$$\hat{g}_{k,n} = \begin{cases} l_{k-n} + l_{k+n}, & k \neq n \\ 1 + l_0 + l_{2n}, & k = n. \end{cases}$$

Furthermore, by using

$$\prod_{k=1}^{N} \cos^2 q_k u \simeq 1 - u^2 \sum_{k=1}^{N} q_k^2$$

the $P_e$ expression given by (26) is approximated by

$$P_e \simeq \left\{1 - \int_{-\infty}^{\infty} (\sin q_0 u/u) \left(1 - u^2 \sum_{k=1}^{N} q_k^2\right) \right.$$

$$\left. \cdot \exp(-\sigma^2 u^2/2)\, du/\pi \right\}/2$$

$$\simeq \operatorname{erfc}(q_0/\sigma) + (2\pi)^{-1/2}(q_0/\sigma^3) \exp(q_0^2/2\sigma^2) \sum_{k=1}^{N} q_k^2$$

$$\simeq \operatorname{erfc}(1/a) + (2\pi)^{-1/2} a \exp(-1/2a^2) \sum_{k=1}^{N} e_k^2. \quad (37)$$

Thus, (37) gives the approximate minimum average probability of error and (36) gives the approximate solution of (25) explicitly in terms of the known $l_{k,n}$, $k = -N,\cdots,N$, $n = -N,\cdots,N$. Finally, we observe that the technique used in this correspondence for the analysis of intersymbol interference in a binary low-pass pulse-communication system is also readily applicable to a binary or a quadriphase bandpass phase-shift-keyed (PSK) communication system.

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## Polyphase Codes With Good Periodic Correlation Properties

DAVID C. CHU

***Abstract*—This correspondence describes the construction of complex codes of the form exp $i\alpha_k$ whose discrete circular autocorrelations are zero for all nonzero lags. There is no restriction on code lengths.**

Polyphase codes with a periodic autocorrelation function that is zero everywhere except at a single maximum per period have been described by Frank and Zadoff [1] and Heimiller [2]. The lengths of such codes are restricted to perfect squares. It will be shown that codes with the same correlation properties can be constructed for any code length. The method is borrowed from the work of Schroeder [3] in connection with synthesis of low peak-factor signals.

Consider a code $\{a_k\}$ of length $N$ composed of unity modulus complex numbers, i.e.,

$$a_k = \exp i\alpha_k, \qquad k = 0,1,\cdots,N-1. \quad (1)$$

The autocorrelation function $\{x_j\}$ is defined as follows:

$$x_0 = \sum_{k=0}^{N-1} a_k a_k^* \quad (2)$$

$$x_j = \sum_{k=0}^{N-j-1} a_k a_{k+j}^* + \sum_{k=N-j}^{N-1} a_k a_{k+j-N}^*, \qquad j = 1,2,\cdots,N-1. \quad (3)$$

It is claimed that for any code length $N$, the phases $\alpha_k$ can be chosen such that for $j = 1,2,\cdots,N-1$, $x_j$ vanishes. The single maximum of magnitude $N$ occurs at $x_0$.

Consider first the case that $N$ is even. We claim that if

$$a_k = \exp i\frac{M\pi k^2}{N}, \quad (4)$$

where $M$ is an integer relatively prime to $N$, then

$$x_j = 0, \qquad j = 1,2,\cdots,N-1.$$

From (3)

$$x_j = \sum_{k=0}^{N-j-1} \exp i\frac{M\pi}{N}[k^2 - (k+j)^2] + \sum_{k=N-j}^{N-1} \exp i\frac{M\pi}{N}$$

$$\cdot [k^2 - (k+j-N)^2], \qquad j = 1,2,\cdots,N-1. \quad (5)$$

Note that

$$\exp i\frac{M\pi}{N}(k+j-N)^2$$

$$= \exp i\frac{M\pi}{N}(k+j)^2 \exp -i2\pi M(k+j) \exp i\pi NM$$

$$= \exp i\frac{M\pi}{N}(k+j)^2, \qquad N \text{ even}.$$

The two summations of (5) may be combined.

$$x_j = \sum_{k=0}^{N-1} \exp i\frac{M\pi}{N}[k^2 - (k+j)^2]$$

$$= \sum_{k=0}^{N-1} \exp i\frac{M\pi}{N}[-2kj - j^2]$$

$$= \exp -i\frac{M\pi j^2}{N} \sum_{k=0}^{N-1} \left[\exp -i\frac{2\pi Mj}{N}\right]^k, \qquad j = 1,2,\cdots,N-1$$

$$= 0. \quad (6)$$

The last step comes from the fact that since $M$ and $N$ are relatively prime, then $\exp -i(2\pi M/N)$ is a primitive $N$th root of unity. Therefore, $\exp -i(2\pi Mj/N)$ is an $N$th root of unity but not equal to 1 for the range of $j$ shown in (6). Finally, we employ the theorem

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The author is with the Hewlett-Packard Company, Santa Clara, Calif. 95050.

$$\sum_{k=0}^{N-1} r^k = 0,$$

where $r$ is an $N$th root of unity and $r \neq 1$.

Consider next the case of $N$ odd. We claim that if

$$a_k = \exp i \frac{M\pi k(k+1)}{N}, \tag{7}$$

where $M$ is relatively prime to $N$, then

$$x_j = 0, \qquad j = 1,2,\cdots, N-1.$$

Substituting (7) into (3), we have

$$x_j = \sum_{k=0}^{N-j-1} \exp i \frac{M\pi}{N} [k(k+1) - (k+j)(k+j+1)] + \sum_{k=N-j}^{N-1} \exp i \frac{M\pi}{N} [k(k+1) - (k+j-N) \cdot (k+j+1-N)]. \tag{8}$$

With some manipulation, one can show that for odd $N$,

$$\exp i \frac{M\pi}{N} [(k+j-N)(k+j+1-N)] = \exp i \frac{M\pi}{N} [(k+j)(k+j+1)] \tag{9}$$

and the two summations in (8) can be combined into one:

$$\begin{aligned} x_j &= \sum_{k=0}^{N-1} \exp i \frac{M\pi}{N} [k(k+1) - (k+j)(k+j+1)] \\ &= \sum_{k=0}^{N-1} \exp i \frac{M\pi}{N} [-2jk - j^2 - j] \\ &= \exp -i \frac{M\pi}{N} [j(j+1)] \sum_{k=0}^{N-1} \left[\exp -i \frac{2\pi Mj}{N}\right]^k \\ &= 0, \qquad j = 1,\cdots, N-1 \text{ and } M \text{ relatively prime to } N. \end{aligned}$$

Thus a code of any length $N$ may be phase coded by either (4) or (7), respectively, depending on whether $N$ is even or odd, and the resulting code will have the desired correlation properties. Trivial variations such as cyclic shifts, addition of a constant to $\alpha_k$, or conjugating the entire code obviously will not affect the autocorrelation function analogously to the aperiodic case [4]. In addition, certain linear phase shifts of the form $\exp i(2\pi qk/N)$, where $q$ is any integer, when introduced into the code also will not affect the correlation. To show this, let the modified sequence be $\{b_k\}$ where

$$b_k = a_k \exp \frac{i2\pi qk}{N}. \tag{10}$$

Substituting $b_k$ for $a_k$ in (3)

$$\begin{aligned} x_j &= \sum_{k=0}^{N-j-1} a_k a_{k+j}^* \exp i \frac{2\pi q}{N} (k - k - j) + \sum_{k=N-j}^{N-1} a_k a_{k+j-N}^* \exp i \frac{2\pi q}{N} (k - k - j + N) \\ &= \exp -i \frac{2\pi qj}{N} \left( \sum_{k=0}^{N-j-1} a_k a_{k+j}^* + \sum_{k=N-j}^{N-1} a_k a_{k+j-N}^* \right) \\ &= 0, \qquad j = 1,2,\cdots, N-1 \end{aligned} \tag{11}$$

as the quantity inside the parentheses vanishes.

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# A Stochastic Model for Burst-Correcting Convolutional Decoders

PHILIPPE PIRET

***Abstract*—A stochastic model is described for the decoder of an optimal burst-correcting convolutional code. From this model, an upper bound is obtained for $\bar{p}$, the error probability per word after decoding.**

## Introduction

Recent papers [5], [6], use stochastic models to analyze decoders. We use here similar methods to upperbound the error probability after decoding of an optimal binary burst-error-correcting convolutional code.

A burst of $r$ consecutive words followed by a guard space $G$ of $r(n+k)/(n-k)$ correct words [1] is always correctable by an optimal type-$B_2$ code. Optimal codes for this bound are well known [2], [3] and the practical construction of a type-$B_2$ code for arbitrary $r$ requires only the construction of a type-$B_2$ code with $r = 1$. This last code is called basic, and codes with $r > 1$ are obtained by interlacing.

A basic code can be defined as the set of all binary semi-infinite sequences orthogonal to the matrix $A$ constructed by indefinitely displacing the binary matrix $B$ by $(n-k)$ rows lower at each time (Fig. 1).

The matrix $B$ defining a code can be divided into $K$ submatrices $B_i$:

$$B = \begin{bmatrix} B_0 \\ B_1 \\ \vdots \\ B_{K-1} \end{bmatrix},$$

where each $B_i$ has $(n-k)$ rows.

We shall consider systematic codes such that the last $(n-k)$ columns of $B_0$ form an identity matrix, and for $i \neq 0$ the last $(n-k)$ columns of $B_i$ are zero. We also restrict our study to nonsingular codes, i.e., codes defined by a matrix $B$ whose first $n$ rows form a nonsingular matrix. This family includes the Berlekamp–Preparata codes as a subclass [2], [3]. Optimal codes can be constructed if $(n-k)$ is a factor of $(n+k)$. We suppose that $(n-k)$ and $n$ are relatively prime. Otherwise a shorter code exists, with the same rate and correcting capability and no larger guard space $G$ [7]. The only possible values of $(n-k)$ are thus 1 and 2 for $n$ odd and 1 for $n$ even.

Let $x$ be a received semi-infinite sequence of the code represented as a column vector; then $Ax$ is the syndrome sequence. Each word of $x$ can be decoded by looking at $N$ elements of $Ax$ [4], where $N$, the number of rows of the matrix $B$, is given

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The author is with the MBLE Research Laboratory, Brussels, Belgium.