE MULT (U, V, M)
        INTEGER P(7), P1(7), U(7), P2(7), V(7), M(7,7)
        DO  12  I = 1,7
12      P (I) = 0
        IJ = 7
20      IF (V(IJ).EQ.0) GO TO 13
        DO  14 I = 1,7
        P (I) = U(I) + P(I)
        N = P(I)
        N1 = MOD (N,2)
        P (I) = N1
14      CONTINUE
13      DO  15 K = 1,7
        DO  16 I = 1,7
        P1 (I) = U(I) * M (I,K)
16      CONTINUE
        P2 (K) = 0
        DO 17  I = 1,7
        P2 (K) = P2(K) + P1 (I)
        N = P2(K)
        N1 = MOD (N,2)
```

```
       P2(K) = N1
17     CONTINUE
15     CONTINUE
       DO  18 L = 1,7
18     U(L) = P2(L)
       IF (IJ.EQ.1)  GO TO 19
       IJ = IJ -1
       GO TO 20
19     DO  21 L = 1,7
21     V (L) = P (L)
       RETURN
       END

C  The subroutine SQUARE squares the polynomial U over
C  GF (2 * * 7) using the irreducible trinomial X * * 7 + X + 1.
       SUBROUTINE SQUARE (U, S)
       INTEGER Z(7), Z1(7), U(7), S(7,7)
       DO  22 J = 1,7
       DO  23 I = 1,7
       Z(I) = U(I) * S(I,J)
23     CONTINUE
       Z1(J) = 0
       DO  24 I = 1,7
       Z1 (J) = Z1(J) + Z(I)
       NUM = Z1 (J)
       NUM1 = MOD (NUM, 2)
       Z1 (J) = NUM1
24     CONTINUE
22     CONTINUE
       DO  25 I = 1,7
25     U(I) = Z1(I)
       RETURN
       END
```

## Appendix 16

### Short Cycling Attack in PKD System over GF (2**7)(RANDCYCLE.F77)

```
C  *  * *  *  *  *  *  *  *  *  *  *  *  *  *  *  *  *  *  *  *
C  This program carries out short cycling attack on the public
C  key distribution over GF (2**7) with irreducible trinomial
C  x ** 7 + x + 1.  The attack consists of repeated encipherings
C  to determine the cycle length and the secret key.  For a
C  fixed base polynomial, it calculates the average cycle length
C  when the secret keys vary over the range 1 to 127.
C  This procedure is repeated for the 127 possible DEC functions
C  of the form DEC (b * y) where b is any one of the 127
C  polynomials over GF (2 **7).  Then, the expected cycle length
C  for the 127 DEC functions is determined.
C  This program can be run with different system base
C  polynomials.
C
C  The program utilizes the subroutines MULT and SQUARE
C  given in CYCLE.F77.  (Appendix 15).
C  *  *  *  *  *  *  *  *  *  *  *  *  *  *  *  *  *  *  *  *

         INTEGER  M(7,7), S(7,7), U(7), V(7), Z1(7), X(7), IC
      +           TEMP1(7), CHECK(350,7),TEMP(7), FLAG, BASE,
      +           TEMP2(7), KEY, U1(7), V1(7), B(7), DEC, KEY1
      +           DEC1, BASE1
         REAL     Q, Q1, QK, SUM, AVR, EXPAVR, EXP

         SUM  = 0.0
         EXP  = 0.0

         WRITE (1,1)
1        FORMAT (1H, 'INPUT THE BASE, SECRET KEY
      + AND DEC POLYNOMIAL B PLEASE')
         READ (1,*) BASE, KEY, DEC
C   Convert BASE, KEY and DEC to 7-bit binary vectors.
         DO  2 L1 = 1,7
         Q = BASE/2
         IR = BASE - INT (Q) * 2
         L2 = 8 - L1
         U (L2) = IR
         BASE = INT (Q)
```

```
2         CONTINUE
101       DEC1 = DEC
501       KEY1 = KEY
          DO  3  L1 = 1, 7
          QK = KEY/2
          Q1 = DEC/2
          IRK = KEY - INT (QK) *2
          IR1 = DEC - INT (Q1) *2
          L2 = 8 - L1
          X(L2) = IRK
          B (L2) = IR1
          KEY = INT (QK)
          DEC = INT (Q1)
3         CONTINUE

C  Compute the cycle length ICC for a given KEY and DEC
          IC = 1
          DO  38 I = 1,7
          TEMP1 (I) = U(I)
38        CONTINUE
          DO 10 I = 1,6
          V (I) = 0
          V (7) = 1
          I = 7
          DO  40 I = 1,7
          TEMP (I) = U (I)
40        CONTINUE
31        IF (X(I).EQ.0)  GO TO 32
          CALL MULT (U, V, M)
          DO  41  II = 1,7
          U (II) = TEMP (II)
41        CONTINUE
32        IF (I.EQ.1) GO TO 33
          CALL SQUARE (U,S)
          DO  42 II = 1,7
          TEMP (II) = U(II)
42        CONTINUE
          I = I -1
          GO TO 31
```

```
33         CALL MULT (B, V, M)
           DO  82  I = 1,7
           B (I) = TEMP2(I)
82         CONTINUE
           DO 51 JJ = 1,7
           CHECK (IC, JJ) = V(JJ)
51         CONTINUE
           IC = IC -1
           IF (IC.EQ.0) GO TO 56
           DO 521 II = 1, IC
           DO 52 JJ = 1,7
           IF (V(JJ).EQ.CHECK (II,JJ)) FLAG = 1
           IF (V(JJ).NE.CHECK (II,JJ)) FLAG = 0
           IF (FLAG.EQ.0) GO TO 56
52         CONTINUE
           IF (FLAG.EQ.1) GO TO 53
521        CONTINUE
56         DO  54 L = 1,7
           X (L) = V(L)
           U(L) = TEMP1(L)
54         CONTINUE
           IC = IC + 2
           IF (IC •EQ.350) GO TO 55
           GO to 50
53         ICC = IC + 1
           DO 499 III = 1, 7
           U (III) = TEMP1(III)
499        CONTINUE
           SUM = SUM + ICC -1
           GO TO 502
           SUM = SUM + IC -1

c  Increment the KEY by 1 and repeat the process.
502        KEY = KEY1 + 1
           KEY1 = KEY
           IF (KEY.EQ.128) GO TO 503
           GO TO 501
```

```
C  When all the 127 possible values for the KEY have been
C  tried, compute the average cycle length for a given DEC
C  and a given base polynomial.

503       AVR = SUM/127
          WRITE (1,477) SUM, AVR
477       FORMAT (1X, 'SUM =', F12.5, 3X, 'AVR = ', F10.5)
          WRITE (1,504)
504       FORMAT (' * * * * * * * * * * * * * * *')

C  Increment the DEC value by 1 and repeat the process of
C  finding different cycle lengths when the KEY takes all the
C  127 possible values

          DEC = DEC1 + 1
          DEC1 = DEC
          KEY  = 1
          KEY1 = KEY
          SUM  = 0.0
          EXP  = EXP + AVR
          IF (DEC.EQ.128) GO TO 102
          GO TO 101

C  When the 127 DEC values of the special form are tried,
C  compute the expected cycle length for the given base
C  polynomial.

102       EXPAVR  =  EXP/127
          WRITE  (1,103) EXPAVR
103       FORMAT (1X, 'THE EXPECTED CYCLE LENGTH IS' F10.5)
          STOP
          END
```

## Appendix 17

### Results of Cycling in PKD System over GF ( 2 * * 7)

A 17.1 Cycle lengths in GF (2 * * 7) for several values of Secret Key, x

| Base a | x = 2 | x = 4 | x = 15 | x = 32 | x = 64 | x = 67 | x = 68 | x = 73 | x = 105 | x = 115 |
|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 74 | 74 | 74 | 74 | 11 | 74 | 74 | 74 | 42 | 42 |
| 3 | 118 | 118 | 118 | 118 | 2 | 118 | 118 | 118 | 118 | 118 |
| 4 | 27 | 53 | 53 | 27 | 53 | 53 | 53 | 27 | 27 | 27 |
| 5 | 59 | 63 | 59 | 59 | 59 | 63 | 59 | 59 | 63 | 59 |
| 6 | 75 | 75 | 38 | 75 | 75 | 75 | 75 | 75 | 11 | 38 |
| 7 | 63 | 13 | 9 | 63 | 9 | 63 | 20 | 63 | 13 | 9 |
| 8 | 11 | 45 | 65 | 45 | 65 | 6 | 65 | 11 | 65 | 65 |
| 9 | 1 | 116 | 116 | 116 | 116 | 116 | 116 | 1 | 116 | 116 |
| 10 | 43 | 3 | 75 | 75 | 43 | 75 | 43 | 43 | 43 | 3 |
| 11 | 12 | 92 | 10 | 92 | 10 | 12 | 2 | 12 | 12 | 92 |
| 12 | 53 | 12 | 34 | 4 | 34 | 24 | 4 | 53 | 12 | 12 |
| 13 | 101 | 8 | 101 | 101 | 13 | 101 | 101 | 101 | 1 | 101 |
| 14 | 113 | 113 | 113 | 113 | 113 | 113 | 113 | 113 | 113 | 12 |
| 15 | 47 | 54 | 54 | 47 | 14 | 47 | 54 | 47 | 47 | 54 |
| 16 | 68 | 9 | 68 | 14 | 9 | 33 | 33 | 68 | 68 | 33 |

| Base a | x = 2 | x = 4 | x = 15 | x = 32 | x = 64 | x =67 | x = 68 | x = 73 | x = 105 | x = 115 |
|---|---|---|---|---|---|---|---|---|---|---|
| 17 | 28 | 89 | 28 | 89 | 28 | 28 | 89 | 28 | 89 | 89 |
| 18 | 40 | 1 | 40 | 40 | 14 | 16 | 11 | 40 | 40 | 14 |
| 19 | 106 | 106 | 106 | 106 | 106 | 106 | 8 | 107 | 106 | 106 |
| 20 | 22 | 30 | 15 | 30 | 30 | 17 | 17 | 22 | 5 | 22 |
| 21 | 107 | 107 | 107 | 107 | 107 | 107 | 107 | 107 | 107 | 7 |
| 22 | 52 | 62 | 11 | 62 | 62 | 61 | 62 | 52 | 52 | 62 |
| 23 | 91 | 91 | 91 | 91 | 91 | 7 | 91 | 91 | 91 | 91 |
| 24 | 25 | 45 | 45 | 25 | 25 | 45 | 50 | 25 | 25 | 50 |
| 25 | 18 | 48 | 51 | 18 | 51 | 51 | 51 | 18 | 18 | 48 |
| 26 | 10 | 50 | 36 | 50 | 26 | 26 | 50 | 10 | 50 | 50 |
| 27 | 58 | 39 | 58 | 39 | 58 | 58 | 58 | 58 | 58 | 3 |
| 28 | 80 | 80 | 80 | 43 | 43 | 43 | 80 | 80 | 80 | 80 |
| 29 | 111 | 111 | 111 | 111 | 111 | 111 | 111 | 111 | 111 | 111 |
| 30 | 49 | 49 | 16 | 16 | 49 | 46 | 51 | 49 | 7 | 51 |
| 31 | 27 | 27 | 58 | 27 | 17 | 5 | 58 | 27 | 27 | 9 |

| Base a | x = 2 | x = 4 | x = 15 | x = 32 | x = 64 | x = 67 | x = 68 | x = 73 | x = 105 | x = 115 |
|---|---|---|---|---|---|---|---|---|---|---|
| 32 | 72 | 19 | 72 | 72 | 72 | 72 | 72 | 72 | 72 | 72 |
| 33 | 66 | 24 | 3 | 66 | 66 | 21 | 21 | 66 | 5 | 66 |
| 34 | 116 | 116 | 116 | 116 | 116 | 116 | 5 | 116 | 116 | 116 |
| 35 | 33 | 2 | 56 | 56 | 56 | 34 | 56 | 33 | 56 | 34 |
| 36 | 92 | 92 | 92 | 92 | 92 | 7 | 21 | 92 | 92 | 92 |
| 37 | 18 | 18 | 69 | 69 | 33 | 33 | 69 | 18 | 69 | 33 |
| 38 | 127 | 127 | 127 | 127 | 127 | 127 | 127 | 127 | 127 | 127 |
| 39 | 50 | 50 | 50 | 50 | 37 | 50 | 50 | 50 | 50 | 14 |
| 40 | 109 | 109 | 109 | 10 | 109 | 109 | 109 | 109 | 109 | 109 |
| 41 | 40 | 69 | 69 | 69 | 40 | 69 | 69 | 40 | 69 | 69 |
| 42 | 57 | 57 | 11 | 50 | 57 | 50 | 57 | 57 | 50 | 50 |
| 43 | 30 | 79 | 30 | 80 | 5 | 30 | 79 | 30 | 30 | 79 |
| 44 | 20 | 44 | 20 | 44 | 26 | 44 | 29 | 20 | 44 | 44 |
| 45 | 42 | 42 | 63 | 63 | 63 | 63 | 63 | 42 | 63 | 63 |
| 46 | 87 | 87 | 36 | 87 | 36 | 87 | 87 | 87 | 87 | 36 |
| 47 | 61 | 12 | 3 | 24 | 24 | 61 | 61 | 61 | 12 | 61 |

| Base a | x = 2 | x = 4 | x = 15 | x = 32 | x = 64 | x = 67 | x = 68 | x = 73 | x = 105 | x = 115 |
|---|---|---|---|---|---|---|---|---|---|---|
| 48 | 91 | 51 | 91 | 91 | 26 | 91 | 91 | 91 | 26 | 26 |
| 49 | 78 | 43 | 78 | 78 | 78 | 78 | 78 | 78 | 78 | 78 |
| 50 | 104 | 18 | 18 | 4 | 104 | 18 | 104 | 104 | 104 | 104 |
| 51 | 72 | 72 | 18 | 48 | 72 | 72 | 48 | 72 | 48 | 72 |
| 52 | 2 | 72 | 21 | 13 | 13 | 21 | 13 | 2 | 72 | 72 |
| 53 | 82 | 15 | 82 | 18 | 82 | 18 | 82 | 82 | 15 | 82 |
| 54 | 31 | 2 | 52 | 52 | 52 | 52 | 2 | 31 | 52 | 52 |
| 55 | 37 | 9 | 21 | 8 | 32 | 21 | 8 | 37 | 37 | 32 |
| 56 | 21 | 21 | 21 | 72 | 72 | 72 | 72 | 21 | 72 | 21 |
| 57 | 40 | 40 | 49 | 49 | 4 | 49 | 49 | 40 | 4 | 49 |
| 58 | 14 | 49 | 63 | 14 | 49 | 63 | 49 | 14 | 14 | 49 |
| 59 | 38 | 27 | 27 | 31 | 38 | 38 | 9 | 38 | 13 | 31 |
| 60 | 17 | 17 | 13 | 26 | 5 | 31 | 31 | 17 | 17 | 5 |
| 61 | 12 | 47 | 22 | 37 | 47 | 87 | 22 | 12 | 37 | 12 |
| 62 | 45 | 45 | 4 | 45 | 66 | 66 | 66 | 45 | 45 | 4 |
| 63 | 126 | 126 | 126 | 126 | 126 | 126 | 126 | 126 | 126 | 126 |

| Base a | x = 2 | x = 4 | x = 15 | x = 32 | x = 64 | x = 67 | x = 68 | x = 73 | x = 105 | x = 115 |
|---|---|---|---|---|---|---|---|---|---|---|
| 64 | 7 | 33 | 46 | 46 | 33 | 46 | 7 | 7 | 10 | 12 |
| 65 | 60 | 54 | 58 | 58 | 58 | 58 | 60 | 60 | 60 | 60 |
| 66 | 75 | 16 | 75 | 16 | 75 | 16 | 75 | 75 | 16 | 3 |
| 67 | 16 | 87 | 87 | 87 | 16 | 87 | 6 | 16 | 87 | 9 |
| 68 | 97 | 97 | 97 | 97 | 15 | 97 | 97 | 97 | 97 | 97 |
| 69 | 12 | 108 | 108 | 108 | 104 | 108 | 108 | 12 | 108 | 108 |
| 70 | 5 | 79 | 10 | 1 | 79 | 79 | 79 | 5 | 8 | 79 |
| 71 | 118 | 114 | 118 | 118 | 9 | 118 | 118 | 118 | 118 | 118 |
| 72 | 88 | 88 | 88 | 88 | 24 | 88 | 88 | 88 | 88 | 88 |
| 73 | 104 | 104 | 104 | 104 | 104 | 104 | 104 | 104 | 104 | 104 |
| 74 | 117 | 117 | 117 | 117 | 117 | 117 | 117 | 117 | 117 | 117 |
| 75 | 103 | 103 | 22 | 103 | 103 | 103 | 103 | 103 | 103 | 103 |
| 76 | 120 | 120 | 120 | 120 | 120 | 120 | 120 | 120 | 120 | 120 |
| 77 | 84 | 84 | 84 | 84 | 84 | 84 | 84 | 84 | 84 | 37 |
| 78 | 28 | 28 | 77 | 77 | 16 | 77 | 28 | 28 | 77 | 77 |
| 79 | 8 | 28 | 28 | 91 | 91 | 91 | 91 | 8 | 91 | 28 |

| Base a | x = 2 | x = 4 | x = 15 | x = 32 | x = 64 | x = 67 | x = 68 | x = 73 | x = 105 | x = 115 |
|---|---|---|---|---|---|---|---|---|---|---|
| 80 | 92 | 8 | 92 | 18 | 92 | 9 | 92 | 92 | 7 | 92 |
| 81 | 121 | 121 | 121 | 121 | 121 | 121 | 121 | 121 | 121 | 121 |
| 82 | 26 | 66 | 29 | 66 | 66 | 26 | 4 | 26 | 66 | 29 |
| 83 | 79 | 40 | 79 | 40 | 40 | 79 | 79 | 79 | 79 | 40 |
| 84 | 21 | 33 | 21 | 64 | 33 | 33 | 64 | 21 | 21 | 64 |
| 85 | 79 | 79 | 79 | 31 | 79 | 11 | 11 | 79 | 79 | 79 |
| 86 | 26 | 46 | 46 | 46 | 51 | 46 | 51 | 26 | 4 | 46 |
| 87 | 81 | 81 | 81 | 81 | 34 | 4 | 34 | 81 | 81 | 81 |
| 88 | 93 | 93 | 93 | 93 | 19 | 93 | 93 | 93 | 93 | 93 |
| 89 | 121 | 121 | 121 | 121 | 121 | 121 | 121 | 121 | 121 | 121 |
| 90 | 92 | 92 | 92 | 15 | 92 | 92 | 92 | 92 | 92 | 92 |
| 91 | 80 | 80 | 13 | 80 | 80 | 31 | 31 | 80 | 80 | 31 |
| 92 | 30 | 79 | 5 | 79 | 79 | 30 | 30 | 30 | 8 | 79 |
| 93 | 118 | 118 | 111 | 118 | 8 | 118 | 118 | 118 | 118 | 118 |
| 94 | 6 | 47 | 47 | 47 | 43 | 47 | 14 | 6 | 14 | 43 |
| 95 | 43 | 14 | 47 | 47 | 43 | 43 | 14 | 43 | 47 | 10 |

| Base a | x = 2 | x = 4 | x = 15 | x = 32 | x = 64 | x = 67 | x = 68 | x = 73 | x = 105 | x = 115 |
|---|---|---|---|---|---|---|---|---|---|---|
| 96 | 17 | 71 | 71 | 21 | 1 | 21 | 71 | 17 | 21 | 71 |
| 97 | 23 | 23 | 27 | 23 | 66 | 23 | 66 | 23 | 66 | 66 |
| 98 | 97 | 26 | 97 | 97 | 26 | 97 | 26 | 97 | 4 | 97 |
| 99 | 15 | 111 | 111 | 111 | 111 | 111 | 111 | 15 | 111 | 111 |
| 100 | 18 | 90 | 90 | 90 | 90 | 15 | 15 | 18 | 18 | 90 |
| 101 | 84 | 84 | 35 | 6 | 84 | 84 | 84 | 84 | 84 | 35 |
| 102 | 34 | 40 | 25 | 25 | 40 | 25 | 25 | 34 | 3 | 40 |
| 103 | 125 | 125 | 125 | 125 | 125 | 125 | 125 | 125 | 125 | 125 |
| 104 | 24 | 24 | 24 | 24 | 95 | 95 | 95 | 24 | 95 | 24 |
| 105 | 48 | 27 | 46 | 48 | 48 | 48 | 48 | 48 | 48 | 48 |
| 106 | 54 | 49 | 49 | 49 | 49 | 23 | 54 | 54 | 49 | 23 |
| 107 | 24 | 40 | 63 | 63 | 63 | 63 | 40 | 24 | 40 | 63 |
| 108 | 60 | 44 | 44 | 60 | 44 | 60 | 44 | 60 | 6 | 1 |
| 109 | 96 | 18 | 96 | 96 | 96 | 18 | 96 | 96 | 96 | 96 |
| 110 | 108 | 108 | 108 | 108 | 108 | 108 | 108 | 108 | 8 | 8 |
| 111 | 57 | 57 | 61 | 57 | 61 | 8 | 61 | 57 | 57 | 61 |

| Base a | x = 2 | x = 4 | x = 15 | x = 32 | x = 64 | x = 67 | x = 68 | x = 73 | x = 105 | x = 115 |
|---|---|---|---|---|---|---|---|---|---|---|
| 112 | 42 | 41 | 41 | 22 | 41 | 42 | 41 | 42 | 42 | 16 |
| 113 | 3 | 90 | 90 | 3 | 90 | 90 | 28 | 3 | 90 | 28 |
| 114 | 123 | 123 | 123 | 123 | 123 | 123 | 123 | 123 | 123 | 124 |
| 115 | 96 | 96 | 8 | 96 | 23 | 96 | 96 | 96 | 96 | 8 |
| 116 | 27 | 25 | 18 | 27 | 2 | 1 | 27 | 27 | 33 | 25 |
| 117 | 104 | 104 | 104 | 2 | 9 | 104 | 104 | 104 | 104 | 104 |
| 118 | 66 | 66 | 66 | 15 | 66 | 66 | 66 | 66 | 15 | 88 |
| 119 | 13 | 97 | 13 | 13 | 97 | 97 | 97 | 13 | 97 | 97 |
| 120 | 115 | 115 | 115 | 115 | 115 | 115 | 115 | 115 | 115 | 115 |
| 121 | 35 | 5 | 35 | 19 | 22 | 22 | 19 | 35 | 35 | 22 |
| 122 | 126 | 126 | 126 | 126 | 126 | 126 | 126 | 126 | 126 | 126 |
| 123 | 52 | 6 | 26 | 52 | 15 | 52 | 23 | 52 | 52 | 15 |
| 124 | 82 | 82 | 82 | 82 | 82 | 82 | 1 | 82 | 9 | 8 |
| 125 | 2 | 66 | 66 | 66 | 13 | 13 | 66 | 2 | 66 | 66 |
| 126 | 75 | 75 | 75 | 75 | 5 | 75 | 75 | 75 | 75 | 75 |
| 127 | 110 | 110 | 110 | 110 | 110 | 110 | 110 | 110 | 110 | 110 |

## A.17.2 Expected cycle lengths for the System Base a = 38, using DEC (b * y)

| Polynomial b<br>(evaluated as<br>a binary vector) | Expected<br>Cycle length | |
|---|---|---|
| 1 | 127.00 | (Refer to Section A 17.1, |
| 2 | 38.87 | a = 38 cycle length = 127) |
| 3 | 54.48 | |
| 4 | 66.39 | |
| 5 | 94.67 | |
| 6 | 43.80 | |
| 7 | 35.87 | |
| 8 | 22.75 | |
| 9 | 47.87 | |
| 10 | 121.11 | |
| 11 | 44.50 | |
| 12 | 87.55 | |
| 13 | 27.17 | |
| 14 | 29.77 | |
| 15 | 113.61 | |
| 16 | 39.87 | |
| 17 | 43.36 | |
| 18 | 64.50 | |
| 19 | 103.19 | |
| 20 | 49.77 | |
| 21 | 109.85 | |
| 22 | 66.54 | |
| 23 | 53.09 | |
| 24 | 127.00 | |
| 25 | 109.95 | |
| 26 | 68.83 | |
| 27 | 76.54 | |
| 28 | 35.76 | |
| 29 | 52.15 | |
| 30 | 51.93 | |
| 31 | 47.80 | |
| 32 | 99.28 | |

| | |
|---|---|
| 33 | 73.16 |
| 34 | 53.17 |
| 35 | 29.39 |
| 36 | 46.40 |
| 37 | 121.11 |
| 38 | 45.44 |
| 39 | 53.02 |
| 40 | 71.49 |
| 41 | 52.94 |
| 42 | 64.42 |
| 43 | 70.09 |
| 44 | 60.95 |
| 45 | 127.00 |
| 46 | 57.27 |
| 47 | 45.50 |
| 48 | 44.00 |
| 49 | 123.05 |
| 50 | 43.02 |
| 51 | 75.60 |
| 52 | 27.30 |
| 53 | 54.87 |
| 54 | 29.71 |
| 55 | 35.60 |
| 56 | 38.28 |
| 57 | 64.00 |
| 58 | 35.71 |
| 59 | 67.99 |
| 60 | 48.39 |
| 61 | 46.89 |
| 62 | 54.40 |
| 63 | 77.71 |
| 64 | 77.71 |
| 65 | 69.69 |
| 66 | 61.60 |
| 67 | 82.09 |
| 68 | 49.79 |
| 69 | 60.18 |
| 70 | 78.09 |

| | |
|---|---|
| 71 | 62.91 |
| 72 | 59.52 |
| 73 | 108.20 |
| 74 | 115.38 |
| 75 | 36.04 |
| 76 | 85.96 |
| 77 | 63.08 |
| 78 | 65.25 |
| 79 | 119.19 |
| 80 | 54.83 |
| 81 | 44.69 |
| 82 | 111.77 |
| 83 | 53.17 |
| 84 | 81.11 |
| 85 | 37.35 |
| 86 | 43.36 |
| 87 | 52.97 |
| 88 | 52.26 |
| 89 | 42.43 |
| 90 | 51.00 |
| 91 | 65.90 |
| 92 | 67.20 |
| 93 | 33.47 |
| 94 | 54.69 |
| 95 | 77.88 |
| 96 | 98.20 |
| 97 | 34.20 |
| 98 | 38.67 |
| 99 | 56.65 |
| 100 | 26.26 |
| 101 | 43.66 |
| 102 | 60.75 |
| 103 | 68.02 |
| 104 | 37.00 |
| 105 | 60.56 |
| 106 | 48.98 |
| 107 | 49.60 |
| 108 | 106.53 |

| | |
|---|---|
| 109 | 87.68 |
| 110 | 55.31 |
| 111 | 54.35 |
| 112 | 78.11 |
| 113 | 33.91 |
| 114 | 48.41 |
| 115 | 52.91 |
| 116 | 43.87 |
| 117 | 42.75 |
| 118 | 39.53 |
| 119 | 40.97 |
| 120 | 75.69 |
| 121 | 61.03 |
| 122 | 60.71 |
| 123 | 41.87 |
| 124 | 70.98 |
| 125 | 32.64 |
| 126 | 32.24 |
| 127 | 69.25 |

### A.17.3 Average Expected Cycle lengths for several values of System Base

| System Base | Average Expected Cycle Length |
|---|---|
| 8 | 67.65 |
| 15 | 65.44 |
| 28 | 63.28 |
| 38 | 61.16 |
| 42 | 65.59 |
| 53 | 64.16 |
| 64 | 66.41 |
| 79 | 61.91 |
| 83 | 66.61 |
| 90 | 63.12 |
| 98 | 64.11 |
| 104 | 64.17 |
| 117 | 63.81 |
| 124 | 61.51 |

## Appendix 18

## Public Key Distribution Program Listing

### A 18.1 Main Program

```
                                          Comments
   PKD    :      LDA  C051
                 STA  COA8
                 LDA @  66
                 STA    FA
                 LDA  COA6
                 AND @  01
                 BEQ    L1
                 JMP NOLINK        Displays NOLINK Error
   L1            LDA @  6E
                 STA  COAD
                 LDA @  14
                 STA  COA4
                 JSR  BAUD         Sets Baud Rate
                 LDA  COAA
                 JSR  TRANS        Displays 'If wishing to Transmit
                                   enter your user ID and the
                                   receiver's user ID'.
   L4            LDA  COA6
                 AND @  03
                 BNE    L3
                 JMP    RX
   L3            LDA  C000
                 BPL    L4
                 JSR  FD6A
                 LDA @  33
                 STA  COAD
                 INX
                 LDY @  00
   L2            CPX @  00
                 BEQ    L5
   K1            LDA  COAB
                 BPL    K1
                 LDA  COAB
                 JSR  DELAY
                 AND @  01
```

```
          BEQ    K1
          LDA  0200,Y
          STA  COAC
          INY
          DEX
          JMP    L2
          LDA  0202
          AND  @  OF
          ASL
          ASL
          ASL
          ASL
          STA    F9
          LDA  @  16
          STA  COAD
K2        LDA  COAB
          JSR  DELAY
          AND  @  02
          BEQ    K2
          LDA  COAA
          CMP  0202
          BEQ    K3
          JMP    K4
K3        LDA  @  33
          STA  COAD
          JSR  SUB8        Stores the public key of B, $y_B$,
                           in 6100 to 610F
          JSR  INPUTKEY    Displays 'Input Secret Key'
          JSR  SUB9        Stores the secret key, $x_A$, in
                           6200 to 620F.
          JSR   EXP        Forms $K_{AB} = (y_B)^{x_A}$ in GF(2**127)
                           and clears 6200 to 620F. $K_{AB}$
                           stored in 6000 to 600F.
          JSR SUB10        Stores the common key $K_{AB}$ in the
                           KEY Register of the Data Security
                           Device.
          JSR  SUB11       Stores the session key, $K_s$, 6700
                           to 6707 in the DATA Register of
                           the Data Security Device.
```

```
             JSR  SUB12      Forms EKAB (Ks) and transmits it
                             to the receiver B.
             INC  6700
             JSR  SUB11      Forms EKAB (Ks +1) for
                             authentication purposes and stores
             JSR  SUB13      it in 6710 to 671F
             LDY  @ 00
             LDA  @ 16
             STA  COAD
L22          LDA  COAB
             JSR  DELAY
             AND  @ 02
             BEQ   L22
             LDA  COAA
             STA  6720,Y
             INY
             CPY  @ 08
             BEQ   L23
             JMP   L22
L23          CPY  @ 00
             BEQ   L24
             LDA  6720,Y     Verify received EKAB (Ks +1)
             CMP  6710,Y
             BNE   L25
             DEY
             JMP   L23
L24          JSR  SUB14      Load Ks into KEY Register of
             JSR  SUB10      Data Security Device
             JSR  SUB15      Clear all temporary memory locations.

             Jump to Block Encryption or
             Stream cipher Feedback or
             Cipher Block Chaining Program
             for DES encrypted data communication.

RX           LDA  @ 16
             STA  COAD
R1           LDA  COAB
             JSR  DELAY
             AND  @ 02
```

```
          BEQ     R1
          LDA   COAA
          AND @  OF
          ASL
          ASL
          ASL
          ASL
          STA     F9
          LDX @  02
T3        CPX @  00
          BEQ     T1
T2        LDA   COAB
          JSR   DELAY
          AND @  02
          BEQ     T2
          LDA   COAA
          STA     FD
          DEX
          JMP     T3
T1        LDA     33
          STA   COAD
          JSR   INPUTKEY
          JSR   SUB9        Stores the secret key of B, x_B, in
                            6200 to 620F.
R2        LDA   COAB
          BPL     R2
          LDA   COAB
          JSR   DELAY
          AND @  01
          BEQ     R2
          LDA     FD
          STA   COAC
          JSR   SUB8       Stores the public key of A, y_A, in
                           6100 to 610F.
          JSR    EXP       Forms K_AB =(y_A)^x_B  in GF (2 **127)
                           and stores it in 6000 to 600F.
          LDA @  16
          STA   COAD
          LDA   COAA
```

```
         LDY @  00
R6       CPY @  08
         BEQ    R5
R4       LDA  COAB
         JSR  DELAY
         AND @  02
         BEQ    R4
         LDA  COAA
         STA  6700,Y
         INY
         JMP    R6
R5       LDA @  0E
         STA  COA1
         JSR  SUB10      Stores the common key K_AB in the
                         KEY Register of the Data Security Device.
         JSR  SUB11      Stores the received encrypted session
                         key K_s in the DATA Register of the
                         Data Security Device.
         JSR  SUB13      Decrypted K_s is stored in 6710
                         to 671F.
         LDY @  00
R9       CPY @  08
         BEQ    R8
         LDA  6710,Y
         EOR  6008,Y
         STA  6700,Y
         INY
         JMP    R9
R8       LDA @  33
         STA  COAD
         INC  6700
         LDA @  06
         STA  COA1
         JSR  SUB11      Encrypts K_s+1 using K_AB and
         JSR  SUB12      transmits the cipher to the sender A.
         JSR  SUB14      Load the session key K_s into the
         LDA @  00       KEY Register of the Data Security
         STA  COA1       Device.
         JSR  SUB10
```

```
                JSR  SUB15        Clear all temporary memory
                                  locations
```

Jump to Block Encryption
or Cipher Feedback program
for encrypted data communication.

## A 18.2 Subroutines

Listings of some important subroutines used by the main program are given below.

```
                                      Comments
SUB8     :    LDY @ 00         Stores the public key in memory
      S2      CPY @ 10         locations 6100 to 610F and in
              BEQ   S1         6500 to 650F
              LDA  (F9),Y
              STA  6100,Y
              STA  6500,Y
              INY
              JMP   S2
      S1      RTS


SUB9     :    LDY @ 00         Stores the secret key in memory
      S4      CPY @ 10         locations 6200 to 620F
              BEQ   S3
              JSR  FDOC
              STA  6200,Y
              JSR  FDFO
              LDA   4E
              STA  6700,Y
              LDA @ 00
              STA  6000,Y
              STA  6400,Y
              INY
              JMP   S4
     S3       LDA @ 01
              STA  600F
     S6       CPY @ 00
              BEQ   S5
              LDA @ 88
```

```
                    JSR  FDF0
                    DEY
                    JMP    S6
         S5         JSR  FD8B
                    RTS

SUB10               LDA  @ 06       Stores the common key $K_{AB}$ in
                    STA  C0A1       the KEY Register of the Data
                    LDA  C0A3       Security Device.
                    AND  @ 10
                    BNE    S7
                    JMP  FF2D
         S7         LDY @  00
         S9         CPY @  08
                    BEQ    S8
                    LDA  6000,Y
                    JSR  PARITY CHECK
                    LDA    FB
                    STA  C0A0
                    LDA  C0A3
                    AND @  20
                    BEQ   S10
                    JMP  FF2D
         S10        INY
                    JMP    S9
          S8        RTS
SUB11      :        LDY  @ 00       Encrypts $K_s$ +1 under the
                    CPY  @ 08       key $K_{AB}$ using the DES
                    BEQ   S11       Block Encryption mode
                    LDA  C0A3
                    AND  @ 40
                    BNE   S12
                    JMP  FF2D
         S12        LDA  6700,Y
                    EOR  6008,Y
                    STA  C0A0
                    JMP   S13
         S11        RTS
```

```
SUB13     :      LDY  @ 00      Stores the encrypted $K_s$ +1 under
    S17          CPY  @ 08      the key $K_{AB}$ in memory locations
                 BEQ   S15      6710 to 671F.
    S16          LDA  COA3
                 BPL   S16
                 LDA  COA2
                 STA  6710,Y
                 INY
                 JMP   S17
    S15          RTS

SUB14     :      DEC  6700      Stores the session key in the
                 LDY  @ 00      memory locations 6000 to 600F
    S22          CPY  @ 08      which is then transferred to the
                 BEQ   S23      KEY Register of the Data Security
                                Device.
                 LDA  6700,Y
                 STA  6000,Y
                 INY
                 JMP   S22
    S23          RTS

SUB15     :      LDY  @ 00      Clears all temporary memory
    S24          CPY  @ 10      locations used in this PKD program.
                 BEQ   S25
                 LDA  @ 00
                 STA  6200,Y
                 STA  6700,Y
                 STA  6710,Y
                 STA  6720,Y
                 STA  6000,Y
                 INY
                 JMP   S24
    S25          RTS

SUB12     :      LDY  @ 00      Used to encrypt either $K_s$ under
    S18          CPY  @ 08      $K_{AB}$ or $K_s$+1 under $K_{AB}$ which
                 BEQ   S21      is then transmitted to the other
    S19          LDA  COAB      end.
```

```
            BPL   S19
            LDA  COAB
            JSR  DELAY
            AND @ 01
            BEQ   S19
S20         LDA  COA3
            BPL   S20
            LDA  COA2
            STA  COAC
            INC
            JMP   S18
S21         RTS
```

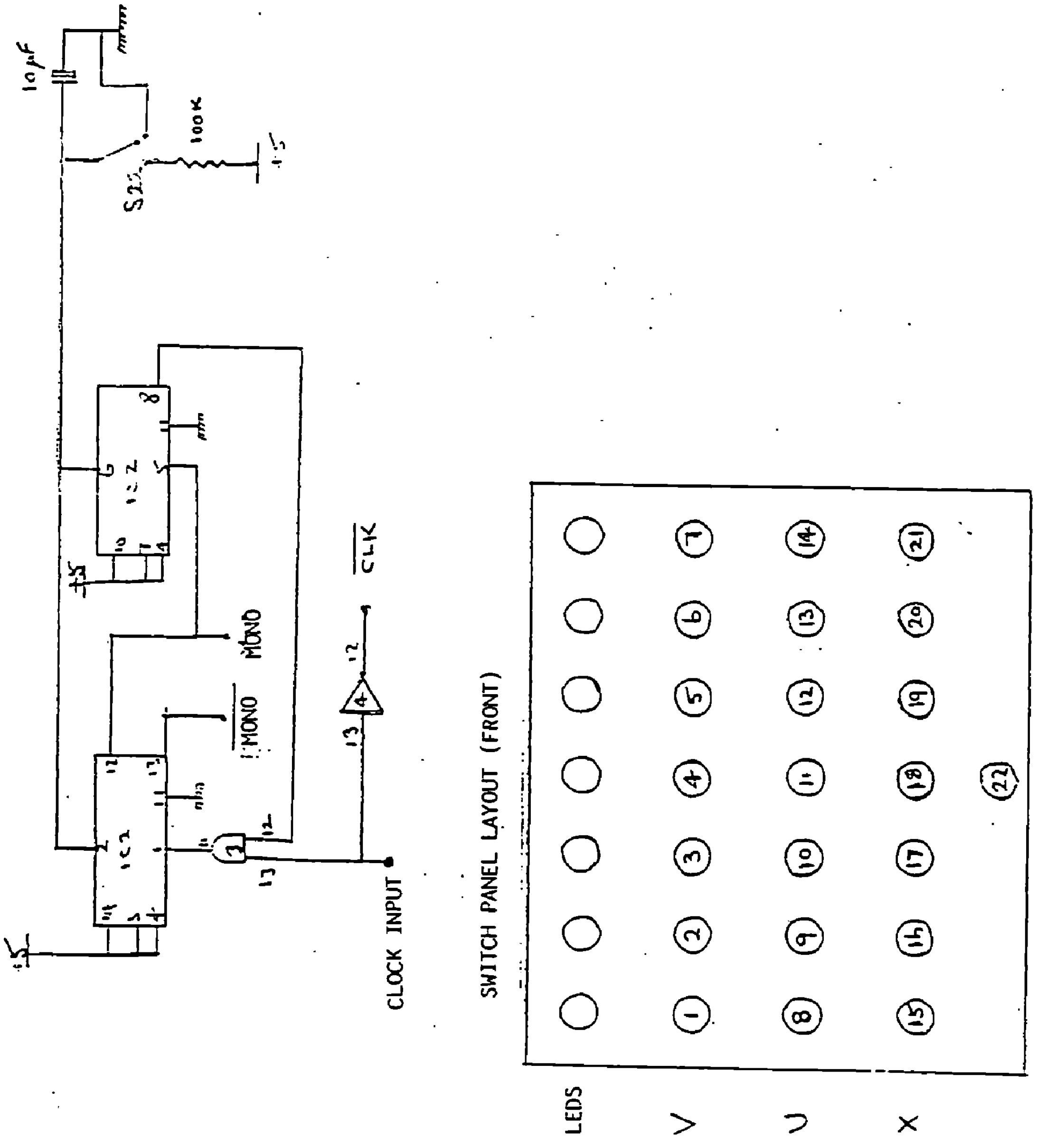
## Appendix 19

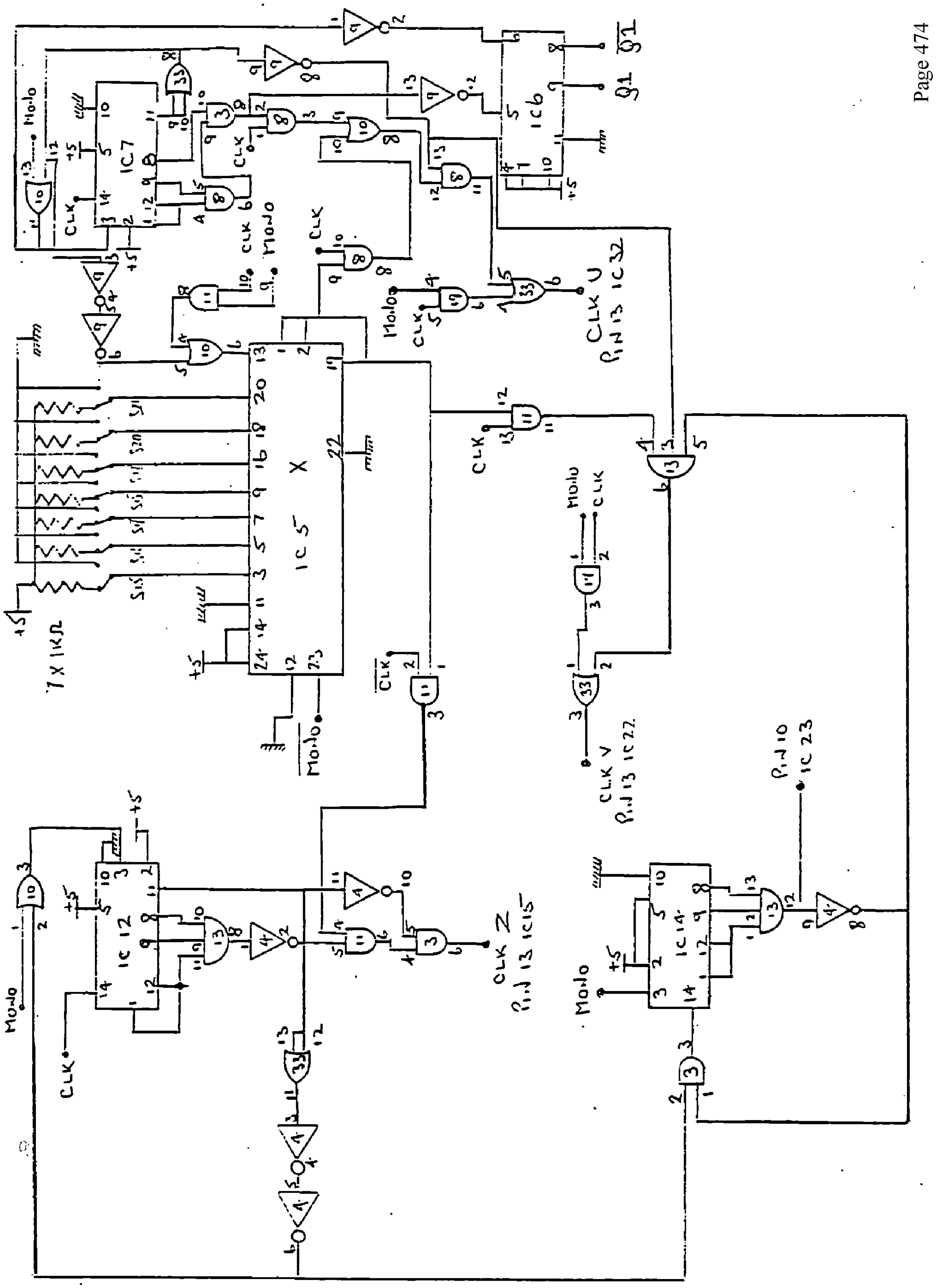
## Determination of A Normal Basis Generator in GF(2 * * 127) (FINDNORBAS. F77)

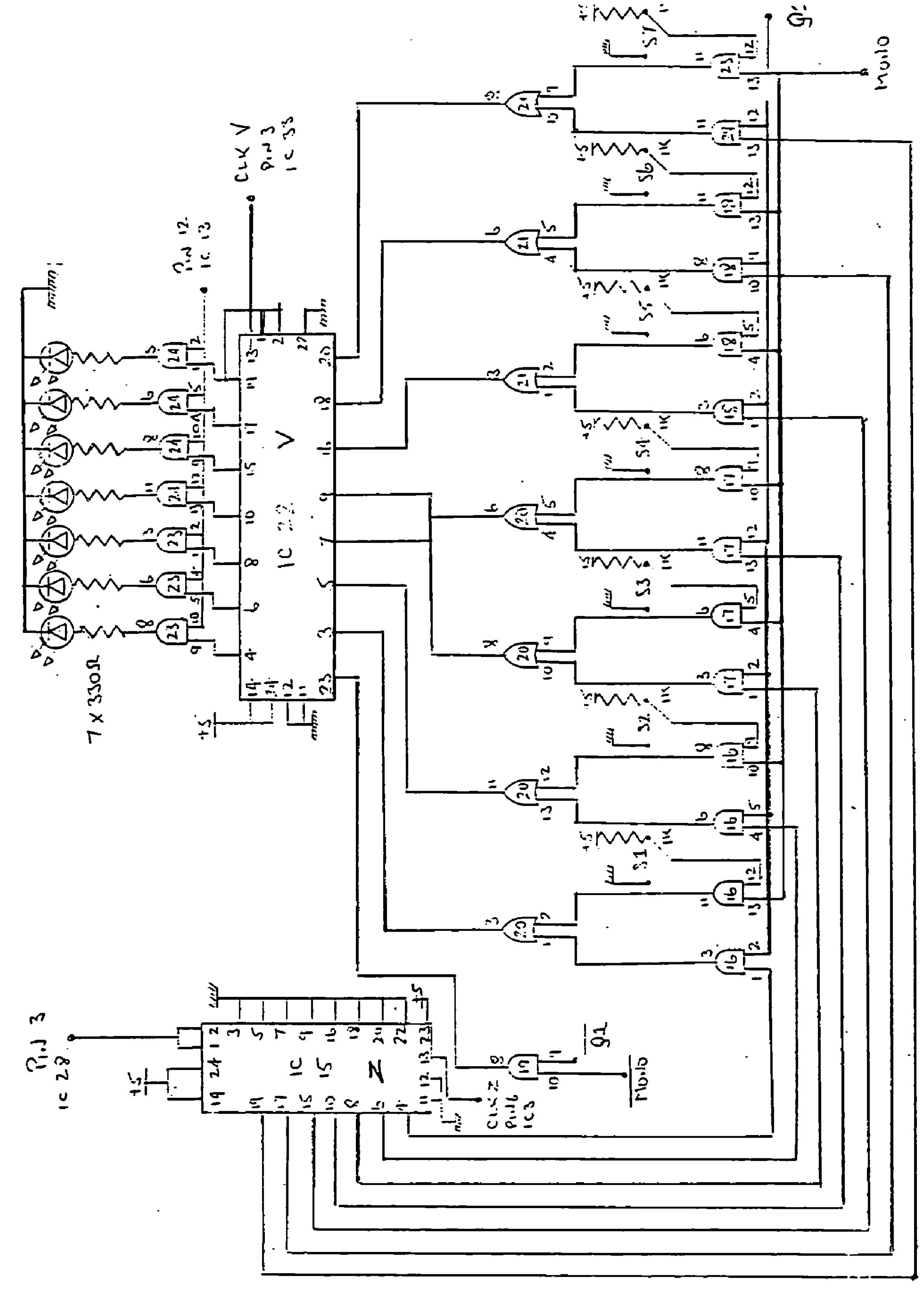
```
C  * * * * * * * * * * * * * * * * * * * * * *
C  This program selects elements of the form v = x * * ILL + 1
C  over GF (2 * * 127) where ILL varies from 1 to 126 and
C  computes the conjugates v * * 2, v * * 4, . . . , v * * (2 * *
C  (127-1)) over GF(2).  All operations are done modulo
C  the irreducible polynomial x * * 127 + x + 1.  Then it
C  determines whether these 127 conjugates are linearly
C  independent by calling the program DETMOD. F77 (Appendix 12).
C  DETMOD. F77 finds the determinant of the 127 x 127 matrix
C  formed by the 127 conjugates.  If the determinant is not
C  equal to zero, then the conjugates form a normal basis
C  over GF (2 * * 127).
C  * * * * * * * * * * * * * * * * * * * * * *

                INTEGER  TEMP(127), B(127, 127), V(127), C(127)
                ILL = 1

C  Choose an element over GF (2 * * 127) of the form
C  x * * ILL + 1
42              DO  1  I = 1, 127
1               V (I)  = 0
                V (ILL) = 1
                V (1)  =  1

C  Compute the 127 conjugates and form the matrix B
                IJ  = 1
                DO  6 IL = 1, 127
                B (IJ, IL) = V(IL)
6               CONTINUE
                IJ = IJ +1
100             I = 1
                C (1) = V(1)
                DO  2 I = 1,63
                J = 64 + I
                K = I + 1
                JJ = 2 * I
                KK = 2 * I + 1
                C (JJ) = V(J)
                L = V(K) + V(J)
                C (KK) = MOD (L,2)
2               CONTINUE
                DO  3  IK = 1, 127
                V (IK) = C(IK)
                B (IJ, IK) = C(IK)
3               CONTINUE
                IJ = IJ + 1
                IF (IJ.EQ.128) GO TO 110
                GO TO 100
C  Find the determinant of the matrix B using DETMOD. F77
                CALL  DETMOD (B, IDET)
                WRITE (1,210) ILL, IDET
210             FORMAT (1X, 'DET = ', I10, 2X, 'ILL = ', I5)
C  If the matrix B is singular over GF(2), increment ILL
C  by one and repeat the procedure until ILL = 128.  If the
C  matrix B is non-singular stop.
                IF (IDET. EQ. 0)  GO TO 810
                STOP
810             ILL = ILL + 1
                IF (ILL.EQ.128) GO TO 221
                GO TO 42
221             STOP
                END
```

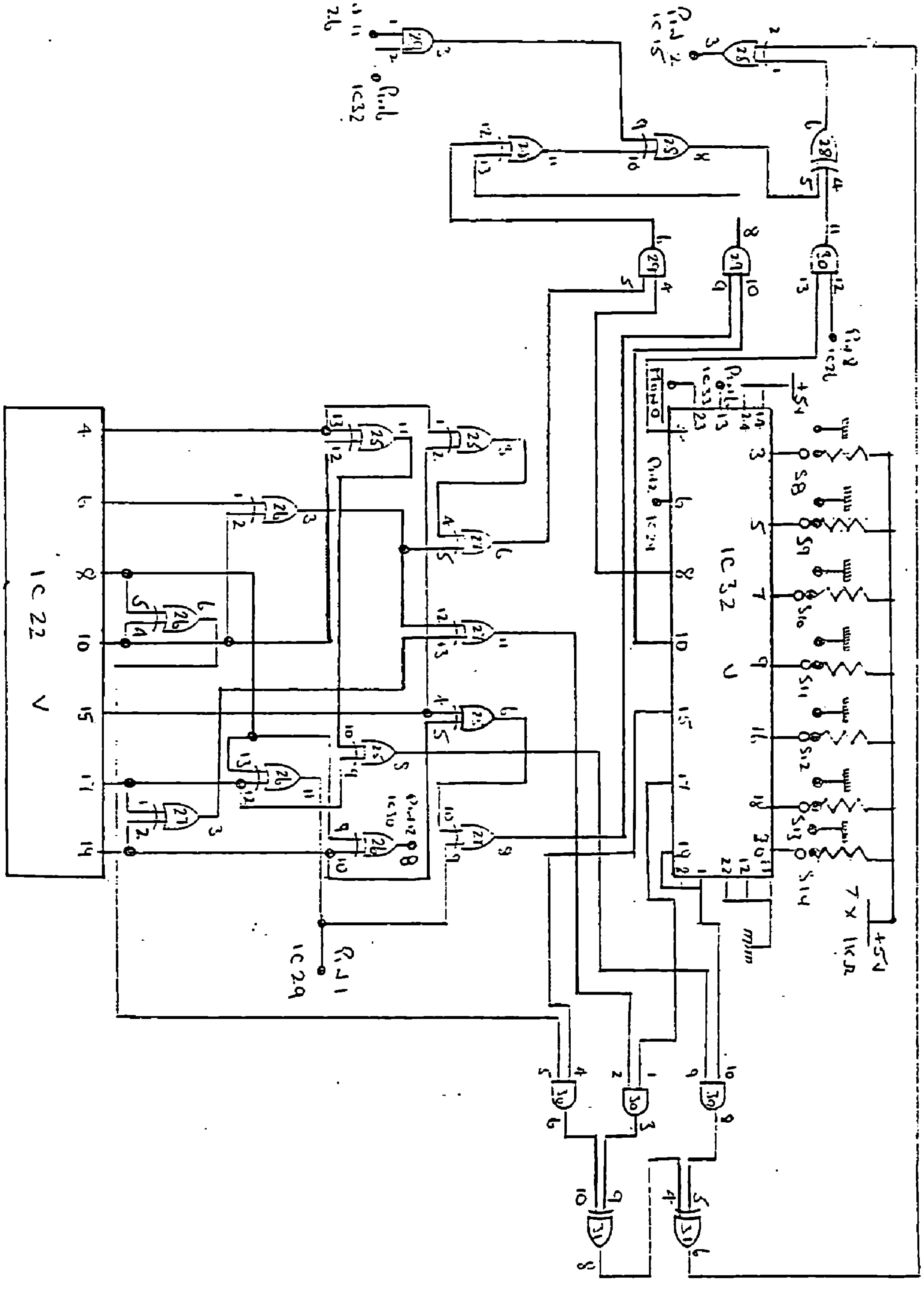
Appendix 20

Normal Basis Exponentiator - Circuit Diagram




IC7
IC6
IC5
IC12
IC14
X
Q1
$\overline{Q1}$
CLK U
Pin 13 IC32
CLK V
Pin 13 IC22
CLK Z
Pin 13 IC15
Pin 10
IC 23
MOJO
$\overline{MOJO}$
CLK
$\overline{CLK}$
+5
7X1KΩ


7 x 330Ω
IC 22
V
CLK V
Pin 3
IC 33
Pin 12
IC 13
Pin 3
IC 28
IC 15
Z
CLK Z
Pin 6
IC 8
Q1
Mono
S1
S2
S3
S4
S5
S6
S7
1K
+5


IC 22
IC 32
IC 29
IC 32
+5V
7 x 1KΩ
S8
S9
S10
S11
S12
S13
S14
25
26
27
28
29
30
31

## Appendix 21

## Multiplier Matrix Program (M-MATRIX.F77)

```
C * * * * * * * * * * * * * * * * * * * * * *
C This program calculates the M-matrix required for implementing
C multiplication over GF (2 * * N) using the normal basis
C generated by the program  FINDNORBAS. F77.  Here N = 127.
C * * * * * * * * * * * * * * * * * * * * * *

        INTEGER  B(127, 127),  U(127), V(127), Z(127), X(127, 127),
     +           S(8128, 127), INV(127, 127), X1(127, 127),
     +           FLAG, TEMP(127), T1, T2, T3, T.
        COMMON/B1/B
        COMMON/B3/S
        COMMON/B6/lNV
        COMMON/B7/U/V/Z/TEMP
        COMMON/B13/X
        COMMON/B14/X1

        N = 127
        N1 = N + 1
        N3 = N - 1
        N2 = 8128

C Read the matrix B which contains the 127 conjugates which
C form the normal basis matrix
        READ (5, *) ((B(I,J), J = 1,N), I = 1,N)
C Read the inverse of matrix B, lNV.
        READ (5, *) ((lNV(I,J), J = 1,N), I = 1,N)

C Form the composite matrix by multiplying the 127 conjugates
C arranged in a column vector with those arranged in a row
C vector.  Then reduce every element of this composite matrix
C modulo X * * 127 + X + 1.

        II = 1
        IP = 1
514     I = IP
```

```
          DO  400 J = 1, N
          U (J) = B (I,J)
          TEMP(J) = B(I,J)
400       CONTINUE
          I = I + 1
          DO  401 J = 1,N
          S (II,J) = B (I,J)
401       CONTINUE
512       DO 402 J = 1, N
          V(J) = B (I,J)
402       CONTINUE
          DO 202 IA = 1,N
          Z (IA) = 0
202       CONTINUE
          DO  220 IA = 1,N
          IF (V(IA).EQ.0)  GO TO 210
          DO 204 IB = 1, N
          IZ = Z (IB) + U(IB)
          Z (IB) = MOD (IZ,2)
204       CONTINUE
210       T1 = U(127)
          T2 = U(1)
          T3 = T1 + T2
          T = MOD (T3,2)
          DO 230 IC = 1, 125
          ID = 128 - IC
          IE = ID - 1
          U (ID) = U(IE)
230       CONTINUE
          U(2) = T
          U(1) = T1
220       CONTINUE
          DO 221 JJ = 1,N
          U(JJ) = TEMP(JJ)
221       CONTINUE
          II = II + 1
          DO 403 J = 1,N
          S(II,J) = Z(J)
403       CONTINUE
```

```
         I = I + 1
         IF (I.EQ.N1) GO TO 511
         GO TO 512
511      IP = IP + 1
         IF (IP.EQ.N) GO TO 513
         II = II + 1
         GO TO 514
513      II = II + 1
         DO 404  J = 1, N
         S(II,J) = B(1,J)
404      CONTINUE

C  Convert every element of the composite matrix S from
C  canonical basis representation to normal basis
C  representation using .INV matrix.

         DO 405 I = 1, N2
         DO 406 J = 1,N
         U(J) = S (I,J)
406      CONTINUE
         DO 407 II = 1,N
         IT = 0
         DO 408 J = 1,N
         IT = IT + U(J) * 1NV(J,II)
         IT = MOD (IT,2)
408      CONTINUE
         V(II) = IT
407      CONTINUE
         DO 409 J = 1,N
         S(I,J) = V(J)
409      CONTINUE
405      CONTINUE

C  Form the M-Matrix from the composite matrix by choosing
C  the coefficient of the conjugate v.

         I = 1
         J = 1
         II = 1
612      X(I,J) = S(II,1)
         X1(I,J) = S(II,2)
         I = I + 1
         II = II + 1
         IF (I.EQ.N1) GO TO 611
         GO TO 612
611      J = J + 1
         I = J
         IF (J.EQ.N1) GO TO 613
         GO TO 612
613      DO 410 I = 1,N
         DO 410 J = 1,N
         X (I,J) = X(J,I)
         X1 (I,J) = X1 (J,I)
410      CONTINUE

         WRITE (6,316)
316      FORMAT (1X, 'THE M-MATRIX IS')
         DO 6 IKK = 1, N
         WRITE (6,315) (X(IKK,JJJ), JJJ = 1,N
315      FORMAT (1X, 11 (I4, 2X))
6        CONTINUE
C  Check whether the M-matrix is correct by rotating the
C  rows one position downward and then shifting the rows
C  one position right and then comparing this M-matrix with
C  the M-matrix formed by choosing the coefficient of
C  conjugate v * * 2.
         DO 411  J = 1, N
         TEMP (J) = X(N,J)
411      CONTINUE
         DO 412 I = 1, N3
         I1 = N1 - 1
         II1 = I1 - 1
         DO 412 J = 1, N
         X(I1,J) = X(II1,J)
412      CONTINUE
         DO 413 J = 1,N
         X(1,J) = TEMP(J)
413      CONTINUE
         DO 414 I = 1,N
         TEMP (I) = X(I,N)
```

```
414         CONTINUE
            DO 415 J = 1, N3
            J1 = 128 -J
            JJ1 = J1 - 1
            DO 415 I = 1, N
            X(I,J1) = X(I,JJ1)
415         CONTINUE
            DO 416 I = 1, N
            X (I,1) = TEMP (I)
416         CONTINUE
            DO 417 I = 1, N
            DO 417 J = 1, N
            IF (X(I,J).EQ.X1(I,J)) GO TO 614
            GO TO 615
614         FLAG  = 1
417         CONTINUE
            WRITE (6,310) FLAG
310         FORMAT (1X, 'MATCHING OK; M-MATRIX CORRECT', I5)
            STOP
615         WRITE (6,311) I,J
311         FORMAT (1X, 'NON MATCHING; M-MATRIX WRONG',
            I5,2X,I5)
            STOP
            END
```

## Appendix 22

### Multiplier Implementation using T-Matrix Approach (T-MATRIX.F77)

```
C  * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *
C  This program calculates the number of EX-OR gates required to
C  implement the T-matrix.  It reads the M matrix as input and
C  forms the T-matrix.  Then it determines the number of'1's in
C  each row of the T-matrix, the number of EX-OR gates required
C  for each row and the total number of EX-OR gates required for
C  the whole T-matrix.
C  * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *

            INTEGER B(127,127), COUNT, COUNT1, TOTAL(127), SUM(127),
          +         NUM, RCOUNT, Z, REM , U(127,127)
            REAL   ANS

            N = 127

C  Read the M matrix
            READ (5,*) ((B(I,J), J = 1, N). I = 1, N)

C  Form the U-matrix from the M-matrix

            DO 10 II = 1, N
            DO 10 JJ = 1, N
10          U (II,JJ) = 0
            DO 2 I = 1, N
            IF (I.EQ.N)  GO TO 3
            DO 4 J = I + 1, N
            U (I,J) = B (I,J)
4           CONTINUE
3           IP = 1
2           CONTINUE

C  Count the number of '1's in each row and store it in
C  TOTAL array.

            DO 500 I = 1, N
            TOTAL (I) = 0
            DO 401 J = 1, N
            IF (U(I,J).EQ.1) GO TO 402
            GO TO 403
402         TOTAL (I) = TOTAL (I) + 1
```

```
403         IP = 1
401         CONTINUE
500         CONTINUE

C  Count the number of EX-OR gates required, NUM, for each row
C  and store it in SUM array.

            DO 400 I = 1, N
            NUM = 0
            RCOUNT = 0
            COUNT1 = TOTAL (I)
            IF (COUNT1.EQ.1) GO TO 600
            IF (COUNT1.EQ.0) GO TO 610
            GO TO 201
600         NUM = 1
            GO TO 601
610         NUM = 0
            GO TO 601
201         ANS = COUNT1/2.0
            COUNT = ABS (ANS)
            REM = COUNT1-COUNT *2.0
            RCOUNT = RCOUNT + REM
            NUM = NUM + COUNT
            IF (COUNT.EQ.1) GO TO 200
            COUNT1 = COUNT
            GO TO 201
200         IF (RCOUNT.EQ.1) GO TO 202
            IF (RCOUNT.EQ.0) GO TO 202
            IF (RCOUNT. GT. 1) GO TO 203
            STOP
203         COUNT1 = RCOUNT + 1
            RCOUNT = 0
            GO TO 201
202         NUM = NUM + RCOUNT
601         SUM(I) = NUM
400         CONTINUE

C  Write the number of EX-OR gates required for each row

            WRITE (6,460) (SUM(JJ), JJ = 1, N)
460         FORMAT (1X, 9(I6, 2X)
C  Calculate the total number of EX-OR gates for the whole
C  M matrix and print the result.
            Z = 0
            DO 450 II = 1, N
            Z = Z + SUM (II)
450         CONTINUE
            WRITE (6,461)Z
461         FORMAT (1X, I10)
            STOP
            END
```

## Appendix 23

### Exclusive-or Gate Count Program (EXORNO.F77)

```
C  * * * * * * * * * * * * * * * * * * * * * * *
C  This program counts the number of EX-OR gates required to
C  implement the M matrix without any optimisation.  It reads
C  the M matrix as input and outputs the total number of '1's
C  in each row of the M matrix, the number of EX-OR gates
C  required for each row and the total number of EX-OR gates
C  for the whole M matrix.
C  * * * * * * * * * * * * * * * * * * * * * * *

            INTEGER B(127,127), COUNT, COUNT1, TOTAL(127),
         +          SUM(127), NUM, RCOUNT, Z, REM
            REAL    ANS

            N = 127

C  Read the M matrix
            READ (5, *) ((B(I,J), J = 1, N), I = 1, N)

C  Count the number of '1's in each row and store it in TOTAL
C  array.
            DO 500 I = 1, N
            TOTAL (I) = 0
            DO 401  J = 1, N
            IF (B(I,J).EQ.1) GO TO 402
            GO TO 403
402         TOTAL (I) = TOTAL (I) + 1
403         IP = 1
401         CONTINUE
500         CONTINUE

C  Count the number of EX-OR gates required, NUM, for each
C  row and store it in SUM array.

            DO 400 I = 1, N
            NUM = 0
            RCOUNT = 0
            COUNT1 = TOTAL (I)
            IF (COUNT1.EQ.1) GO TO 600
            GO TO 201
```

```
600         NUM = 1
            GO TO 601
201         ANS = COUNT1/2.0
            COUNT = ABS (ANS)
            REM = COUNT1-COUNT * 2.0
            RCOUNT = RCOUNT + REM
            NUM = NUM + COUNT
            IF (COUNT.EQ.1) GO TO 200
            COUNT1 = COUNT
            GO TO 201
200         IF (RCOUNT.EQ.1) GO TO 202
            IF (RCOUNT.EQ.0) GO TO 202
            IF (RCOUNT.GT.1) GO TO 203
            STOP
203         COUNT1 = RCOUNT + 1
            RCOUNT = 0
            GO TO 201
202         NUM = NUM + RCOUNT
601         SUM (I) = NUM
400         CONTINUE
c  Write the number of EX-OR gates required for each row
            WRITE (6,460) (SUM(JJ), JJ = 1, N)
460         FORMAT (1X, 9 (I6, 2X))
c  Calculate the total number of EX-OR gates for the whole
c  M matrix and print the result.
            Z = 0
            DO 450 II = 1, N
            Z = Z + SUM (II)
450         CONTINUE
            WRITE (6,461) Z
461         FORMAT (1X, I10)
            STOP
            END
```

## Appendix 24

### Inverse Matrix over GF (2) Program (INVMOD. F77)

```
C  *  *  *  *  *  *  *  *  *  *  *  *  *  *  *  *  *  *  *  *  *
C  This program calculates the inverse of a given NxN matrix
C  over GF(2)
C  *  *  *  *  *  *  *  *  *  *  *  *  *  *  *  *  *  *  *  *  *


            INTEGER  B(127,254), TEMP (254), XM, X, XM2, XM3,
                     XM4, XM5, Y, XM6, XM7, XM8, NX, NY, LZ
            N = 127
            READ (1, *)  ((B(I,J), J = 1, N). I = 1, N)

            I = 1
            NX = N + 1
            NY = 2 * N
C  Form the identity matrix in columns NX to NY and rows
C  1 to N
            DO 81 IK = 1, N
            DO 81 IJ = NX, NY
            B (IK, IJ) = 0
81          CONTINUE
            DO 80 J = NX, NY
            B (I,J) = 1
            I = I + 1
80          CONTINUE
C  Compute the inverse matrix by performing elementary row
C  operations and store the inverse in columns NX to NY and
C  rows 1 to N
            L = 1
            K = 2
70          L1 = L
            XM = B(L,L)
            XM = ABS (XM)
            XM = MOD (XM,2)
            IF (XM.EQ.0) GO TO 215
```

```
214         DO 95 J = L, NY
            XM2 = B(L,J)
            XM2 = ABS (XM2)
            XM2 = MOD (XM2,2)
            XM2 = XM2/XM
95          CONTINUE
            DO 141 I = K,N
            X = B (I,L)
            X = ABS (X)
            X = MOD (X,2)
            IF (X.EQ.0) GO TO 141
            DO 140 J = L, NY
            XM3 = B(I,J)
            XM4 = B(L,J)
            XM3 = ABS (XM3)
            XM4 = ABS (XM4)
            XM3 = MOD (XM3,2)
            XM4 = MOD (XM4,2)
            B(I,J) = XM3 - XM4 *  X
            IF (B(I,J).EQ. -1) B(I,J) =1
140         CONTINUE
141         CONTINUE
            L = L + 1
            K = K + 1
            IF (L-N) 70, 190, 190
            XM = B(L,L)
            XM = ABS (XM)
            XM = MOD (XM,2)
            DO 195 J = L, NY
            XM6 = B(L,J)
            XM6 = ABS (XM6)
            XM6 = MOD (XM6,2)
            XM6 = XM6/XM
195         CONTINUE
            L = N
235         LZ =L  -1
            DO 291  K = 1, LZ
            I = L - K
            Y = B (I,L)
            Y = ABS (Y)
            Y = MOD (Y,2)
            IF (Y.EQ.0) GO TO 291
            DO 290 J = L, NY
            XM7 = B(I,J)
            XM7 = ABS (XM7)
            XM7 = MOD (XM7,2)
            XM8 = B(L,J)
            XM8 = ABS (XM8,2)
            B(I,J) = XM7 - XM8 * Y
            IF (B(I,J).EQ.- 1) B(I,J) = 1
290         CONTINUE
291         CONTINUE
            L = L - 1
            IF (L-1) 320, 320, 235
320         WRITE (1,330)
330         FORMAT (1X, 'THE INVERSE MATRIX IS')
            WRITE (1,331) (B(II,J), J = NX, NY)
331         FORMAT (1X, 11 (I5, 2X))
            CONTINUE
            STOP
215         L1 = L1 + 1
            N1 = N + 1
            IF (L1.EQ.N1) GO TO 212
            XM = B(L1,L)
            XM = ABS (XM)
            XM = MOD (XM,2)
            IF (XM.EQ.0) GO TO 215
            DO 96 J = L, NY
            TEMP (J) = B(L,J)
            B(L,J) = B(L1,J)
            B (L1,J) = TEMP (J)
96          CONTINUE
            GO TO 214
212         WRITE (1,813)
813         FORMAT (1X, 'THE MATRIX IS SINGULAR')
            STOP
            END
```

## Appendix 25.

## Matrix based Public Key Distribution Program (PKDEXT. F77)

```
C * * * * * * * * * * * * * * * * * * * * *
C This program is used to design a matrix based Public
C key distribution system based on exponentiation over GF(p)
C where p is a prime.
C This program essentially consists of a number of calls to
C different subroutines (programs) given in the previous
C appendices and hence only the main steps are given here.
C * * * * * * * * * * * * * * * * * * * * *


        INTEGER B1(2,2), B2(3,3), X(5,5), XINV(5,5),
     +          M(5,5),  A(5,5)
        p = 5
C Input the companion matrix B1 and check the order of B1
C using MATEXP. F77.
        READ (1,*)  ((B1(I,J), J = 1,2), I = 1,2)
        CALL MATEXP. F77
C Input the companion matrix B2 and check the order of B2
C using MATEXP. F77
        READ (1,*) ((B2(I,J), J = 1,3), I = 1,3)
        CALL MATEXP. F77
C Select an arbitary X matrix and find the determinant
C of X to see if it is non-singular modulo p using DETMOD.F77
        READ (1, *)  ((X(I,J), J = 1,5), I = 1,5)
        CALL DETMOD. F77
C Find the inverse of X using INVMOD.F77
        CALL INVMOD. F77
C Multiply the X matrix with the composite B1 B2 matrix
C and the XINV matrix using MULPKD. F77 which is given as
C a subroutine in MATEXP. F77.
        CALL MULPKD  (X, B1, B2, XINV, A)
C The A matrix is therefore the base matrix in the public
C key distribution system.  The system is then implemented
C using MATEXP. F77.
```

## Appendix 26

### Dickson Polynomials based PK System Program (DPOLY. F77)

```
C * * * * * * * * * * * * * * * * * * * * * *
C This program evaluates the coefficients of Dickson
C polynomials for GN(x) for given N over Z/mZ.  The value of N
C specified in this program is equal to the degree of
C polynomial plus 1 and the modulus m is equal to the product
C of two primes p and q.  The program then evaluates the
C function GN(x) for all x, 1 to m-1.
C * * * * * * * * * * * * * * * * * * * * * *

            INTEGER * 4  G0(462), G1(462), G2(462), H(462)

            N  = 6

C Use the recursive function to evaluate GN(x).

            N3 = N + 1
            N1 = N - 1
            N2 = N1 -1
            IFLAG = 0
            G0(1) = 2
            G0(2) = 0
            G1(1) = 0
            G1 (2) = 1
            DO  1  II = 3, N
            G0 (II) = 0
            G1 (II) = 0
            G2 (II) = 0
1           CONTINUE
            DO 2  I = 1, N2
            DO 3  J = 1, N
3           H(J) = G1 (J)
            DO 4 J = 1, N1
            K = N3 - J
            L = K - 1
            G1 (K) = G1 (L)
4           CONTINUE
            G1(1) = 0
            DO 5 J = 1, N
            G2(J) = G1(J) - GO(J)
            IM = G2(J)
            IF (IM.LT.0) IFLAG = 1
            IN = ABS (IM)
            IT = MOD (IN, 35)
            IF (IFLAG.EQ.1) GO TO 12
            G2 (J) = IT
            GO TO 13
12          G2(J) = 35 - IT
13          IFLAG = 0
            G1(J) = G2(J)
            GO(J) = H(J)
5           CONTINUE
2           CONTINUE

C  Evaluate the function for IX equal to 1 to 34.

            IX = 1
121         IZ = 0
            DO 16 IJ = 1, N
            JJ = G2 (I5)
            IF (JJ.EQ.0) GO TO 111
            IJ1  =  IJ -1
            IY = 1
            IF (IJ1.EQ.0) GO TO 112
            DO 17 II = 1, IJ1
            IY = IX *  IY
            IY = MOD (IY, 35)
17          CONTINUE
112         IY = IY *  JJ
            IY = MOD (IY, 35)
            IZ = IZ + IY
            IZ = MOD (IZ, 35)
111         IKK = 1
16          CONTINUE
```

```
            WRITE (1, 18) IX, IZ
18          FORMAT (1X, 'IX =', I16, 2X, 'IZ = ', I16)
            IX = IX + 1
            IF (IX.EQ.35)  GO TO 120
            GO TO 121
120         STOP
            END
```

## Papers Published/Submitted

# Secure communications between microcomputer systems

**P W Sanders and V Varadharajan** describe a security interface unit that uses the DES to encrypt sensitive data

*A data security communications interface unit has been developed to allow data transfer between Apple terminals in either plain or encrypted format under user control. The unit employs the Data Encryption Standard algorithm and has a degree of sophistication sufficient to meet most user needs. The unit uses the 6502 microprocessor to control encryption, decryption and communications. In addition to the transfer of encrypted data, the interface also provides a facility for storing encrypted program and data files locally in the Apple disc system. Further, the encryption system has been designed to allow storage and retrieval of completely encrypted or partly encrypted frames of information on the Prestel database. The interface has been tested extensively using several DES modes of operation.*

*Keywords: data communications, security, encryption, Data Encryption Standard, Prestel*

Data security has never been more significant than it is today, owing to the expanding role of distributed computation, distributed databases and telecommunications applications such as electronic mail and electronic funds transfer. Converging computer and communications technologies have resulted in a dramatic increase in the volume and speed of information collection and distribution. Greater information transfer in turn implies a greater risk of exposure of sensitive or confidential information to unauthorized users, owing to the ready availability of inexpensive miniature intercepting devices. These have resulted in an increased interest in computer data security, not only in the military and political area but also in the field of commerce. This has motivated research, particularly in the art of cryptography, which forms the central technique of communications security.

This article describes the design of an encryption interface unit employing the Data Encryption Standard[1] adopted by the US National Bureau of Standards. It has been designed primarily for the Apple microcomputer, which is commonly used as an intelligent terminal in communications networks. The purpose of this interface is threefold.

- It allows secure data communications in a point-to-point configuration.
- It provides a local storage facility in the Apple disc system for the encrypted program and datafiles.
- It allows storage and retrieval of encrypted or partly encrypted information on the Prestel viewdata system.

## SYSTEM DESIGN

The encryption system configuration for point-to-point data communications is shown in Figure 1. The communications link is half duplex, allowing transmission in either direction, but not simultaneously, with datarates ranging from 50–1 200 bit/s, suitable for transmission over standard telephone lines. The encryption unit is incorporated as an in-built feature of the terminal, this being superior to a stand-alone arrangement for reasons of access control, since the former technique greatly reduces the chances of

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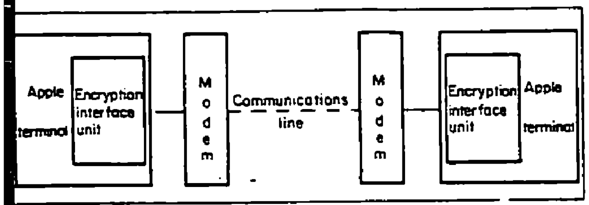


*Figure 1. Point-to-point system configuration*

detection between the terminal and encryption unit, where the text is in plain form. As shown in Figure 1, the plain text from the keyboard is encrypted by the interface and the cipher is transmitted over the standard telephone line via the modem.

A schematic diagram of the encryption interface showing different functional blocks is shown in Figure 2. The unit is designed around the 6502 microprocessor with associated random access memory (RAM) and read only memory (ROM) of the Apple system. Encryption and decryption of data is carried out by a large-scale integration device using the DES algorithm. Data communications is handled by a dual enhanced communications controller element (DEUCE). This controller contains two independent asynchronous receiver/transmitter channels and two independent generators providing possible transmission rates up to 19 200 bit/s. The encryption unit contains the necessary EIA RS232-compatible circuits for interfacing it to the modem, and a PROM is provided to hold the necessary programs for the operation of the system.

The interface can operate in three different formats; plain, encrypted or a mixture of both. The users at both ends of the communications link initially choose a datarate from the range of 50–1 200 bit/s and then select one of the three modes. In the case of the plain mode, the data transfer between the users will be in plain form. If either of the other two modes is chosen, then the secret DES key has to be entered. It is assumed that the parties concerned have preknowledge of the key, a necessity for proper communication of the encrypted data. Any eight alphanumerical characters of the keyboard can be used to form the 64 bit key required for the DES algorithm. It is essential that the key should be chosen randomly (for example, by some form of random number generator), so that it may not be easily guessed by any cryptanalyst. The key is displayed on the VDU to verify the correctness of entry but the display is erased immediately after the last character is input, to avoid detection by others during the course of communication. Many users would probably use some easy-to-remember phrase or number combination for developing the key, and in such cases the phrase can be converted to a form suitable for DES using a good

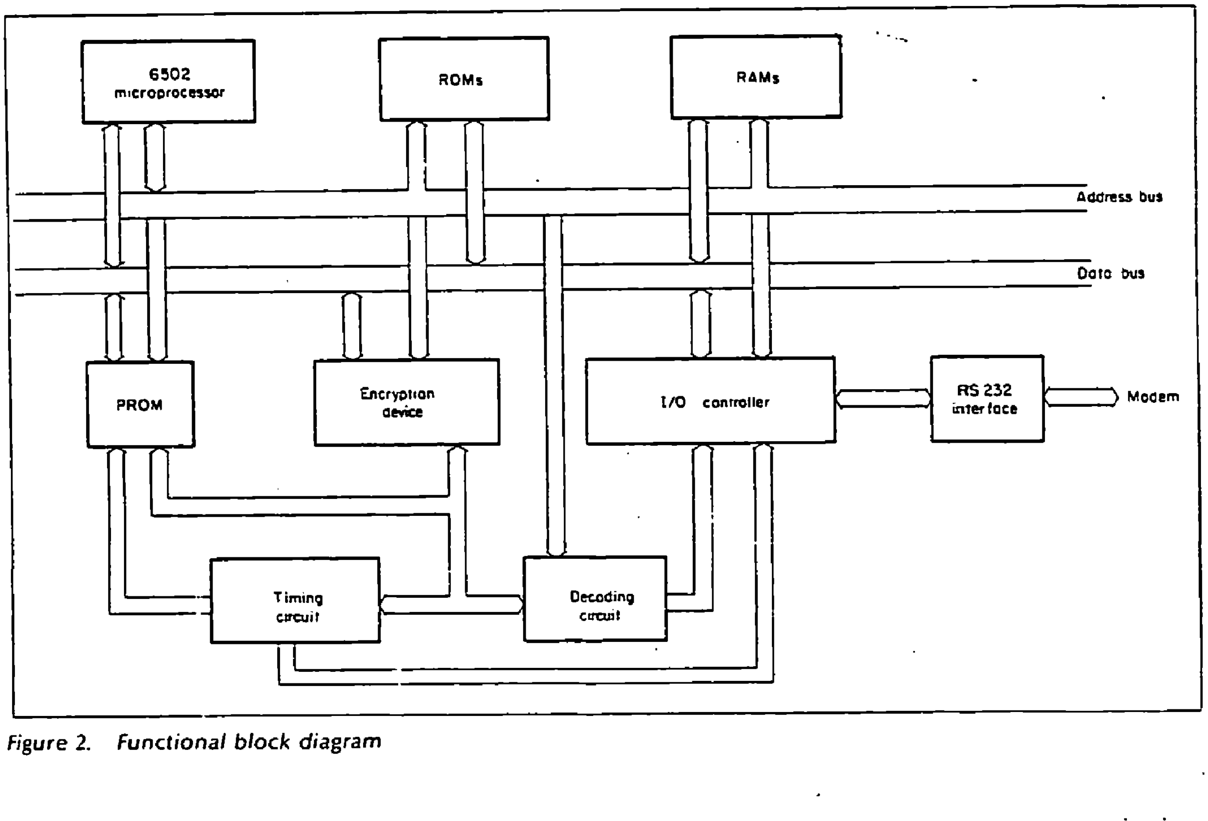


*Figure 2. Functional block diagram*

'hashing' function. Further minor software changes can be made to provide for multiple DES key encryption, to achieve higher levels of security or to modify facilities.

When the terminals are set ready to accept data, the user who presses a key first gains control of the line, and thus will be able to transmit data. A maximum of 256 bytes of message can be input to the system from the keyboard at one time. The message is encrypted using one of the several DES modes mentioned below, and then transmitted over the line. If the key is to be altered for the next message, the system must be reset. Further, data stored in memory can be encrypted and transmitted in a similar fashion.

In the case of a mixture of plain and encrypted information, the input from the terminal will always be in a plain form until a change to the encrypted mode is initiated by typing a special character (CNTRL-A). The unit is automatically returned to the plain format after 8 characters (64 bit). Alternatively, another special character (CNTRL-B) can be used to return to the plain format, but then one must ensure that the CNTRL-B character does not occur within the enciphered data to ensure unambiguous decryption at the receiving end. This can be achieved by using multiple CNTRL-B characters to indicate the end of encrypted text. The larger the number of such characters, the smaller the probability that they occur in the enciphered data, and the smaller the ambiguity in decryption, but this unfortunately increases the number of redundant characters in the transmitted data.

One of the requirements of this mixture format is that in a multiuser network, the plain information must only be deciphered correctly by the user with the right key. There may be cases where the encryption algorithm transforms a noncontrol character to a control character and vice versa. As a control character is not displayed by the Apple, this results in a line of text with parts of it encrypted at the transmitting end not producing a line of text at the receiving end when using the wrong key. As a certain amount of delay is required for special control characters such as CNTRL-G (Bell) or CNTRL-J (line feed), this can cause an over-run error at the receiving end, even in the reception of plain text when using the wrong key. Software has been written to overcome such situations.

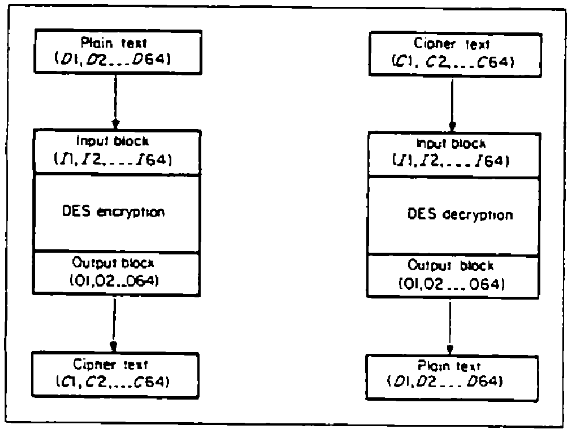


*Figure 3. Electronic code-book mode*

Several modes of the Data Encryption Standard have been investigated[2] using the developed encryption system, namely the electronic code book (ECB), cipher block changing (CBC) and stream cipher feedback (CFB).

The ECB allowed a transformation of a 64 bit plain text word into a cipher text word of the same length, as shown in Figure 3. In this mode, the information is encrypted in integral multiples of 64 bits. The last block is padded with random bits prior to encryption to build it up to 64 bits. During decryption, the padding is taken into account so that the random bits are discarded after decryption. A critical weakness of this mode (Figure 4)* is that a given plain text always produces the same cipher text under the same key. Thus, the compromise of the plain text underlying any cipher text block results in the compromise of all repetitions of this same text for the remainder of the cryptographic period. This problem is often referred to as the code-book analysis problem.

In the CBC mode, a plain text block is exclusive-ored with the previous cipher text block, prior to encryption, as shown in Figure 5. In the first encryption cycle, the plain text block is exclusive-ored with a block of pseudorandom bits called the initialization vector (IV). Mathematically, the scheme can be expressed as follows:

if the $i^{\text{th}}$ plain text and cipher text blocks are $x(i)$ and $y(i)$
and the initialization and feedback vectors are $U(1)$, $U(2) \ldots U(n-1)$
where $U(1) = z =$ initialization vector
$U(i) = y(i-1)$

then the encipherment and decipherment become

$$y(i) = f_k\{x(i) + y(i-1)\} \qquad i \geq 1$$
$$x(i) = f_k^{-1}[y(i) + y(i-1)] \qquad i \geq 1$$

where $x(0) = y(0) = z$, $f$ defines the cryptographic function and the subscript designates the particular key.

The security of this mode depends, amongst other factors, upon the management of the initialization vector. CBC reproduces the same cipher text wherever the same plain text is encrypted under a fixed key and initialization vector. In the ECB mode, the cipher text repetition is found to occur at block level, whereas in the CBC mode the cipher text repetition is at message level. The CBC mode not only reduces the code-book analysis problem but also provides a limited error extension characteristic which is valuable in protecting

*The ASCII character set has been extended using Hershey[3] characters to indicate all 256 possible codes produced by encipherment.

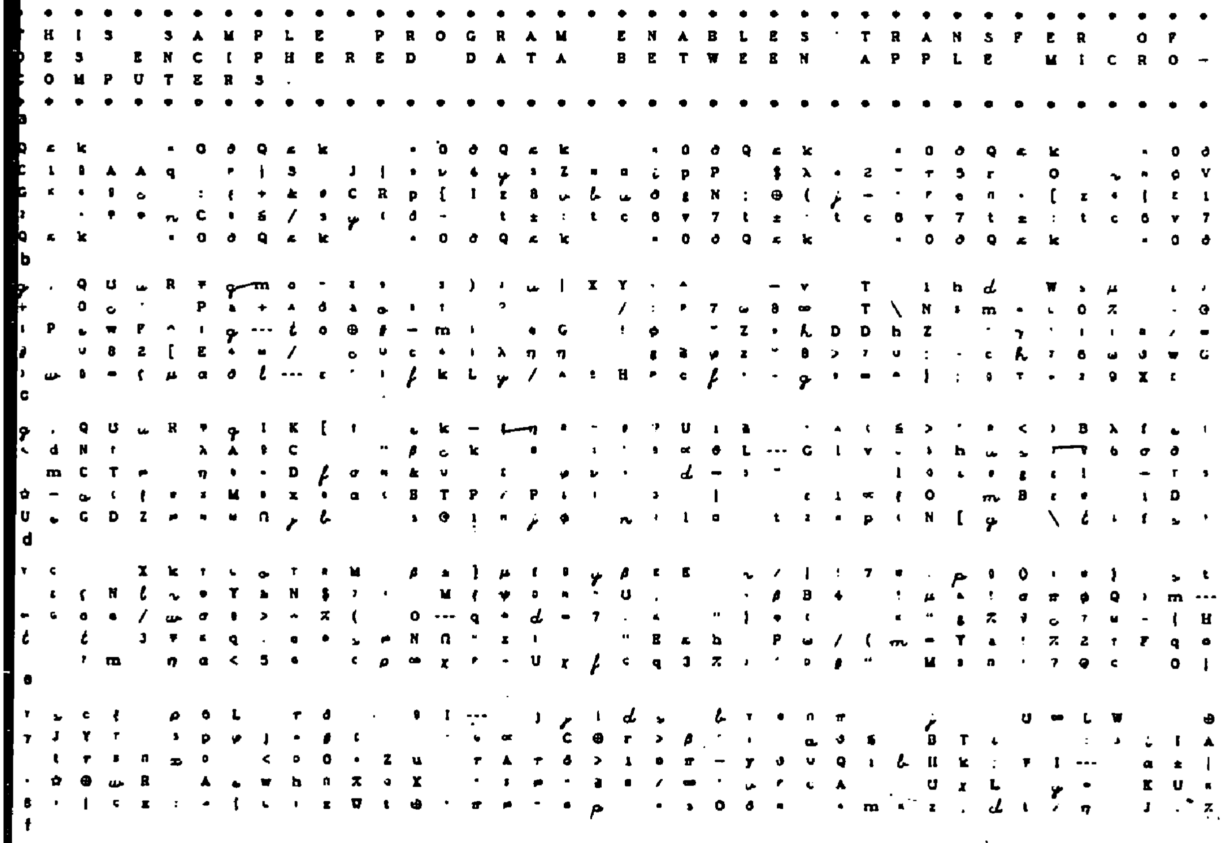

HIS SAMPLE PROGRAM ENABLES TRANSFER OF
ES ENCIPHERED DATA BETWEEN APPLE MICRO-
OMPUTERS.
b
c
d
e
f

*Figure 4. Examples of encrypted text*

*a Plaintext; b electronic code book; c cipher block chaining; d cipher block chaining with plaintext feedback; e stream cipher feedback; f stream cipher feedback with plaintext feedback*

against fradulent data alteration. One or more errors in a single cipher text block affects the decryption of two blocks, namely the block in which the error occurs and the succeeding block, but they synchronize thereafter (see Figure 6). This self-synchronizing scheme is particularly suitable when noise is present on the communications lines.

To provide for error propagation throughout the message, a slight variation of the CBC technique is implemented. Here the input plain text is modified by making it a function of both the previous plain text and the cipher text blocks prior to encryption, as shown in Figure 7. That is

$$y(i) = f_k[x(i) + U(i)] \quad i \geq 1$$
$$x(i) = f_k^{-1}[y(i) + U(i)] \quad i \geq 1$$

where

$$U(i) = \begin{cases} z & i = 1 \\ x\ (i-1) + y(i-1) & i > 1 \end{cases}$$

This scheme represents a general block cipher, and if any portion of the cipher becomes garbled, the decryption of all subsequent blocks until the end of the message is garbled (Figure 6). This technique is used for the purpose of message authentication.

While the CBC technique overcomes the codebook analysis problem, the problem of padding of the last block still remains. A stream cipher mode is therefore implemented to cope with this problem. With this technique, the DES is used as a random number generator. The output of the DES is exclusive-ored with plain text to form the cipher text. The decryption process operates the same way as encryption, with the exact pseudorandom stream of encrypting bits being generated. An 8 bit CFB implementation is shown in Figure 8, although any number of the 64 bits can be used. With this stream cipher technique, the plain text is encrypted character by character and not in blocks. An error in the cipher text character affects not only the decryption of the garbled cipher text but also the eight succeeding characters until the bit error is shifted out

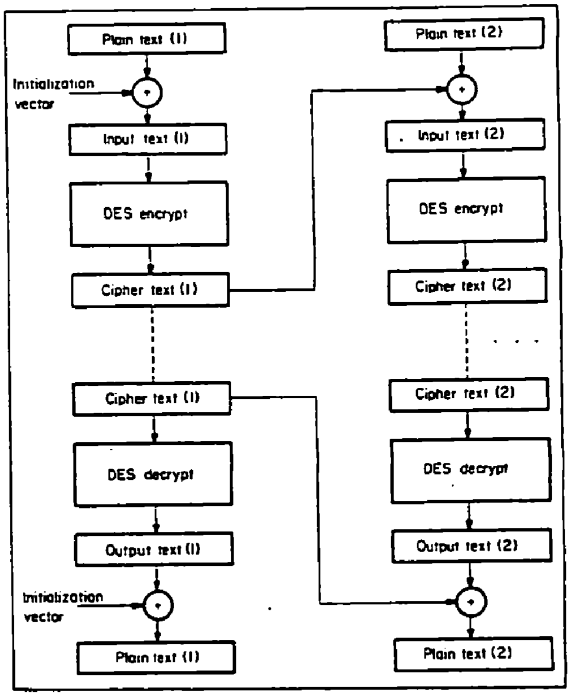


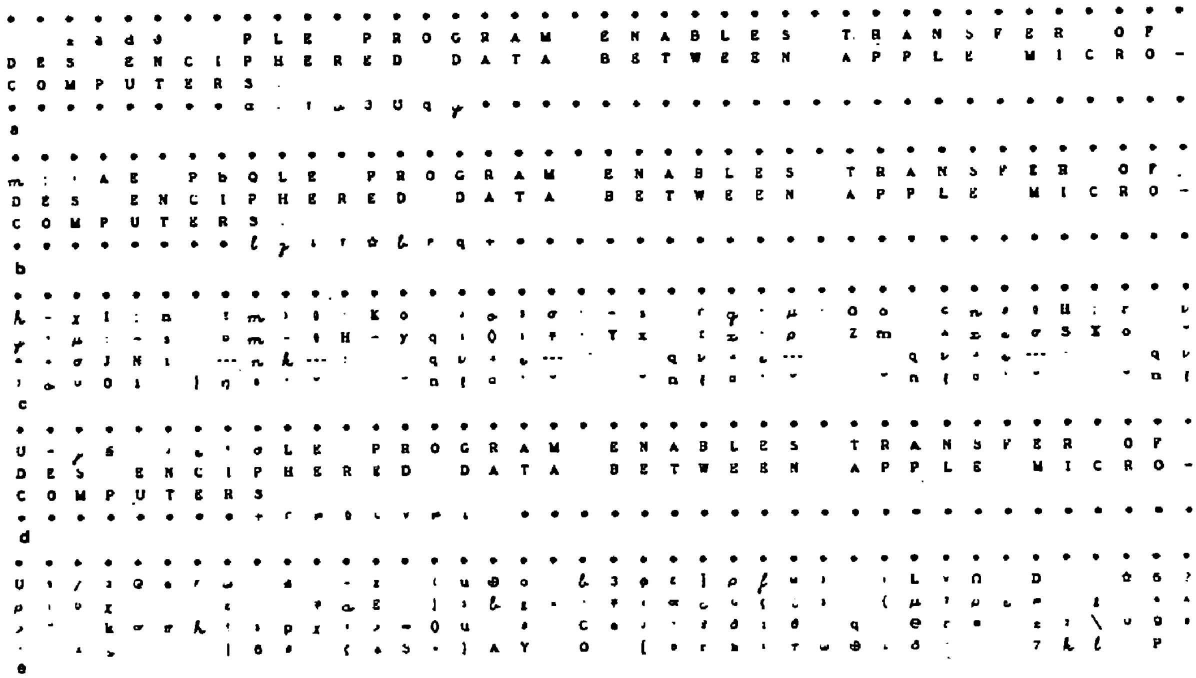
*Figure 5. Cipher block chaining mode*

of the CFB input block (Figure 6). Again this is a self-synchronizing scheme.

To provide for error propagation, a slight variation of this technique has also been implemented by providing a feedback from the initialization vector, in addition to the feedback from the cipher text as shown in Figure 9. This scheme represents a general stream cipher and corruption of a single bit of cipher text will cause each subsequent bit of recovered plain text to be in error (Figure 6).

The encryption system can also be used to provide storage of encrypted information locally in the Apple disc system. In this case, instead of transmitting the encrypted information to another user over a communications line, it is stored in memory.

Theoretically, any of the DES modes discussed above could be used, but when a file is encrypted, recovery from an error must be affected with cipher text alone. If a ciphering procedure with error propagation is used for file security, subsequent inability to read a fraction of the cipher text because of damage either to the physical medium or to the recorded bits may prevent all the following cipher text from being deciphered. Therefore a self-synchronizing approach is desirable for file encryption. This therefore leaves the two chaining modes CBC and CFB, either of which could be employed. If cipher feedback on 8 bit characters is used, then the maximum speed will be one eighth of the block mode speed, and hence, the



```
* * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *
    z a d J     P L E   P R O G R A M   E N A B L E S   T R A N S F E R   O F
D E S   E N C I P H E R E D   D A T A   B E T W E E N   A P P L E   M I C R O -
C O M P U T E R S
* * * * * * * * a   i   3 U q   * * * * * * * * * * * * * * * * * * * * * * * *
a
* * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *
m :   A E   P b Q L E   P R O G R A M   E N A B L E S   T R A N S F E R   O F .
D E S   E N C I P H E R E D   D A T A   B E T W E E N   A P P L E   M I C R O -
C O M P U T E R S .
* * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *
b
* * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *
[illegible]
c
* * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *
U -   6     O L E   P R O G R A M   E N A B L E S   T R A N S F E R   O F
D E S   E N C I P H E R E D   D A T A   B E T W E E N   A P P L E   M I C R O -
C O M P U T E R S
* * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *
d
* * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *
[illegible]
e
```

*Figure 6. Error characteristics*

*a Electronic code book; b cipher block chaining; c cipher block chaining with plaintext feedback; d stream cipher feedback; e stream cipher feedback with plaintext feedback*

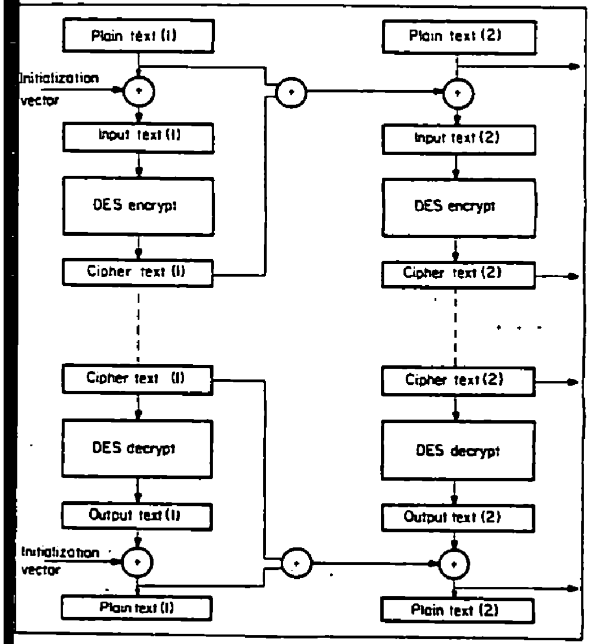


*Figure 7. Cipher block chaining with plain text feedback*

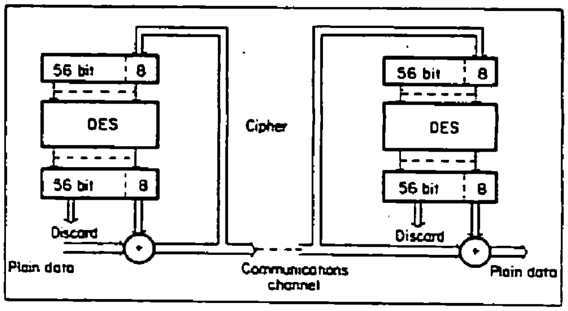


*Figure 8. Cipher feedback mode*

throughput is correspondingly lower. On the other hand, with CBC, the problem of padding exists, since it is a block cipher.

The program performs encryption and decryption of either an Applesoft BASIC file or Integer BASIC file, or an input datafile from the keyboard. The encrypted file is automatically stored on the disc under the file name provided by the user. The encrypted file can be loaded back from the disc at a later time and decrypted to give the original file, provided the same key has been used.

## SECURITY IN PRESTEL VIEWDATA SYSTEM

The encryption system has been incorporated into the Prestel network, allowing transfer of encrypted as well as plain information between an Apple microcomputer and a Prestel viewdata computer. This allows secure storage and retrieval of sensitive information, such as bank statements or legal documents.

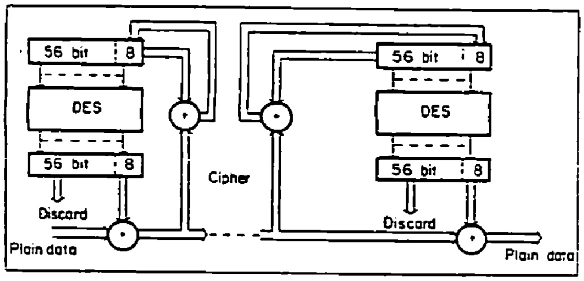


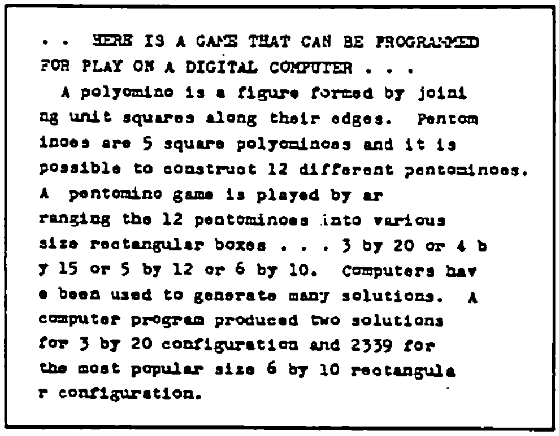
*Figure 9. Stream cipher with cipher text and plain text feedback*

The basic unit of information on Prestel is a frame which consists of up to a maximum of 960 characters. One or more frames are linked together to form a page, and these pages of information form the Prestel database. A natural choice for encryption would therefore be a complete frame, but there may be instances where the encipherment of a section of a frame, or even a few characters is required. The header information at the start of the frame can be used to indicate that encipherment has been used on that frame.

As in the case of file security, the two possible modes of DES suitable for this application are cipher block chaining (CBC) and stream cipher feedback (CFB). Since it is required to encrypt parts of a frame down to individual characters, only the cipher feedback mode, which allows character-by-character encryption, can be used. Further, if the CBC mode is used, when parts of a frame are encrypted, it is likely to require padding for each encrypted part. This in turn will result in cryptogram extension, and will pose a problem when storing the enciphered frame on the Prestel database, since each frame is limited to a maximum of 960 characters.

The encrypted information passes through the Prestel control unit, which rejects any of the control characters present in the cipher text. There is therefore a need to prevent the occurrence of these control characters in the encrypted information. That is, the data format is restricted to satisfy the Prestel computer protocols. This can be achieved by using 6 bit cipher feedback without altering the existing Prestel software. The 64 character codes chosen for encipherment are 0–9, A–Z, a–z, space and period. All other codes are transparent and hence bypass encryption. This is, of course, a weakness, but work is currently being carried out to expand this to a full character and graphics set. The output codes are reformed into the same range as the input, thus preserving the one-to-one relationship between transmission and reception. As we were mainly interested in enciphering alphanumerical characters present in the frame, the above set of input codes was found to be adequate for the purpose.

The system is connected to the public switched telephone network via a modem in the usual manner. The Prestel number is dialled, and when the Prestel computer responds by sending a continuous data tone of high frequency, the data switch is pressed for the modem to take control of the line. The terminal becomes ready for data transfer.



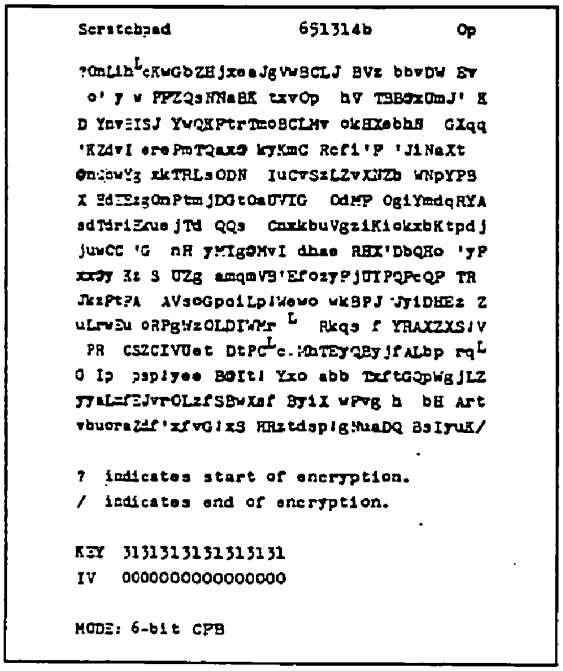
```
. .  HERE IS A GAME THAT CAN BE PROGRAMMED
FOR PLAY ON A DIGITAL COMPUTER . . .
  A polyomino is a figure formed by joini
ng unit squares along their edges.  Pentom
inoes are 5 square polyominoes and it is
possible to construct 12 different pentominoes.
A  pentomino game is played by ar
ranging the 12 pentominoes into various
size rectangular boxes . . . 3 by 20 or 4 b
y 15 or 5 by 12 or 6 by 10.  Computers hav
e been used to generate many solutions.  A
computer program produced two solutions
for 3 by 20 configuration and 2339 for
the most popular size 6 by 10 rectangula
r configuration.
```

*Figure 10. Prestel page in plain form*

After entering the secret DES key in the normal fashion, the user has the choice as to when to set the interface into the decryption mode. This enables him to decipher only those pages which have some enciphered data and to read the other Prestel pages in plain form. Only the user with the right key and correct initialization vector will be able to obtain the original plain text. This software implementation allows changes in the initialization vector during communication, whereas to change the key, the system needs to be reset and restarted. This allows every user to have a single secret key, although he may use any number of different initialization vectors.

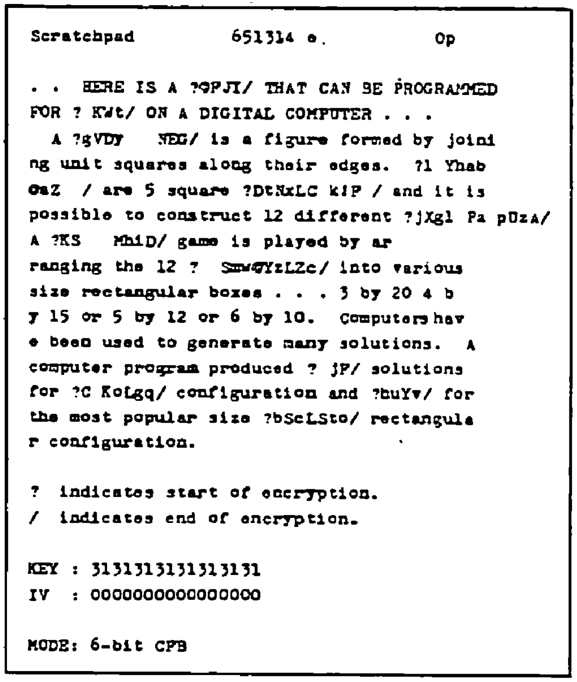
In the editing mode, the user is able to enter, amend, copy and delete encrypted as well as plain frames in Prestel. From the user point of view, it is essential that the encryption operations must be as simple as possible. Start (?) and stop (/) markers are used to indicate the beginning and end of enciphered data. The CNTRL-A key is used to set the interface unit to the encryption state. All subsequent characters input are encrypted under the CFB mode. The CNTRL-B key automatically returns the interface to the plain mode, and so allows encryption of even a single byte of data. Examples of completely encrypted and partly encrypted frames are shown in Figures 10, 11 and 12.

A stand-alone unit working on the same principles has also been developed for any computer connected between the RS232 interface output and the corres-



```
Scratchpad                651314b          Op

?OnLlhLcKwGbZHjxeaJgVwBCLJ BVz bbvDW Ev
 o' y w FPZQsNNaBK txvOp  bV TBBGxOmJ' K
D YnvEISJ YwQKPtrTmoBCLHv okHXebhH  GXqq
'KZdvI erePmTQax9 kyKmC Rcfi'P 'JiNaXt
OnQowYz zkTRLsODN  IuCvSzLZvXNZb WNpYPB
X EdEEzgOnPtmjDGtOaUVIG  OdMP OgiYmdqRYA
sdTdriEcuejTd QQs  CnzkbuVgziKiokxbKtpdj
juwCC 'G  nH yMIg9MvI dbae RHX'DbQHo 'yP
xx9y Kz S UZg amqmVB'EfozyPjUTPQPcQP TR
JkzPtPA  AVsoGpoiLp!Wewo wkBPJ Jy1DHEz Z
uLrwEu oRPgWzOLDIWMr L  Rkqs f YRAXZXS!V
 PR  CSZCIVUet DtPCLc.MnTEyQByjfALbp rqL
O Ip  psp!yee BOIt! Yxo abb TxftGQpWgjLZ
yyaL=fEJvrOLzfSBwXsf ByiX wPvg h  bH Art
vbucraZdf'xfvG!xS HRztdsp!gNuaDQ BsIyuK/

?  indicates start of encryption.
/  indicates end of encryption.

KEY  3131313131313131
IV   0000000000000000

MODE: 6-bit CFB
```

*Figure 11. Encrypted Prestel page*



```
Scratchpad           651314 e.          Op

. .  HERE IS A ?9PJI/ THAT CAN BE PROGRAMMED
FOR ? KWt/ ON A DIGITAL COMPUTER . . .
  A ?gVDy   NEG/ is a figure formed by joini
ng unit squares along their edges.  ?1 Yhab
OaZ  / are 5 square ?DtNxLC k!P / and it is
possible to construct 12 different ?jXgl Pa pUzA/
A ?KS   MhiD/ game is played by ar
ranging the 12 ?  SmwGYzLZc/ into various
size rectangular boxes . . . 3 by 20 4 b
y 15 or 5 by 12 or 6 by 10.  Computers hav
e been used to generate many solutions.  A
computer program produced ? jP/ solutions
for ?C KoLgq/ configuration and ?buYv/ for
the most popular size ?bScLSto/ rectangula
r configuration.

?  indicates start of encryption.
/  indicates end of encryption.

KEY : 3131313131313131
IV  : 0000000000000000

MODE: 6-bit CFB
```

*Figure 12. Partly encrypted Prestel page*

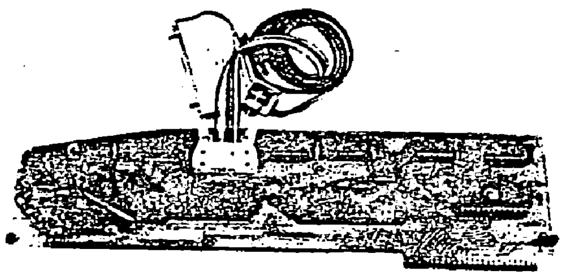


*Figure 13. Communications card providing secure storage and transmission between Apple microcomputers and any other computer*

ponding input of the modem. Figure 13 shows the communications board designed for the Apple microcomputer.

## REFERENCES

1 *Data encryption standard* FIPS Publ. 46, National Bureau of Standards, USA (January 1977)

2 *DES modes of operation* FIPS Publ. 81, National Bureau of Standards, USA (September 1980)

3 **Buckley, D C** *Hershey characters* Plymouth Polytechnic Computer Centre, Internal Handout No. U5.8-14 (February 1983)

## References

1 MARCATILI, E. A. J.: 'Optical subpicosecond gate'. *Appl. Opt.*, 1980, 19, pp. 1468–1476
2 HAUS, H. A., KIRSCH, S. T., MATHYSSEK, K., and LEONBERGER, F. J.: 'Picosecond optical sampling', *IEEE J. Quantum Electron.*, 1980, QE-16, pp. 870–874
3 LIU, P. L., and MARCATILI, E. A. J.: 'Computer simulation of the velocity matched gate'. *Appl. Opt.*, 1981, 20, pp. 862–866
4 ALFERNESS, R. C., KOROTKY, S. K., and MARCATILI, E. A. J.: 'Velocity matching techniques for integrated-optic traveling-wave switch/modulators'. *IEEE J. Quantum Electron.*, 1984, QE-20, (to be published)
5 ALFERNESS, R. C., JOYNER, C. H., BUHL, L. L., and KOROTKY, S. K.: 'High-speed traveling-wave directional coupler switch/modulator for $\lambda = 1{\cdot}32\,\mu$m', *ibid.*, 1983, QE-19, pp. 1339–1341
6 LEE, T. P., BURRUS, C. A., OGAWA, K., and DENTAI, A. G.: 'Very-high-speed back-illuminated InGaAs/InP PIN punch-through photodiode'. *Electron. Lett.*, 1981, 17, pp. 431–432
7 UEHARA, S.: 'Calibration of optical modulator frequency response with application to signal level control', *Appl. Opt.*, 1978, 17, pp. 68–71
8 SUETA, T., and IZUTSU, M.: 'High speed guided-wave optical components'. 3rd Int. Conf. on integ. opt. and opt. Fiber comm., San Francisco, Paper TUM2, 1981

# PUBLIC KEY DISTRIBUTION IN MATRIX RINGS

*Indexing terms: Codes, Public key systems*

An extension of the Diffie-Hellman public key distribution system to matrix rings is described. Using rings of non-singular matrices over $Z/pZ$ and upper triangular matrices with invertible elements along the diagonal over $Z/pZ$, it is shown that the number of possible secret keys is much greater for a given prime $p$ compared to the original system. An outline of a method to construct the base matrix used in the system is given.

*Introduction:* Diffie and Hellman[1] first proposed the idea of public key distribution in which two parties exchanging only public information over an insecure channel could establish a secret key for use in a conventional cryptosystem such as the data encryption standard.[2] They proposed a system based on the exponential function $f: x \rightarrow a^x$ over GF($p$), where $p$ is a very large prime and $a$ is a primitive element in GF($p$). The security of this system depends on the difficulty of computing logarithms over GF($p$). Pohlig and Hellman[3] investigated an algorithm for computing logarithms over GF($p$) and proved it to be efficient when $p - 1$ consists of small prime factors only, but computationally infeasible when $p - 1$ contains at least one large prime factor.

Here we investigate an extension of the exponentiation system to matrix rings.

*Extended exponentiation system:* The group formed by only the nonsingular matrices of order $n$, $M_n$, is considered as the ring of all $n \times n$ matrices over a finite field containing nilpotent elements when $n > 1$.

To form a public key distribution system, it is required to choose an element $A \in M_n(Z/pZ)$, where $p$ is a very large prime such that

$$A^r \equiv I \pmod p$$

where $r$ is the order of $A$, the base matrix.

The base matrix $A$, the prime $p$ and the order $r$ are to be made public. Each user chooses a random number $x_i$ less than $r$ and generates a public matrix $C_i$, where

$$C_i \equiv A^{x_i} \pmod p$$

Two users can arrive at the same key in the same way as in the Diffie-Hellman system. For instance, if user 1 wishes to initiate an interchange of secret information with user 2, he first extracts the public matrix $C_2$ of user 2. Then he computes

$$C_2^{x_1} \pmod p$$

User 2 obtains the public key $C_1$ of user 1 and computes

$$C_1^{x_2} \pmod p$$

It can be seen that this process yields the common key $K_{12}$, where

$$K_{12} = K_{21} = C_1^{x_2}(\text{mod } p) = C_2^{x_1}(\text{mod } p)$$
$$= A^{x_1 x_2}(\text{mod } p)$$

This can be used in the connection protocol of a DES based system to establish the session key. With the Diffie–Hellman system the maximum number of secret keys possible is limited to $p - 1$, whereas with this extended system it depends on the order of the base matrix: the larger the value of $r$, the greater the number of users that the system can support. Again the security of this system is dependent on the difficulty of computing logarithms over GF($p$).

*Design of base matrix A:* The system designer needs to construct a matrix $A$ in $M_n(F_p)$ and determine the order $r$. One method of construction of $A$ with a given order is outlined below.

Consider an irreducible polynomial $f(x)$ of degree $m$ for which $\lambda$ is a root ($\lambda \in F_q$):

$$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_{m-1} x^{m-1} \qquad a_i \in F_p$$

Regarding $F_q$ as an $m$-dimensional vector space over $F_p$ with basis $\{1, \lambda, \lambda^2 \ldots \lambda^{m-1}\}$, let $T$ represent the following linear transformation on $F_q$:

$$T: x \rightarrow \lambda x$$

Under $T$,

$$1 \rightarrow \lambda, \lambda \rightarrow \lambda^2, \ldots, \lambda^m \rightarrow -a_{m-1}\lambda^{m-1} \ldots -a_0$$

Hence the matrix representation of the linear transformation $T$ relative to the basis $\{1, \lambda, \ldots, \lambda^{m-1}\}$ is given by the companion matrix:

$$B = \begin{pmatrix} 0 & 1 & 0 \cdots\cdots 0 \\ 0 & 0 & 1 \cdots\cdots 0 \\ 0 & 0 & 0 \;\; 1 \cdots 0 \\ \vdots & & \\ -a_{m-1} & -a_{m-2} & \cdots\cdots\cdots -a_0 \end{pmatrix}$$

Linear independence of $I, T, T^2 \ldots T^{m-1}$ implies that $I, B, B^2 \ldots B^{m-1}$ are linearly independent. Since $f(x) = 0$, we have $f(B) = 0$. But $f(x)$ and degree $m$ and so the linear independence implies that $f(x)$ is the minimum function of $B$.

Hence the order of the matrix $B$ is equal to $p^m - 1$, and

$$B^{p^m-1} \equiv I \pmod p$$

Thus the system designer can choose irreducible polynomials of degrees $m_1, m_2 \ldots m_s$ over GF($p$) and form the composite matrix $B$ as shown below:

$$B = \begin{pmatrix} B_1 & & & \\ & B_2 & & \\ & & \ddots & \\ & & & B_s \end{pmatrix}$$

where the order of $B_1$ is equal to $p^{m_i} - 1$ for $1 < i < m_s$. The order of the matrix $B$ is then given by the expression

$$l_{cm}\{(p^{m_1} - 1), (p^{m_2} - 1), \ldots, (p^{m_s} - 1)\}$$

The matrix $A$ to be used in the public key distribution system can then be obtained by conjugating $B$ with an arbitrary matrix $Y$ belonging to $M_n(F_p)$:

$$A = YBY^{-1}$$

The order of $A$ is the same as that of $B$ and they are of dimension $n$, where $n$ is given by

$$n = \sum_{i=1}^{t} m_i$$

The outlined method has been implemented on the prime computer system and an example is given.

On the other hand, if we choose the base matrix $A$ from the ring of upper triangular matrices over $Z/pZ$, where $p$ is a prime, then the maximum order of such a matrix is equal to $(p-1)p$, which can be obtained by having nonzero elements along the main super diagonal. This can be shown as follows.

Partitioning $A$ into a diagonal matrix $D$ and an upper triangular nilpotent matrix $U$, i.e. $A = U + D$, then we see that $D \,.\, U$, $U \,.\, D$ and $U^2$ are also upper triangular nilpotent. Then, inductively, if

$$A^l = D^l + U_l$$

we have

$$\begin{aligned} A^{l+1} &= (A + U)(D^l + U_l) \\ &= D^{l+1} + \underbrace{UD^l + DU_l + UU_l}_{\text{nilpotent upper triangular}} \\ &= D^{l+1} + U_{l+1} \end{aligned}$$

Hence $A^{\phi(p)} = A^{p-1} = I + U_\phi$, where $\phi$ is the Euler function, and $(I + U_\phi)^{p^t} \equiv I$ for some $t$.

If $p$ is assumed to be greater than $n-1$, we have $t = 1$. Thus the order of $A$ is $(p-1)p$.

*Example:* Let $p = 5$. Let $f_1(x) = x^2 + x + 1$. $f_1(x)$ is irreducible over $Z/5Z$. The matrix $B_1$ is therefore given by

$$B = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \equiv \begin{pmatrix} 0 & 1 \\ 4 & 4 \end{pmatrix} \pmod 5$$

and

$$\begin{pmatrix} 0 & 1 \\ 4 & 4 \end{pmatrix}^{5^2-1} \equiv I \pmod 5$$

Let $f_2(x) = x^3 + 3x^2 + x + 2$. $f_2(x)$ is irreducible over $Z/5Z$. Hence the matrix $B_2$ is given by

$$B_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -1 & -3 \end{pmatrix} \equiv \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 3 & 4 & 2 \end{pmatrix} \pmod 5$$

and

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 3 & 4 & 2 \end{pmatrix}^{5^3-1} \equiv I \pmod 5$$

Now we need to choose $Y$ and $Y^{-1}$ such that

$$\begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} Y^{-1} = A \in M_n(F_p)$$

Let $x$ be an arbitary nonsingular matrix given below:

$$\begin{bmatrix} 2 & 3 & 1 & 0 & 1 \\ 1 & 2 & 2 & 3 & 1 \\ 1 & 0 & 3 & 2 & 4 \\ 1 & 2 & 1 & 4 & 3 \\ 2 & 4 & 0 & 1 & 1 \end{bmatrix}$$

(The determinant of $Y$ is 4 (mod 5) and hence $x$ is nonsingular.) The inverse of $Y$ is given by

$$Y^{-1} = \begin{bmatrix} 0 & 2 & 1 & 3 & 0 \\ 0 & 1 & 2 & 2 & 0 \\ 3 & 3 & 1 & 4 & 3 \\ 2 & 2 & 4 & 2 & 4 \\ 3 & 0 & 1 & 4 & 2 \end{bmatrix}$$

and hence

$$A = \begin{bmatrix} 0 & 2 & 0 & 4 & 3 \\ 1 & 1 & 3 & 1 & 3 \\ 4 & 0 & 0 & 3 & 2 \\ 3 & 3 & 2 & 4 & 4 \\ 1 & 2 & 4 & 1 & 1 \end{bmatrix}$$

The order of $A$ is equal to

$$l_{cm}\{(5^2 - 1)(5^3 - 1)\} = 744$$

and

$$A^{744} \equiv I \pmod 5$$

Hence the key space is $2 \leq x \leq 743$, where $A^x \equiv C \pmod 5$.

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*5th March 1984*

## References

1 DIFFIE, W., and HELLMAN, M. E.: 'New directions in cryptography', *IEEE Trans.*, 1976, IT-22, pp. 644–654

2 Data Encryption Standard, FIPS Publ. 46, National Bureau of Standards, USA, Jan. 1977

3 POHLIG, S. G., and HELLMAN, M. E.: 'An improved algorithm for computing logarithms over GF(*p*) and its cryptographic significance', *IEEE Trans.*, 1979, IT-24, pp. 106–110

[*] VARADHARAJAN, V., and ODONI, R. W. K.: 'Extension of RSA cryptosystem to matrix rings' (unpublished)

## ERRATUM

### Authors' correction

SPHICOPOULOS, T., TEODORIDIS, V., and GARDIOL, F.: 'Tractable form of the dyadic Green function for application to multilayered isotropic media', *Electron. Lett.*, 1983, 19, (24), pp. 1055–1056

Throughout the letter, TM and TE must be transposed

Submitted to the journal CRYPTOLOGIA.

# Extension of RSA Cryptosystem to Matrix Rings

## Abstract

A generalization of the RSA cryptosystem in the ring of matrices over Z/mZ is presented. It is shown that factorization of the modulus m is needed to compute the exponent of the group formed by either non-singular matrix messages or upper triangular matrices including diagonal elements thus offering the same level of security as the RSA system. The latter method employing the triangular matrices as messages seems to be more practical than the use of arbitrary non-singular matrix messages. The scheme is as suitable for privacy and authentication as its predececessor.

## 1. Introduction

Assume R is a finite ring with unity which is associative but not necessarily commutative. Suppose that members of the ring R are used as messages and that r∈R is enciphered as $r^e$ where e is the published encrypting exponent. The trapdoor property can be stated as follows:-

There exists some integer n>0 such that $r^{n+1} = r$ for all r∈R. These rings are to be referred to as trapdoor rings. For instance, in Z/pZ, $r^p = r$ for all r∈R. More generally, if we let R = $F_q$ = GF(q), the field of q elements where q is a prime power ($p^k$), then $r^q = r$ for all r∈R. Further if R and S are any two such trapdoor rings, then the direct sum (or product) R ⊕ S consisting of vectors (r,s) with r∈R and s∈S is another trapdoor ring say T. The number of elements in the ring T is equal to the product of the number of elements in R and S. This above procedure can be applied repeatedly taking vectors of arbitrarily many components each taken from some finite field. Considering finite fields $F_{q_i}$ for 1⩽i⩽j, where $q_i$'s can be the same or different, the trapdoor ring R is formed by all vectors x = ($x_1$, ..., $x_j$), where $x_i \varepsilon F_{q_i}$ for 1⩽i⩽j. The ring R consists of $q_1 q_2 \ldots q_j$ elements and the equality $r^{n+1} = r$ is obeyed for all r∈R, where n is equal to $(q_1-1)(q_2-1) \ldots (q_j-1)$ or any multiple of it.

There are many finite rings which are not trapdoor rings. Consider for instance, R = $Z/p^2Z$ where p is a prime. Then $p^2 \equiv p^3 \equiv \ldots \equiv 0$ in the ring R but p ≠ 0 in the ring R. So the property that $p^{n+1} \equiv p$ is not satisfied for any n>0. More generally, for a ring R to be a trapdoor ring, it is necessary that R have no nilpotent elements except zero. However, if we take an integer m to be a square free positive integer say $m = p_1 \dots p_j$ where all $p_i$'s are distinct primes, then the ring $R = Z/mZ$ is a trapdoor ring. This ring can in fact be regarded as a direct sum of $F_{p_1} \oplus F_{p_2} \oplus \dots \oplus F_{p_j}$ as described above. If $j = 2$, then this becomes the standard trapdoor ring used by the RSA cryptosystem[1]. This is a consequence of theorems of Wedderburn [2] on finite semisimple rings and skew fields.

The original RSA scheme derived its message space from the ring of integers modulo m, Z/mZ, where m is the product of two large distinct primes p and q. Here we investigate other finite systems that might serve as a basis for an extended RSA cryptosystem. Essential background material in group and ring theory can be found in [3].

## 2. Matrices over Z/mZ

If the ring of all n x n matrices over the ring Z/mZ is considered, it is seen that the ring contains nilpotent elements when $n>1$. To overcome this problem, initially only the group $M_n$ formed by the non-singular matrices of order n, is selected to form the message space of this extended system. But the use of such arbitrary non-singular matrices as messages poses further problems as the sender has no control over the matrix elements but must accept what the plaintext dictates. That is, the sender cannot always ensure that his messages will form non-singular matrices over Z/mZ. However, to begin with, the use of non-singular matrices as a possible message space is investigated in our matrix based RSA system.

Let us first consider the finite group formed by matrices of order n whose determinants are relatively prime to p and whose elements are in Z/pZ (p prime). The order of the group formed by these elements can easily be shown [4] to be equal to $N_p$ where

$$N_p = (p^n-1)(p^n-p) \dots (p^n-p^{n-1}) \qquad (1)$$

If such non-singular matrices with elements over Z/pZ are used as messages then one can form a conventional cryptographic system where the secret key contains the modulus p itself. The encrypting (e) and decrypting (d) exponents can then be determined using

$$ed = 1 \pmod{N_p} \qquad (2)$$

The encrypting key is therefore (e, p, n) and the decrypting key is (d,p,n). None of these keys can be made public and the encryption and decryption procedures are as in the RSA system.

$$\left.\begin{array}{l} C = M^e \pmod{p} \\ \text{and} \\ M = C^d \pmod{p} \end{array}\right\} \qquad (3)$$

The above system can be modified to include the public key property as follows: Suppose the modulus used in the system is a composite number m whose factorization is given by

$$m = \prod_{j=1}^{s} p_j^{r_j} \qquad (4)$$

Then, using the Chinese Remainder Theorem, the order $N_m$ [5-8] of the multiplicative group formed by non-singular matrices of order n over Z/mZ is given by

$$N_m = \prod_{j=1}^{s} N_{p_j}^{r_j} \qquad (5)$$

where

$$N_{p_j}^{r_j} = p_j^{(r_j-1)n^2} (p_j^n-1)(p_j^n-p_j) \ldots (p_j^n-p_j^{n-1}) \qquad (6)$$

Now as in the RSA cryptosystem if we take m to be the product of two distinct primes p and q, then the expressions (5) and (6) simplify to

$$N_m = N_p.N_q \qquad (7)$$

$$= (p^n-1)(p^n-p) \ldots (p^n-p^{n-1})(q^n-1) \ldots (q^n-q^{n-1}) \qquad (8)$$

Therefore for a message matrix $M \varepsilon M_n (Z/mZ)$

$$M^{N_m} \equiv I \pmod{m} \qquad (9)$$

and hence the coding exponents can be calculated using

$$ed \equiv 1 \pmod{N_m} \qquad (10)$$

The expression for the order $N_m$ depends on the structure of m, that is, on its prime factors. This can therefore be used to form a public key cryptosystem whose security is the same as that of the RSA system.

Although the order $N_m$ can be used in finding e and d usually the order is a very large number. For instance, even for small primes such as p = 13 and q = 23, the order is approximately $1.6 \times 10^{22}$ for 3 x 3 non-singular matrices. Therefore, it is desirable to find the exponent EXP of the group, i.e., the least integer greater than zero such that

$$M^{EXP} \equiv I \pmod{m} \qquad (11)$$

EXP is a divisor of the order $N_m$ of the group.

Let us first consider the exponent of the group formed by the non-singular matrices over Z/pZ, $M_n(Z/pZ)$.

Let the exponent be $\ell$ such that

$$A^{\ell} \equiv I \text{ for all } A \in M_n(Z/pZ) \qquad (12)$$

Assume that $p > n$. $A^{\ell} \equiv I \pmod{p}$ implies that $x^{\ell}-1$ is divisible by the minimum polynomial of A. As A ranges over the non-singular matrices of order n over Z/pZ, $x^{\ell}-1$ must be divisible by every monic irreducible polynomial $p(x)$ $(\neq x)$ of degree $\leq n$ in Z/pZ. Every irreducible polynomial $p(x)(\neq x)$ of degree u divides $x^{p^u-1}-1$. Thus $x^{\ell}-1$ must be divisible by $x^{p^u-1}-1$

But

$$x^b-1 \equiv 0 \pmod{x^a-1} \qquad (13)$$

implies $a|b$.

Hence,

$$\ell = 0 \pmod{p^u-1} \text{ for } 1 \leq u \leq n \qquad (14)$$

Therefore,

$$\ell = 0 \pmod{\ell cm\{p-1, \ldots, p^n-1\}} \text{ certainly} \qquad (15)$$

Furthermore, the matrix A given by

$$A = I + \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \cdot & \ddots & \ddots & \text{\Large 0} & \cdot \\ \cdot & & \ddots & \ddots & \cdot \\ \cdot & \text{\Large 0} & & \ddots & 0 \\ 0 & \cdot & \cdot & \cdot & 0 \end{bmatrix} . \qquad (16)$$

$$\text{satisfies } A^p = I \neq A \quad (p>n) \qquad (17)$$

That is, A has order p and hence $p | \ell$.
Hence the exponent of $GL(n,p), p>n$, is given by

$$\ell = p \ \ell cm \{ p-1, p^2-1, \ldots, p^n-1\} \qquad (18)$$

Now for any $A \in M_n(Z/pZ)$, using Jordan's Canonical form, there exists a non-singular matrix E such that

$$E^{-1}AE = \begin{bmatrix} B_1 & & & \\ & B_2 & & \\ & & \ddots & \\ & & & B_r \end{bmatrix} \qquad (19)$$

$$= D$$

Each block $B_i$ is of the form

$$B_i = \begin{bmatrix} \lambda_i & 1 & & O \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ O & & & \lambda_i \end{bmatrix} \qquad (20)$$

i.e. $B_i = \lambda_i I_i + N_i$ for some upper triangular nilpotent matrix $N_i$.

where $\lambda_i$'s are non-zero in $F_{p^{r_i}}$ for some $r_i \leq n$.

If the order of D is k, that is, $D^k \equiv I \pmod p$, then as $D = E^{-1}AE$, this gives

$$(E^{-1}AE)^k = (E^{-1}AE)(E^{-1}AE) \ldots \text{ k times}$$
$$= E^{-1}A^kE$$

$$\text{Thus } A^k \equiv I \pmod p \qquad (21)$$

Order of A = order of D = k = $\ell cm$ of orders of $B_i$.
If $N_i = O$, then order of $B_i$ is a divisor of $p^{r_i}-1$.
Hence the order of A divides $\ell cm \{p-1, \ldots, p^n-1\}$.
Otherwise, $B_i^p = \lambda_i^p I_i$ will have such an order and hence the order of A divides $p \ \ell cm \{p-1, \ldots, p^n-1\}$

A multiple of p in the expression for the exponent $\ell$ is expected as the order given by (1) contains multiples of p.

Similarly,

$$A^s = I \pmod{q} \text{ for all } A \text{ in } M_n(Z/qZ)$$

where $s = q\ \ell cm\ \{q-1,\ q^2-1,\ \ldots,\ q^n-1\}$ (22)

Therefore exponent EXP of the group GL(n,m) where m = p.q is given by

$$EXP = \ell cm\ (p\ \ell cm\ \{p-1,\ \ldots,\ p^n-1\},\ q\ \ell cm\ \{q-1,\ \ldots,\ q^n-1\}) \quad (23)$$

Now let us extend this argument to non-square free modulus m. First consider a matrix A in $M_n$ $(Z/p^2Z)$. Let $\theta$ be the natural homomorphism from $Z/p^2Z$ onto $Z/pZ$ (p prime). From the above argument

$$\theta(A)^{EXP} \equiv I \text{ in } M_n(Z/pZ) \quad (24)$$

$$\equiv \theta(I)$$

Therefore

$$\theta(A^t - I) \equiv 0 \pmod{p} \text{ where } t = EXP$$

This means that every entry in $A^t - I$ is some multiple of prime p and hence

$$A^t - I = p\ B \text{ for some matrix } B$$

That is,

$$A^t = I + pB \quad (25)$$

$$A^{tp} = (I+pB)^p$$

Using the binomial theorem,

$$A^{tp} = I + \binom{p}{1} pB + \binom{p}{2} p^2 B^2 + \ldots$$

$$\equiv I \pmod{p}$$

Therefore considering in general a matrix in $M_n\ (Z/p^h Z)$

$$M_n\ (Z/p^h Z) \longrightarrow M_n\ (Z/pZ)$$

$$\theta : \quad A \longrightarrow \theta(A)$$

If $\theta(A)$ has order t, then A has order t or pt or $p^2 t$ ... or $p^{h-1} t$.

If

$$m = \prod_{i=1}^{s} p_i^{r_i} \qquad (26)$$

then

$$\left.\begin{array}{l} EXP = \ell cm\ \{v_1, v_2, \ldots, v_s\} \\ \text{where } v_i = p_i^{r_i - 1}(w_i) \\ \text{and } \quad w_i = p_i \ \ \ell cm(p_i - 1, p_i^2 - 1, \ldots, p_i^n - 1) \end{array}\right\} \qquad (27)$$

(assuming $p_i$ is greater than n for all i.)

Again from (27), it is clearly seen that the exponent EXP depends upon the prime factor composition of m.

From cryptography point of view, the use of such non-singular matrices may be considered impractical as the sender has no control over the elements in the message matrix. The sender is faced with the problem of determining whether a plaintext message matrix is non-singular or not: if it is singular, he cannot encrypt that particular message. This is not acceptable for cryptographic application. However, the proportion of non-singular matrices over Z/mZ, where for instance m = pq and p and q are large district primes, is very much close to 1.

One common approach to obtain an arbitrary non-singular matrix over the reals is to have the diagonal entries of the matrix message much bigger than the corresponding entries in the row and the column. But this diagonal dominance does not always ensure that the matrix will be non-singular when working over finite rings. For instance, consider the matrix A given below which is 'diagonally dominant'.

$$A = \begin{pmatrix} p-1 & 1 \\ 1 & p-1 \end{pmatrix} \pmod p \qquad (28)$$

Det A = $p^2 - 2p \equiv 0 \pmod p$. Hence diagonal dominance is not applicable over finite rings.

Alternatively, let us now consider the set of upper triangular matrices as a possible choice of our message space. If the diagonal entries are made unity, this ensures that the matrix is invertible as the determinant is relatively prime to the modulus m.

Let M represent such an upper triangular message matrix. We can partition the matrix M into I+N where N is a nilpotent matrix and I is the identity matrix. If M is in $U_n(Z/pZ)$ then $(I+N)^p = I$ as $N^p = 0$ assuming $p \geqslant n-1$. The order of the group formed by these upper triangular matrices if $p^{n(n-1)/2}$. The order becomes $m^{n(n-1)/2}$ when considering matrices $U_n$ over Z/mZ. Here we see that the order of the group depends only on m and not on the factorization of m. Hence this is not suitable for a public key system.

On the other hand, if we alter the message space to contain upper triangular matrices with diagonal entries prime to m, then such messages are invertible modulo m. This is not a serious problem as in practice m is a product of large primes and the diagonal elements can be chosen to be relatively smaller integers. Now the order of the group formed by such matrices is determined as follows:

Considering a nxn matrix, it is required that all the n diagonal entries must be coprime to m. The number of integers less than m and coprime to m is given by the Euler totient function $\emptyset(m)$.

The remaining $\frac{1}{2}n(n-1)$ superdiagonal entries of the matrix may take any value modulo m. Therefore the order is equal to $m^{n(n-1)/2}\emptyset(m)^n$
The vital difference between this and the one calculated above is that now the order of the group is dependent on the prime factors of m. Hence the modulus m needs to be factorized before the decryption

exponent d can be calculated using ed = 1 mod (order). As for the set of non-singular matrices, one can determine the exponent of the group formed by these upper triangular matrices with diagonal entries prime to m. The exponent can be used instead of the order in finding e and d.

First consider a square free modulus m. Let

$$m = \prod_{j=1}^{s} p_j \qquad (29)$$

Let us first consider a message M in Z/pZ whose diagonal elements are relatively prime to p.

$$M = \begin{pmatrix} \alpha_{11} & & ? \\ & \ddots & \\ O & & \alpha_{nn} \end{pmatrix} \text{ where } (d_{ii}, p\ ) = 1 \text{ for all } i,\ 1 \leqslant i \leqslant n \qquad (30)$$

Partitioning M into a diagonal matrix D and an upper triangular nilpotent matrix U, that is, M = D+U, then it is seen that D·U, U·D and $U^2$ are also upper triangular nilpotent. Then inductively,

$$\text{if } M^r = D^r + U_r \qquad (31)$$

we have $M^{r+1} = (D+U)(D^r+U)$

$$= D^{r+1} + \underbrace{U.D^r + D.U_r + U.U_r}_{\text{nilpotent upper triangular}}$$

$$M^{r+1} = D^{r+1} + U_{r+1} \qquad (32)$$

Hence

$$M^{\emptyset(p)} \equiv I + U_{\emptyset} \text{ (mod p)} \qquad (33)$$

and

$$(I + U_{\emptyset})^p \equiv I + U_{\emptyset}^p \text{ (mod p)}$$

$$(I + U_{\emptyset})^{p^2} \equiv I + U_{\emptyset}^{p^2} \text{ (mod p)}$$

$$\vdots \qquad\qquad \vdots$$

etc

Thus

$$M^{p^{t}\emptyset(p)} \equiv I \pmod{p} \text{ for some } t \qquad (34)$$

If $p \geqslant n-1$, then $t = 1$

Therefore, the exponent of the group formed by upper triangular matrices (with invertible elements along the diagonal) over Z/pZ is equal to $\emptyset(p).p$

Now if

$$m = \prod_{j=1}^{s} p_j \qquad (35)$$

then the exponent divides

$$\ell\text{cm } \{\emptyset(p_1)p_1,\ \emptyset(p_2)p_2,\ \ldots\ ,\ \emptyset(p_j)p_j\} \qquad (36)$$

(Note that if the diagonal entries are unity then the exponent is equal to m or 2m, if m is even or odd respectively.)

Let us now consider a non-square free modulus m given by

$$m = \prod_{j=1}^{s} p_j^{r_j} \qquad (37)$$

First consider an nxn upper triangular non-singular matrix M over $Z/p^r$ Z, $(p > n)$. Again let $M = D + U$ where D is a diagonal matrix and U is an upper triangular nilpotent matrix over $Z/p^rZ$. Following the argument given above, it is seen that

$$M^{\emptyset(p^r)} \equiv I + U_\emptyset \quad (\text{mod } p^r) \qquad (38)$$

where $U_\emptyset$ is some upper triangular nilpotent matrix.

Hence

$$(I + U_\emptyset)^p \equiv I + p\, U_{\emptyset_1} \quad (\text{mod } p^r)$$

$$(I + pU_{\emptyset_1})^p \equiv I + p^2\, U_{\emptyset_2} \quad (\text{mod } p^r)$$

$$\vdots \qquad\qquad \vdots$$

$$(I + p\, U_{\emptyset_{r-1}})^p \equiv (I + U_\emptyset)^{p^r} \equiv I + p^r\, U_{\emptyset_r} \equiv I \ (\text{mod } p^r)$$

Therefore $M^{\emptyset(p^r)p^r} \equiv I \pmod{p^r}$ (39)

Hence the exponent of the group formed by upper triangular non-singular matrices over Z/mZ, where m is given by (37), divides

$$\ell cm \{\emptyset(p_1^{r_1})p_1^{r_1}, \emptyset(p_2^{r_2})p_2^{r_2}, \ldots, \emptyset(p_s^{r_s})p_s^{r_s}\} \qquad (40)$$

## 3. System Design and Operation

The designer randomly chooses large primes $p_1$ to $p_s$ for some $s \geqslant 2$ following the guidelines suggested in [1] and forms their product m. The primes need not be necessarily distinct. As in the case of the RSA system, these primes are kept secret. As the upper triangular non-singular matrix messages can be selected arbitrarily a public key cryptosystem using such a message space has been implemented. The coding exponents e and d are determined using

$$ed = 1 \pmod{EXP} \qquad (41)$$

where EXP can be equal to (40).

The public encryption key is given by (e,m,n) and the secret decryption key is (d,m,n) where n denotes the dimension of matrix messages.

The message is divided into blocks of integers less than the modulus m and a sequence of nxn upper triangular matrices is constructed by arranging the integers in order as they occur, left to right and top to bottom. The encryption procedure consists of raising each of these upper triangular matrices to the power e. This has been implemented using the well known square and multiply technique [9]. The ciphertext produced consists of a sequence of nxn upper triangular matrices over Z/mZ. Each of these matrices is transmitted to the receiver by sending the n(n+1)/2 ciphertext matrix elements (excluding the lower triangular zeroes) in order as they occur in the matrix, left to right, top to bottom, with a space separating the elements. The receiver recovers the original message by first reconstructing the sequence of nxn upper triangular cipher matrices and then raising them to the power d modulo m.

Such an extended RSA matrix system using upper triangular message matrices has been simulated on the Prime computer. An example showing the various parameters is given in Figure 1.

## 4. Discussion

One can see that the RSA system can be extended to matrix messages provided the message space is restricted to avoid nilpotent elements. In this paper, the group of non-singular matrices over Z/mZ and the group of upper triangular matrices with diagonal elements coprime to m over Z/mZ have been investigated. In both these cases, factorization of the modulus m into primes is required to compute the exponent thus providing the same level of security as the RSA system. But the first case, employing arbitrary non-singular matrices as the message space does not seem to be suitable for a practical cryptosystem as it is not possible to restrict arbitrary plaintext matrices to be non-singular. On the other hand, in the second case employing upper triangular non-singular matrices, it appears that the messages can be selected arbitrarily in practice. The exponent of the group formed by such upper triangular nxn matrices over Z/mZ divides $\ell cm\ (\emptyset(p_1{}^{r_1})p_1{}^{r_1}, \ldots, \emptyset(p_s{}^{r_s})p_s{}^{r_s})$ where $m = \prod_{j=1}^{s} p_j{}^{r_j}$ with $p_j > n$ for all $1 \leq j \leq s$.

Further two points are worth mentioning regarding this extended matrix system. Firstly, it is seen that non-square free modulus can be used which is not possible with the RSA system. Secondly, the use of a matrix as a message may allow large amounts of data to be processed within one encryption/decruption cycle. Whether this is an advantage depends upon the ease with which matrix manipulations can be carried out. Computational savings can be achieved if transform techniques are used to reduce the number of scalar multiplications involved in a matrix multiplication.

### Example

Let the modulus $m = p.q = 41.29 = 1189$

Exponent of the group formed by 3x3 upper triangular non-singular matrices over Z/1189Z, divides

$\ell cm\ \{40.41,\ 28.29\} = 332920$

Choosing the encrypting exponent, $e = 1317$, the decrypting exponent d can be calculated using Euclid's algorithm and is equal to 117293. That is,

$$1317\ .\ 117293 \equiv 1\ (\mathrm{mod}\ 332920)$$

Let us assume that the plaintext message to be encrypted is 232677205141. In this example, the message is divided into 2-digit blocks of integers less than m. Starting from right to left as

(23 26 77 20 51 41)

The upper triangular plaintext message matrices become

$$P_1 = \begin{bmatrix} 90 & 23 & 26 \\ 0 & 50 & 77 \\ 0 & 0 & 48 \end{bmatrix} \quad \text{and } P_2 = \begin{bmatrix} 31 & 20 & 51 \\ 0 & 215 & 41 \\ 0 & 0 & 289 \end{bmatrix}$$

where the diagonal elements are arbitrarily chosen to be relatively prime to 1189.

The ciphertext message matrices are then given by

$$C_1 = \begin{bmatrix} 90 & 23 & 26 \\ 0 & 50 & 77 \\ 0 & 0 & 48 \end{bmatrix}^{1317} \equiv \begin{bmatrix} 1105 & 458 & 1070 \\ 0 & 50 & 251 \\ 0 & 0 & 831 \end{bmatrix} \pmod{1189}$$

$$C_2 = \begin{bmatrix} 31 & 20 & 51 \\ 0 & 215 & 41 \\ 0 & 0 & 289 \end{bmatrix}^{1317} \equiv \begin{bmatrix} 843 & 774 & 660 \\ 0 & 592 & 41 \\ 0 & 0 & 405 \end{bmatrix} \pmod{1189}$$

These ciphertext matrices are transmitted to the receiver as
(1105 458 1070 50 251 831 843 774 660 592 41 405)

The receiver reconstructs the matrices $C_1$ and $C_2$ and computes $C_1^d$ (modm) and $C_2^d$ (modm) to obtain $P_1$ and $P_2$ and hence the plaintext message. That is,

$$C_1^d = \begin{bmatrix} 1105 & 458 & 1070 \\ 0 & 50 & 251 \\ 0 & 0 & 831 \end{bmatrix}^{117293} \equiv \begin{bmatrix} 90 & 23 & 26 \\ 0 & 50 & 77 \\ 0 & 0 & 48 \end{bmatrix} \pmod{1189} = P_1$$

$$C_2^d = \begin{bmatrix} 843 & 774 & 660 \\ 0 & 592 & 41 \\ 0 & 0 & 405 \end{bmatrix}^{117293} \equiv \begin{bmatrix} 31 & 20 & 51 \\ 0 & 215 & 41 \\ 0 & 0 & 289 \end{bmatrix} \pmod{1189} \equiv P_2$$

Figure 1

References

1. Rivest, R.L., Shamir, A., and Adleman, L. 'A Method for obtaining signatures and public key cryptosystems', Comm. ACM, 1978, 21, pp. 120-126.

2. Rédei, L., Algebra Vol. 1, Pergamon Press, International Series of Monographs in Pure and Applied Mathematics, 1967.

3. Albert, A.A., Fundamental Concepts of Higher Algebra, The University of Chicago Press, Phoenix Science Series, 1961.

4. Jordan, C., Traité des Substitutions, 1870.

5. Farahat, H.K., 'The multiplicative group of a ring', Math-Zeitschr., 87, 1965, pp. 378-384.

6. Brawley, J.V., 'The number of non-singular matrices over a finite ring with identity', 1971 (unpublished).

7. Roby Norbert, 'Sur le cardinal du Groupe GL(n,A) où A est anneau fini', Annals Acad. Brasil, 49, 1977, pp.15-18.

8. McDonald, B.R. 'Involutory matrices over finite Local rings', Canadian Journal Math., 24, 1972, pp.369-378.

9. Knuth, D.E., 'The Art of Computer Programming, Vol.2: Seminumerical Algorithms, 2nd Edition, Addison- Wesley, 1981.

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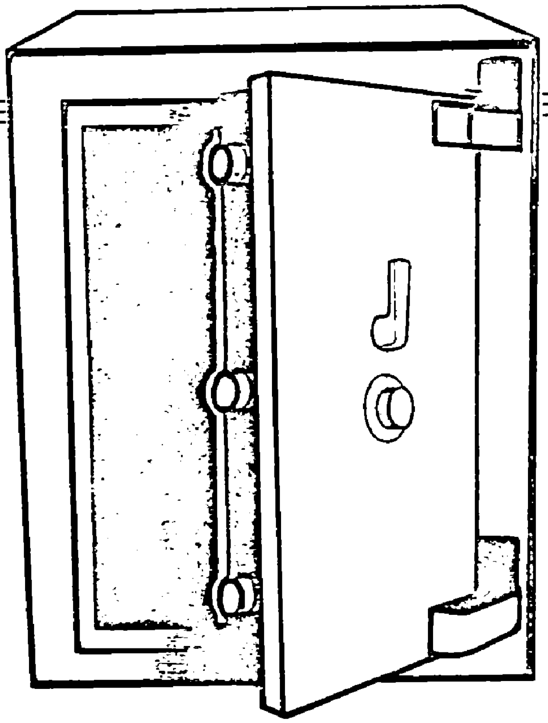


# DES can keep your data safe

TH the advent of electronic mail, ctronic funds transfer and the tributed office, an increasing amount of ıfidential data is being stored on disc d transmitted between terminals and nputers.

The information may be financial, ıere a high degree of security is required, it may consist of sensitive commercial alings, personnel records, legal docu-ents, etc., which could cause difficulties problems if they fell into the wrong nds.

It is not too difficult to take a copy of a sc, or listen in on telephone lines with e cheap, yet sophisticated communica-n equipment presently available and en examine the information at one's sure.

The relatively simple precautions, if ıy, employed in many data systems ove a minor obstacle to the determined ıttack". The following describes the eration of a unit whereby the Apple ter-inal user can reduce this possible curity problem to a minimum.

Data encryption moved out of the ilitary and political spheres in 1976 hen an IBM encryption algorithm was ccepted by the American National ureau of Standards as the encryption echanism for all federal non-military pplications. It is known as the Data ncryption Standard (DES) and is likely to e adopted as such in this country. The ard shown in the photograph opposite tilises this standard.

The algorithm changes the plain data ıto an unintelligible cipher form under the ontrol of a key; the change being to such n extent that the transformation is entirely different for each key value.

The algorithm, which has been made public for almost 10 years now and has not yet been cracked, uses multiple modulo 2 addition, permutations and substitutions to ensure that the key or plain text cannot be discovered from the cipher text.

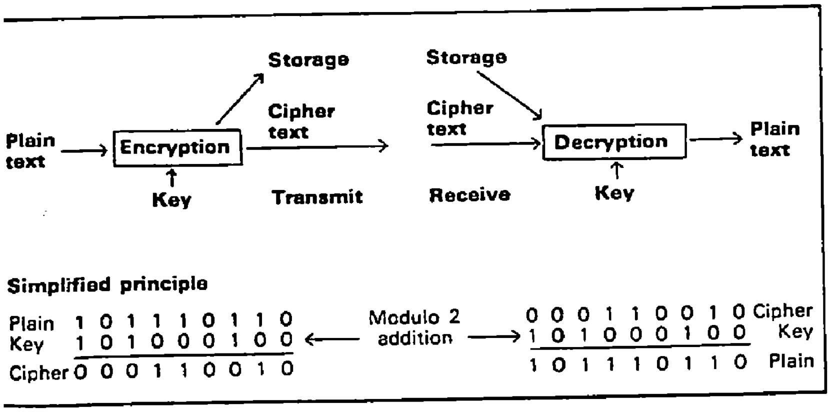
The very simple modulo 2 example in Figure I shows the difficulty of retrieving the original information when the "carrys" are lost unless the correct key is available.

The standard uses a 56 bit key with 64 bit blocks of plain text to produce, after 16 "rounds", 64 bit cipher blocks of data giving a "key space" of $2^{56}$ – or $10^{17}$ different key combinations. This is an extremely large number, and ensures that if each key combination was tried in turn by a fast computer it would take many years before all were covered.

The key can frequently be changed or multiple encryption made with different keys to maximise security. This alogrithm has recently become available in large scale integrated circuit form.

The encryption card can operate in one of three modes.

- Point to point communication between Apple terminals not containing a host computer.
- Local storage of encrypted programs



Simplified principle

```
Plain  1 0 1 1 1 0 1 1 0             Modulo 2      0 0 0 1 1 0 0 1 0 Cipher
Key    1 0 1 0 0 0 1 0 0 <------     addition ---> 1 0 1 0 0 0 1 0 0  Key
       -----------------                           -----------------
Cipher 0 0 0 1 1 0 0 1 0                           1 0 1 1 1 0 1 1 0  Plain
```

Figure I

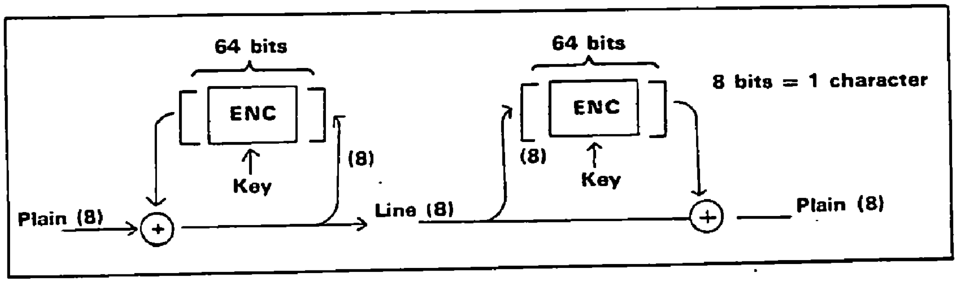


*Figure II*

ıd data files on the Apple disc system.
Communication between an Apple ter-
inal and a host computer for storage and
trieval of encrypted information in its
emory bank. The operation of these
odes is controlled by programs stored in
OM on the card, the appropriate
·ogram being selected by a CALL com-
and after the card has been activated
ith a POKE instruction.

In the secure storage mode Applesoft
· Integer Basic data is passed through
ıe encryption chip after it has been in-
ialised with the chosen key, consisting of
ıy eight character combination, includ-
ıg control characters, from the keyboard.

The encrypted form is then stored on a
isc and can be catalogued in the normal
ıanner.

If this cipher text is later downloaded
om the disc and listed complete rubbish
ill result. Often the listing continues to
ın because the end of line/file pointers
re never reached, or it is maybe much
horter in length than the original because
ıe control characters in the encrypted
orm are not displayed.

When operating in the communication
ıodes, a suitable modem must be used to
onvert the digital codes from the card
ıto equivalent speech signals for
ansmission. The card incorporates a
ommunication controller that allows full
luplex, asynchronous transmission at the
CITT standard rates to 9.6 k bit/sec: the
equired rate being selected from the key-
ıoard.

Different rates for transmission and
eception allow asymmetric duplex opera-
ion, such as is used on the Prestel data
ıases and other viewdata systems.

The DES algorithm can be used in a
ıumber of ways to obtain the required
ecurity but the method chosen by the
ard is a stream cipher feedback arrange-
nent as shown in Figure II.

In this case the DES is used as a
ɔseudo random number generator, its

## By P.W. SANDERS and V. VARADHARAJAN

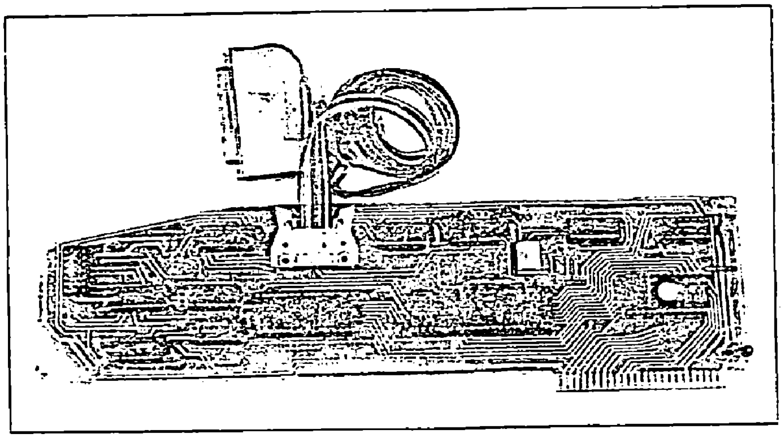
output being continuously exclusive OR'-ed (modulo 2 addition) with the plain text. The decryption process operates in the same manner – an identical pseudo random stream of bits being generated in synchronism to retrieve the plain text after another exclusive OR operation. This method has a number of advantages when used in the above applications:

□ There is a chaining of the encryption – the output cipher text character becomes dependent upon the previous eight characters of cipher and plain text, as well as the key, which removes the problem of straight forward substitution of characters to alter the message without knowledge of the key.

□ The system has a self-synchronising property when confronted with transmission errors. A error within a cipher text character affects that character and the following eight characters only on decryption, after which the system automatically becomes resynchronised.

□ The arrangement is very flexible, allowing a mixture of plain and encrypted characters under keyboard control. With block cipher, eight character blocks must always be used, necessitating "padding" if less than eight encrypted characters are needed. With this "byte" feedback a single character can be efficiently encrypted.

A stand-alone unit has been developed for any computer that has the same facilities as the card, but being connected between the RS 232 interface output of the computer and the corresponding input of the modem.



*The encryption card*