
DIGITAL COMMUNICATIONS

Third Edition

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McGraw-Hill, Inc.

New York St. Louis San Francisco Auckland Bogotá Caracas Lisbon
London Madrid Mexico City Milan Montreal New Delhi
San Juan Singapore Sydney Tokyo Toronto

This book was set in Times Roman.
The editors were George T. Hoffman and John M. Morriss;
the production supervisor was Leroy A. Young.
The cover was designed by Tana Kamine.
Project supervision was done by The Universities Press (Belfast) Ltd.
R. R. Donnelley & Sons Company was printer and binder.

DIGITAL COMMUNICATIONS

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This book is printed on acid-free paper.

3 4 5 6 7 8 9 0 DOC/DOC 9 9 8 7

ISBN 0-07-051726-6

Library of Congress Cataloging-in-Publication Data

Proakis, John G.

Digital communications / John G. Proakis.—3rd ed.

p. cm.—(McGraw-Hill series in electrical and computer engineering. Communications and signal processing)

Includes bibliographical references and index.

ISBN 0-07-051726-6

I. Digital communications. I. Title. II. Series.

TK5103.7.P76 1995

621.382—dc20

94-41620

CONTENTS

Preface	xix
1 Introduction	1
1-1 Elements of a Digital Communication System	1
1-2 Communication Channels and Their Characteristics	3
1-3 Mathematical Models for Communication Channels	11
1-4 A Historical Perspective in the Development of Digital Communications	13
1-5 Overview of the Book	16
1-6 Bibliographical Notes and References	16
2 Probability and Stochastic Processes	17
2-1 Probability	17
2-1-1 Random Variables, Probability Distributions, and Probability Densities	22
2-1-2 Functions of Random Variables	28
2-1-3 Statistical Averages of Random Variables	33
2-1-4 Some Useful Probability Distributions	37
2-1-5 Upper bounds on the Tail Probability	53
2-1-6 Sums of Random Variables and the Central Limit Theorem	58
2-2 Stochastic Processes	62
2-2-1 Statistical Averages	64
2-2-2 Power Density Spectrum	67
2-2-3 Response of a Linear Time-Invariant System to a Random Input Signal	68
2-2-4 Sampling Theorem for Band-Limited Stochastic Processes	72
2-2-5 Discrete-Time Stochastic Signals and Systems	74
2-2-6 Cyclostationary Processes	75
2-3 Bibliographical Notes and References	77
Problems	77

3	Source Coding	82
3-1	Mathematical Models for Information	82
3-2	A Logarithmic Measure of Information	84
3-2-1	Average Mutual Information and Entropy	87
3-2-2	Information Measures for Continuous Random Variables	91
3-3	Coding for Discrete Sources	93
3-3-1	Coding for Discrete Memoryless Sources	94
3-3-2	Discrete Stationary Sources	103
3-3-3	The Lempel–Ziv Algorithm	106
3-4	Coding for Analog Sources—Optimum Quantization	108
3-4-1	Rate-Distortion Function	108
3-4-2	Scalar Quantization	113
3-4-3	Vector Quantization	118
3-5	Coding Techniques for Analog Sources	125
3-5-1	Temporal Waveform Coding	125
3-5-2	Spectral Waveform Coding	136
3-5-3	Model-Based Source Coding	138
3-6	Bibliographical Notes and References	144
	Problems	144
4	Characterization of Communication Signals and Systems	152
4-1	Representation of Bandpass Signals and Systems	152
4-1-1	Representation of Bandpass Signals	153
4-1-2	Representation of Linear Bandpass Systems	157
4-1-3	Response of a Bandpass System to a Bandpass Signal	157
4-1-4	Representation of Bandpass Stationary Stochastic Processes	159
4-2	Signal Space Representation	163
4-2-1	Vector Space Concepts	163
4-2-2	Signal Space Concepts	165
4-2-3	Orthogonal Expansions of Signals	165
4-3	Representation of Digitally Modulated Signals	173
4-3-1	Memoryless Modulation Methods	174
4-3-2	Linear Modulation with Memory	186
4-3-3	Nonlinear Modulation Methods with Memory	190
4-4	Spectral Characteristics of Digitally Modulated Signals	203
4-4-1	Power Spectra of Linearly Modulated Signals	204
4-4-2	Power Spectra of CPFSK and CPM Signals	209
4-4-3	Power Spectra of Modulated Signals with Memory	220
4-5	Bibliographical Notes and References	223
	Problems	224
5	Optimum Receivers for the Additive White Gaussian Noise Channel	233
5-1	Optimum Receiver for Signals Corrupted by AWGN	233
5-1-1	Correlation Demodulator	234
5-1-2	Matched-Filter Demodulator	238

5-1-3	The Optimum Detector	244
5-1-4	The Maximum-Likelihood Sequence Detector	249
5-1-5	A Symbol-by-Symbol MAP Detector for Signals with Memory	254
5-2	Performance of the Optimum Receiver for Memoryless Modulation	257
5-2-1	Probability of Error for Binary Modulation	257
5-2-2	Probability of Error for M -ary Orthogonal Signals	260
5-2-3	Probability of Error for M -ary Biorthogonal Signals	264
5-2-4	Probability of Error for Simplex Signals	266
5-2-5	Probability of Error for M -ary Binary-Coded Signals	266
5-2-6	Probability of Error for M -ary PAM	267
5-2-7	Probability of Error for M -ary PSK	269
5-2-8	Differential PSK (DPSK) and its Performance	274
5-2-9	Probability of Error for QAM	278
5-2-10	Comparison of Digital Modulation Methods	282
5-3	Optimum Receiver for CPM Signals	284
5-3-1	Optimum Demodulation and Detection of CPM	285
5-3-2	Performance of CPM Signals	290
5-3-3	Symbol-by-Symbol Detection of CPM Signals	296
5-4	Optimum Receiver for Signals with Random Phase in AWGN Channel	301
5-4-1	Optimum Receiver for Binary Signals	302
5-4-2	Optimum Receiver for M -ary Orthogonal Signals	308
5-4-3	Probability of Error for Envelope Detection of M -ary Orthogonal Signals	308
5-4-4	Probability of Error for Envelope Detection of Correlated Binary Signals	312
5-5	Regenerative Repeaters and Link Budget Analysis	313
5-5-1	Regenerative Repeaters	314
5-5-2	Communication Link Budget Analysis	316
5-6	Bibliographical Notes and References	319
	Problems	320
6	Carrier and Symbol Synchronization	333
6-1	Signal Parameter Estimation	333
6-1-1	The Likelihood Function	335
6-1-2	Carrier Recovery and Symbol Synchronization in Signal Demodulation	336
6-2	Carrier Phase Estimation	337
6-2-1	Maximum-Likelihood Carrier Phase Estimation	339
6-2-2	The Phase-Locked Loop	341
6-2-3	Effect of Additive Noise on the Phase Estimate	343
6-2-4	Decision-Directed Loops	347
6-2-5	Non-Decision-Directed Loops	350
6-3	Symbol Timing Estimation	358
6-3-1	Maximum-Likelihood Timing Estimation	359
6-3-2	Non-Decision-Directed Timing Estimation	361

6-4	Joint Estimation of Carrier Phase and Symbol Timing	365
6-5	Performance Characteristics of ML Estimators	367
6-6	Bibliographical Notes and References	370
	Problems	371
7	Channel Capacity and Coding	374
7-1	Channel Models and Channel Capacity	375
7-1-1	Channel Models	375
7-1-2	Channel Capacity	380
7-1-3	Achieving Channel Capacity with Orthogonal Signals	387
7-1-4	Channel Reliability Functions	389
7-2	Random Selection of Codes	390
7-2-1	Random Coding Based on M -ary Binary-Coded Signals	390
7-2-2	Random Coding Based on M -ary Multi-amplitude Signals	397
7-2-3	Comparison of R_n^* with the Capacity of the AWGN Channel	399
7-3	Communication System Design Based on the Cutoff Rate	400
7-4	Bibliographical Notes and References	406
	Problems	406
8	Block and Convolutional Channel Codes	413
8-1	Linear Block Codes	413
8-1-1	The Generator Matrix and the Parity Check Matrix	417
8-1-2	Some Specific Linear Block Codes	421
8-1-3	Cyclic Codes	423
8-1-4	Optimum Soft-Decision Decoding of Linear Block Codes	436
8-1-5	Hard-Decision Decoding	445
8-1-6	Comparison of Performance between Hard-Decision and Soft-Decision Decoding	456
8-1-7	Bounds on Minimum Distance of Linear Block Codes	461
8-1-8	Nonbinary Block Codes and Concatenated Block Codes	464
8-1-9	Interleaving of Coded Data for Channels with Burst Errors	468
8-2	Convolutional Codes	470
8-2-1	The Transfer Function of a Convolutional Code	477
8-2-2	Optimum Decoding of Convolutional Codes— The Viterbi Algorithm	483
8-2-3	Probability of Error for Soft-Decision Decoding	486
8-2-4	Probability of Error for Hard-Decision Decoding	489
8-2-5	Distance Properties of Binary Convolutional Codes	492
8-2-6	Nonbinary Dual- k Codes and Concatenated Codes	492
8-2-7	Other Decoding Algorithms for Convolutional Codes	500
8-2-8	Practical Considerations in the Application of Convolutional Codes	506
8-3	Coded Modulation for Bandwidth-Constrained Channels	511
8-4	Bibliographical Notes and References	526
	Problems	528

9	Signal Design for Band-Limited Channels	534
9-1	Characterization of Band-Limited Channels	534
9-2	Signal Design for Band-Limited Channels	540
9-2-1	Design of Band-Limited Signals for No Intersymbol Interference—The Nyquist Criterion	542
9-2-2	Design of Band-Limited Signals with Controlled ISI—Partial-Response Signals	548
9-2-3	Data Detection for Controlled ISI	551
9-2-4	Signal Design for Channels with Distortion	557
9-3	Probability of Error in Detection of PAM	561
9-3-1	Probability of Error for Detection of PAM with Zero ISI	561
9-3-2	Probability of Error for Detection of Partial-Response Signals	562
9-3-3	Probability of Error for Optimum Signals in Channel with Distortion	565
9-4	Modulation Codes for Spectrum Shaping	566
9-5	Bibliographical Notes and References	576
	Problems	576
10	Communication through Band-Limited Linear Filter Channels	583
10-1	Optimum Receiver for Channels with ISI and AWGN	584
10-1-1	Optimum Maximum-Likelihood Receiver	584
10-1-2	A Discrete-Time Model for a Channel with ISI	586
10-1-3	The Viterbi Algorithm for the Discrete-Time White Noise Filter Model	589
10-1-4	Performance of MLSE for Channels with ISI	593
10-2	Linear Equalization	601
10-2-1	Peak Distortion Criterion	602
10-2-2	Mean Square Error (MSE) Criterion	607
10-2-3	Performance Characteristics of the MSE Equalizer	612
10-2-4	Fractionally Spaced Equalizer	617
10-3	Decision-Feedback Equalization	621
10-3-1	Coefficient Optimization	621
10-3-2	Performance Characteristics of DFE	622
10-3-3	Predictive Decision-Feedback Equalizer	626
10-4	Bibliographical Notes and References	628
	Problems	628
11	Adaptive Equalization	636
11-1	Adaptive Linear Equalizer	636
11-1-1	The Zero-Forcing Algorithm	637
11-1-2	The LMS algorithm	639
11-1-3	Convergence Properties of the LMS Algorithm	642
11-1-4	Excess MSE Due to Noisy Gradient Estimates	644
11-1-5	Baseband and Passband Linear Equalizers	648
11-2	Adaptive Decision-Feedback Equalizer	649
11-2-1	Adaptive Equalization of Trellis-Coded Signals	650

11-3	An Adaptive Channel Estimator for ML Sequence Detection	652
11-4	Recursive Least-Squares Algorithms for Adaptive Equalization	654
11-4-1	Recursive Least-Squares (Kalman) Algorithm	656
11-4-2	Linear Prediction and the Lattice Filter	660
11-5	Self-Recovering (Blind) Equalization	664
11-5-1	Blind Equalization Based on Maximum-Likelihood Criterion	664
11-5-2	Stochastic Gradient Algorithms	668
11-5-3	Blind Equalization Algorithms Based on Second- and Higher-Order Signal Statistics	673
11-6	Bibliographical Notes and References	675
	Problems	676
12	Multichannel and Multicarrier Systems	680
12-1	Multichannel Digital Communication in AWGN Channels	680
12-1-1	Binary Signals	682
12-1-2	M -ary Orthogonal Signals	684
12-2	Multicarrier Communications	686
12-2-1	Capacity of a Non-Ideal Linear Filter Channel	687
12-2-2	An FFT-Based Multicarrier System	689
12-3	Bibliographical Notes and References	692
	Problems	693
13	Spread Spectrum Signals for Digital Communications	695
13-1	Model of Spread Spectrum Digital Communication System	697
13-2	Direct Sequence Spread Spectrum Signals	698
13-2-1	Error Rate Performance of the Decoder	702
13-2-2	Some Applications of DS Spread Spectrum Signals	712
13-2-3	Effect of Pulsed Interference on DS Spread Spectrum Systems	717
13-2-4	Generation of PN Sequences	724
13-3	Frequency-Hopped Spread Spectrum Signals	729
13-3-1	Performance of FH Spread Spectrum Signals in AWGN Channel	732
13-3-2	Performance of FH Spread Spectrum Signals in Partial-Band Interference	734
13-3-3	A CDMA System Based on FH Spread Spectrum Signals	741
13-4	Other Types of Spread Spectrum Signals	743
13-5	Synchronization of Spread Spectrum Signals	744
13-6	Bibliographical Notes and References	752
	Problems	753
14	Digital Communication through Fading Multipath Channels	758
14-1	Characterization of Fading Multipath Channels	759
14-1-1	Channel Correlation Functions and Power Spectra	762
14-1-2	Statistical Models for Fading Channels	767

14-2	The Effect of Characteristics on the Choice of a Channel Model	770
14-3	Frequency-Nonselective, Slowly Fading Channel	772
14-4	Diversity Techniques for Fading Multipath Channels	777
14-4-1	Binary Signals	778
14-4-2	Multiphase Signals	785
14-4-3	M -ary Orthogonal Signals	787
14-5	Digital Signaling over a Frequency-Selective, Slowly Fading Channel	795
14-5-1	A Tapped-Delay-Line Channel Model	795
14-5-2	The RAKE Demodulator	797
14-5-3	Performance of RAKE Receiver	798
14-6	Coded Waveforms for Fading Channels	806
14-6-1	Probability of Error for Soft-Decision Decoding of Linear Binary Block Codes	808
14-6-2	Probability of Error for Hard-Decision Decoding of Linear Binary Block Codes	811
14-6-3	Upper Bounds on the Performance of Convolutional Codes for a Raleigh Fading Channel	811
14-6-4	Use of Constant-Weight Codes and Concatenated Codes for a Fading Channel	814
14-6-5	System Design Based on the Cutoff Rate	825
14-6-6	Trellis-Coded Modulation	830
14-7	Bibliographical Notes and References	832
	Problems	833
15	Multuser Communications	840
15-1	Introduction to Multiple Access Techniques	840
15-2	Capacity of Multiple Access Methods	843
15-3	Code-Division Multiple Access	849
15-3-1	CDMA Signal and Channel Models	849
15-3-2	The Optimum Receiver	851
15-3-3	Suboptimum Detectors	854
15-3-4	Performance Characteristics of Detectors	859
15-4	Random Access Methods	862
15-4-1	ALOHA System and Protocols	863
15-4-2	Carrier Sense Systems and Protocols	867
15-5	Bibliographical Notes and References	872
	Problems	873
Appendix A	The Levinson–Durbin Algorithm	879
Appendix B	Error Probability for Multichannel Binary Signals	882

Appendix C	Error Probabilities for Adaptive Reception of M-phase Signals	887
C-1	Mathematical Model for an M -phase Signaling Communications System	887
C-2	Characteristic Function and Probability Density Function of the Phase θ	889
C-3	Error Probabilities for Slowly Rayleigh Fading Channels	891
C-4	Error Probabilities for Time-Invariant and Ricean Fading Channels	893
Appendix D	Square-Root Factorization	897
	References and Bibliography	899
	Index	917

INTRODUCTION

In this book, we present the basic principles that underlie the analysis and design of digital communication systems. The subject of digital communications involves the transmission of information in digital form from a source that generates the information to one or more destinations. Of particular importance in the analysis and design of communication systems are the characteristics of the physical channels through which the information is transmitted. The characteristics of the channel generally affect the design of the basic building blocks of the communication system. Below, we describe the elements of a communication system and their functions.

1-1 ELEMENTS OF A DIGITAL COMMUNICATION SYSTEM

Figure 1-1-1 illustrates the functional diagram and the basic elements of a digital communication system. The source output may be either an analog signal, such as audio or video signal, or a digital signal, such as the output of a teletype machine, that is discrete in time and has a finite number of output characters. In a digital communication system, the messages produced by the source are converted into a sequence of binary digits. Ideally, we should like to represent the source output (message) by as few binary digits as possible. In other words, we seek an efficient representation of the source output that results in little or no redundancy. The process of efficiently converting the output of either an analog or digital source into a sequence of binary digits is called *source encoding* or *data compression*.

The sequence of binary digits from the source encoder, which we call the

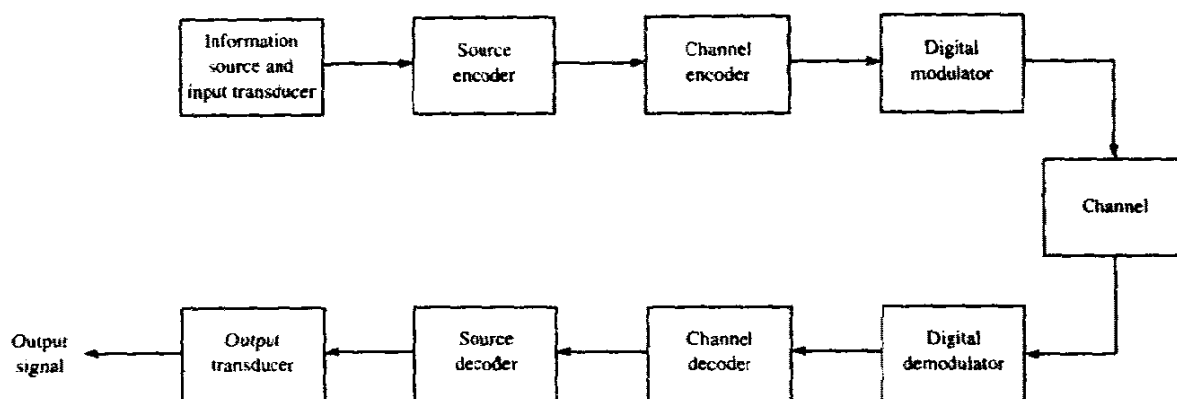


FIGURE 1-1-1 Basic elements of a digital communication system.

information sequence, is passed to the *channel encoder*. The purpose of the channel encoder is to introduce, in a controlled manner, some redundancy in the binary information sequence that can be used at the receiver to overcome the effects of noise and interference encountered in the transmission of the signal through the channel. Thus, the added redundancy serves to increase the reliability of the received data and improves the fidelity of the received signal. In effect, redundancy in the information sequence aids the receiver in decoding the desired information sequence. For example, a (trivial) form of encoding of the binary information sequence is simply to repeat each binary digit m times, where m is some positive integer. More sophisticated (nontrivial) encoding involves taking k information bits at a time and mapping each k -bit sequence into a unique n -bit sequence, called a *code word*. The amount of redundancy introduced by encoding the data in this manner is measured by the ratio n/k . The reciprocal of this ratio, namely k/n , is called the rate of the code or, simply, the *code rate*.

The binary sequence at the output of the channel encoder is passed to the *digital modulator*, which serves as the interface to the communications channel. Since nearly all of the communication channels encountered in practice are capable of transmitting electrical signals (waveforms), the primary purpose of the digital modulator is to map the binary information sequence into signal waveforms. To elaborate on this point, let us suppose that the coded information sequence is to be transmitted one bit at a time at some uniform rate R bits/s. The digital modulator may simply map the binary digit 0 into a waveform $s_0(t)$ and the binary digit 1 into a waveform $s_1(t)$. In this manner, each bit from the channel encoder is transmitted separately. We call this *binary modulation*. Alternatively, the modulator may transmit b coded information bits at a time by using $M = 2^b$ distinct waveforms $s_i(t)$, $i = 0, 1, \dots, M - 1$, one waveform for each of the 2^b possible b -bit sequences. We call this *M-ary modulation* ($M > 2$). Note that a new b -bit sequence enters the modulator

every b/R seconds. Hence, when the channel bit rate R is fixed, the amount of time available to transmit one of the M waveforms corresponding to a b -bit sequence is b times the time period in a system that uses binary modulation.

The *communication channel* is the physical medium that is used to send the signal from the transmitter to the receiver. In wireless transmission, the channel may be the atmosphere (free space). On the other hand, telephone channels usually employ a variety of physical media, including wire lines, optical fiber cables, and wireless (microwave radio). Whatever the physical medium used for transmission of the information, the essential feature is that the transmitted signal is corrupted in a random manner by a variety of possible mechanisms, such as additive *thermal noise* generated by electronic devices, man-made noise, e.g., automobile ignition noise, and atmospheric noise, e.g., electrical lightning discharges during thunderstorms.

At the receiving end of a digital communications system, the *digital demodulator* processes the channel-corrupted transmitted waveform and reduces the waveforms to a sequence of numbers that represent estimates of the transmitted data symbols (binary or M -ary). This sequence of numbers is passed to the channel decoder, which attempts to reconstruct the original information sequence from knowledge of the code used by the channel encoder and the redundancy contained in the received data.

A measure of how well the demodulator and decoder perform is the frequency with which errors occur in the decoded sequence. More precisely, the average probability of a bit-error at the output of the decoder is a measure of the performance of the demodulator–decoder combination. In general, the probability of error is a function of the code characteristics, the types of waveforms used to transmit the information over the channel, the transmitter power, the characteristics of the channel, i.e., the amount of noise, the nature of the interference, etc., and the method of demodulation and decoding. These items and their effect on performance will be discussed in detail in subsequent chapters.

As a final step, when an analog output is desired, the source decoder accepts the output sequence from the channel decoder and, from knowledge of the source encoding method used, attempts to reconstruct the original signal from the source. Due to channel decoding errors and possible distortion introduced by the source encoder and, perhaps, the source decoder, the signal at the output of the source decoder is an approximation to the original source output. The difference or some function of the difference between the original signal and the reconstructed signal is a measure of the distortion introduced by the digital communication system.

1-2 COMMUNICATION CHANNELS AND THEIR CHARACTERISTICS

As indicated in the preceding discussion, the communication channel provides the connection between the transmitter and the receiver. The physical channel

may be a pair of wires that carry the electrical signal, or an optical fiber that carries the information on a modulated light beam, or an underwater ocean channel in which the information is transmitted acoustically, or free space over which the information-bearing signal is radiated by use of an antenna. Other media that can be characterized as communication channels are data storage media, such as magnetic tape, magnetic disks, and optical disks.

One common problem in signal transmission through any channel is additive noise. In general, additive noise is generated internally by components such as resistors and solid-state devices used to implement the communication system. This is sometimes called *thermal noise*. Other sources of noise and interference may arise externally to the system, such as interference from other users of the channel. When such noise and interference occupy the same frequency band as the desired signal, its effect can be minimized by proper design of the transmitted signal and its demodulator at the receiver. Other types of signal degradations that may be encountered in transmission over the channel are signal attenuation, amplitude and phase distortion, and multipath distortion.

The effects of noise may be minimized by increasing the power in the transmitted signal. However, equipment and other practical constraints limit the power level in the transmitted signal. Another basic limitation is the available channel bandwidth. A bandwidth constraint is usually due to the physical limitations of the medium and the electronic components used to implement the transmitter and the receiver. These two limitations result in constraining the amount of data that can be transmitted reliably over any communications channel as we shall observe in later chapters. Below, we describe some of the important characteristics of several communication channels.

Wireline Channels The telephone network makes extensive use of wire lines for voice signal transmission, as well as data and video transmission. Twisted-pair wire lines and coaxial cable are basically guided electromagnetic channels that provide relatively modest bandwidths. Telephone wire generally used to connect a customer to a central office has a bandwidth of several hundred kilohertz (kHz). On the other hand, coaxial cable has a usable bandwidth of several megahertz (MHz). Figure 1-2-1 illustrates the frequency range of guided electromagnetic channels, which include waveguides and optical fibers.

Signals transmitted through such channels are distorted in both amplitude and phase and further corrupted by additive noise. Twisted-pair wireline channels are also prone to crosstalk interference from physically adjacent channels. Because wireline channels carry a large percentage of our daily communications around the country and the world, much research has been performed on the characterization of their transmission properties and on methods for mitigating the amplitude and phase distortion encountered in signal transmission. In Chapter 9, we describe methods for designing optimum transmitted signals and their demodulation; in Chapters 10 and 11, we

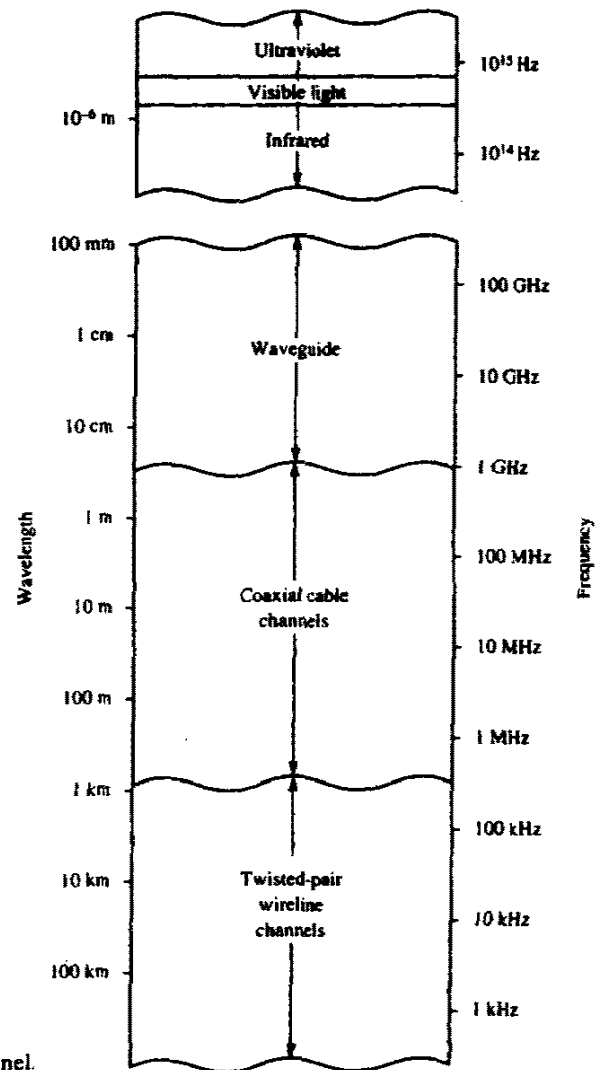


FIGURE 1-2-1 Frequency range for guided wire channel.

consider the design of channel equalizers that compensate for amplitude and phase distortion on these channels.

Fiber Optic Channels Optical fibers offer the communications system designer a channel bandwidth that is several orders of magnitude larger than coaxial cable channels. During the past decade, optical fiber cables have been developed that have a relatively low signal attenuation, and highly reliable photonic devices have been developed for signal generation and signal detection. These technological advances have resulted in a rapid deployment of optical fiber channels, both in domestic telecommunication systems as well as for trans-Atlantic and trans-Pacific communications. With the large bandwidth

available on fiber optic channels, it is possible for telephone companies to offer subscribers a wide array of telecommunication services, including voice, data, facsimile, and video.

The transmitter or modulator in a fiber optic communication system is a light source, either a light-emitting diode (LED) or a laser. Information is transmitted by varying (modulating) the intensity of the light source with the message signal. The light propagates through the fiber as a light wave and is amplified periodically (in the case of digital transmission, it is detected and regenerated by repeaters) along the transmission path to compensate for signal attenuation. At the receiver, the light intensity is detected by a photodiode, whose output is an electrical signal that varies in direct proportion to the power of the light impinging on the photodiode. Sources of noise in fiber optic channels are photodiodes and electronic amplifiers.

It is envisioned that optical fiber channels will replace nearly all wireline channels in the telephone network by the turn of the century.

Wireless Electromagnetic Channels In wireless communication systems, electromagnetic energy is coupled to the propagation medium by an antenna which serves as the radiator. The physical size and the configuration of the antenna depend primarily on the frequency of operation. To obtain efficient radiation of electromagnetic energy, the antenna must be longer than $\frac{1}{10}$ of the wavelength. Consequently, a radio station transmitting in the AM frequency band, say at $f_c = 1$ MHz (corresponding to a wavelength of $\lambda = c/f_c = 300$ m), requires an antenna of at least 30 m. Other important characteristics and attributes of antennas for wireless transmission are described in Chapter 5.

Figure 1-2-2 illustrates the various frequency bands of the electromagnetic spectrum. The mode of propagation of electromagnetic waves in the atmosphere and in free space may be subdivided into three categories, namely, ground-wave propagation, sky-wave propagation, and line-of-sight (LOS) propagation. In the VLF and audio frequency bands, where the wavelengths exceed 10 km, the earth and the ionosphere act as a waveguide for electromagnetic wave propagation. In these frequency ranges, communication signals practically propagate around the globe. For this reason, these frequency bands are primarily used to provide navigational aids from shore to ships around the world. The channel bandwidths available in these frequency bands are relatively small (usually 1–10% of the center frequency), and hence the information that is transmitted through these channels is of relatively slow speed and generally confined to digital transmission. A dominant type of noise at these frequencies is generated from thunderstorm activity around the globe, especially in tropical regions. Interference results from the many users of these frequency bands.

Ground-wave propagation, as illustrated in Fig. 1-2-3, is the dominant mode of propagation for frequencies in the MF band (0.3–3 MHz). This is the frequency band used for AM broadcasting and maritime radio broadcasting. In AM broadcasting, the range with groundwave propagation of even the more

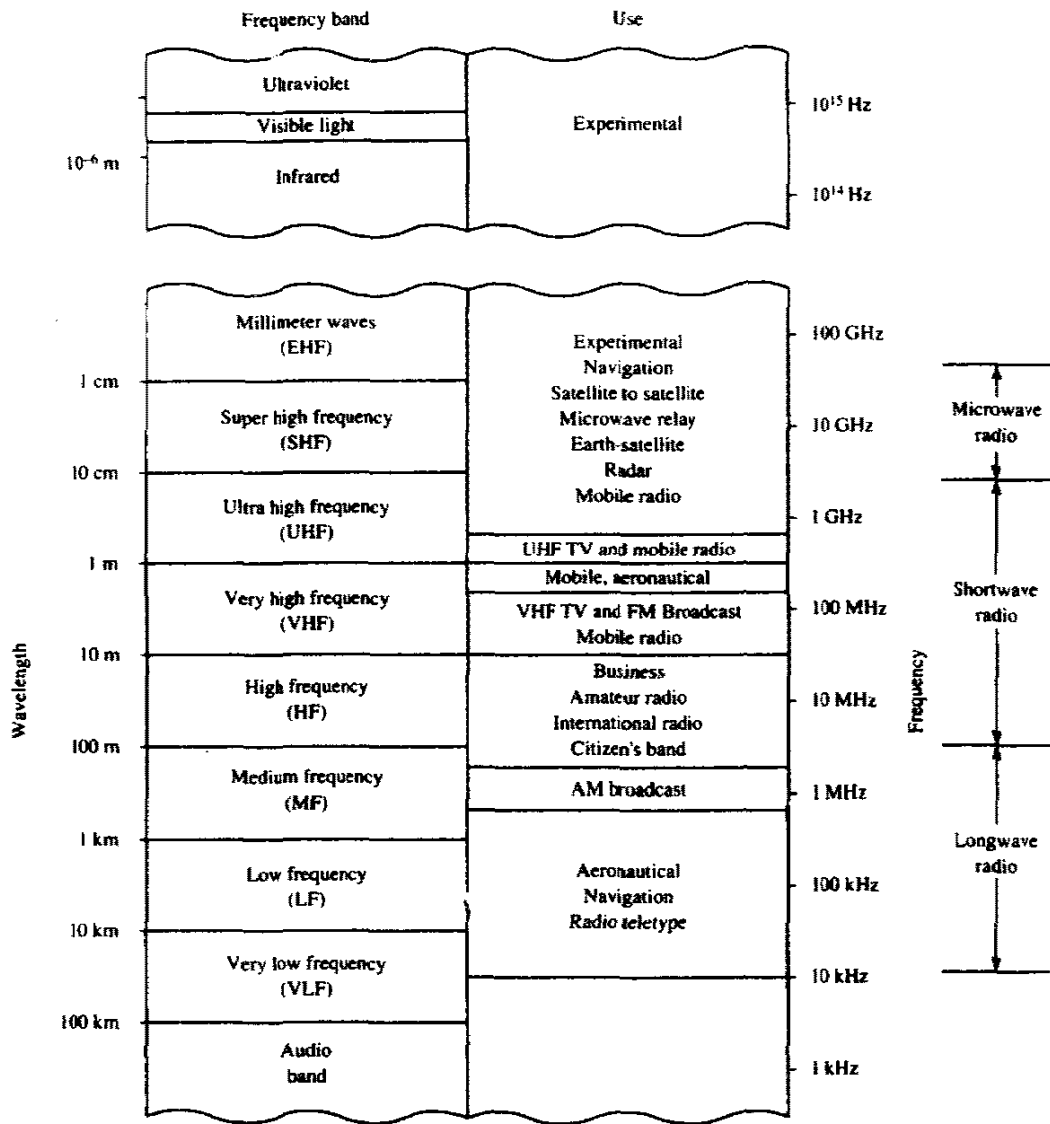
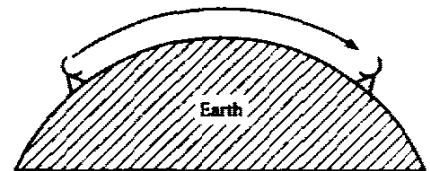


FIGURE 1-2-2 Frequency range for wireless electromagnetic channels. [Adapted from Carlson (1975), 2nd edition, © McGraw-Hill Book Company Co. Reprinted with permission of the publisher.]

FIGURE 1-2-3 Illustration of ground-wave propagation.



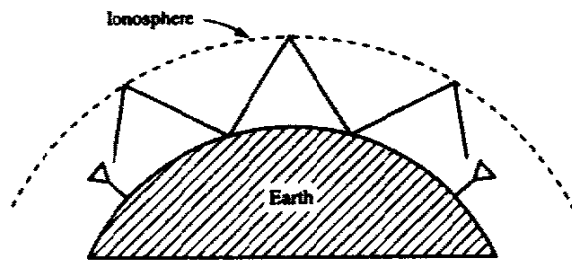


FIGURE 1-2-4 Illustration of sky-wave propagation.

powerful radio stations is limited to about 150 km. Atmospheric noise, man-made noise, and thermal noise from electronic components at the receiver are dominant disturbances for signal transmission in the MF band.

Sky-wave propagation, as illustrated in Fig. 1-2-4 results from transmitted signals being reflected (bent or refracted) from the ionosphere, which consists of several layers of charged particles ranging in altitude from 50 to 400 km above the surface of the earth. During the daytime hours, the heating of the lower atmosphere by the sun causes the formation of the lower layers at altitudes below 120 km. These lower layers, especially the D-layer, serve to absorb frequencies below 2 MHz, thus severely limiting sky-wave propagation of AM radio broadcast. However, during the night-time hours, the electron density in the lower layers of the ionosphere drops sharply and the frequency absorption that occurs during the daytime is significantly reduced. As a consequence, powerful AM radio broadcast stations can propagate over large distances via sky wave over the F-layer of the ionosphere, which ranges from 140 to 400 km above the surface of the earth.

A frequently occurring problem with electromagnetic wave propagation via sky wave in the HF frequency range is *signal multipath*. Signal multipath occurs when the transmitted signal arrives at the receiver via multiple propagation paths at different delays. It generally results in intersymbol interference in a digital communication system. Moreover, the signal components arriving via different propagation paths may add destructively, resulting in a phenomenon called *signal fading*, which most people have experienced when listening to a distant radio station at night when sky wave is the dominant propagation mode. Additive noise at HF is a combination of atmospheric noise and thermal noise.

Sky-wave ionospheric propagation ceases to exist at frequencies above approximately 30 MHz, which is the end of the HF band. However, it is possible to have ionospheric scatter propagation at frequencies in the range 30–60 MHz, resulting from signal scattering from the lower ionosphere. It is also possible to communicate over distances of several hundred miles by use of tropospheric scattering at frequencies in the range 40–300 MHz. Troposcatter results from signal scattering due to particles in the atmosphere at altitudes of 10 miles or less. Generally, ionospheric scatter and tropospheric scatter

involve large signal propagation losses and require a large amount of transmitter power and relatively large antennas.

Frequencies above 30 MHz propagate through the ionosphere with relatively little loss and make satellite and extraterrestrial communications possible. Hence, at frequencies in the VHF band and higher, the dominant mode of electromagnetic propagation is line-of-sight (LOS) propagation. For terrestrial communication systems, this means that the transmitter and receiver antennas must be in direct LOS with relatively little or no obstruction. For this reason, television stations transmitting in the VHF and UHF frequency bands mount their antennas on high towers to achieve a broad coverage area.

In general, the coverage area for LOS propagation is limited by the curvature of the earth. If the transmitting antenna is mounted at a height h m above the surface of the earth, the distance to the radio horizon, assuming no physical obstructions such as mountains, is approximately $d = \sqrt{15h}$ km. For example, a TV antenna mounted on a tower of 300 m in height provides a coverage of approximately 67 km. As another example, microwave radio relay systems used extensively for telephone and video transmission at frequencies above 1 GHz have antennas mounted on tall towers or on the top of tall buildings.

The dominant noise limiting the performance of a communication system in VHF and UHF frequency ranges is thermal noise generated in the receiver front end and cosmic noise picked up by the antenna. At frequencies in the SHF band above 10 GHz, atmospheric conditions play a major role in signal propagation. For example, at 10 GHz, the attenuation ranges from about 0.003 dB/km in light rain to about 0.3 dB/km in heavy rain. At 100 GHz, the attenuation ranges from about 0.1 dB/km in light rain to about 6 dB/km in heavy rain. Hence, in this frequency range, heavy rain introduces extremely high propagation losses that can result in service outages (total breakdown in the communication system).

At frequencies above the EHF (extremely high frequency) band, we have the infrared and visible light regions of the electromagnetic spectrum, which can be used to provide LOS optical communication in free space. To date, these frequency bands have been used in experimental communication systems, such as satellite-to-satellite links.

Underwater Acoustic Channels Over the past few decades, ocean exploration activity has been steadily increasing. Coupled with this increase is the need to transmit data, collected by sensors placed under water, to the surface of the ocean. From there, it is possible to relay the data via a satellite to a data collection center.

Electromagnetic waves do not propagate over long distances under water except at extremely low frequencies. However, the transmission of signals at such low frequencies is prohibitively expensive because of the large and powerful transmitters required. The attenuation of electromagnetic waves in water can be expressed in terms of the *skin depth*, which is the distance a signal is attenuated by $1/e$. For sea water, the skin depth $\delta = 250/\sqrt{f}$, where f is

expressed in Hz and δ is in m. For example, at 10 kHz, the skin depth is 2.5 m. In contrast, acoustic signals propagate over distances of tens and even hundreds of kilometers.

An underwater acoustic channel is characterized as a multipath channel due to signal reflections from the surface and the bottom of the sea. Because of wave motion, the signal multipath components undergo time-varying propagation delays that result in signal fading. In addition, there is frequency-dependent attenuation, which is approximately proportional to the square of the signal frequency. The sound velocity is nominally about 1500 m/s, but the actual value will vary either above or below the nominal value depending on the depth at which the signal propagates.

Ambient ocean acoustic noise is caused by shrimp, fish, and various mammals. Near harbors, there is also man-made acoustic noise in addition to the ambient noise. In spite of this hostile environment, it is possible to design and implement efficient and highly reliable underwater acoustic communication systems for transmitting digital signals over large distances.

Storage Channels Information storage and retrieval systems constitute a very significant part of data-handling activities on a daily basis. Magnetic tape, including digital audio tape and video tape, magnetic disks used for storing large amounts of computer data, optical disks used for computer data storage, and compact disks are examples of data storage systems that can be characterized as communication channels. The process of storing data on a magnetic tape or a magnetic or optical disk is equivalent to transmitting a signal over a telephone or a radio channel. The readback process and the signal processing involved in storage systems to recover the stored information are equivalent to the functions performed by a receiver in a telephone or radio communication system to recover the transmitted information.

Additive noise generated by the electronic components and interference from adjacent tracks is generally present in the readback signal of a storage system, just as is the case in a telephone or a radio communication system.

The amount of data that can be stored is generally limited by the size of the disk or tape and the density (number of bits stored per square inch) that can be achieved by the write/read electronic systems and heads. For example, a packing density of 10^9 bits per square inch has been recently demonstrated in an experimental magnetic disk storage system. (Current commercial magnetic storage products achieve a much lower density.) The speed at which data can be written on a disk or tape and the speed at which it can be read back are also limited by the associated mechanical and electrical subsystems that constitute an information storage system.

Channel coding and modulation are essential components of a well-designed digital magnetic or optical storage system. In the readback process, the signal is demodulated and the added redundancy introduced by the channel encoder is used to correct errors in the readback signal.

1-3 MATHEMATICAL MODELS FOR COMMUNICATION CHANNELS

In the design of communication systems for transmitting information through physical channels, we find it convenient to construct mathematical models that reflect the most important characteristics of the transmission medium. Then, the mathematical model for the channel is used in the design of the channel encoder and modulator at the transmitter and the demodulator and channel decoder at the receiver. Below, we provide a brief description of the channel models that are frequently used to characterize many of the physical channels that we encounter in practice.

The Additive Noise Channel The simplest mathematical model for a communication channel is the additive noise channel, illustrated in Fig. 1-3-1. In this model, the transmitted signal $s(t)$ is corrupted by an additive random noise process $n(t)$. Physically, the additive noise process may arise from electronic components and amplifiers at the receiver of the communication system, or from interference encountered in transmission (as in the case of radio signal transmission).

If the noise is introduced primarily by electronic components and amplifiers at the receiver, it may be characterized as thermal noise. This type of noise is characterized statistically as a *gaussian noise process*. Hence, the resulting mathematical model for the channel is usually called the *additive gaussian noise channel*. Because this channel model applies to a broad class of physical communication channels and because of its mathematical tractability, this is the predominant channel model used in our communication system analysis and design. Channel attenuation is easily incorporated into the model. When the signal undergoes attenuation in transmission through the channel, the received signal is

$$r(t) = \alpha s(t) + n(t) \quad (1-3-1)$$

where α is the attenuation factor.

The Linear Filter Channel In some physical channels, such as wireline telephone channels, filters are used to ensure that the transmitted signals do not exceed specified bandwidth limitations and thus do not interfere with one another. Such channels are generally characterized mathematically as linear filter channels with additive noise, as illustrated in Fig. 1-3-2. Hence, if the

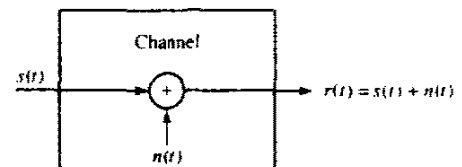


FIGURE 1-3-1 The additive noise channel.

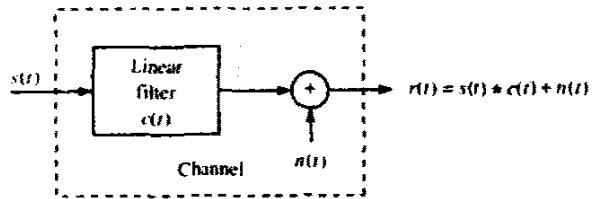


FIGURE 1-3-2 The linear filter channel with additive noise.

channel input is the signal $s(t)$, the channel output is the signal

$$\begin{aligned} r(t) &= s(t) \star c(t) + n(t) \\ &= \int_{-\infty}^{\infty} c(\tau) s(t - \tau) d\tau + n(t) \end{aligned} \quad (1-3-2)$$

where $c(t)$ is the impulse response of the linear filter and \star denotes convolution.

The Linear Time-Variant Filter Channel Physical channels such as underwater acoustic channels and ionospheric radio channels that result in time-variant multipath propagation of the transmitted signal may be characterized mathematically as time-variant linear filters. Such linear filters are characterized by a time-variant channel impulse response $c(\tau; t)$, where $c(\tau; t)$ is the response of the channel at time t due to an impulse applied at time $t - \tau$. Thus, τ represents the "age" (elapsed-time) variable. The linear time-variant filter channel with additive noise is illustrated in Fig. 1-3-3. For an input signal $s(t)$, the channel output signal is

$$\begin{aligned} r(t) &= s(t) \star c(\tau; t) + n(t) \\ &= \int_{-\infty}^{\infty} c(\tau; t) s(t - \tau) d\tau + n(t) \end{aligned} \quad (1-3-3)$$

A good model for multipath signal propagation through physical channels, such as the ionosphere (at frequencies below 30 MHz) and mobile cellular radio channels, is a special case of (1-3-3) in which the time-variant impulse response has the form

$$c(\tau; t) = \sum_{k=1}^L a_k(t) \delta(\tau - \tau_k) \quad (1-3-4)$$

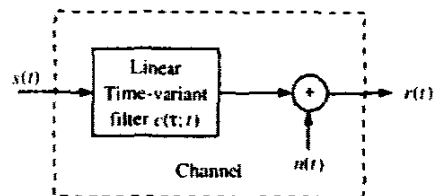


FIGURE 1-3-3 Linear time-variant filter channel with additive noise.

where the $\{a_k(t)\}$ represents the possibly time-variant attenuation factor for the L multipath propagation paths and $\{\tau_k\}$ are the corresponding time delays. If (1-3-4) is substituted into (1-3-3), the received signal has the form

$$r(t) = \sum_{k=1}^L a_k(t)s(t - \tau_k) + n(t) \quad (1-3-5)$$

Hence, the received signal consists of L multipath components, where each component is attenuated by $\{a_k(t)\}$ and delayed by $\{\tau_k\}$.

The three mathematical models described above adequately characterize the great majority of the physical channels encountered in practice. These three channel models are used in this text for the analysis and design of communication systems.

1-4 A HISTORICAL PERSPECTIVE IN THE DEVELOPMENT OF DIGITAL COMMUNICATIONS

It is remarkable that the earliest form of electrical communication, namely *telegraphy*, was a digital communication system. The electric telegraph was developed by Samuel Morse and was demonstrated in 1837. Morse devised the variable-length binary code in which letters of the English alphabet are represented by a sequence of dots and dashes (code words). In this code, more frequently occurring letters are represented by short code words, while letters occurring less frequently are represented by longer code words. Thus, the *Morse code* was the precursor of the variable-length source coding methods described in Chapter 3.

Nearly 40 years later, in 1875, Emile Baudot devised a code for telegraphy in which every letter was encoded into fixed-length binary code words of length 5. In the *Baudot code*, binary code elements are of equal length and designated as mark and space.

Although Morse is responsible for the development of the first electrical digital communication system (telegraphy), the beginnings of what we now regard as modern digital communications stem from the work of Nyquist (1924), who investigated the problem of determining the maximum signaling rate that can be used over a telegraph channel of a given bandwidth without intersymbol interference. He formulated a model of a telegraph system in which a transmitted signal has the general form

$$s(t) = \sum_n a_n g(t - nT) \quad (1-4-1)$$

where $g(t)$ represents a basic pulse shape and $\{a_n\}$ is the binary data sequence of $\{\pm 1\}$ transmitted at a rate of $1/T$ bits/s. Nyquist set out to determine the optimum pulse shape that was bandlimited to W Hz and maximized the bit rate under the constraint that the pulse caused no intersymbol interference at the

sampling time k/T , $k = 0, \pm 1, \pm 2, \dots$. His studies led him to conclude that the maximum pulse rate is $2W$ pulses/s. This rate is now called the *Nyquist rate*. Moreover, this pulse rate can be achieved by using the pulses $g(t) = (\sin 2\pi Wt)/2\pi Wt$. This pulse shape allows recovery of the data without intersymbol interference at the sampling instants. Nyquist's result is equivalent to a version of the sampling theorem for bandlimited signals, which was later stated precisely by Shannon (1948). The sampling theorem states that a signal of bandwidth W can be reconstructed from samples taken at the Nyquist rate of $2W$ samples/s using the interpolation formula

$$s(t) = \sum_n s\left(\frac{n}{2W}\right) \frac{\sin [2\pi W(t - n/2W)]}{2\pi W(t - n/2W)} \quad (1-4-2)$$

In light of Nyquist's work, Hartley (1928) considered the issue of the amount of data that can be transmitted reliably over a bandlimited channel when multiple amplitude levels are used. Due to the presence of noise and other interference, Hartley postulated that the receiver can reliably estimate the received signal amplitude to some accuracy, say A_s . This investigation led Hartley to conclude that there is a maximum data rate that can be communicated reliably over a bandlimited channel when the maximum signal amplitude is limited to A_{\max} (fixed power constraint) and the amplitude resolution is A_s .

Another significant advance in the development of communications was the work of Wiener (1942), who considered the problem of estimating a desired signal waveform $s(t)$ in the presence of additive noise $n(t)$, based on observation of the received signal $r(t) = s(t) + n(t)$. This problem arises in signal demodulation. Wiener determined the linear filter whose output is the best mean-square approximation to the desired signal $s(t)$. The resulting filter is called the *optimum linear (Wiener) filter*.

Hartley's and Nyquist's results on the maximum transmission rate of digital information were precursors to the work of Shannon (1948a,b), who established the mathematical foundations for information transmission and derived the fundamental limits for digital communication systems. In his pioneering work, Shannon formulated the basic problem of reliable transmission of information in statistical terms, using probabilistic models for information sources and communication channels. Based on such a statistical formulation, he adopted a logarithmic measure for the information content of a source. He also demonstrated that the effect of a transmitter power constraint, a bandwidth constraint, and additive noise can be associated with the channel and incorporated into a single parameter, called the *channel capacity*. For example, in the case of an additive white (spectrally flat) gaussian noise interference, an ideal bandlimited channel of bandwidth W has a capacity C given by

$$C = W \log_2 \left(1 + \frac{P}{WN_0} \right) \text{ bits/s} \quad (1-4-3)$$

where P is the average transmitted power and N_0 is the power spectral density of the additive noise. The significance of the channel capacity is as follows: If the information rate R from the source is less than C ($R < C$) then it is theoretically possible to achieve reliable (error-free) transmission through the channel by appropriate coding. On the other hand, if $R > C$, reliable transmission is not possible regardless of the amount of signal processing performed at the transmitter and receiver. Thus, Shannon established basic limits on communication of information, and gave birth to a new field that is now called *information theory*.

Another important contribution to the field of digital communication is the work of Kotelnikov (1947), who provided a coherent analysis of the various digital communication systems based on a geometrical approach. Kotelnikov's approach was later expanded by Wozencraft and Jacobs (1965).

Following Shannon's publications, came the classic work of Hamming (1950) on error-detecting and error-correcting codes to combat the detrimental effects of channel noise. Hamming's work stimulated many researchers in the years that followed, and a variety of new and powerful codes were discovered, many of which are used today in the implementation of modern communication systems.

The increase in demand for data transmission during the last three to four decades, coupled with the development of more sophisticated integrated circuits, has led to the development of very efficient and more reliable digital communication systems. In the course of these developments, Shannon's original results and the generalization of his results on maximum transmission limits over a channel and on bounds on the performance achieved have served as benchmarks for any given communication system design. The theoretical limits derived by Shannon and other researchers that contributed to the development of information theory serve as an ultimate goal in the continuing efforts to design and develop more efficient digital communication systems.

There have been many new advances in the area of digital communications following the early work of Shannon, Kotelnikov, and Hamming. Some of the most notable developments are the following:

- The development of new block codes by Muller (1954), Reed (1954), Reed and Solomon (1960), Bose and Ray-Chaudhuri (1960a,b), and Goppa (1970, 1971).
- The development of concatenated codes by Forney (1966).
- The development of computationally efficient decoding of BCH codes, e.g., the Berlekamp–Massey algorithm (see Chien, 1964; Berlekamp, 1968).
- The development of convolutional codes and decoding algorithms by Wozencraft and Reiffen (1961), Fano (1963), Zigangirov (1966), Jelinek (1969), Forney (1970, 1972), and Viterbi (1967, 1971).
- The development of trellis-coded modulation by Ungerboeck (1982), Forney *et al.* (1984), Wei (1987), and others.
- The development of efficient source encodings algorithms for data

compression, such as those devised by Ziv and Lempel (1977, 1978) and Linde *et al.* (1980).

1-5 OVERVIEW OF THE BOOK

Chapter 2 presents a brief review of the basic notions in the theory of probability and random processes. Our primary objectives in this chapter are to present results that are used throughout the book and to establish some necessary notation.

In Chapter 3, we provide an introduction to source coding for discrete and analog sources. Included in this chapter are the Huffman coding algorithm and the Lempel–Ziv algorithm for discrete sources, and scalar and vector quantization techniques for analog sources.

Chapter 4 treats the characterization of communication signals and systems from a mathematical viewpoint. Included in this chapter is a geometric representation of signal waveforms used for digital communications.

Chapters 5–8 are focused on modulation/demodulation and channel coding/decoding for the additive, white gaussian noise channel. The emphasis is on optimum demodulation and decoding techniques and their performance.

The design of efficient modulators and demodulators for linear filter channels with distortion is treated in Chapters 9–11. The focus is on signal design and on channel equalization methods to compensate for the channel distortion.

The final four chapters treat several more specialized topics. Chapter 12 treats multichannel and multicarrier communication systems. Chapter 13 is focused on spread spectrum signals for digital communications and their performance characteristics. Chapter 14 provides a in-depth treatment of communication through fading multipath channels. Included in this treatment is a description of channel characterization, signal design and demodulation techniques and their performance, and coding/decoding techniques and their performance. The last chapter of the book is focused on multiuser communication systems and multiple access methods.

1-6 BIBLIOGRAPHICAL NOTES AND REFERENCES

There are several historical treatments regarding the development of radio and telecommunications during the past century. These may be found in the books by McMahan (1984), Millman (1984), and Ryder and Fink (1984). We have already cited the classical works of Nyquist (1924), Hartley (1928), Kotelnikov (1947), Shannon (1948), and Hamming (1950), as well as some of the more important advances that have occurred in the field since 1950. The collected papers by Shannon have been published by IEEE Press in a book edited by Sloane and Wyner (1993). Other collected works published by the IEEE Press that might be of interest to the reader are *Key Papers in the Development of Coding Theory*, edited by Berlekamp (1974), and *Key Papers in the Development of Information Theory*, edited by Slepian (1974).

PROBABILITY AND STOCHASTIC PROCESSES

The theory of probability and stochastic processes is an essential mathematical tool in the design of digital communication systems. This subject is important in the statistical modeling of sources that generate the information, in the digitization of the source output, in the characterization of the channel through which the digital information is transmitted, in the design of the receiver that processes the information-bearing signal from the channel, and in the evaluation of the performance of the communication system. Our coverage of this rich and interesting subject is brief and limited in scope. We present a number of definitions and basic concepts in the theory of probability and stochastic processes and we derive several results that are important in the design of efficient digital communication systems and in the evaluation of their performance.

We anticipate that most readers have had some prior exposure to the theory of probability and stochastic processes, so that our treatment serves primarily as a review. Some readers, however, who have had no previous exposure may find the presentation in this chapter extremely brief. These readers will benefit from additional reading of engineering-level treatments of the subject found in the texts by Davenport and Root (1958), Davenport (1970), Papoulis (1984), Helstrom (1991), and Leon-Garcia (1994).

2-1 PROBABILITY

Let us consider an experiment, such as the rolling of a die, with a number of possible outcomes. The sample space S of the experiment consists of the set of all possible outcomes. In the case of the die,

$$S = \{1, 2, 3, 4, 5, 6\} \tag{2-1-1}$$

where the integers 1, . . . , 6 represent the number of dots on the six faces of the die. These six possible outcomes are the sample points of the experiment. An event is a subset of S , and may consist of any number of sample points. For example, the event A defined as

$$A = \{2, 4\} \quad (2-1-2)$$

consists of the outcomes 2 and 4. The complement of the event A , denoted by \bar{A} , consists of all the sample points in S that are not in A and, hence,

$$\bar{A} = \{1, 3, 5, 6\} \quad (2-1-3)$$

Two events are said to be mutually exclusive if they have no sample points in common—that is, if the occurrence of one event excludes the occurrence of the other. For example, if A is defined as in (2-1-2) and the event B is defined as

$$B = \{1, 3, 6\} \quad (2-1-4)$$

then A and B are mutually exclusive events. Similarly, A and \bar{A} are mutually exclusive events.

The union (sum) of two events is an event that consists of all the sample points in the two events. For example, if B is the event defined in (2-1-4) and C is the event defined as

$$C = \{1, 2, 3\} \quad (2-1-5)$$

then, the union of B and C , denoted by $B \cup C$, is the event

$$\begin{aligned} D &= B \cup C \\ &= \{1, 2, 3, 6\} \end{aligned} \quad (2-1-6)$$

Similarly, $A \cup \bar{A} = S$, where S is the entire sample space or the certain event. On the other hand, the intersection of two events is an event that consists of the points that are common to the two events. Thus, if $E = B \cap C$ represents the intersection of the events B and C , defined by (2-1-4) and (2-1-5), respectively, then

$$E = \{1, 3\}$$

When the events are mutually exclusive, the intersection is the null event, denoted as \emptyset . For example, $A \cap B = \emptyset$, and $A \cap \bar{A} = \emptyset$. The definitions of union and intersection are extended to more than two events in a straightforward manner.

Associated with each event A contained in S is its probability $P(A)$. In the assignment of probabilities to events, we adopt an axiomatic viewpoint. That

is, we postulate that the probability of the event A satisfies the condition $P(A) \geq 0$. We also postulate that the probability of the sample space (certain event) is $P(S) = 1$. The third postulate deals with the probability of mutually exclusive events. Suppose that $A_i, i = 1, 2, \dots$, are a (possibly infinite) number of events in the sample space S such that

$$A_i \cap A_j = \emptyset \quad i \neq j = 1, 2, \dots$$

Then the probability of the union of these mutually exclusive events satisfies the condition

$$P\left(\bigcup_i A_i\right) = \sum_i P(A_i) \quad (2-1-7)$$

For example, in a roll of a fair die, each possible outcome is assigned the probability $\frac{1}{6}$. The event A defined by (2-1-2) consists of two mutually exclusive subevents or outcomes, and, hence, $P(A) = \frac{2}{6} = \frac{1}{3}$. Also, the probability of the event $A \cup B$, where A and B are the mutually exclusive events defined by (2-1-2) and (2-1-4), respectively, is $P(A) + P(B) = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}$.

Joint Events and Joint Probabilities Instead of dealing with a single experiment, let us perform two experiments and consider their outcomes. For example, the two experiments may be two separate tosses of a single die or a single toss of two dice. In either case, the sample space S consists of the 36 two-tuples (i, j) where $i, j = 1, 2, \dots, 6$. If the dice are fair, each point in the sample space is assigned the probability $\frac{1}{36}$. We may now consider joint events, such as $\{i \text{ is even, } j = 3\}$, and determine the associated probabilities of such events from knowledge of the probabilities of the sample points.

In general, if one experiment has the possible outcomes $A_i, i = 1, 2, \dots, n$, and the second experiment has the possible outcomes $B_j, j = 1, 2, \dots, m$, then the combined experiment has the possible joint outcomes $(A_i, B_j), i = 1, 2, \dots, n, j = 1, 2, \dots, m$. Associated with each joint outcome (A_i, B_j) is the joint probability $P(A_i, B_j)$ which satisfies the condition

$$0 \leq P(A_i, B_j) \leq 1$$

Assuming that the outcomes $B_j, j = 1, 2, \dots, m$, are mutually exclusive, it follows that

$$\sum_{j=1}^m P(A_i, B_j) = P(A_i) \quad (2-1-8)$$

Similarly, if the outcomes $A_i, i = 1, 2, \dots, n$, are mutually exclusive then

$$\sum_{i=1}^n P(A_i, B_j) = P(B_j) \quad (2-1-9)$$

Furthermore, if all the outcomes of the two experiments are mutually exclusive then

$$\sum_{i=1}^n \sum_{j=1}^m P(A_i, B_j) = 1 \quad (2-1-10)$$

The generalization of the above treatment to more than two experiments is straightforward.

Conditional Probabilities Consider a combined experiment in which a joint event occurs with probability $P(A, B)$. Suppose that the event B has occurred and we wish to determine the probability of occurrence of the event A . This is called the *conditional probability* of the event A given the occurrence of the event B and is defined as

$$P(A | B) = \frac{P(A, B)}{P(B)} \quad (2-1-11)$$

provided $P(B) > 0$. In a similar manner, the probability of the event B conditioned on the occurrence of the event A is defined as

$$P(B | A) = \frac{P(A, B)}{P(A)} \quad (2-1-12)$$

provided $P(A) > 0$. The relations in (2-1-11) and (2-1-12) may also be expressed as

$$P(A, B) = P(A | B)P(B) = P(B | A)P(A) \quad (2-1-13)$$

The relations in (2-1-11), (2-1-12), and (2-1-13) also apply to a single experiment in which A and B are any two events defined on the sample space S and $P(A, B)$ is interpreted as the probability of the $A \cap B$. That is, $P(A, B)$ denotes the simultaneous occurrence of A and B . For example, consider the events B and C given by (2-1-4) and (2-1-5), respectively, for the single toss of a die. The joint event consists of the sample points $\{1, 3\}$. The conditional probability of the event C given that B occurred is

$$P(C | B) = \frac{\frac{2}{6}}{\frac{3}{6}} = \frac{2}{3}$$

In a single experiment, we observe that when two events A and B are mutually exclusive, $A \cap B = \emptyset$ and, hence, $P(A | B) = 0$. Also, if A is a subset of B then $A \cap B = A$ and, hence,

$$P(A | B) = \frac{P(A)}{P(B)}$$

On the other hand, if B is a subset of A , we have $A \cap B = B$ and, hence,

$$P(A | B) = \frac{P(B)}{P(B)} = 1$$

An extremely useful relationship for conditional probabilities is Bayes' theorem, which states that if A_i , $i = 1, 2, \dots, n$, are mutually exclusive events such that

$$\bigcup_{i=1}^n A_i = S$$

and B is an arbitrary event with nonzero probability then

$$\begin{aligned} P(A_i | B) &= \frac{P(A_i, B)}{P(B)} \\ &= \frac{P(B | A_i)P(A_i)}{\sum_{j=1}^n P(B | A_j)P(A_j)} \end{aligned} \quad (2-1-14)$$

We use this formula in Chapter 5 to derive the structure of the optimum receiver for a digital communication system in which the events A_i , $i = 1, 2, \dots, n$, represent the possible transmitted messages in a given time interval, $P(A_i)$ represent their *a priori* probabilities, B represents the received signal, which consists of the transmitted message (one of the A_i) corrupted by noise, and $P(A_i | B)$ is the *a posteriori* probability of A_i conditioned on having observed the received signal B .

Statistical Independence The statistical independence of two or more events is another important concept in probability theory. It usually arises when we consider two or more experiments or repeated trials of a single experiment. To explain this concept, we consider two events A and B and their conditional probability $P(A | B)$, which is the probability of occurrence of A given that B has occurred. Suppose that the occurrence of A does not depend on the occurrence of B . That is,

$$P(A | B) = P(A) \quad (2-1-15)$$

Substitution of (2-1-15) into (2-1-13) yields the result

$$P(A, B) = P(A)P(B) \quad (2-1-16)$$

That is, the joint probability of the events A and B factors into the product of

the elementary or marginal probabilities $P(A)$ and $P(B)$. When the events A and B satisfy the relation in (2-1-16), they are said to be *statistically independent*.

For example, consider two successive experiments in tossing a die. Let A represent the even-numbered sample points $\{2, 4, 6\}$ in the first toss and B represent the even-numbered possible outcomes $\{2, 4, 6\}$ in the second toss. In a fair die, we assign the probabilities $P(A) = \frac{1}{2}$ and $P(B) = \frac{1}{2}$. Now, the joint probability of the joint event "even-numbered outcome on the first toss and even-numbered outcome on the second toss" is just the probability of the nine pairs of outcomes (i, j) , $i = 2, 4, 6, j = 2, 4, 6$, which is $\frac{1}{4}$. Also,

$$P(A, B) = P(A)P(B) = \frac{1}{4}$$

Thus, the events A and B are statistically independent. Similarly, we may say that the outcomes of the two experiments are statistically independent.

The definition of statistical independence can be extended to three or more events. Three statistically independent events A_1 , A_2 , and A_3 must satisfy the following conditions:

$$\begin{aligned} P(A_1, A_2) &= P(A_1)P(A_2) \\ P(A_1, A_3) &= P(A_1)P(A_3) \\ P(A_2, A_3) &= P(A_2)P(A_3) \\ P(A_1, A_2, A_3) &= P(A_1)P(A_2)P(A_3) \end{aligned} \quad (2-1-17)$$

In the general case, the events A_i , $i = 1, 2, \dots, n$, are statistically independent provided that the probabilities of the joint events taken 2, 3, 4, \dots , and n at a time factor into the product of the probabilities of the individual events.

2-1-1 Random Variables, Probability Distributions, and Probability Densities

Given an experiment having a sample space S and elements $s \in S$, we define a function $X(s)$ whose domain is S and whose range is a set of numbers on the real line. The function $X(s)$ is called a *random variable*. For example, if we flip a coin the possible outcomes are head (H) and tail (T), so S contains two points labeled H and T. Suppose we define a function $X(s)$ such that

$$X(s) = \begin{cases} 1 & (s = H) \\ -1 & (s = T) \end{cases} \quad (2-1-18)$$

Thus we have mapped the two possible outcomes of the coin-flipping

experiment into the two points (± 1) on the real line. Another experiment is the toss of a die with possible outcomes $S = \{1, 2, 3, 4, 5, 6\}$. A random variable defined on this sample space may be $X(s) = s$, in which case the outcomes of the experiment are mapped into the integers $1, \dots, 6$, or, perhaps, $X(s) = s^2$, in which case the possible outcomes are mapped into the integers $\{1, 4, 9, 16, 25, 36\}$. These are examples of discrete random variables.

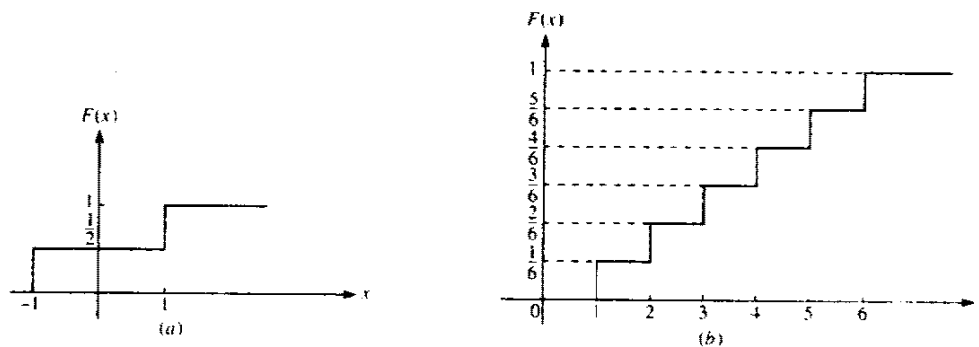
Although we have used as examples experiments that have a finite set of possible outcomes, there are many physical systems (experiments) that generate continuous outputs (outcomes). For example, the noise voltage generated by an electronic amplifier has a continuous amplitude. Consequently, the sample space S of voltage amplitudes $v \in S$ is continuous and so is the mapping $X(v) = v$. In such a case, the random variable† X is said to be a *continuous random variable*.

Given a random variable X , let us consider the event $\{X \leq x\}$ where x is any real number in the interval $(-\infty, \infty)$. We write the probability of this event as $P(X \leq x)$ and denote it simply by $F(x)$, i.e.,

$$F(x) = P(X \leq x) \quad (-\infty < x < \infty) \tag{2-1-19}$$

The function $F(x)$ is called the *probability distribution function* of the random variable X . It is also called the *cumulative distribution function* (cdf). Since $F(x)$ is a probability, its range is limited to the interval $0 \leq F(x) \leq 1$. In fact, $F(-\infty) = 0$ and $F(\infty) = 1$. For example, the discrete random variable generated by flipping a fair coin and defined by (2-1-18) has the cdf shown in Fig. 2-1-1(a). There are two discontinuities or jumps in $F(x)$, one at $x = -1$ and one at $x = 1$. Similarly, the random variable $X(s) = s$ generated by tossing a fair die has the cdf shown in Fig. 2-1-1(b). In this case $F(x)$ has six jumps, one at each of the points $x = 1, \dots, 6$.

FIGURE 2-1-1 Examples of the cumulative distribution functions of two discrete random variables.



† The random variable $X(s)$ will be written simply as X .

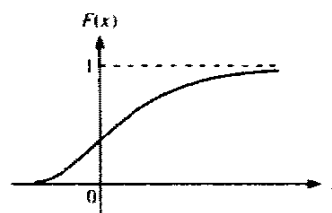


FIGURE 2-1-2 An example of the cumulative distribution function of a continuous random variable.

The cdf of a continuous random variable typically appears as shown in Fig. 2-1-2. This is a smooth, nondecreasing function of x . In some practical problems, we may also encounter a random variable of a mixed type. The cdf of such a random variable is a smooth, nondecreasing function in certain parts of the real line and contains jumps at a number of discrete values of x . An example of such a cdf is illustrated in Fig. 2-1-3.

The derivative of the cdf $F(x)$, denoted as $p(x)$, is called the *probability density function* (pdf) of the random variable X . Thus, we have

$$p(x) = \frac{dF(x)}{dx} \quad (-\infty < x < \infty) \quad (2-1-20)$$

or, equivalently

$$F(x) = \int_{-\infty}^x p(u) du \quad (-\infty < x < \infty) \quad (2-1-21)$$

Since $F(x)$ is a nondecreasing function, it follows that $p(x) \geq 0$. When the random variable is discrete or of a mixed type, the pdf contains impulses at the points of discontinuity of $F(x)$. In such cases, the discrete part of $p(x)$ may be expressed as

$$p(x) = \sum_{i=1}^n P(X = x_i) \delta(x - x_i) \quad (2-1-22)$$

where x_i , $i = 1, 2, \dots, n$, are the possible discrete values of the random

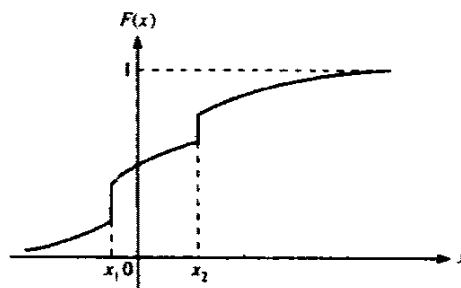


FIGURE 2-1-3 An example of the cumulative distribution function of a random variable of a mixed type.

variable; $P(X = x_i)$, $i = 1, 2, \dots, n$, are the probabilities, and $\delta(x)$ denotes an impulse at $x = 0$.

Often we are faced with the problem of determining the probability that a random variable X falls in an interval (x_1, x_2) , where $x_2 > x_1$. To determine the probability of this event, let us begin with the event $\{X \leq x_2\}$. The event can always be expressed as the union of two mutually exclusive events $\{X \leq x_1\}$ and $\{x_1 < X \leq x_2\}$. Hence the probability of the event $\{X \leq x_2\}$ can be expressed as the sum of the probabilities of the mutually exclusive events. Thus we have

$$P(X \leq x_2) = P(X \leq x_1) + P(x_1 < X \leq x_2)$$

$$F(x_2) = F(x_1) + P(x_1 < X \leq x_2)$$

or, equivalently,

$$\begin{aligned} P(x_1 < X \leq x_2) &= F(x_2) - F(x_1) \\ &= \int_{x_1}^{x_2} p(x) dx \end{aligned} \quad (2-1-23)$$

In other words, the probability of the event $\{x_1 < X \leq x_2\}$ is simply the area under the pdf in the range $x_1 < X \leq x_2$.

Multiple Random Variables, Joint Probability Distributions, and Joint Probability Densities In dealing with combined experiments or repeated trials of a single experiment, we encounter multiple random variables and their cdfs and pdfs. Multiple random variables are basically multidimensional functions defined on a sample space of a combined experiment. Let us begin with two random variables X_1 and X_2 , each of which may be continuous, discrete, or mixed. The joint cumulative distribution function (joint cdf) for the two random variables is defined as

$$\begin{aligned} F(x_1, x_2) &= P(X_1 \leq x_1, X_2 \leq x_2) \\ &= \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} p(u_1, u_2) du_1 du_2 \end{aligned} \quad (2-1-24)$$

where $p(x_1, x_2)$ is the joint probability density function (joint pdf). The latter may also be expressed in the form

$$p(x_1, x_2) = \frac{\partial^2}{\partial x_1 \partial x_2} F(x_1, x_2) \quad (2-1-25)$$

When the joint pdf $p(x_1, x_2)$ is integrated over one of the variables, we obtain the pdf of the other variable. That is,

$$\begin{aligned} \int_{-\infty}^{\infty} p(x_1, x_2) dx_1 &= p(x_2) \\ \int_{-\infty}^{\infty} p(x_1, x_2) dx_2 &= p(x_1) \end{aligned} \quad (2-1-26)$$

The pdfs $p(x_1)$ and $p(x_2)$ obtained from integrating over one of the variables are called *marginal pdfs*. Furthermore, if $p(x_1, x_2)$ is integrated over both variables, we obtain

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(x_1, x_2) dx_1 dx_2 = F(\infty, \infty) = 1 \quad (2-1-27)$$

We also note that $F(-\infty, -\infty) = F(-\infty, x_2) = F(x_1, -\infty) = 0$.

The generalization of the above expressions to multidimensional random variables is straightforward. Suppose that X_i , $i = 1, 2, \dots, n$, are random variables with a joint cdf defined as

$$\begin{aligned} F(x_1, x_2, \dots, x_n) &= P(X_1 \leq x_1, X_2 \leq x_2, \dots, X_n \leq x_n) \\ &= \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} \dots \int_{-\infty}^{x_n} p(u_1, u_2, \dots, u_n) du_1 du_2 \dots du_n \end{aligned} \quad (2-1-28)$$

where $p(x_1, x_2, \dots, x_n)$ is the joint pdf. By taking the partial derivatives of $F(x_1, x_2, \dots, x_n)$ given by (2-1-28), we obtain

$$p(x_1, x_2, \dots, x_n) = \frac{\partial^n}{\partial x_1 \partial x_2 \dots \partial x_n} F(x_1, x_2, \dots, x_n) \quad (2-1-29)$$

Any number of variables in $p(x_1, x_2, \dots, x_n)$ can be eliminated by integrating over these variables. For example, integration over x_2 and x_3 yields

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(x_1, x_2, x_3, \dots, x_n) dx_2 dx_3 = p(x_1, x_4, \dots, x_n) \quad (2-1-30)$$

It also follows that $F(x_1, \infty, \infty, x_4, \dots, x_n) = F(x_1, x_4, x_5, \dots, x_n)$ and

$$F(x_1, -\infty, -\infty, x_4, \dots, x_n) = 0.$$

Conditional Probability Distribution Functions Let us consider two random variables X_1 and X_2 with joint pdf $p(x_1, x_2)$. Suppose that we wish to determine the probability that the random variable $X_1 \leq x_1$ conditioned on

$$x_2 - \Delta x_2 < X_2 \leq x_2$$

where Δx_2 is some positive increment. That is, we wish to determine the probability of the event $(X_1 \leq x_1 | x_2 - \Delta x_2 < X_2 \leq x_2)$. Using the relations established earlier for the conditional probability of an event, the probability of the event $(X_1 \leq x_1 | x_2 - \Delta x_2 < X_2 \leq x_2)$ can be expressed as the probability

of the joint event $(X_1 \leq x_1, x_2 - \Delta x_2 < X_2 \leq x_2)$ divided by the probability of the event $(x_2 - \Delta x_2 < X_2 \leq x_2)$. Thus

$$\begin{aligned} P(X_1 \leq x_1 \mid x_2 - \Delta x_2 < X_2 \leq x_2) &= \frac{\int_{-\infty}^{x_1} \int_{x_2 - \Delta x_2}^{x_2} p(u_1, u_2) du_1 du_2}{\int_{x_2 - \Delta x_2}^{x_2} p(u_2) du_2} \\ &= \frac{F(x_1, x_2) - F(x_1, x_2 - \Delta x_2)}{F(x_2) - F(x_2 - \Delta x_2)} \end{aligned} \quad (2-1-31)$$

Assuming that the pdfs $p(x_1, x_2)$ and $p(x_2)$ are continuous functions over the interval $(x_2 - \Delta x_2, x_2)$, we may divide both numerator and denominator in (2-1-31) by Δx_2 and take the limit as $\Delta x_2 \rightarrow 0$. Thus we obtain

$$\begin{aligned} P(X_1 \leq x_1 \mid X_2 = x_2) &\equiv F(x_1 \mid x_2) = \frac{\partial F(x_1, x_2) / \partial x_2}{\partial F(x_2) / \partial x_2} \\ &= \frac{\partial [\int_{-\infty}^{x_1} \int_{-\infty}^{x_2} p(u_1, u_2) du_1 du_2] / \partial x_2}{\partial [\int_{-\infty}^{x_2} p(u_2) du_2] / \partial x_2} \\ &= \frac{\int_{-\infty}^{x_1} p(u_1, x_2) du_1}{p(x_2)} \end{aligned} \quad (2-1-32)$$

which is the conditional cdf of the random variable X_1 given the random variable X_2 . We observe that $F(-\infty \mid x_2) = 0$ and $F(\infty \mid x_2) = 1$. By differentiating (2-1-32) with respect to x_1 , we obtain the corresponding pdf $p(x_1 \mid x_2)$ in the form

$$p(x_1 \mid x_2) = \frac{p(x_1, x_2)}{p(x_2)} \quad (2-1-33)$$

Alternatively, we may express the joint pdf $p(x_1, x_2)$ in terms of the conditional pdfs, $p(x_1 \mid x_2)$ or $p(x_2 \mid x_1)$, as

$$\begin{aligned} p(x_1, x_2) &= p(x_1 \mid x_2)p(x_2) \\ &= p(x_2 \mid x_1)p(x_1) \end{aligned} \quad (2-1-34)$$

The extension of the relations given above to multidimensional random variables is also easily accomplished. Beginning with the joint pdf of the random variables $X_i, i = 1, 2, \dots, n$, we may write

$$p(x_1, x_2, \dots, x_n) = p(x_1, x_2, \dots, x_k \mid x_{k+1}, \dots, x_n)p(x_{k+1}, \dots, x_n) \quad (2-1-35)$$

where k is any integer in the range $1 < k < n$. The joint conditional cdf corresponding to the pdf $p(x_1, x_2, \dots, x_k \mid x_{k+1}, \dots, x_n)$ is

$$\begin{aligned} F(x_1, x_2, \dots, x_k \mid x_{k+1}, \dots, x_n) \\ = \frac{\int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_k} p(u_1, u_2, \dots, u_k, x_{k+1}, \dots, x_n) du_1 du_2 \cdots du_k}{p(x_{k+1}, \dots, x_n)} \end{aligned} \quad (2-1-36)$$

This conditional cdf satisfies the properties previously established for these functions, such as

$$\begin{aligned} F(\infty, x_2, \dots, x_k | x_{k+1}, \dots, x_n) &= F(x_2, x_3, \dots, x_k | x_{k+1}, \dots, x_n) \\ F(-\infty, x_2, \dots, x_k | x_{k+1}, \dots, x_n) &= 0 \end{aligned}$$

Statistically Independent Random Variables. We have already defined statistical independence of two or more events of a sample space S . The concept of statistical independence can be extended to random variables defined on a sample space generated by a combined experiment or by repeated trials of a single experiment. If the experiments result in mutually exclusive outcomes, the probability of an outcome in one experiment is independent of an outcome in any other experiment. That is, the joint probability of the outcomes factors into a product of the probabilities corresponding to each outcome. Consequently, the random variables corresponding to the outcomes in these experiments are independent in the sense that their joint pdf factors into a product of marginal pdfs. Hence the multidimensional random variables are statistically independent if and only if

$$F(x_1, x_2, \dots, x_n) = F(x_1)F(x_2) \cdots F(x_n) \quad (2-1-37)$$

or, alternatively,

$$p(x_1, x_2, \dots, x_n) = p(x_1)p(x_2) \cdots p(x_n) \quad (2-1-38)$$

2-1-2 Functions of Random Variables

A problem that arises frequently in practical applications of probability is the following. Given a random variable X , which is characterized by its pdf $p(x)$, determine the pdf of the random variable $Y = g(X)$, where $g(X)$ is some given function of X . When the mapping g from X to Y is one-to-one, the determination of $p(y)$ is relatively straightforward. However, when the mapping is not one-to-one, as is the case, for example, when $Y = X^2$, we must be very careful in our derivation of $p(y)$.

Example 2-1-1

Consider the random variable Y defined as

$$Y = aX + b \quad (2-1-39)$$

where a and b are constants. We assume that $a > 0$. If $a < 0$, the approach is similar (see Problem 2-3). We note that this mapping, illustrated in Fig. 2-1-4(a) is linear and monotonic. Let $F_X(x)$ and $F_Y(y)$ denote the cdfs for X and Y , respectively.† Then

$$\begin{aligned} F_Y(y) &= P(Y \leq y) = P(aX + b \leq y) = P\left(X \leq \frac{y-b}{a}\right) \\ &= \int_{-\infty}^{(y-b)/a} p_X(x) dx = F_X\left(\frac{y-b}{a}\right) \end{aligned} \quad (2-1-40)$$

† To avoid confusion in changing variables, subscripts are used in the respective pdfs and cdfs.

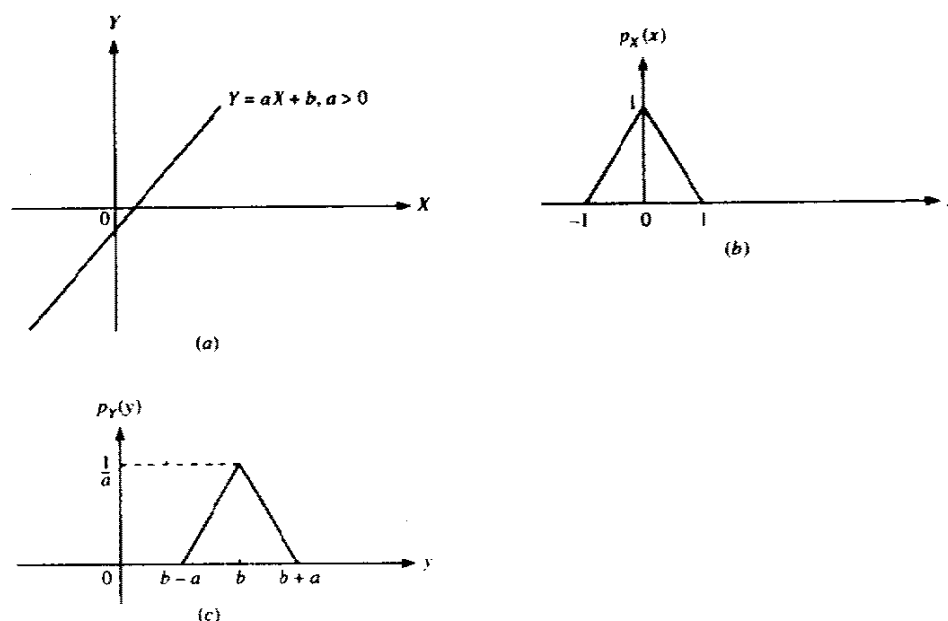


FIGURE 2-1-4 A linear transformation of a random variable X and an example of the corresponding pdfs of X and Y .

By differentiating (2-1-40) with respect to y , we obtain the relationship between the respective pdfs. It is

$$p_Y(y) = \frac{1}{a} p_X\left(\frac{y-b}{a}\right) \quad (2-1-41)$$

Thus (2-1-40) and (2-1-41) specify the cdf and pdf of the random variable Y in terms of the cdf and pdf of the random variable X for the linear transformation in (2-1-39). To illustrate this mapping for a specific pdf $p_X(x)$, consider the one shown in Fig. 2-1-4(b). The pdf $p_Y(y)$ that results from the mapping in (2-1-39) is shown in Fig. 2-1-4(c).

Example 2-1-2

Consider the random variable Y defined as

$$Y = aX^3 + b, \quad a > 0 \quad (2-1-42)$$

As in Example 2-1-1, the mapping between X and Y is one-to-one. Hence

$$\begin{aligned} F_Y(y) &= P(Y \leq y) = P(aX^3 + b \leq y) \\ &= P\left[X \leq \left(\frac{y-b}{a}\right)^{1/3}\right] = F_X\left[\left(\frac{y-b}{a}\right)^{1/3}\right] \end{aligned} \quad (2-1-43)$$

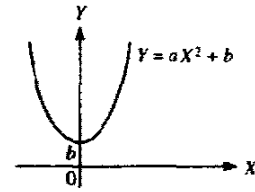


FIGURE 2-1-5 A quadratic transformation of the random variable X .

Differentiation of (2-1-43) with respect to y yields the desired relationship between the two pdfs as

$$p_Y(y) = \frac{1}{3a[(y-b)/a]^{2/3}} p_X\left[\left(\frac{y-b}{a}\right)^{1/3}\right] \quad (2-1-44)$$

Example 2-1-3

The random variable Y is defined as

$$Y = aX^2 + b, \quad a > 0 \quad (2-1-45)$$

In contrast to Examples 2-1-1 and 2-1-2, the mapping between X and Y , illustrated in Fig. 2-1-5, is not one-to-one. To determine the cdf of Y , we observe that

$$\begin{aligned} F_Y(y) &= P(Y \leq y) = P(aX^2 + b \leq y) \\ &= P\left(|X| \leq \sqrt{\frac{y-b}{a}}\right) \end{aligned}$$

Hence

$$F_Y(y) = F_X\left(\sqrt{\frac{y-b}{a}}\right) - F_X\left(-\sqrt{\frac{y-b}{a}}\right) \quad (2-1-46)$$

Differentiating (2-1-46) with respect to y , we obtain the pdf of Y in terms of the pdf of X in the form

$$p_Y(y) = \frac{p_X[\sqrt{(y-b)/a}]}{2a\sqrt{(y-b)/a}} + \frac{p_X[-\sqrt{(y-b)/a}]}{2a\sqrt{(y-b)/a}} \quad (2-1-47)$$

In Example 2-1-3, we observe that the equation $g(x) = ax^2 + b = y$ has two real solutions,

$$\begin{aligned} x_1 &= \sqrt{\frac{y-b}{a}} \\ x_2 &= -\sqrt{\frac{y-b}{a}} \end{aligned}$$

and that $p_Y(y)$ consists of two terms corresponding to these two solutions. That is,

$$p_Y(y) = \frac{p_X[x_1 = \sqrt{(y-b)/a}]}{|g'[x_1 = \sqrt{(y-b)/a}]|} + \frac{p_X[x_2 = -\sqrt{(y-b)/a}]}{|g'[x_2 = -\sqrt{(y-b)/a}]|} \quad (2-1-48)$$

where $g'(x)$ denotes the first derivative of $g(x)$.

In the general case, suppose that x_1, x_2, \dots, x_n are the real roots of the equation $g(x) = y$. Then the pdf of the random variable $Y = g(X)$ may be expressed as

$$p_Y(y) = \sum_{i=1}^n \frac{p_X(x_i)}{|g'(x_i)|} \quad (2-1-49)$$

where the roots $x_i, i = 1, 2, \dots, n$, are functions of y .

Now let us consider functions of multidimensional random variables. Suppose that $X_i, i = 1, 2, \dots, n$, are random variables with joint pdf $p_X(x_1, x_2, \dots, x_n)$, and let $Y_i, i = 1, 2, \dots, n$, be another set of n random variables related to the X_i by the functions

$$Y_i = g_i(X_1, X_2, \dots, X_n), \quad i = 1, 2, \dots, n \quad (2-1-50)$$

We assume that the $g_i(X_1, X_2, \dots, X_n), i = 1, 2, \dots, n$, are single-valued functions with continuous partial derivatives and invertible. By "invertible" we mean that the $X_i, i = 1, 2, \dots, n$, can be expressed as functions of $Y_i, i = 1, 2, \dots, n$, in the form

$$X_i = g_i^{-1}(Y_1, Y_2, \dots, Y_n), \quad i = 1, 2, \dots, n \quad (2-1-51)$$

where the inverse functions are also assumed to be single-valued with continuous partial derivatives. The problem is to determine the joint pdf of $Y_i, i = 1, 2, \dots, n$, denoted by $p_Y(y_1, y_2, \dots, y_n)$, given the joint pdf $p_X(x_1, x_2, \dots, x_n)$.

To determine the desired relation, let R_X be the region in the n -dimensional space of the random variables $X_i, i = 1, 2, \dots, n$, and let R_Y be the (one-to-one) mapping of R_X defined by the functions $Y_i = g_i(X_1, X_2, \dots, X_n)$. Clearly,

$$\begin{aligned} & \iint_{R_Y} \cdots \int p_Y(y_1, y_2, \dots, y_n) dy_1 dy_2 \cdots dy_n \\ &= \iint_{R_X} \cdots \int p_X(x_1, x_2, \dots, x_n) dx_1 dx_2 \cdots dx_n \quad (2-1-52) \end{aligned}$$

By making a change in variables in the multiple integral on the right-hand side of (2-1-52) with the substitution

$$x_i = g_i^{-1}(y_1, y_2, \dots, y_n) \equiv g_i^{-1}, \quad i = 1, 2, \dots, n$$

we obtain

$$\begin{aligned} & \iint_{\mathcal{R}_Y} \cdots \int p_Y(y_1, y_2, \dots, y_n) dy_1 dy_2 \cdots dy_n \\ &= \iint_{\mathcal{R}_X} \cdots \int p_X(x_1 = g_1^{-1}, x_2 = g_2^{-1}, \dots, x_n = g_n^{-1}) |J| dy_1 dy_2 \cdots dy_n \end{aligned} \quad (2-1-53)$$

where J denotes the jacobian of the transformation, defined by the determinant

$$J = \begin{vmatrix} \frac{\partial g_1^{-1}}{\partial y_1} & \frac{\partial g_2^{-1}}{\partial y_1} & \cdots & \frac{\partial g_n^{-1}}{\partial y_1} \\ \vdots & \vdots & & \vdots \\ \frac{\partial g_1^{-1}}{\partial y_n} & \frac{\partial g_2^{-1}}{\partial y_n} & \cdots & \frac{\partial g_n^{-1}}{\partial y_n} \end{vmatrix} \quad (2-1-54)$$

Consequently, the desired relation for the joint pdf of the $Y_i, i = 1, 2, \dots, n$, is

$$p_Y(y_1, y_2, \dots, y_n) = p_X(x_1 = g_1^{-1}, x_2 = g_2^{-1}, \dots, x_n = g_n^{-1}) |J| \quad (2-1-55)$$

Example 2-1-4

An important functional relation between two sets of n -dimensional random variables that frequently arises in practice is the linear transformation

$$Y_i = \sum_{j=1}^n a_{ij} X_j, \quad i = 1, 2, \dots, n \quad (2-1-56)$$

where the $\{a_{ij}\}$ are constants. It is convenient to employ the matrix form for the transformation, which is

$$\mathbf{Y} = \mathbf{A}\mathbf{X} \quad (2-1-57)$$

where \mathbf{X} and \mathbf{Y} are n -dimensional vectors and \mathbf{A} is an $n \times n$ matrix. We assume that \mathbf{A} is nonsingular. Then \mathbf{A} is invertible and, hence,

$$\mathbf{X} = \mathbf{A}^{-1}\mathbf{Y} \quad (2-1-58)$$

Equivalently, we have

$$X_i = \sum_{j=1}^n b_{ij} Y_j, \quad i = 1, 2, \dots, n \quad (2-1-59)$$

where $\{b_{ij}\}$ are the elements of the inverse matrix \mathbf{A}^{-1} . The jacobian of this transformation is $J = 1/\det \mathbf{A}$. Hence

$$\begin{aligned} & p_Y(y_1, y_2, \dots, y_n) \\ &= p_X\left(x_1 = \sum_{j=1}^n b_{1j} y_j, x_2 = \sum_{j=1}^n b_{2j} y_j, \dots, x_n = \sum_{j=1}^n b_{nj} y_j\right) \frac{1}{|\det \mathbf{A}|} \end{aligned} \quad (2-1-60)$$

2-1-3 Statistical Averages of Random Variables

Averages play an important role in the characterization of the outcomes of experiments and the random variables defined on the sample space of the experiments. Of particular interest are the first and second moments of a single random variable and the joint moments, such as the correlation and covariance, between any pair of random variables in a multidimensional set of random variables. Also of great importance are the characteristic function for a single random variable and the joint characteristic function for a multidimensional set of random variables. This section is devoted to the definition of these important statistical averages.

First we consider a single random variable X characterized by its pdf $p(x)$. The *mean* or *expected value* of X is defined as

$$E(X) \equiv m_x = \int_{-\infty}^{\infty} xp(x) dx \quad (2-1-61)$$

where $E(\quad)$ denotes expectation (statistical averaging). This is the first moment of the random variable X . In general, the n th moment is defined as

$$E(X^n) = \int_{-\infty}^{\infty} x^n p(x) dx \quad (2-1-62)$$

Now, suppose that we define a random variable $Y = g(X)$, where $g(X)$ is some arbitrary function of the random variable X . The expected value of Y is

$$E(Y) = E[g(X)] = \int_{-\infty}^{\infty} g(x)p(x) dx \quad (2-1-63)$$

In particular, if $Y = (X - m_x)^n$ where m_x is the mean value of X , then

$$E(Y) = E[(X - m_x)^n] = \int_{-\infty}^{\infty} (x - m_x)^n p(x) dx \quad (2-1-64)$$

This expected value is called the n th *central moment* of the random variable X , because it is a moment taken relative to the mean. When $n = 2$, the central moment is called the *variance* of the random variable and denoted as σ_x^2 . That is,

$$\sigma_x^2 = \int_{-\infty}^{\infty} (x - m_x)^2 p(x) dx \quad (2-1-65)$$

This parameter provides a measure of the dispersion of the random variable X . By expanding the term $(x - m_x)^2$ in the integral of (2-1-65) and noting that the expected value of a constant is equal to the constant, we obtain the expression that relates the variance to the first and second moments, namely,

$$\begin{aligned} \sigma_x^2 &= E(X^2) - [E(X)]^2 \\ &= E(X^2) - m_x^2 \end{aligned} \quad (2-1-66)$$

In the case of two random variables, X_1 and X_2 , with joint pdf $p(x_1, x_2)$, we define the *joint moment* as

$$E(X_1^k X_2^n) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1^k x_2^n p(x_1, x_2) dx_1 dx_2 \quad (2-1-67)$$

and the *joint central moment* as

$$\begin{aligned} E[(X_1 - m_1)^k (X_2 - m_2)^n] \\ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x_1 - m_1)^k (x_2 - m_2)^n p(x_1, x_2) dx_1 dx_2 \end{aligned} \quad (2-1-68)$$

where $m_i = E(X_i)$. Of particular importance to us are the joint moment and joint central moment corresponding to $k = n = 1$. These joint moments are called the *correlation* and the *covariance* of the random variables X_1 and X_2 , respectively.

In considering multidimensional random variables, we can define joint moments of any order. However, the moments that are most useful in practical applications are the correlations and covariances between pairs of random variables. To elaborate, suppose that $X_i, i = 1, 2, \dots, n$, are random variables with joint pdf $p(x_1, x_2, \dots, x_n)$. Let $p(x_i, x_j)$ be the joint pdf of the random variables X_i and X_j . Then the correlation between X_i and X_j is given by the joint moment

$$E(X_i X_j) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_i x_j p(x_i, x_j) dx_i dx_j \quad (2-1-69)$$

and the covariance of X_i and X_j is

$$\begin{aligned} \mu_{ij} &\equiv E[(X_i - m_i)(X_j - m_j)] \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x_i - m_i)(x_j - m_j) p(x_i, x_j) dx_i dx_j \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_i x_j p(x_i, x_j) dx_i dx_j - m_i m_j \\ &= E(X_i X_j) - m_i m_j \end{aligned} \quad (2-1-70)$$

The $n \times n$ matrix with elements μ_{ij} is called the *covariance matrix* of the random variables $X_i, i = 1, 2, \dots, n$. We shall encounter the covariance matrix in our discussion of jointly gaussian random variables in Section 2-1-4.

Two random variables are said to be *uncorrelated* if $E(X_i X_j) = E(X_i)E(X_j) = m_i m_j$. In that case, the covariance $\mu_{ij} = 0$. We note that when X_i and X_j are statistically independent, they are also uncorrelated. However, if X_i and X_j are uncorrelated, they are not necessarily statistically independent.

Two random variables are said to be *orthogonal* if $E(X_i X_j) = 0$. We note that this condition holds when X_i and X_j are uncorrelated and either one or both of the random variables have zero mean.

Characteristic Functions The *characteristic function* of a random variable X is defined as the statistical average

$$E(e^{jvX}) \equiv \psi(jv) = \int_{-\infty}^{\infty} e^{jvx} p(x) dx \quad (2-1-71)$$

where the variable v is real and $j = \sqrt{-1}$. We note that $\psi(jv)$ may be described as the Fourier transform† of the pdf $p(x)$. Hence the inverse Fourier transform is

$$p(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \psi(jv) e^{-jvx} dv \quad (2-1-72)$$

One useful property of the characteristic function is its relation to the moments of the random variable. We note that the first derivative of (2-1-71) with respect to v yields

$$\frac{d\psi(jv)}{dv} = j \int_{-\infty}^{\infty} x e^{jvx} p(x) dx$$

By evaluating the derivative at $v = 0$, we obtain the first moment (mean)

$$E(X) = m_x = -j \left. \frac{d\psi(jv)}{dv} \right|_{v=0} \quad (2-1-73)$$

The differentiation process can be repeated, so that the n th derivative of $\psi(jv)$ evaluated at $v = 0$ yields the n th moment

$$E(X^n) = (-j)^n \left. \frac{d^n \psi(jv)}{dv^n} \right|_{v=0} \quad (2-1-74)$$

Thus the moments of a random variable can be determined from the characteristic function. On the other hand, suppose that the characteristic function can be expanded in a Taylor series about the point $v = 0$. That is,

$$\psi(jv) = \sum_{n=0}^{\infty} \left[\left. \frac{d^n \psi(jv)}{dv^n} \right|_{v=0} \right] \frac{v^n}{n!} \quad (2-1-75)$$

Using the relation in (2-1-74) to eliminate the derivative in (2-1-75), we obtain

† Usually the Fourier transform of a function $g(u)$ is defined as $G(v) = \int_{-\infty}^{\infty} g(u) e^{-jvu} du$, which differs from (2-1-71) by the negative sign in the exponential. This is a trivial difference, however, so we call the integral in (2-1-71) a Fourier transform.

an expression for the characteristic function in terms of its moments in the form

$$\psi(j\nu) = \sum_{n=0}^{\infty} E(X^n) \frac{(j\nu)^n}{n!} \quad (2-1-76)$$

The characteristic function provides a simple method for determining the pdf of a sum of statistically independent random variables. To illustrate this point, let X_i , $i = 1, 2, \dots, n$, be a set of n statistically independent random variables and let

$$Y = \sum_{i=1}^n X_i \quad (2-1-77)$$

The problem is to determine the pdf of Y . We shall determine the pdf of Y by first finding its characteristic function and then computing the inverse Fourier transform. Thus

$$\begin{aligned} \psi_Y(j\nu) &= E(e^{j\nu Y}) \\ &= E\left[\exp\left(j\nu \sum_{i=1}^n X_i\right)\right] \\ &= E\left[\prod_{i=1}^n (e^{j\nu X_i})\right] \\ &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left(\prod_{i=1}^n e^{j\nu x_i}\right) p(x_1, x_2, \dots, x_n) dx_1 dx_2 \cdots dx_n \quad (2-1-78) \end{aligned}$$

Since the random variables are statistically independent, $p(x_1, x_2, \dots, x_n) = p(x_1)p(x_2) \cdots p(x_n)$, and, hence, the n th-order integral in (2-1-78) reduces to a product of n single integrals, each corresponding to the characteristic function of one of the X_i . Hence,

$$\psi_Y(j\nu) = \prod_{i=1}^n \psi_{X_i}(j\nu) \quad (2-1-79)$$

If, in addition to their statistical independence, the X_i are identically distributed then all the $\psi_{X_i}(j\nu)$ are identical. Consequently,

$$\psi_Y(j\nu) = [\psi_X(j\nu)]^n \quad (2-1-80)$$

Finally, the pdf of Y is determined from the inverse Fourier transform of $\psi_Y(j\nu)$, given by (2-1-72).

Since the characteristic function of the sum of n statistically independent random variables is equal to the product of the characteristic functions of the individual random variables X_i , $i = 1, 2, \dots, n$, it follows that, in the transform domain, the pdf of Y is the n -fold convolution of the pdfs of the X_i . Usually the n -fold convolution is more difficult to perform than the characteristic function method described above in determining the pdf of Y .

When working with n -dimensional random variables, it is appropriate to define an n -dimensional Fourier transform of the joint pdf. In particular, if

X_i , $i = 1, 2, \dots, n$, are random variables with pdf $p(x_1, x_2, \dots, x_n)$, the n -dimensional characteristic function is defined as

$$\begin{aligned} \psi(jv_1, jv_2, \dots, jv_n) &= E \left[\exp \left(j \sum_{i=1}^n v_i X_i \right) \right] \\ &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp \left(j \sum_{i=1}^n v_i x_i \right) p(x_1, x_2, \dots, x_n) dx_1 dx_2 \cdots dx_n \end{aligned} \quad (2-1-81)$$

Of special interest is the two-dimensional characteristic function

$$\psi(jv_1, jv_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{j(v_1 x_1 + v_2 x_2)} p(x_1, x_2) dx_1 dx_2 \quad (2-1-82)$$

We observe that the partial derivatives of $\psi(jv_1, jv_2)$ with respect to v_1 and v_2 can be used to generate the joint moments. For example, it is easy to show that

$$E(X_1 X_2) = - \left. \frac{\partial^2 \psi(jv_1, jv_2)}{\partial v_1 \partial v_2} \right|_{v_1=v_2=0} \quad (2-1-83)$$

Higher-order moments are generated in a straightforward manner.

2-1-4 Some Useful Probability Distributions

In subsequent chapters, we shall encounter several different types of random variables. In this section we list these frequently encountered random variables, their pdfs, their cdfs, and their moments. We begin with the binomial distribution, which is the distribution of a discrete random variable, and then we present the distributions of several continuous random variables.

Binomial Distribution Let X be a discrete random variable that has two possible values, say $X = 1$ or $X = 0$, with probabilities p and $1 - p$, respectively. The pdf of X is shown in Fig. 2-1-6. Now, suppose that

$$Y = \sum_{i=1}^n X_i$$

where the X_i , $i = 1, 2, \dots, n$, are statistically independent and identically

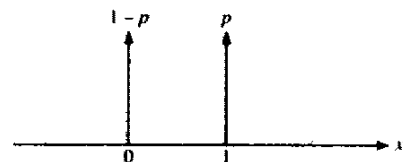


FIGURE 2-1-6 The probability distribution function of X .

distributed random variables with the pdf shown in Fig. 2-1-6. What is the probability distribution function of Y ?

To answer this question, we observe that the range of Y is the set of integers from 0 to n . The probability that $Y = 0$ is simply the probability that all the $X_i = 0$. Since the X_i are statistically independent,

$$P(Y = 0) = (1 - p)^n$$

The probability that $Y = 1$ is simply the probability that one $X_i = 1$ and the rest of the $X_i = 0$. Since this event can occur in n different ways,

$$P(Y = 1) = np(1 - p)^{n-1}$$

To generalize, the probability that $Y = k$ is the probability that k of the X_i are equal to one and $n - k$ are equal to zero. Since there are

$$\binom{n}{k} \equiv \frac{n!}{k!(n-k)!} \quad (2-1-84)$$

different combinations that result in the event $\{Y = k\}$, it follows that

$$P(Y = k) = \binom{n}{k} p^k (1 - p)^{n-k} \quad (2-1-85)$$

where $\binom{n}{k}$ is the binomial coefficient. Consequently, the pdf of Y may be expressed as

$$\begin{aligned} p(y) &= \sum_{k=0}^n P(Y = k) \delta(y - k) \\ &= \sum_{k=0}^n \binom{n}{k} p^k (1 - p)^{n-k} \delta(y - k) \end{aligned} \quad (2-1-86)$$

The cdf of Y is

$$\begin{aligned} F(y) &= P(Y \leq y) \\ &= \sum_{k=0}^{[y]} \binom{n}{k} p^k (1 - p)^{n-k} \end{aligned} \quad (2-1-87)$$

where $[y]$ denotes the largest integer m such that $m \leq y$. The cdf in (2-1-87) characterizes a binomially distributed random variable.

The first two moments of Y are

$$\begin{aligned} E(Y) &= np \\ E(Y^2) &= np(1 - p) + n^2 p^2 \\ \sigma^2 &= np(1 - p) \end{aligned} \quad (2-1-88)$$

and the characteristic function is

$$\psi(jv) = (1 - p + pe^{jv})^n \quad (2-1-89)$$

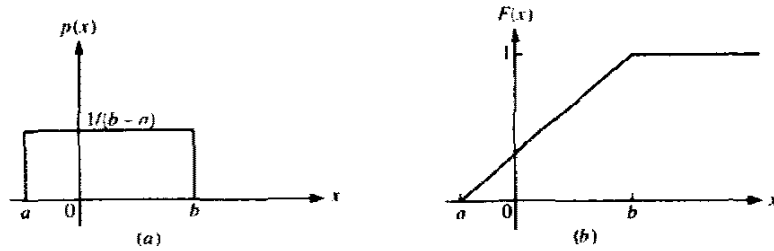


FIGURE 2-1-7 The pdf and cdf of a uniformly distributed random variable.

Uniform Distribution The pdf and cdf of a uniformly distributed random variable X are shown in Fig. 2-1-7. The first two moments of X are

$$\begin{aligned} E(X) &= \frac{1}{2}(a + b) \\ E(X^2) &= \frac{1}{3}(a^2 + b^2 + ab) \\ \sigma^2 &= \frac{1}{12}(a - b)^2 \end{aligned} \quad (2-1-90)$$

and the characteristic function is

$$\psi(j\nu) = \frac{e^{j\nu b} - e^{j\nu a}}{j\nu(b - a)} \quad (2-1-91)$$

Gaussian (Normal) Distribution The pdf of a gaussian or normally distributed random variable is

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-m_x)^2/2\sigma^2} \quad (2-1-92)$$

where m_x is the mean and σ^2 is the variance of the random variable. The cdf is

$$\begin{aligned} F(x) &= \int_{-\infty}^x p(u) du \\ &= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^x e^{-(u-m_x)^2/2\sigma^2} du \\ &= \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{(x-m_x)/\sqrt{2}\sigma} e^{-t^2} dt \\ &= \frac{1}{2} + \frac{1}{2} \operatorname{erf}\left(\frac{x-m_x}{\sqrt{2}\sigma}\right) \end{aligned} \quad (2-1-93)$$

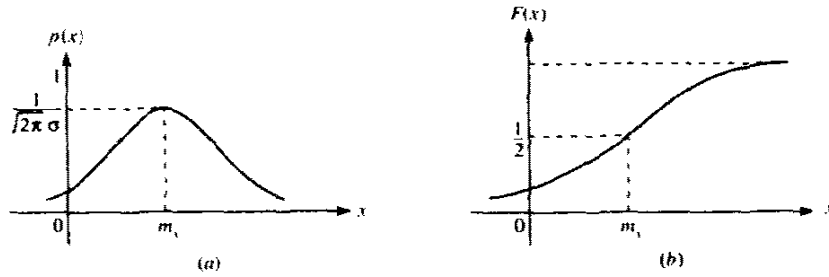


FIGURE 2-1-8 The pdf and cdf of a gaussian-distributed random variable.

where $\text{erf}(x)$ denotes the error function, defined as

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \quad (2-1-94)$$

The pdf and cdf are illustrated in Fig. 2-1-8.

The cdf $F(x)$ may also be expressed in terms of the complementary error function. That is,

$$F(x) = 1 - \frac{1}{2} \text{erfc}\left(\frac{x - m_x}{\sqrt{2}\sigma}\right)$$

where

$$\begin{aligned} \text{erfc}(x) &= \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt \\ &= 1 - \text{erf}(x) \end{aligned} \quad (2-1-95)$$

We note that $\text{erf}(-x) = -\text{erf}(x)$, $\text{erfc}(-x) = 2 - \text{erfc}(x)$, $\text{erf}(0) = \text{erfc}(\infty) = 0$, and $\text{erf}(\infty) = \text{erfc}(0) = 1$. For $x > m_x$, the complementary error function is proportional to the area under the tail of the gaussian pdf. For large values of x , the complementary error function $\text{erfc}(x)$ may be approximated by the asymptotic series

$$\text{erfc}(x) = \frac{e^{-x^2}}{x\sqrt{\pi}} \left(1 - \frac{1}{2x^2} + \frac{1 \cdot 3}{2^2 x^4} - \frac{1 \cdot 3 \cdot 5}{2^3 x^6} + \dots \right) \quad (2-1-96)$$

where the approximation error is less than the last term used.

The function that is frequently used for the area under the tail of the gaussian pdf is denoted by $Q(x)$ and defined as

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-t^2/2} dt, \quad x \geq 0 \quad (2-1-97)$$

By comparing (2-1-95) with (2-1-97), we find

$$Q(x) = \frac{1}{2} \text{erfc}\left(\frac{x}{\sqrt{2}}\right) \quad (2-1-98)$$

The characteristic function of a gaussian random variable with mean m , and variance σ^2 is

$$\begin{aligned}\psi(j\nu) &= \int_{-\infty}^{\infty} e^{j\nu x} \left[\frac{1}{\sqrt{2\pi}\sigma} e^{-(x-m)^2/2\sigma^2} \right] dx \\ &= e^{j\nu m - (1/2)\nu^2\sigma^2}\end{aligned}\quad (2-1-99)$$

The central moments of a gaussian random variable are

$$E[(X - m)^k] \equiv \mu_k = \begin{cases} 1 \cdot 3 \cdots (k-1)\sigma^k & (\text{even } k) \\ 0 & (\text{odd } k) \end{cases} \quad (2-1-100)$$

and the ordinary moments may be expressed in terms of the central moments as

$$E(X^k) = \sum_{i=0}^k \binom{k}{i} m^i \mu_{k-i} \quad (2-1-101)$$

The sum of n statistically independent gaussian random variables is also a gaussian random variable. To demonstrate this point, let

$$Y = \sum_{i=1}^n X_i \quad (2-1-102)$$

where the X_i , $i = 1, 2, \dots, n$, are statistically independent gaussian random variables with means m_i and variances σ_i^2 . Using the result in (2-1-79), we find that the characteristic function of Y is

$$\begin{aligned}\psi_Y(j\nu) &= \prod_{i=1}^n \psi_{X_i}(j\nu) \\ &= \prod_{i=1}^n e^{j\nu m_i - \nu^2 \sigma_i^2 / 2} \\ &= e^{j\nu m_Y - \nu^2 \sigma_Y^2 / 2}\end{aligned}\quad (2-1-103)$$

where

$$\begin{aligned}m_Y &= \sum_{i=1}^n m_i \\ \sigma_Y^2 &= \sum_{i=1}^n \sigma_i^2\end{aligned}\quad (2-1-104)$$

Therefore, Y is gaussian-distributed with mean m_Y and variance σ_Y^2 .

Chi-Square Distribution A chi-square-distributed random variable is related to a gaussian-distributed random variable in the sense that the former can be viewed as a transformation of the latter. To be specific, let $Y = X^2$, where X is a gaussian random variable. Then Y has a chi-square distribution. We distinguish between two types of chi-square distributions. The first is called a

central chi-square distribution and is obtained when X has zero mean. The second is called a non-central chi-square distribution, and is obtained when X has a nonzero mean.

First we consider the central chi-square distribution. Let X be gaussian-distributed with zero mean and variance σ^2 . Since $Y = X^2$, the result given in (2-1-47) applies directly with $a = 1$ and $b = 0$. Thus we obtain the pdf of Y in the form

$$p_Y(y) = \frac{1}{\sqrt{2\pi y} \sigma} e^{-y/2\sigma^2}, \quad y \geq 0 \quad (2-1-105)$$

The cdf of Y is

$$\begin{aligned} F_Y(y) &= \int_0^y p_Y(u) du \\ &= \frac{1}{\sqrt{2\pi} \sigma} \int_0^y \frac{1}{\sqrt{u}} e^{-u/2\sigma^2} du \end{aligned} \quad (2-1-106)$$

which cannot be expressed in closed form. The characteristic function, however, can be determined in closed form. It is

$$\psi(j\nu) = \frac{1}{(1 - j2\nu\sigma^2)^{1/2}} \quad (2-1-107)$$

Now, suppose that the random variable Y is defined as

$$Y = \sum_{i=1}^n X_i^2 \quad (2-1-108)$$

where the X_i , $i = 1, 2, \dots, n$, are statistically independent and identically distributed gaussian random variables with zero mean and variance σ^2 . As a consequence of the statistical independence of the X_i , the characteristic function of Y is

$$\psi_Y(j\nu) = \frac{1}{(1 - j2\nu\sigma^2)^{n/2}} \quad (2-1-109)$$

The inverse transform of this characteristic function yields the pdf

$$p_Y(y) = \frac{1}{\sigma^n 2^{n/2} \Gamma(\frac{1}{2}n)} y^{n/2-1} e^{-y/2\sigma^2}, \quad y \geq 0 \quad (2-1-110)$$

where $\Gamma(p)$ is the gamma function, defined as

$$\begin{aligned} \Gamma(p) &= \int_0^\infty t^{p-1} e^{-t} dt, \quad p > 0 \\ \Gamma(p) &= (p-1)!, \quad p \text{ an integer, } p > 0 \\ \Gamma(\frac{1}{2}) &= \sqrt{\pi}, \quad \Gamma(\frac{3}{2}) = \frac{1}{2}\sqrt{\pi} \end{aligned} \quad (2-1-111)$$

This pdf, which is a generalization of (2-1-105), is called a *chi-square* (or

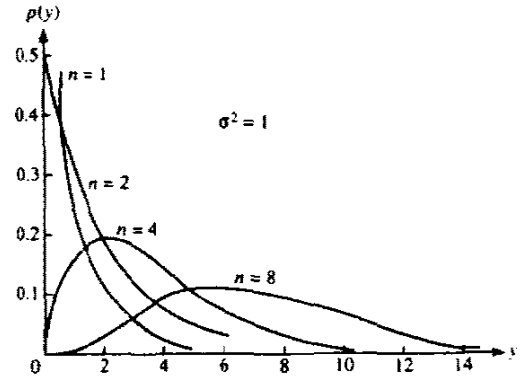


FIGURE 2-1-9 The pdf of a chi-square-distributed random variable for several degrees of freedom.

gamma) pdf with n degrees of freedom. It is illustrated in Fig. 2-1-9. The case $n = 2$ yields the exponential distribution.

The first two moments of Y are

$$\begin{aligned} E(Y) &= n\sigma^2 \\ E(Y^2) &= 2n\sigma^4 + n^2\sigma^4 \\ \sigma_y^2 &= 2n\sigma^4 \end{aligned} \quad (2-1-112)$$

The cdf of Y is

$$F_Y(y) = \int_0^y \frac{1}{\sigma^n 2^{n/2} \Gamma(\frac{1}{2}n)} u^{n/2-1} e^{-u/2\sigma^2} du, \quad y \geq 0 \quad (2-1-113)$$

This integral can be easily manipulated into the form of the incomplete gamma function, which is tabulated by Pearson (1965). When n is even, the integral in (2-1-113) can be expressed in closed form. Specifically, let $m = \frac{1}{2}n$, where m is an integer. Then, by repeated integration by parts, we obtain

$$F_Y(y) = 1 - e^{-y/2\sigma^2} \sum_{k=0}^{m-1} \frac{1}{k!} \left(\frac{y}{2\sigma^2} \right)^k, \quad y \geq 0 \quad (2-1-114)$$

Let us now consider a noncentral chi-square distribution, which results from squaring a gaussian random variable having a nonzero mean. If X is gaussian with mean m_x and variance σ^2 , the random variable $Y = X^2$ has the pdf

$$p_Y(y) = \frac{1}{\sqrt{2\pi y} \sigma} e^{-(y+m_x^2)/2\sigma^2} \cosh\left(\frac{\sqrt{y} m_x}{\sigma^2}\right), \quad y \geq 0 \quad (2-1-115)$$

which is obtained by applying the result in (2-1-47) to the gaussian pdf given by (2-1-92). The characteristic function corresponding to this pdf is

$$\psi_Y(j\nu) = \frac{1}{(1 - j2\nu\sigma^2)^{1/2}} e^{jm_x^2\nu/(1 - j2\nu\sigma^2)} \quad (2-1-116)$$

To generalize these results, let Y be the sum of squares of gaussian random variables as defined by (2-1-108). The X_i , $i = 1, 2, \dots, n$, are assumed to be statistically independent with means m_i , $i = 1, 2, \dots, n$, and identical variances equal to σ^2 . Then the characteristic function of Y , obtained from (2-1-116) by applying the relation in (2-1-79), is

$$\psi_Y(jv) = \frac{1}{(1 - j2v\sigma^2)^{n/2}} \exp\left(\frac{jv \sum_{i=1}^n m_i^2}{1 - j2v\sigma^2}\right) \quad (2-1-117)$$

This characteristic function can be inverse-Fourier-transformed to yield the pdf

$$p_Y(y) = \frac{1}{2\sigma^2} \left(\frac{y}{s^2}\right)^{(n-2)/4} e^{-(s^2+y)/2\sigma^2} I_{n/2-1}\left(\sqrt{y} \frac{s}{\sigma^2}\right), \quad y \geq 0 \quad (2-1-118)$$

where, by definition,

$$s^2 = \sum_{i=1}^n m_i^2 \quad (2-1-119)$$

and $I_\alpha(x)$ is the α th-order modified Bessel function of the first kind, which may be represented by the infinite series

$$I_\alpha(x) = \sum_{k=0}^{\infty} \frac{(x/2)^{\alpha+2k}}{k! \Gamma(\alpha + k + 1)}, \quad x \geq 0 \quad (2-1-120)$$

The pdf given by (2-1-118) is called the *noncentral chi-square pdf with n degrees of freedom*. The parameter s^2 is called the *noncentrality parameter of the distribution*.

The cdf of the noncentral chi square with n degrees of freedom is

$$F_Y(y) = \int_0^y \frac{1}{2\sigma^2} \left(\frac{u}{s^2}\right)^{(n-2)/4} e^{-(s^2+u)/2\sigma^2} I_{n/2-1}\left(\sqrt{u} \frac{s}{\sigma^2}\right) du \quad (2-1-121)$$

There is no closed-form expression for this integral. However, when $m = \frac{1}{2}n$ is an integer, the cdf can be expressed in terms of the generalized Marcum's Q function, which is defined as

$$\begin{aligned} Q_m(a, b) &= \int_b^\infty x \left(\frac{x}{a}\right)^{m-1} e^{-(x^2+a^2)/2} I_{m-1}(ax) dx \\ &= Q_1(a, b) + e^{(a^2+b^2)/2} \sum_{k=1}^{m-1} \left(\frac{b}{a}\right)^k I_k(ab) \end{aligned} \quad (2-1-122)$$

where

$$Q_1(a, b) = e^{-(a^2+b^2)/2} \sum_{k=0}^{\infty} \left(\frac{a}{b}\right)^k I_k(ab), \quad b > a > 0 \quad (2-1-123)$$

If we change the variable of integration in (2-1-121) from u to x , where

$$x^2 = u/\sigma^2$$

and let $a^2 = s^2/\sigma^2$, then it is easily shown that

$$F_Y(y) = 1 - Q_m\left(\frac{s}{\sigma}, \frac{\sqrt{y}}{\sigma}\right) \quad (2-1-124)$$

Finally, we state that the first two moments of a noncentral chi-square-distributed random variable are

$$\begin{aligned} E(Y) &= n\sigma^2 + s^2 \\ E(Y^2) &= 2n\sigma^4 + 4\sigma^2s^2 + (n\sigma^2 + s^2)^2 \\ \sigma_Y^2 &= 2n\sigma^4 + 4\sigma^2s^2 \end{aligned} \quad (2-1-125)$$

Rayleigh Distribution The Rayleigh distribution is frequently used to model the statistics of signals transmitted through radio channels such as cellular radio. This distribution is closely related to the central chi-square distribution. To illustrate this point, let $Y = X_1^2 + X_2^2$ where X_1 and X_2 are zero-mean statistically independent gaussian random variables, each having a variance σ^2 . From the discussion above, it follows that Y is chi-square-distributed with two degrees of freedom. Hence, the pdf of Y is

$$p_Y(y) = \frac{1}{2\sigma^2} e^{-y/2\sigma^2}, \quad y \geq 0 \quad (2-1-126)$$

Now, suppose we define a new random variable

$$R = \sqrt{X_1^2 + X_2^2} = \sqrt{Y} \quad (2-1-127)$$

Making a simple change of variable in the pdf of (2-1-126), we obtain the pdf of R in the form

$$p_R(r) = \frac{r}{\sigma^2} e^{-r^2/2\sigma^2}, \quad r \geq 0 \quad (2-1-128)$$

This is the pdf of a Rayleigh-distributed random variable. The corresponding cdf is

$$\begin{aligned} F_R(r) &= \int_0^r \frac{u}{\sigma^2} e^{-u^2/2\sigma^2} du \\ &= 1 - e^{-r^2/2\sigma^2}, \quad r \geq 0 \end{aligned} \quad (2-1-129)$$

The moments of R are

$$E(R^k) = (2\sigma^2)^{k/2} \Gamma(1 + \frac{1}{2}k) \quad (2-1-130)$$

and the variance is

$$\sigma_r^2 = (2 - \frac{1}{2}\pi)\sigma^2 \quad (2-1-131)$$

The characteristic function of the Rayleigh-distributed random variable is

$$\psi_R(j\nu) = \int_0^\infty \frac{r}{\sigma^2} e^{-r^2/2\sigma^2} e^{j\nu r} dr \quad (2-1-132)$$

This integral may be expressed as

$$\begin{aligned} \psi_R(j\nu) &= \int_0^\infty \frac{r}{\sigma^2} e^{-r^2/2\sigma^2} \cos \nu r dr + j \int_0^\infty \frac{r}{\sigma^2} e^{-r^2/2\sigma^2} \sin \nu r dr \\ &= {}_1F_1(1, \frac{1}{2}; -\frac{1}{2}\nu^2\sigma^2) + j\sqrt{\frac{1}{2}\pi} \nu\sigma^2 e^{-\nu^2\sigma^2/2} \end{aligned} \quad (2-1-133)$$

where ${}_1F_1(1, \frac{1}{2}; -a)$ is the confluent hypergeometric function, which is defined as

$${}_1F_1(\alpha, \beta; x) = \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+k)\Gamma(\beta)x^k}{\Gamma(\alpha)\Gamma(\beta+k)k!}, \quad \beta \neq 0, -1, -2, \dots \quad (2-1-134)$$

Beaulieu (1990) has shown that ${}_1F_1(1, \frac{1}{2}; -a)$ may be expressed as

$${}_1F_1(1, \frac{1}{2}; -a) = -e^{-a} \sum_{k=0}^{\infty} \frac{a^k}{(2k-1)k!} \quad (2-1-135)$$

As a generalization of the above expression, consider the random variable

$$R = \sqrt{\sum_{i=1}^n X_i^2} \quad (2-1-136)$$

where the X_i , $i = 1, 2, \dots, n$, are statistically independent, identically distributed zero mean gaussian random variables. The random variable R has a generalized Rayleigh distribution. Clearly, $Y = R^2$ is chi-square-distributed with n degrees of freedom. Its pdf is given by (2-1-110). A simple change in variable in (2-1-110) yields the pdf of R in the form

$$p_R(r) = \frac{r^{n-1}}{2^{(n-2)/2}\sigma^n\Gamma(\frac{1}{2}n)} e^{-r^2/2\sigma^2}, \quad r \geq 0 \quad (2-1-137)$$

As a consequence of the functional relationship between the central chi-square and the Rayleigh distributions, the corresponding cdfs are similar. Thus, for any n , the cdf of R can be put in the form of the incomplete gamma function. In the special case when n is even, i.e., $n = 2m$, the cdf of R can be expressed in the closed form

$$F_R(r) = 1 - e^{-r^2/2\sigma^2} \sum_{k=0}^{m-1} \frac{1}{k!} \left(\frac{r^2}{2\sigma^2}\right)^k, \quad r \geq 0 \quad (2-1-138)$$

Finally, we state that the k th moment of R is

$$E(R^k) = (2\sigma^2)^{k/2} \frac{\Gamma(\frac{1}{2}(n+k))}{\Gamma(\frac{1}{2}n)}, \quad k \geq 0 \quad (2-1-139)$$

which holds for any integer n .

Rice Distribution Just as the Rayleigh distribution is related to the central chi-square distribution, the Rice distribution is related to the noncentral chi-square distribution. To illustrate this relation, let $Y = X_1^2 + X_2^2$, where X_1 and X_2 are statistically independent gaussian random variables with means m_i , $i = 1, 2$, and common variance σ^2 . From the previous discussion, we know that Y has a noncentral chi-square distribution with noncentrality parameter $s^2 = m_1^2 + m_2^2$. The pdf of Y , obtained from (2-1-118) for $n = 2$, is

$$p_Y(y) = \frac{1}{2\sigma^2} e^{-(s^2+y)/2\sigma^2} I_0\left(\sqrt{y} \frac{s}{\sigma^2}\right), \quad y \geq 0 \quad (2-1-140)$$

Now, we define a new random variable $R = \sqrt{Y}$. The pdf of R , obtained from (2-1-140) by a simple change of variable, is

$$p_R(r) = \frac{r}{\sigma^2} e^{-(r^2+s^2)/2\sigma^2} I_0\left(\frac{rs}{\sigma^2}\right), \quad r \geq 0 \quad (2-1-141)$$

This is the pdf of a Ricean-distributed random variable. As will be shown in Chapter 5, this pdf characterizes the statistics of the envelope of a signal corrupted by additive narrowband gaussian noise. It is also used to model the signal statistics of signals transmitted through some radio channels. The cdf of R is easily obtained by specializing the results in (2-1-124) to the case $m = 1$. This yields

$$F_R(r) = 1 - Q_1\left(\frac{s}{\sigma}, \frac{r}{\sigma}\right), \quad r \geq 0 \quad (2-1-142)$$

where $Q_1(a, b)$ is defined by (2-1-123).

As a generalization of the expressions given above, let R be defined as in (2-1-136) where the X_i , $i = 1, 2, \dots, n$ are statistically independent gaussian random variables with means m_i , $i = 1, 2, \dots, n$, and identical variances equal to σ^2 . The random variable $R^2 = Y$ has a noncentral chi-square distribution with n degrees of freedom and noncentrality parameter s^2 given by (2-1-119). Its pdf is given by (2-1-118). Hence the pdf of R is

$$p_R(r) = \frac{r^{n/2}}{\sigma^2 s^{(n-2)/2}} e^{-(r^2+s^2)/2\sigma^2} I_{n/2-1}\left(\frac{rs}{\sigma^2}\right), \quad r \geq 0 \quad (2-1-143)$$

and the corresponding cdf is

$$F_R(r) = P(R \leq r) = P(\sqrt{Y} \leq r) = P(Y \leq r^2) = F_Y(r^2) \quad (2-1-144)$$

where $F_Y(r^2)$ is given by (2-1-121). In the special case where $m = \frac{1}{2}n$ is an integer, we have

$$F_R(r) = 1 - Q_m\left(\frac{s}{\sigma}, \frac{r}{\sigma}\right), \quad r \geq 0 \quad (2-1-145)$$

which follows from (2-1-124). Finally, we state that the k th moment of R is

$$E(R^k) = (2\sigma^2)^{k/2} e^{-s^2/2\sigma^2} \frac{\Gamma(\frac{1}{2}(n+k))}{\Gamma(\frac{1}{2}n)} {}_1F_1\left(\frac{n+k}{2}, \frac{n}{2}; \frac{s^2}{2\sigma^2}\right), \quad k \geq 0 \quad (2-1-146)$$

where ${}_1F_1(\alpha, \beta; x)$ is the confluent hypergeometric function.

Nakagami m -Distribution Both the Rayleigh distribution and the Rice distribution are frequently used to describe the statistical fluctuations of signals received from a multipath fading channel. These channel models are considered in Chapter 14. Another distribution that is frequently used to characterize the statistics of signals transmitted through multipath fading channels is the Nakagami m -distribution. The pdf for this distribution is given by Nakagami (1960) as

$$p_R(r) = \frac{2}{\Gamma(m)} \left(\frac{m}{\Omega}\right)^m r^{2m-1} e^{-mr^2/\Omega} \quad (2-1-147)$$

where Ω is defined as

$$\Omega = E(R^2) \quad (2-1-148)$$

and the parameter m is defined as the ratio of moments, called the *fading figure*,

$$m = \frac{\Omega^2}{E[(R^2 - \Omega)^2]}, \quad m \geq \frac{1}{2} \quad (2-1-149)$$

A normalized version of (2-1-147) may be obtained by defining another random variable $X = R/\sqrt{\Omega}$ (see Problem 2-15). The n th moment of R is

$$E(R^n) = \frac{\Gamma(m + \frac{1}{2}n)}{\Gamma(m)} \left(\frac{\Omega}{m}\right)^{n/2}$$

By setting $m = 1$, we observe that (2-1-147) reduces to a Rayleigh pdf. For

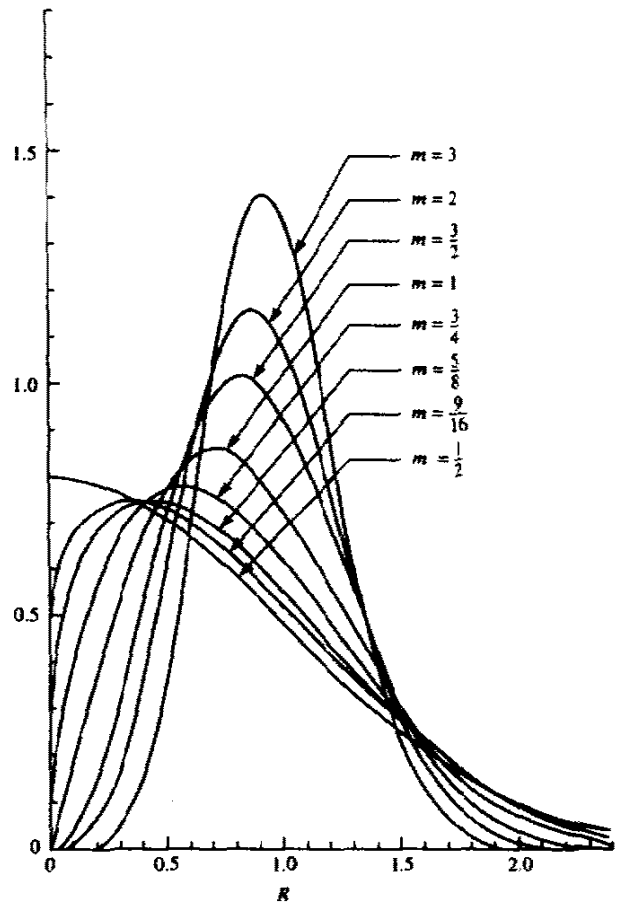


FIGURE 2-1-10 The m -distributed pdf, shown with $\Omega = 1$. m is the fading figure. (Miyagaki et al. 1978.)

values of m in the range $\frac{1}{2} \leq m \leq 1$, we obtain pdfs that have larger tails than a Rayleigh-distributed random variable. For values of $m > 1$, the tail of the pdf decays faster than that of the Rayleigh. Figure 2-1-10 illustrates the pdfs for different values of m .

Multivariate Gaussian Distribution Of the many multivariate or multi-dimensional distributions that can be defined, the multivariate gaussian distribution is the most important and the one most likely to be encountered in practice. We shall briefly introduce this distribution and state its basic properties.

Let us assume that $X_i, i = 1, 2, \dots, n$, are gaussian random variables with means $m_i, i = 1, 2, \dots, n$, variances $\sigma_i^2, i = 1, 2, \dots, n$, and covariances $\mu_{ij}, i, j = 1, 2, \dots, n$. Clearly, $\mu_{ii} = \sigma_i^2, i = 1, 2, \dots, n$. Let \mathbf{M} denote the $n \times n$

covariance matrix with elements $\{\mu_{ij}\}$, let \mathbf{X} denote the $n \times 1$ column vector of random variables, and let \mathbf{m}_x denote the $n \times 1$ column vector of mean values m_i , $i = 1, 2, \dots, n$. The joint pdf of the gaussian random variables X_i , $i = 1, 2, \dots, n$, is defined as

$$p(x_1, x_2, \dots, x_n) = \frac{1}{(2\pi)^{n/2}(\det \mathbf{M})^{1/2}} \exp \left[-\frac{1}{2}(\mathbf{x} - \mathbf{m}_x)' \mathbf{M}^{-1}(\mathbf{x} - \mathbf{m}_x) \right] \quad (2-1-150)$$

where \mathbf{M}^{-1} denotes the inverse of \mathbf{M} and \mathbf{x}' denotes the transpose of \mathbf{x} .

The characteristic function corresponding to this n -dimensional joint pdf is

$$\psi(j\mathbf{v}) = E(e^{j\mathbf{v}'\mathbf{x}})$$

where \mathbf{v} is an n -dimensional vector with elements v_i , $i = 1, 2, \dots, n$. Evaluation of this n -dimensional Fourier transform yields the result

$$\psi(j\mathbf{v}) = \exp(j\mathbf{m}_x'\mathbf{v} - \frac{1}{2}\mathbf{v}'\mathbf{M}\mathbf{v}) \quad (2-1-151)$$

An important special case of (2-1-150) is the bivariate or two-dimensional gaussian pdf. The mean \mathbf{m}_x and the covariance matrix \mathbf{M} for this case are

$$\mathbf{m}_x = \begin{bmatrix} m_1 \\ m_2 \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} \sigma_1^2 & \mu_{12} \\ \mu_{12} & \sigma_2^2 \end{bmatrix} \quad (2-1-152)$$

where the joint central moment μ_{12} is defined as

$$\mu_{12} = E[(X_1 - m_1)(X_2 - m_2)]$$

It is convenient to define a normalized covariance

$$\rho_{ij} = \frac{\mu_{ij}}{\sigma_i \sigma_j}, \quad i \neq j \quad (2-1-153)$$

where ρ_{ij} satisfies the condition $0 \leq |\rho_{ij}| \leq 1$. When dealing with the two-dimensional case, it is customary to drop the subscripts on μ_{12} and ρ_{12} . Hence the covariance matrix is expressed as

$$\mathbf{M} = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix} \quad (2-1-154)$$

Its inverse is

$$\mathbf{M}^{-1} = \frac{1}{\sigma_1^2\sigma_2^2(1-\rho^2)} \begin{bmatrix} \sigma_2^2 & -\rho\sigma_1\sigma_2 \\ -\rho\sigma_1\sigma_2 & \sigma_1^2 \end{bmatrix} \quad (2-1-155)$$

and $\det \mathbf{M} = \sigma_1^2 \sigma_2^2 (1 - \rho^2)$. Substitution for \mathbf{M}^{-1} into (2-1-150) yields the desired bivariate gaussian pdf in the form

$$p(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \times \exp \left[-\frac{\sigma_2^2(x_1 - m_1)^2 - 2\rho\sigma_1\sigma_2(x_1 - m_1)(x_2 - m_2) + \sigma_1^2(x_2 - m_2)^2}{2\sigma_1^2\sigma_2^2(1-\rho^2)} \right] \quad (2-1-156)$$

We note that when $\rho = 0$, the joint pdf $p(x_1, x_2)$ in (2-1-156) factors into the product $p(x_1)p(x_2)$, where $p(x_i)$, $i = 1, 2$, are the marginal pdfs. Since ρ is a measure of the correlation between X_1 and X_2 , we have shown that when the gaussian random variables X_1 and X_2 are uncorrelated, they are also statistically independent. This is an important property of gaussian random variables, which does not hold in general for other distributions. It extends to n -dimensional gaussian random variables in a straightforward manner. That is, if $\rho_{ij} = 0$ for $i \neq j$ then the random variables X_i , $i = 1, 2, \dots, n$ are uncorrelated and, hence, statistically independent.

Now, let us consider a linear transformation of n gaussian random variables X_i , $i = 1, 2, \dots, n$, with mean vector \mathbf{m}_x and covariance matrix \mathbf{M} . Let

$$\mathbf{Y} = \mathbf{A}\mathbf{X} \quad (2-1-157)$$

where \mathbf{A} is a nonsingular matrix. As shown previously, the jacobian of this transformation is $J = 1/\det \mathbf{A}$. Since $\mathbf{X} = \mathbf{A}^{-1}\mathbf{Y}$, we may substitute for \mathbf{X} in (2-1-150) and, thus, we obtain the joint pdf of \mathbf{Y} in the form

$$p(\mathbf{y}) = \frac{1}{(2\pi)^{n/2}(\det \mathbf{M})^{1/2} \det \mathbf{A}} \exp \left[-\frac{1}{2}(\mathbf{A}^{-1}\mathbf{y} - \mathbf{m}_x)' \mathbf{M}^{-1}(\mathbf{A}^{-1}\mathbf{y} - \mathbf{m}_x) \right] \\ = \frac{1}{(2\pi)^{n/2}(\det \mathbf{Q})^{1/2}} \exp \left[-\frac{1}{2}(\mathbf{y} - \mathbf{m}_y)' \mathbf{Q}^{-1}(\mathbf{y} - \mathbf{m}_y) \right] \quad (2-1-158)$$

where the vector \mathbf{m}_y and the matrix \mathbf{Q} are defined as

$$\mathbf{m}_y = \mathbf{A}\mathbf{m}_x \\ \mathbf{Q} = \mathbf{A}\mathbf{M}\mathbf{A} \quad (2-1-159)$$

Thus we have shown that a linear transformation of a set of jointly gaussian random variables results in another set of jointly gaussian random variables.

Suppose that we wish to perform a linear transformation that results in n statistically independent gaussian random variables. How should the matrix \mathbf{A} be selected? From our previous discussion, we know that the gaussian random

variables are statistically independent if they are pairwise-uncorrelated, i.e., if the covariance matrix \mathbf{Q} is diagonal. Therefore, we must have

$$\mathbf{A}\mathbf{M}\mathbf{A}' = \mathbf{D} \quad (2-1-160)$$

where \mathbf{D} is a diagonal matrix. The matrix \mathbf{M} is a covariance matrix; hence, it is positive definite. One solution is to select \mathbf{A} to be an orthogonal matrix ($\mathbf{A}' = \mathbf{A}^{-1}$) consisting of columns that are the eigenvectors of the covariance matrix \mathbf{M} . Then \mathbf{D} is a diagonal matrix with diagonal elements equal to the eigenvalues of \mathbf{M} .

Example 2-1-5

Consider the bivariate gaussian pdf with covariance matrix

$$\mathbf{M} = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}$$

Let us determine the transformation \mathbf{A} that will result in uncorrelated random variables. First, we solve for the eigenvalues of \mathbf{M} . The characteristic equation is

$$\begin{aligned} \det(\mathbf{M} - \lambda\mathbf{I}) &= 0 \\ (1 - \lambda)^2 - \frac{1}{4} &= 0 \\ \lambda &= \frac{3}{2}, \frac{1}{2} \end{aligned}$$

Next we determine the two eigenvectors. If \mathbf{a} denotes an eigenvector, we have

$$(\mathbf{M} - \lambda\mathbf{I})\mathbf{a} = 0$$

With $\lambda_1 = \frac{3}{2}$ and $\lambda_2 = \frac{1}{2}$, we obtain the eigenvectors

$$\mathbf{a}_1 = \begin{bmatrix} \sqrt{\frac{1}{2}} \\ \sqrt{\frac{1}{2}} \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} \sqrt{\frac{1}{2}} \\ -\sqrt{\frac{1}{2}} \end{bmatrix}$$

Therefore,

$$\mathbf{A} = \sqrt{\frac{1}{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

It is easily verified that $\mathbf{A}^{-1} = \mathbf{A}'$ and that

$$\mathbf{A}\mathbf{M}\mathbf{A}' = \mathbf{D}$$

where the diagonal elements of \mathbf{D} are $\frac{3}{2}$ and $\frac{1}{2}$.

2-1-5 Upper Bounds on the Tail Probability

In evaluating the performance of a digital communication system, it is often necessary to determine the area under the tail of the pdf. We refer to this area as the *tail probability*. In this section, we present two upper bounds on the tail probability. The first, obtained from the Chebyshev inequality, is rather loose. The second, called the *Chernoff bound*, is much tighter.

Chebyshev Inequality Suppose that X is an arbitrary random variable with finite mean m_x and finite variance σ_x^2 . For any positive number δ ,

$$P(|X - m_x| \geq \delta) \leq \frac{\sigma_x^2}{\delta^2} \quad (2-1-161)$$

This relation is called the *Chebyshev inequality*. The proof of this bound is relatively simple. We have

$$\begin{aligned} \sigma_x^2 &= \int_{-\infty}^{\infty} (x - m_x)^2 p(x) dx \geq \int_{|x - m_x| \geq \delta} (x - m_x)^2 p(x) dx \\ &\geq \delta^2 \int_{|x - m_x| \geq \delta} p(x) dx = \delta^2 P(|X - m_x| \geq \delta) \end{aligned}$$

Thus the validity of the inequality is established.

It is apparent that the Chebyshev inequality is simply an upper bound on the area under the tails of the pdf $p(y)$, where $Y = X - m_x$, i.e., the area of $p(y)$ in the intervals $(-\infty, -\delta)$ and (δ, ∞) . Hence, the Chebyshev inequality may be expressed as

$$1 - [F_Y(\delta) - F_Y(-\delta)] \leq \frac{\sigma_x^2}{\delta^2} \quad (2-1-162)$$

or, equivalently, as

$$1 - [F_X(m_x + \delta) - F_X(m_x - \delta)] \leq \frac{\sigma_x^2}{\delta^2} \quad (2-1-163)$$

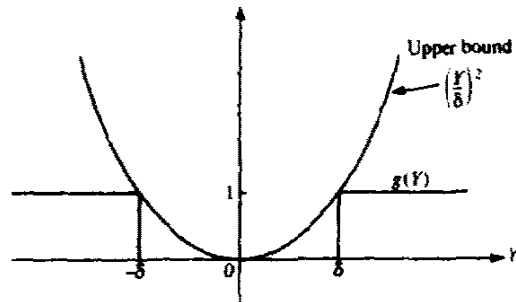
There is another way to view the Chebyshev bound. Working with the zero mean random variable $Y = X - m_x$, for convenience, suppose we define a function $g(Y)$ as

$$g(Y) = \begin{cases} 1 & (|Y| \geq \delta) \\ 0 & (|Y| < \delta) \end{cases} \quad (2-1-164)$$

Since $g(Y)$ is either 0 or 1 with probabilities $P(|Y| < \delta)$ and $P(|Y| \geq \delta)$, respectively, its mean value is

$$E[g(Y)] = P(|Y| \geq \delta) \quad (2-1-165)$$

FIGURE 2-1-11 A quadratic upper bound on $g(Y)$ used in obtaining the tail probability (Chebyshev bound).



Now suppose that we upper-bound $g(Y)$ by the quadratic $(Y/\delta)^2$, i.e.,

$$g(Y) \leq \left(\frac{Y}{\delta}\right)^2 \quad (2-1-166)$$

The graph of $g(Y)$ and the upper bound are shown in Fig. 2-1-11. It follows that

$$E[g(Y)] \leq E\left(\frac{Y^2}{\delta^2}\right) = \frac{E(Y^2)}{\delta^2} = \frac{\sigma_y^2}{\delta^2} = \frac{\sigma_x^2}{\delta^2}$$

Since $E[g(Y)]$ is the tail probability, as seen from (2-1-165), we have obtained the Chebyshev bound.

For many practical applications, the Chebyshev bound is extremely loose. The reason for this may be attributed to the looseness of the quadratic $(Y/\delta)^2$ in overbounding $g(Y)$. There are certainly many other functions that can be used to overbound $g(Y)$. Below, we use an exponential bound to derive an upper bound on the tail probability that is extremely tight.

Chernoff Bound The Chebyshev bound given above involves the area under the two tails of the pdf. In some applications we are interested only in the area under one tail, either in the interval (δ, ∞) or in the interval $(-\infty, \delta)$. In such a case we can obtain an extremely tight upper bound by overbounding the function $g(Y)$ by an exponential having a parameter that can be optimized to yield as tight an upper bound as possible. Specifically, we consider the tail probability in the interval (δ, ∞) . The function $g(Y)$ is overbounded as

$$g(Y) \leq e^{\nu(Y-\delta)} \quad (2-1-167)$$

where $g(Y)$ is now defined as

$$g(Y) = \begin{cases} 1 & (Y \geq \delta) \\ 0 & (Y < \delta) \end{cases} \quad (2-1-168)$$

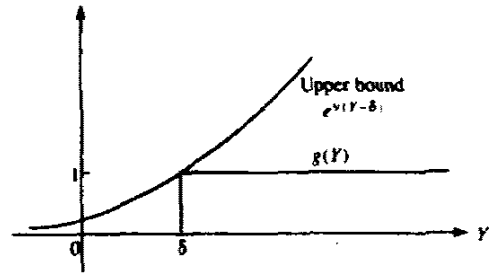


FIGURE 2-1-12 An exponential upper bound on $g(Y)$ used in obtaining the tail probability (Chernoff bound).

and $v \geq 0$ is the parameter to be optimized. The graph of $g(Y)$ and the exponential upper bound are shown in Fig. 2-1-12.

The expected value of $g(Y)$ is

$$E[g(Y)] = P(Y \geq \delta) \leq E(e^{v(Y-\delta)}) \quad (2-1-169)$$

This bound is valid for any $v \geq 0$. The tightest upper bound is obtained by selecting the value of v that minimizes $E(e^{v(Y-\delta)})$. A necessary condition for a minimum is

$$\frac{d}{dv} E(e^{v(Y-\delta)}) = 0 \quad (2-1-170)$$

But the order of differentiation and expectation can be interchanged, so that

$$\begin{aligned} \frac{d}{dv} E(e^{v(Y-\delta)}) &= E\left(\frac{d}{dv} e^{v(Y-\delta)}\right) \\ &= E[(Y - \delta)e^{v(Y-\delta)}] \\ &= e^{-v\delta}[E(Ye^{vY}) - \delta E(e^{vY})] = 0 \end{aligned}$$

Therefore the value of v that gives the tightest upper bound is the solution to the equation

$$E(Ye^{vY}) - \delta E(e^{vY}) = 0 \quad (2-1-171)$$

Let \hat{v} be the solution of (2-1-171). Then, from (2-1-169), the upper bound on the one-sided tail probability is

$$P(Y \geq \delta) \leq e^{-\hat{v}\delta} E(e^{\hat{v}Y}) \quad (2-1-172)$$

This is the Chernoff bound for the upper tail probability for a discrete or a continuous random variable having a zero mean.† This bound may be used to show that $Q(x) \leq e^{-x^2/2}$, where $Q(x)$ is the area in the tail of the gaussian pdf (see Problem 2-18).

† Note that $E(e^{vY})$ for real v is not the characteristic function of Y . It is called the *moment generating function* of Y .

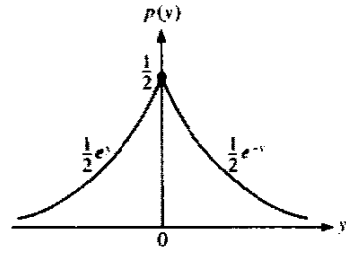


FIGURE 2-1-13 The pdf of a Laplace-distributed random variable.

An upper bound on the lower tail probability can be obtained in a similar manner, with the result that

$$P(Y \leq \delta) \leq e^{-\hat{\nu}\delta} E(e^{\hat{\nu}Y}) \quad (2-1-173)$$

where $\hat{\nu}$ is the solution to (2-1-171) and $\delta < 0$.

Example 2-1-6

Consider the (Laplace) pdf

$$p(y) = \frac{1}{2} e^{-\nu|y|} \quad (2-1-174)$$

which is illustrated in Fig. 2-1-13. Let us evaluate the upper tail probability from the Chernoff bound and compare it with the true tail probability, which is

$$P(Y \geq \delta) = \int_{\delta}^{\infty} \frac{1}{2} e^{-\nu y} dy = \frac{1}{2} e^{-\delta} \quad (2-1-175)$$

To solve (2-1-171) for $\hat{\nu}$, we must determine the moments $E(Ye^{\nu Y})$ and $E(e^{\nu Y})$. For the pdf in (2-1-174), we find that

$$E(Ye^{\nu Y}) = \frac{2\nu}{(\nu+1)^2(\nu-1)^2} \quad (2-1-176)$$

$$E(e^{\nu Y}) = \frac{1}{(1+\nu)(1-\nu)}$$

Substituting these moments into (2-1-171), we obtain the quadratic equation

$$\nu^2 \delta + 2\nu - \delta = 0$$

which has the solutions

$$\hat{\nu} = \frac{-1 \pm \sqrt{1 + \delta^2}}{\delta} \quad (2-1-177)$$

Since $\hat{\nu}$ must be positive, one of the two solutions is discarded. Thus

$$\hat{\nu} = \frac{-1 + \sqrt{1 + \delta^2}}{\delta} \quad (2-1-178)$$

Finally, we evaluate the upper bound in (2-1-172) by eliminating $E(e^{\varphi Y})$ using the second relation in (2-1-176) and by substituting for φ from (2-1-178). The result is

$$P(Y \geq \delta) \leq \frac{\delta^2}{2(-1 + \sqrt{1 + \delta^2})} e^{1 - \sqrt{1 + \delta^2}} \quad (2-1-179)$$

For $\delta \gg 1$, (2-1-179) reduces to

$$P(Y \geq \delta) \leq \frac{\delta}{2} e^{-\delta} \quad (2-1-180)$$

We note that the Chernoff bound decreases exponentially as δ increases. Consequently, it approximates closely the exact tail probability given by (2-1-175). In contrast, the Chebyshev upper bound for the upper tail probability obtained by taking one-half of the probability in the two tails (due to symmetry in the pdf) is

$$P(Y \geq \delta) \leq \frac{1}{\delta^2}$$

Hence, this bound is extremely loose.

When the random variable has a nonzero mean, the Chernoff bound can be extended as we now demonstrate. If $Y = X - m_x$, we have

$$P(Y \geq \delta) = P(X - m_x \geq \delta) = P(X \geq m_x + \delta) = P(X \geq \delta_m)$$

where, by definition, $\delta_m = m_x + \delta$. Since $\delta > 0$, it follows that $\delta_m > m_x$. Let $g(X)$ be defined as

$$g(X) = \begin{cases} 1 & (X \geq \delta_m) \\ 0 & (X < \delta_m) \end{cases} \quad (2-1-181)$$

and upper-bounded as

$$g(X) \leq e^{\varphi(X - \delta_m)} \quad (2-1-182)$$

From this point, the derivation parallels the steps contained in (2-1-169)–(2-1-172). The final result is

$$P(X \geq \delta_m) \leq e^{-\varphi \delta_m} E(e^{\varphi X}) \quad (2-1-183)$$

where $\delta_m > m_x$ and φ is the solution to the equation

$$E(Xe^{\varphi X}) - \delta_m E(e^{\varphi X}) = 0 \quad (2-1-184)$$

In a similar manner, we can obtain the Chernoff bound for the lower tail probability. For $\delta < 0$, we have

$$P(X - m_x \leq \delta) = P(X \leq m_x + \delta) = P(X \leq \delta_m) \leq E(e^{\varphi(X - \delta_m)}) \quad (2-1-185)$$

From our previous development, it is apparent that (2-1-185) results in the bound

$$P(X \leq \delta_m) \leq e^{-\varphi \delta_m} E(e^{\varphi X}) \quad (2-1-186)$$

where $\delta_m < m_x$ and φ is the solution to (2-1-184).

2-1-6 Sums of Random Variables and the Central Limit Theorem

We have previously considered the problem of determining the pdf of a sum of n statistically independent random variables. In this section, we again consider the sum of statistically independent random variables, but our approach is different and is independent of the particular pdf of the random variables in the sum. To be specific, suppose that X_i , $i = 1, 2, \dots, n$, are statistically independent and identically distributed random variables, each having a finite mean m_x and a finite variance σ_x^2 . Let Y be defined as the normalized sum, called the *sample mean*:

$$Y = \frac{1}{n} \sum_{i=1}^n X_i \quad (2-1-187)$$

First we shall determine upper bounds on the tail probabilities of Y and then we shall prove a very important theorem regarding the pdf of Y in the limit as $n \rightarrow \infty$.

The random variable Y defined in (2-1-187) is frequently encountered in estimating the mean of a random variable X from a number of observations X_i , $i = 1, 2, \dots, n$. In other words, the X_i , $i = 1, 2, \dots, n$, may be considered as independent samples drawn from a distribution $F_X(x)$, and Y is the estimate of the mean m_x .

The mean of Y is

$$\begin{aligned} E(Y) &= m_y = \frac{1}{n} \sum_{i=1}^n E(X_i) \\ &= m_x \end{aligned}$$

The variance of Y is

$$\begin{aligned} \sigma_y^2 &= E(Y^2) - m_y^2 = E(Y^2) - m_x^2 \\ &= \frac{1}{n^2} \sum_{i=1}^n \sum_{j=1}^n E(X_i X_j) - m_x^2 \\ &= \frac{1}{n^2} \sum_{i=1}^n E(X_i^2) + \frac{1}{n^2} \sum_{i=1}^n \sum_{\substack{j=1 \\ i \neq j}}^n E(X_i) E(X_j) - m_x^2 \\ &= \frac{1}{n} (\sigma_x^2 + m_x^2) + \frac{1}{n^2} n(n-1) m_x^2 - m_x^2 \\ &= \frac{\sigma_x^2}{n} \end{aligned}$$

When Y is viewed as an estimate for the mean m_x , we note that its expected value is equal to m_x and its variance decreases inversely with the number of

samples n . As n approaches infinity, the variance σ_y^2 approaches zero. An estimate of a parameter (in this case the mean m_x) that satisfies the conditions that its expected value converges to the true value of the parameter and the variance converges to zero as $n \rightarrow \infty$ is said to be a *consistent estimate*.

The tail probability of the random variable Y can be upper-bounded by use of the bounds presented in Section 2-1-5. The Chebyshev inequality applied to Y is

$$P(|Y - m_y| \geq \delta) \leq \frac{\sigma_y^2}{\delta^2} \quad (2-1-188)$$

$$P\left(\left|\frac{1}{n} \sum_{i=1}^n X_i - m_x\right| \geq \delta\right) \leq \frac{\sigma_x^2}{n\delta^2}$$

In the limit as $n \rightarrow \infty$, (2-1-188) becomes

$$\lim_{n \rightarrow \infty} P\left(\left|\frac{1}{n} \sum_{i=1}^n X_i - m_x\right| \geq \delta\right) = 0 \quad (2-1-189)$$

Therefore, the probability that the estimate of the mean differs from the true mean m_x by more than δ ($\delta > 0$) approaches zero as n approaches infinity. This statement is a form of the law of large numbers. Since the upper bound converges to zero relatively slowly, i.e., inversely with n , the expression in (2-1-188) is called the *weak law of large numbers*.

The Chernoff bound applied to the random variable Y yields an exponential dependence of n , and thus provides a tighter upper bound on the one-sided tail probability. Following the procedure developed in Section 2-1-5, we can determine that the tail probability for y is

$$\begin{aligned} P(Y - m_y \geq \delta) &= P\left(\frac{1}{n} \sum_{i=1}^n X_i - m_x \geq \delta\right) \\ &= P\left(\sum_{i=1}^n X_i \geq n\delta_m\right) \leq E\left\{\exp\left[v\left(\sum_{i=1}^n X_i - n\delta_m\right)\right]\right\} \end{aligned} \quad (2-1-190)$$

where $\delta_m = m_x + \delta$ and $\delta > 0$. But the X_i , $i = 1, 2, \dots, n$, are statistically independent and identically distributed. Hence,

$$\begin{aligned} E\left\{\exp\left[v\left(\sum_{i=1}^n X_i - n\delta_m\right)\right]\right\} &= e^{-vn\delta_m} E\left[\exp\left(v \sum_{i=1}^n X_i\right)\right] \\ &= e^{-vn\delta_m} \prod_{i=1}^n E(e^{vX_i}) \\ &= [e^{-v\delta_m} E(e^{vX})]^n \end{aligned} \quad (2-1-191)$$

where X denotes any one of the X_i . The parameter v that yields the tightest upper bound is obtained by differentiating (2-1-191) and setting the derivative equal to zero. This yields the equation

$$E(Xe^{vX}) - \delta_m E(e^{vX}) = 0 \quad (2-1-192)$$

Let the solution of (2-1-192) be denoted by $\hat{\nu}$. Then, the bound on the upper tail probability is

$$P\left(\frac{1}{n} \sum_{i=1}^n X_i \geq \delta_m\right) \leq [e^{-\hat{\nu}\delta_m} E(e^{\hat{\nu}X})]^n, \quad \delta_m > m_x \quad (2-1-193)$$

In a similar manner, we find that the lower tail probability is upper-bounded as

$$P(Y \leq \delta_m) \leq [e^{-\hat{\nu}\delta_m} E(e^{\hat{\nu}X})]^n, \quad \delta_m < m_x \quad (2-1-194)$$

where $\hat{\nu}$ is the solution to (2-1-192).

Example 2-1-7

Let X_i , $i = 1, 2, \dots, n$, be a set of statistically independent random variables defined as

$$X_i = \begin{cases} 1 & \text{with probability } p < \frac{1}{2} \\ -1 & \text{with probability } 1 - p \end{cases}$$

We wish to determine a tight upper bound on the probability that the sum of the X_i is greater than zero. Since $p < \frac{1}{2}$, we note that the sum will have a negative value for the mean; hence we seek the upper tail probability. With $\delta_m = 0$ in (2-1-193), we have

$$P\left(\sum_{i=1}^n X_i \geq 0\right) \leq [E(e^{\hat{\nu}X})]^n \quad (2-1-195)$$

where $\hat{\nu}$ is the solution to the equation

$$E(Xe^{\hat{\nu}X}) = 0 \quad (2-1-196)$$

Now

$$E(Xe^{\hat{\nu}X}) = -(1-p)e^{-\hat{\nu}} + pe^{\hat{\nu}} = 0$$

Hence

$$\hat{\nu} = \ln\left(\sqrt{\frac{1-p}{p}}\right) \quad (2-1-197)$$

Furthermore,

$$E(e^{\hat{\nu}X}) = pe^{\hat{\nu}} + (1-p)e^{-\hat{\nu}}$$

Therefore the bound in (2-1-195) becomes

$$\begin{aligned} P\left(\sum_{i=1}^n X_i \geq 0\right) &\leq [pe^{\hat{\nu}} + (1-p)e^{-\hat{\nu}}]^n \\ &\leq \left[p\sqrt{\frac{1-p}{p}} + (1-p)\sqrt{\frac{p}{1-p}}\right]^n \\ &\leq [4p(1-p)]^{n/2} \end{aligned} \quad (2-1-198)$$

We observe that the upper bound decays exponentially with n , as expected.

In contrast, if the Chebyshev bound were evaluated, the tail probability would decrease inversely with n .

Central Limit Theorem We conclude this section with an extremely useful theorem concerning the cdf of a sum of random variables in the limit as the number of terms in the sum approaches infinity. There are several versions of this theorem. We shall prove the theorem for the case in which the random variables X_i , $i = 1, 2, \dots, n$, being summed are statistically independent and identically distributed, each having a finite mean m_x and a finite variance σ_x^2 . For convenience, we define the normalized random variable

$$U_i = \frac{X_i - m_x}{\sigma_x}, \quad i = 1, 2, \dots, n$$

Thus U_i has a zero mean and unit variance. Now, let

$$Y = \frac{1}{\sqrt{n}} \sum_{i=1}^n U_i \quad (2-1-199)$$

Since each term in the sum has a zero mean and unit variance, it follows that the normalized (by $1/\sqrt{n}$) random variable Y has zero mean and unit variance. We wish to determine the cdf of Y in the limit as $n \rightarrow \infty$.

The characteristic function of Y is

$$\begin{aligned} \psi_Y(j\nu) &= E(e^{j\nu Y}) = E \left[\exp \left(\frac{j\nu \sum_{i=1}^n U_i}{\sqrt{n}} \right) \right] \\ &= \prod_{i=1}^n \psi_{U_i} \left(\frac{j\nu}{\sqrt{n}} \right) \\ &= \left[\psi_U \left(\frac{j\nu}{\sqrt{n}} \right) \right]^n \end{aligned} \quad (2-1-200)$$

where U denotes any of the U_i , which are identically distributed. Now, let us expand the characteristic function of U in a Taylor series. The expansion yields

$$\psi_U \left(j \frac{\nu}{\sqrt{n}} \right) = 1 + j \frac{\nu}{\sqrt{n}} E(U) - \frac{\nu^2}{n2!} E(U^2) + \frac{(j\nu)^3}{(\sqrt{n})^3 3!} E(U^3) - \dots \quad (2-1-201)$$

Since $E(U) = 0$ and $E(U^2) = 1$, (2-1-201) simplifies to

$$\psi_U \left(\frac{j\nu}{\sqrt{n}} \right) = 1 - \frac{\nu^2}{2n} + \frac{1}{n} R(\nu, n) \quad (2-1-202)$$

where $R(\nu, n)/n$ denotes the remainder. We note that $R(\nu, n)$ approaches

zero as $n \rightarrow \infty$. Substitution of (2-1-202) into (2-1-200) yields the characteristic function of Y in the form

$$\psi_Y(jv) = \left[1 - \frac{v^2}{2n} + \frac{R(v, n)}{n} \right]^n \quad (2-1-203)$$

Taking the natural logarithm of (2-1-203), we obtain

$$\ln \psi_Y(jv) = n \ln \left[1 - \frac{v^2}{2n} + \frac{R(v, n)}{n} \right] \quad (2-1-204)$$

For small values of x , $\ln(1+x)$ can be expanded in the power series

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \dots$$

This expansion applied to (2-1-204) yields

$$\ln \psi_Y(jv) = n \left[-\frac{v^2}{2n} + \frac{R(v, n)}{n} - \frac{1}{2} \left(-\frac{v^2}{2n} + \frac{R(v, n)}{n} \right)^2 + \dots \right] \quad (2-1-205)$$

Finally, when we take the limit as $n \rightarrow \infty$, (2-1-205) reduces to $\lim_{n \rightarrow \infty} \ln \psi_Y(jv) = -\frac{1}{2}v^2$, or, equivalently,

$$\lim_{n \rightarrow \infty} \psi_Y(jv) = e^{-v^2/2} \quad (2-1-206)$$

But, this is just the characteristic function of a gaussian random variable with zero mean and unit variance. Thus we have the important result that the sum of statistically independent and identically distributed random variables with finite mean and variance approaches a gaussian cdf as $n \rightarrow \infty$. This result is known as the *central limit theorem*.

Although we assumed that the random variables in the sum are identically distributed, the assumption can be relaxed provided that additional restrictions are imposed on the properties of the random variables. There is one variation of the theorem, for example, in which the assumption of identically distributed random variables is abandoned in favor of a condition on the third absolute moment of the random variables in the sum. For a discussion of this and other variations of the central limit theorem, the reader is referred to the book by Cramer (1946).

2-2 STOCHASTIC PROCESSES

Many of the random phenomena that occur in nature are functions of time. For example, the meteorological phenomena such as the random fluctuations in air temperature and air pressure are functions of time. The thermal noise voltages generated in the resistors of an electronic device such as a radio receiver are also a function of time. Similarly, the signal at the output of a source that generates information is characterized as a random signal that

varies with time. An audio signal that is transmitted over a telephone channel is an example of such a signal. All these are examples of stochastic (random) processes. In our study of digital communications, we encounter stochastic processes in the characterization and modeling of signals generated by information sources, in the characterization of communication channels used to transmit the information, in the characterization of noise generated in a receiver, and in the design of the optimum receiver for processing the received random signal.

At any given time instant, the value of a stochastic process, whether it is the value of the noise voltage generated by a resistor or the amplitude of the signal generated by an audio source, is a random variable. Thus, we may view a stochastic process as a random variable indexed by the parameter t . We shall denote such a process by $X(t)$. In general, the parameter t is continuous, whereas X may be either continuous or discrete, depending on the characteristics of the source that generates the stochastic process.

The noise voltage generated by a single resistor or a single information source represents a single realization of the stochastic process. Hence, it is called a *sample function* of the stochastic process. The set of all possible sample functions, e.g., the set of all noise voltage waveforms generated by resistors, constitute an ensemble of sample functions or, equivalently, the stochastic process $X(t)$. In general, the number of sample functions in the ensemble is assumed to be extremely large; often it is infinite.

Having defined a stochastic process $X(t)$ as an ensemble of sample functions, we may consider the values of the process at any set of time instants $t_1 > t_2 > t_3 > \dots > t_n$ where n is any positive integer. In general, the random variables $X_i \equiv X(t_i)$, $i = 1, 2, \dots, n$, are characterized statistically by their joint pdf $p(x_{t_1}, x_{t_2}, \dots, x_{t_n})$. Furthermore, all the probabilistic relations defined in Section 2-1 for multidimensional random variables carry over to the random variables X_i , $i = 1, 2, \dots, n$.

Stationary Stochastic Processes As indicated above, the random variables X_i , $i = 1, 2, \dots, n$, obtained from the stochastic process $X(t)$ for any set of time instants $t_1 > t_2 > t_3 > \dots > t_n$ and any n are characterized statistically by the joint pdf $p(x_{t_1}, x_{t_2}, \dots, x_{t_n})$. Let us consider another set of n random variables $X_{i+t} \equiv X(t_i + t)$, $i = 1, 2, \dots, n$, where t is an arbitrary time shift. These random variables are characterized by the joint pdf $p(x_{t_1+t}, x_{t_2+t}, \dots, x_{t_n+t})$. The joint pdfs of the random variables X_i and X_{i+t} , $i = 1, 2, \dots, n$, may or may not be identical. When they are identical, i.e., when

$$p(x_{t_1}, x_{t_2}, \dots, x_{t_n}) = p(x_{t_1+t}, x_{t_2+t}, \dots, x_{t_n+t}) \quad (2-2-1)$$

for all t and all n , the stochastic process is said to be *stationary in the strict sense*. That is, the statistics of a stationary stochastic process are invariant to any translation of the time axis. On the other hand, when the joint pdfs are different, the stochastic process is *nonstationary*.

2-2-1 Statistical Averages

Just as we have defined statistical averages for random variables, we may similarly define statistical averages for a stochastic process. Such averages are also called *ensemble averages*. Let $X(t)$ denote a random process and let $X_i \equiv X(t_i)$. The n th moment of the random variable X_i is defined as

$$E(X_i^n) = \int_{-\infty}^{\infty} x_i^n p(x_i) dx_i \quad (2-2-2)$$

In general, the value of the n th moment will depend on the time instant t_i if the pdf of X_i depends on t_i . When the process is stationary, however, $p(x_{i+t}) = p(x_i)$ for all t . Hence, the pdf is independent of time, and, as a consequence, the n th moment is independent of time.

Next we consider the two random variables $X_i \equiv X(t_i)$, $i = 1, 2$. The correlation between X_{t_1} and X_{t_2} is measured by the joint moment

$$E(X_{t_1} X_{t_2}) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_{t_1} x_{t_2} p(x_{t_1}, x_{t_2}) dx_{t_1} dx_{t_2} \quad (2-2-3)$$

Since this joint moment depends on the time instants t_1 and t_2 , it is denoted by $\phi(t_1, t_2)$. The function $\phi(t_1, t_2)$ is called the *autocorrelation function* of the stochastic process. When the process $X(t)$ is stationary, the joint pdf of the pair (X_{t_1}, X_{t_2}) is identical to the joint pdf of the pair (X_{t_1+t}, X_{t_2+t}) for any arbitrary t . This implies that the autocorrelation function of $X(t)$ does not depend on the specific time instants t_1 and t_2 , but, instead, it depends on the time difference $t_1 - t_2$. Thus, for a stationary stochastic process, the joint moment in (2-2-3) is

$$E(X_{t_1} X_{t_2}) = \phi(t_1, t_2) = \phi(t_1 - t_2) = \phi(\tau) \quad (2-2-4)$$

where $\tau = t_1 - t_2$ or, equivalently, $t_2 = t_1 - \tau$. If we let $t_2 = t_1 + \tau$, we have

$$\phi(-\tau) = E(X_{t_1} X_{t_1+\tau}) = E(X_{t_1} X_{t_1-\tau}) = \phi(\tau)$$

Therefore, $\phi(\tau)$ is an even function. We also note that $\phi(0) = E(X_t^2)$ denotes the average power in the process $X(t)$.

There exist nonstationary processes with the property that the mean value of the process is independent of time (a constant) and where the autocorrelation function satisfies the condition that $\phi(t_1, t_2) = \phi(t_1 - t_2)$. Such a process is called *wide-sense stationary*. Consequently, wide-sense stationarity is a less stringent condition than strict-sense stationarity. When reference is made to a stationary stochastic process in any subsequent discussion in which correlation functions are involved, the less stringent condition (wide-sense stationarity) is implied.

Related to the autocorrelation function is the autocovariance function of a stochastic process, which is defined as

$$\begin{aligned} \mu(t_1, t_2) &= E\{(X_{t_1} - m(t_1))(X_{t_2} - m(t_2))\} \\ &= \phi(t_1, t_2) - m(t_1)m(t_2) \end{aligned} \quad (2-2-5)$$

where $m(t_1)$ and $m(t_2)$ are the means of X_{t_1} and X_{t_2} , respectively. When the process is stationary, the autocovariance function simplifies to

$$\mu(t_1, t_2) = \mu(t_1 - t_2) = \mu(\tau) = \phi(\tau) - m^2 \quad (2-2-6)$$

where $\tau = t_1 - t_2$.

Higher-order joint moments of two or more random variables derived from a stochastic process $X(t)$ are defined in an obvious manner. With the possible exception of the gaussian random process, for which higher-order moments can be expressed in terms of first and second moments, high-order moments are encountered very infrequently in practice.

Averages for a Gaussian Process Suppose that $X(t)$ is a gaussian random process. Hence, at time instants $t = t_i$, $i = 1, 2, \dots, n$, the random variables X_i , $i = 1, 2, \dots, n$, are jointly gaussian with mean values $m(t_i)$, $i = 1, 2, \dots, n$, and autocovariances

$$\mu(t_i, t_j) = E[(X_i - m(t_i))(X_j - m(t_j))], \quad i, j = 1, 2, \dots, n \quad (2-2-7)$$

If we denote the $n \times n$ covariance matrix with elements $\mu(t_i, t_j)$ by \mathbf{M} and the vector of mean values by \mathbf{m}_x , then the joint pdf of the random variables X_i , $i = 1, 2, \dots, n$ is given by (2-1-150).

If the gaussian process is stationary then $m(t_i) = m$ for all t_i and $\mu(t_i, t_j) = \mu(t_i - t_j)$. We observe that the gaussian random process is completely specified by the mean and autocovariance functions. Since the joint gaussian pdf depends only on these two moments, it follows that if the gaussian process is wide-sense stationary, it is also strict-sense stationary. Of course, the converse is always true for any stochastic process.

Averages for Joint Stochastic Processes Let $X(t)$ and $Y(t)$ denote two stochastic processes and let $X_i \equiv X(t_i)$, $i = 1, 2, \dots, n$, and $Y_j \equiv Y(t'_j)$, $j = 1, 2, \dots, m$, represent the random variables at times $t_1 > t_2 > t_3 > \dots > t_n$ and $t'_1 > t'_2 > \dots > t'_m$, respectively. The two processes are characterized statistically by their joint pdf

$$p(x_{t_1}, x_{t_2}, \dots, x_{t_n}, y_{t'_1}, y_{t'_2}, \dots, y_{t'_m})$$

for any set of time instants t_1, t_2, \dots, t_n , t'_1, t'_2, \dots, t'_m and for any positive integer values of n and m .

The *cross-correlation function* of $X(t)$ and $Y(t)$, denoted by $\phi_{xy}(t_1, t_2)$, is defined as the joint moment

$$\phi_{xy}(t_1, t_2) = E(X_{t_1} Y_{t_2}) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_{t_1} y_{t_2} p(x_{t_1}, y_{t_2}) dx_{t_1} dy_{t_2} \quad (2-2-8)$$

and the *cross-covariance* is

$$\mu_{xy}(t_1, t_2) = \phi_{xy}(t_1, t_2) - m_x(t_1)m_y(t_2) \quad (2-2-9)$$

When the processes are jointly and individually stationary, we have $\phi_{xy}(t_1, t_2) = \phi_{xy}(t_1 - t_2)$ and $\mu_{xy}(t_1, t_2) = \mu_{xy}(t_1 - t_2)$. In this case, we note that

$$\phi_{xy}(-\tau) = E(X_{t_1} Y_{t_1+\tau}) = E(X_{t_1-\tau} Y_{t_1}) = \phi_{yx}(\tau) \quad (2-2-10)$$

The stochastic processes $X(t)$ and $Y(t)$ are said to be *statistically independent* if and only if

$$p(x_{t_1}, x_{t_2}, \dots, x_{t_n}, y_{t_1}, y_{t_2}, \dots, y_{t_m}) = p(x_{t_1}, x_{t_2}, \dots, x_{t_n})p(y_{t_1}, y_{t_2}, \dots, y_{t_m})$$

for all choices of t_i and t'_i and for all positive integers n and m . The processes are said to be *uncorrelated* if

$$\phi_{xy}(t_1, t_2) = E(X_{t_1})E(Y_{t_2})$$

Hence,

$$\mu_{xy}(t_1, t_2) = 0$$

A *complex-valued stochastic process* $Z(t)$ is defined as

$$Z(t) = X(t) + jY(t) \quad (2-2-11)$$

where $X(t)$ and $Y(t)$ are stochastic processes. The joint pdf of the random variables $Z_i \equiv Z(t_i)$, $i = 1, 2, \dots$, is given by the joint pdf of the components (X_{t_i}, Y_{t_i}) , $i = 1, 2, \dots, n$. Thus, the pdf that characterizes Z_{t_i} , $i = 1, 2, \dots, n$, is

$$p(x_{t_1}, x_{t_2}, \dots, x_{t_n}, y_{t_1}, y_{t_2}, \dots, y_{t_n})$$

The complex-valued stochastic process $Z(t)$ is encountered in the representation of narrowband bandpass noise in terms of its equivalent lowpass components. An important characteristic of such a process is its autocorrelation function. The function is defined as

$$\begin{aligned} \phi_{zz}(t_1, t_2) &= \frac{1}{2}E(Z_{t_1} Z_{t_2}^*) \\ &= \frac{1}{2}E[(X_{t_1} + jY_{t_1})(X_{t_2} - jY_{t_2})] \\ &= \frac{1}{2}\{\phi_{xx}(t_1, t_2) + \phi_{yy}(t_1, t_2) + j[\phi_{yx}(t_1, t_2) - \phi_{xy}(t_1, t_2)]\} \end{aligned} \quad (2-2-12)$$

where $\phi_{xx}(t_1, t_2)$ and $\phi_{yy}(t_1, t_2)$ are the autocorrelation functions of $X(t)$ and $Y(t)$, respectively, and $\phi_{yx}(t_1, t_2)$ and $\phi_{xy}(t_1, t_2)$ are the cross-correlation functions. The factor of $\frac{1}{2}$ in the definition of the autocorrelation function of a complex-valued stochastic process is an arbitrary but mathematically convenient normalization factor, as we will demonstrate in our treatment of such processes in Chapter 4.

When the processes $X(t)$ and $Y(t)$ are jointly and individually stationary, the autocorrelation function of $Z(t)$ becomes

$$\phi_{zz}(t_1, t_2) = \phi_{zz}(t_1 - t_2) = \phi_{zz}(\tau)$$

where $t_2 = t_1 - \tau$. Also, the complex conjugate of (2-2-12) is

$$\phi_{zz}^*(\tau) = \frac{1}{2}E(Z_{t_1}^* Z_{t_1-\tau}) = \frac{1}{2}E(Z_{t_1+\tau}^* Z_{t_1}) = \phi_{zz}(-\tau) \quad (2-2-13)$$

Hence, $\phi_{zz}(\tau) = \phi_{zz}^*(-\tau)$.

Now, suppose that $Z(t) = X(t) + jY(t)$ and $W(t) = U(t) + jV(t)$ are two complex-valued stochastic processes. The cross-correlation function of $Z(t)$ and $W(t)$ is defined as

$$\begin{aligned}\phi_{zw}(t_1, t_2) &= \frac{1}{2}E(Z_{t_1}W_{t_2}^*) \\ &= \frac{1}{2}E[(X_{t_1} + jY_{t_1})(U_{t_2} - jV_{t_2})] \\ &= \frac{1}{2}\{\phi_{xu}(t_1, t_2) + \phi_{yv}(t_1, t_2) + j[\phi_{yu}(t_1, t_2) - \phi_{xv}(t_1, t_2)]\} \quad (2-2-14)\end{aligned}$$

When $X(t)$, $Y(t)$, $U(t)$, and $V(t)$ are pairwise-stationary, the cross-correlation functions in (2-2-14) become functions of the time difference $\tau = t_1 - t_2$. Furthermore,

$$\phi_{zw}^*(\tau) = \frac{1}{2}E(Z_{t_1}^*W_{t_1-\tau}) = \frac{1}{2}E(Z_{t_1+\tau}^*W_{t_1}) = \phi_{wz}(-\tau) \quad (2-2-15)$$

2-2-2 Power Density Spectrum

The frequency content of a signal is a very basic characteristic that distinguishes one signal from another. In general, a signal can be classified as having either a finite (nonzero) average power (infinite energy) or finite energy. The frequency content of a finite energy signal is obtained as the Fourier transform of the corresponding time function. If the signal is periodic, its energy is infinite and, consequently, its Fourier transform does not exist. The mechanism for dealing with periodic signals is to represent them in a Fourier series. With such a representation, the Fourier coefficients determine the distribution of power at the various discrete frequency components.

A stationary stochastic process is an infinite energy signal, and, hence, its Fourier transform does not exist. The spectral characteristic of a stochastic signal is obtained by computing the Fourier transform of the autocorrelation function. That is, the distribution of power with frequency is given by the function

$$\Phi(f) = \int_{-\infty}^{\infty} \phi(\tau)e^{-j2\pi f\tau} d\tau \quad (2-2-16)$$

The inverse Fourier transform relationship is

$$\phi(\tau) = \int_{-\infty}^{\infty} \Phi(f)e^{j2\pi f\tau} df \quad (2-2-17)$$

We observe that

$$\begin{aligned}\phi(0) &= \int_{-\infty}^{\infty} \Phi(f) df \\ &= E(|X_t|^2) \geq 0\end{aligned} \quad (2-2-18)$$

Since $\phi(0)$ represents the average power of the stochastic signal, which is the area under $\Phi(f)$, $\Phi(f)$ is the distribution of power as a function of frequency. Therefore, $\Phi(f)$ is called the *power density spectrum* of the stochastic process.

If the stochastic process is real, $\phi(\tau)$ is real and even, and, hence $\Phi(f)$ is real and even. On the other hand, if the process is complex, $\phi(\tau) = \phi^*(-\tau)$ and, hence

$$\begin{aligned}\Phi^*(f) &= \int_{-\infty}^{\infty} \phi^*(\tau) e^{j2\pi f\tau} d\tau = \int_{-\infty}^{\infty} \phi^*(-\tau) e^{-j2\pi f\tau} d\tau \\ &= \int_{-\infty}^{\infty} \phi(\tau) e^{-j2\pi f\tau} d\tau = \Phi(f)\end{aligned}\quad (2-2-19)$$

Therefore, $\Phi(f)$ is real.

The definition of a power density spectrum can be extended to two jointly stationary stochastic processes $X(t)$ and $Y(t)$, which have a cross-correlation function $\phi_{xy}(\tau)$. The Fourier transform of $\phi_{xy}(\tau)$, i.e.,

$$\Phi_{xy}(f) = \int_{-\infty}^{\infty} \phi_{xy}(\tau) e^{-j2\pi f\tau} d\tau \quad (2-2-20)$$

is called the *cross-power density spectrum*. If we conjugate both sides of (2-2-20), we have

$$\begin{aligned}\Phi_{xy}^*(f) &= \int_{-\infty}^{\infty} \phi_{xy}^*(\tau) e^{j2\pi f\tau} d\tau = \int_{-\infty}^{\infty} \phi_{xy}^*(-\tau) e^{-j2\pi f\tau} d\tau \\ &= \int_{-\infty}^{\infty} \phi_{yx}(\tau) e^{-j2\pi f\tau} d\tau = \Phi_{yx}(f)\end{aligned}\quad (2-2-21)$$

This relation holds in general. However, if $X(t)$ and $Y(t)$ are real stochastic processes,

$$\Phi_{xy}^*(f) = \int_{-\infty}^{\infty} \phi_{xy}(\tau) e^{j2\pi f\tau} d\tau = \Phi_{xy}(-f) \quad (2-2-22)$$

By combining the result in (2-2-21) with the result in (2-2-22), we find that the cross-power density spectrum of two real processes satisfies the condition

$$\Phi_{yx}(f) = \Phi_{xy}(-f) \quad (2-2-23)$$

2-2-3 Response of a Linear Time-Invariant System to a Random Input Signal

Consider a linear time-invariant system (filter) that is characterized by its impulse response $h(t)$ or, equivalently, by its frequency response $H(f)$, where $h(t)$ and $H(f)$ are a Fourier transform pair. Let $x(t)$ be the input signal to the system and let $y(t)$ denote the output signal. The output of the system may be expressed in terms of the convolution integral as

$$y(t) = \int_{-\infty}^{\infty} h(\tau) x(t - \tau) d\tau \quad (2-2-24)$$

Now, suppose that $x(t)$ is a sample function of a stationary stochastic process $X(t)$. Then, the output $y(t)$ is a sample function of a stochastic process $Y(t)$. We wish to determine the mean and autocorrelation functions of the output.

Since convolution is a linear operation performed on the input signal $x(t)$, the expected value of the integral is equal to the integral of the expected value. Thus, the mean value of $Y(t)$ is

$$\begin{aligned} m_y &= E\{Y(t)\} = \int_{-\infty}^{\infty} h(\tau)E\{X(t-\tau)\}d\tau \\ &= m_x \int_{-\infty}^{\infty} h(\tau)d\tau = m_x H(0) \end{aligned} \quad (2-2-25)$$

where $H(0)$ is the frequency response of the linear system at $f = 0$. Hence, the mean value of the output process is a constant.

The autocorrelation function of the output is

$$\begin{aligned} \phi_{yy}(t_1, t_2) &= \frac{1}{2}E\{Y_{t_1} Y_{t_2}^*\} \\ &= \frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\beta)h^*(\alpha)E\{X(t_1-\beta)X^*(t_2-\alpha)\}d\alpha d\beta \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\beta)h^*(\alpha)\phi_{xx}(t_1-t_2+\alpha-\beta)d\alpha d\beta \end{aligned}$$

The last step indicates that the double integral is a function of the time difference $t_1 - t_2$. In other words, if the input process is stationary, the output is also stationary. Hence

$$\phi_{yy}(\tau) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h^*(\alpha)h(\beta)\phi_{xx}(\tau+\alpha-\beta)d\alpha d\beta \quad (2-2-26)$$

By evaluating the Fourier transform of both sides of (2-2-26), we obtain the power density spectrum of the output process in the form

$$\begin{aligned} \Phi_{yy}(f) &= \int_{-\infty}^{\infty} \phi_{yy}(\tau)e^{-j2\pi f\tau}d\tau \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h^*(\alpha)h(\beta)\phi_{xx}(\tau+\alpha-\beta)e^{-j2\pi f\tau}d\tau d\alpha d\beta \\ &= \Phi_{xx}(f)|H(f)|^2 \end{aligned} \quad (2-2-27)$$

Thus, we have the important result that the power density spectrum of the output signal is the product of the power density spectrum of the input multiplied by the magnitude squared of the frequency response of the system.

When the autocorrelation function $\phi_{yy}(\tau)$ is desired, it is usually easier to determine the power density spectrum $\Phi_{yy}(f)$ and then to compute the inverse transform. Thus, we have

$$\begin{aligned}\phi_{yy}(\tau) &= \int_{-\infty}^{\infty} \Phi_{yy}(f) e^{j2\pi f\tau} df \\ &= \int_{-\infty}^{\infty} \Phi_{xx}(f) |H(f)|^2 e^{j2\pi f\tau} df\end{aligned}\quad (2-2-28)$$

We observe that the average power in the output signal is

$$\phi_{yy}(0) = \int_{-\infty}^{\infty} \Phi_{xx}(f) |H(f)|^2 df \quad (2-2-29)$$

Since $\phi_{yy}(0) = E(|Y_t|^2)$, it follows that

$$\int_{-\infty}^{\infty} \Phi_{xx}(f) |H(f)|^2 df \geq 0$$

Suppose we let $|H(f)|^2 = 1$ for any arbitrarily small interval $f_1 \leq f \leq f_2$, and $H(f) = 0$ outside this interval. Then,

$$\int_{f_1}^{f_2} \Phi_{xx}(f) df \geq 0$$

But this is possible if and only if $\Phi_{xx}(f) \geq 0$ for all f .

Example 2-2-1

Suppose that the lowpass filter illustrated in Fig. 2-2-1 is excited by a stochastic process $x(t)$ having a power density spectrum

$$\Phi_{xx}(f) = \frac{1}{2}N_0 \quad \text{for all } f$$

A stochastic process having a flat power density spectrum is called *white*

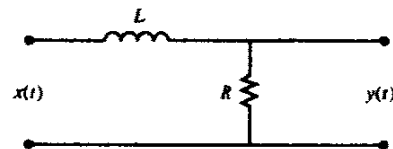
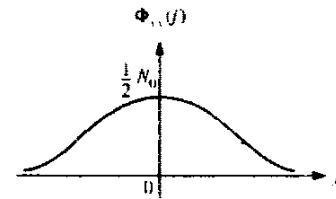


FIGURE 2-2-1 An example of a lowpass filter.

FIGURE 2-2-2 The power density spectrum of the lowpass filter output when the input is white noise.



noise. Let us determine the power density spectrum of the output process. The transfer function of the lowpass filter is

$$H(f) = \frac{R}{R + j2\pi fL} = \frac{1}{1 + j2\pi fL/R}$$

and, hence,

$$|H(f)|^2 = \frac{1}{1 + (2\pi L/R)^2 f^2} \quad (2-2-30)$$

The power density spectrum of the output process is

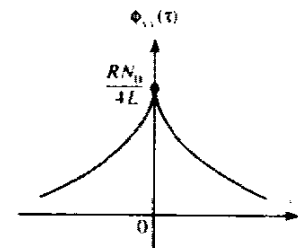
$$\Phi_{yy}(f) = \frac{N_0}{2} \frac{1}{1 + (2\pi L/R)^2 f^2} \quad (2-2-31)$$

This power density spectrum is illustrated in Fig. 2-2-2. Its inverse Fourier transform yields the autocorrelation function

$$\begin{aligned} \phi_{yy}(\tau) &= \int_{-\infty}^{\infty} \frac{N_0}{2} \frac{1}{1 + (2\pi L/R)^2 f^2} e^{j2\pi f\tau} df \\ &= \frac{RN_0}{4L} e^{-(R/L)|\tau|} \end{aligned} \quad (2-2-32)$$

The autocorrelation function $\phi_{yy}(\tau)$ is shown in Fig. 2-2-3. We observe that the second moment of the process $Y(t)$ is $\phi_{yy}(0) = RN_0/4L$.

FIGURE 2-2-3 The autocorrelation function of the output of the lowpass filter for a white-noise input.



As a final exercise, we determine the cross-correlation function between $y(t)$ and $x(t)$, where $x(t)$ denotes the input and $y(t)$ denotes the output of the linear system. We have

$$\begin{aligned}\phi_{yx}(t_1, t_2) &= \frac{1}{2}E(Y_1 X_2^*) = \frac{1}{2} \int_{-\infty}^{\infty} h(\alpha) E[X(t_1 - \alpha) X^*(t_2)] d\alpha \\ &= \int_{-\infty}^{\infty} h(\alpha) \phi_{xx}(t_1 - t_2 - \alpha) d\alpha = \phi_{yx}(t_1 - t_2)\end{aligned}$$

Hence, the stochastic processes $X(t)$ and $Y(t)$ are jointly stationary. With $t_1 - t_2 = \tau$, we have

$$\phi_{yx}(\tau) = \int_{-\infty}^{\infty} h(\alpha) \phi_{xx}(\tau - \alpha) d\alpha \quad (2-2-33)$$

Note that the integral in (2-2-33) is a convolution integral. Hence in the frequency domain the relation (2-2-33) becomes

$$\Phi_{yx}(f) = \Phi_{xx}(f)H(f) \quad (2-2-34)$$

We observe that if the input process is white noise, the cross correlation of the input with the output of the system yields the impulse response $h(t)$ to within a scale factor.

2-2-4 Sampling Theorem for Band-Limited Stochastic Processes

Recall that a deterministic signal $s(t)$ that has a Fourier transform $S(f)$ is called band-limited if $S(f) = 0$ for $|f| > W$, where W is the highest frequency contained in $s(t)$. Such a signal is uniquely represented by samples of $s(t)$ taken at a rate of $f_s \geq 2W$ samples/s. The minimum rate $f_N = 2W$ samples/s is called the *Nyquist rate*. Sampling below the Nyquist rate results in frequency aliasing.

The band-limited signal sampled at the Nyquist rate can be reconstructed from its samples by use of the interpolation formula

$$s(t) = \sum_{n=-\infty}^{\infty} s\left(\frac{n}{2W}\right) \frac{\sin\left[2\pi W\left(t - \frac{n}{2W}\right)\right]}{2\pi W\left(t - \frac{n}{2W}\right)} \quad (2-2-35)$$

where $\{s(n/2W)\}$ are the samples of $s(t)$ taken at $t = n/2W$, $n = 0, \pm 1, \pm 2, \dots$. Equivalently, $s(t)$ can be reconstructed by passing the sampled signal through

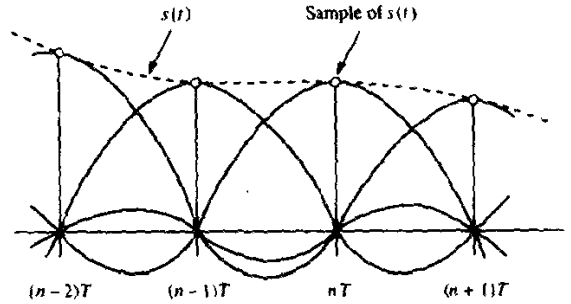


FIGURE 2-2-4 Signal reconstruction based on ideal interpolation.

an ideal low-pass filter with impulse response $h(t) = (\sin 2\pi Wt)/2\pi Wt$. Figure 2-2-4 illustrates the signal reconstruction process based on ideal interpolation.

A stationary stochastic process $X(t)$ is said to be *band-limited* if its power density spectrum $\Phi(f) = 0$ for $|f| > W$. Since $\Phi(f)$ is the Fourier transform of the autocorrelation function $\phi(\tau)$, it follows that $\phi(\tau)$ can be represented as

$$\phi(\tau) = \sum_{n=-\infty}^{\infty} \phi\left(\frac{n}{2W}\right) \frac{\sin\left[2\pi W\left(\tau - \frac{n}{2W}\right)\right]}{2\pi W\left(\tau - \frac{n}{2W}\right)} \quad (2-2-36)$$

where $\{\phi(n/2W)\}$ are samples of $\phi(\tau)$ taken at $\tau = n/2W$, $n = 0, \pm 1, \pm 2, \dots$

Now, if $X(t)$ is a band-limited stationary stochastic process then $X(t)$ can be represented as

$$X(t) = \sum_{n=-\infty}^{\infty} X\left(\frac{n}{2W}\right) \frac{\sin\left[2\pi W\left(t - \frac{n}{2W}\right)\right]}{2\pi W\left(t - \frac{n}{2W}\right)} \quad (2-2-37)$$

where $\{X(n/2W)\}$ are samples of $X(t)$ taken at $t = n/2W$, $n = 0, \pm 1, \pm 2, \dots$. This is the sampling representation for a stationary stochastic process. The samples are random variables that are described statistically by appropriate joint probability density functions. The signal representation in (2-2-37) is easily established by showing that (Problem 2-17)

$$E\left\{\left[X(t) - \sum_{n=-\infty}^{\infty} X\left(\frac{n}{2W}\right) \frac{\sin\left[2\pi W\left(t - \frac{n}{2W}\right)\right]}{2\pi W\left(t - \frac{n}{2W}\right)}\right]^2\right\} = 0 \quad (2-2-38)$$

Hence, equality between the sampling representation and the stochastic process $X(t)$ holds in the sense that the mean square error is zero.

2-2-5 Discrete-Time Stochastic Signals and Systems

The characterization of continuous-time stochastic signals given above can be easily carried over to discrete-time stochastic signals. Such signals are usually obtained by uniformly sampling a continuous-time stochastic process.

A discrete-time stochastic process $X(n)$ consists of an ensemble of sample sequences $\{x(n)\}$. The statistical properties of $X(n)$ are similar to the characterization of $X(t)$ with the restriction that n is now an integer (time) variable. Hence, the m th moment of $X(n)$ is defined as

$$E[X_n^m] = \int_{-\infty}^{\infty} X_n^m p(X_n) dX_n \quad (2-2-39)$$

and the autocorrelation sequence is

$$\phi(n, k) = \frac{1}{2} E(X_n X_k^*) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} X_n X_k^* p(X_n, X_k) dX_n dX_k \quad (2-2-40)$$

Similarly, the autocovariance sequence is

$$\mu(n, k) = \phi(n, k) - E(X_n)E(X_k^*) \quad (2-2-41)$$

For a stationary process, we have $\phi(n, k) \equiv \phi(n - k)$, $\mu(n, k) \equiv \mu(n - k)$, and

$$\mu(n - k) = \phi(n - k) - |m_x|^2 \quad (2-2-42)$$

where $m_x = E(X_n)$ is the mean value.

As in the case of continuous-time stochastic processes, a discrete-time stationary process has infinite energy but a finite average power, which is given as

$$E(|X_n|^2) = \phi(0) \quad (2-2-43)$$

The power density spectrum for the discrete-time process is obtained by computing the Fourier transform of $\phi(n)$. Since $\phi(n)$ is a discrete-time sequence, the Fourier transform is defined as

$$\Phi(f) = \sum_{n=-\infty}^{\infty} \phi(n) e^{-j2\pi fn} \quad (2-2-44)$$

and the inverse transform relationship is

$$\phi(n) = \int_{-1/2}^{1/2} \Phi(f) e^{j2\pi fn} df \quad (2-2-45)$$

We make the observation that the power density spectrum $\Phi(f)$ is periodic with a period $f_p = 1$. In other words, $\Phi(f + k) = \Phi(f)$ for $k = \pm 1, \pm 2, \dots$. This is a characteristic of the Fourier transform of any discrete-time sequence such as $\phi(n)$.

Finally, let us consider the response of a discrete-time, linear time-invariant system to a stationary stochastic input signal. The system is characterized in

the time domain by its unit sample response $h(n)$ and in the frequency domain by the frequency response $H(f)$, where

$$H(f) = \sum_{n=-\infty}^{\infty} h(n)e^{-j2\pi fn} \quad (2-2-46)$$

The response of the system to the stationary stochastic input signal $X(n)$ is given by the convolution sum

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k) \quad (2-2-47)$$

The mean value of the output of the system is

$$m_y = E[y(n)] = \sum_{k=-\infty}^{\infty} h(k)E[x(n-k)] \quad (2-2-48)$$

$$m_y = m_x \sum_{k=-\infty}^{\infty} h(k) = m_x H(0)$$

where $H(0)$ is the zero frequency (dc) gain of the system.

The autocorrelation sequence for the output process is

$$\begin{aligned} \phi_{yy}(k) &= \frac{1}{2}E[y^*(n)y(n+k)] \\ &= \frac{1}{2} \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} h^*(i)h(j)E[x^*(n-i)x(n+k-j)] \\ &= \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} h^*(i)h(j)\phi_{xx}(k-j+i) \end{aligned} \quad (2-2-49)$$

This is the general form for the autocorrelation sequence of the system output in terms of the autocorrelation of the system input and the unit sample response of the system. By taking the Fourier transform of $\phi_{yy}(k)$ and substituting the relation in (2-2-49), we obtain the corresponding frequency domain relationship

$$\Phi_{yy}(f) = \Phi_{xx}(f) |H(f)|^2 \quad (2-2-50)$$

which is identical to (2-2-27) except that in (2-2-50) the power density spectra $\Phi_{yy}(f)$ and $\Phi_{xx}(f)$ and the frequency response $H(f)$ are periodic functions of frequency with period $f_p = 1$.

2-2-6 Cyclostationary Processes

In dealing with signals that carry digital information we encounter stochastic processes that have statistical averages that are periodic. To be specific, let us consider a stochastic process of the form

$$X(t) = \sum_{n=-\infty}^{\infty} a_n g(t-nT) \quad (2-2-51)$$

where $\{a_n\}$ is a (discrete-time) sequence of random variables with mean $m_a = E(a_n)$ for all n and autocorrelation sequence $\phi_{aa}(k) = \frac{1}{2}E(a_n^* a_{n+k})$. The signal $g(t)$ is deterministic. The stochastic process $X(t)$ represents the signal for several different types of linear modulation techniques which are introduced in Chapter 4. The sequence $\{a_n\}$ represents the digital information sequence (of symbols) that is transmitted over the communication channel and $1/T$ represents the rate of transmission of the information symbols.

Let us determine the mean and autocorrelation function of $X(t)$. First, the mean value is

$$\begin{aligned} E[X(t)] &= \sum_{n=-\infty}^{\infty} E(a_n)g(t-nT) \\ &= m_a \sum_{n=-\infty}^{\infty} g(t-nT) \end{aligned} \quad (2-5-52)$$

We observe that the mean is time-varying. In fact, it is periodic with period T . The autocorrelation function of $X(t)$ is

$$\begin{aligned} \phi_{xx}(t+\tau, t) &= \frac{1}{2}E[X(t+\tau)X^*(t)] \\ &= \frac{1}{2} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} E(a_n^* a_m) g^*(t-nT) g(t+\tau-mT) \\ &= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \phi_{aa}(m-n) g^*(t-nT) g(t+\tau-mT) \end{aligned} \quad (2-2-53)$$

Again, we observe that

$$\phi_{xx}(t+\tau+kT, t+kT) = \phi_{xx}(t+\tau, t) \quad (2-2-54)$$

for $k = \pm 1, \pm 2, \dots$. Hence, the autocorrelation function of $X(t)$ is also periodic with period T .

Such a stochastic process is called *cyclostationary* or *periodically stationary*. Since the autocorrelation function depends on both the variables t and τ , its frequency domain representation requires the use of a two-dimensional Fourier transform.

Since it is highly desirable to characterize such signals by their power density spectrum, an alternative approach is to compute the *time-average autocorrelation function* over a single period, defined as

$$\bar{\phi}_{xx}(\tau) = \frac{1}{T} \int_{-T/2}^{T/2} \phi_{xx}(t+\tau, t) dt \quad (2-2-55)$$

Thus, we eliminate the time dependence by dealing with the average autocorrelation function. Now, the fourier transform of $\bar{\phi}_{xx}(\tau)$ yields the

average power density spectrum of the cyclostationary stochastic process. This approach allows us to simply characterize cyclostationary processes in the frequency domain in terms of the power spectrum. That is, the power density spectrum is

$$\Phi_{xx}(f) = \int_{-\infty}^{\infty} \bar{\phi}_{xx}(\tau) e^{-j2\pi f\tau} d\tau \quad (2-2-56)$$

2-3 BIBLIOGRAPHICAL NOTES AND REFERENCES

In this chapter we have provided a review of basic concepts and definitions in the theory of probability and stochastic processes. As stated in the opening paragraph, this theory is an important mathematical tool in the statistical modeling of information sources, communication channels, and in the design of digital communication systems. Of particular importance in the evaluation of communication system performance is the Chernoff bound. This bound is frequently used in bounding the probability of error of digital communication systems that employ coding in the transmission of information. Our coverage also highlighted a number of probability distributions and their properties, which are frequently encountered in the design of digital communication systems.

The texts by Davenport and Root (1958), Davenport (1970), Papoulis (1984) Pebbles (1987), Helstrom (1991) and Leon-Garcia (1994) provide *engineering-oriented treatments of probability and stochastic processes*. A more mathematical treatment of probability theory may be found in the text by Loève (1955). Finally, we cite the book by Miller (1964), which treats multidimensional gaussian distributions.

PROBLEMS

- 2-1 One experiment has four mutually exclusive outcomes A_i , $i = 1, 2, 3, 4$, and a second experiment has three mutually exclusive outcomes B_j , $j = 1, 2, 3$. The joint probabilities $P(A_i, B_j)$ are

$$\begin{aligned} P(A_1, B_1) &= 0.10, & P(A_1, B_2) &= 0.08, & P(A_1, B_3) &= 0.13 \\ P(A_2, B_1) &= 0.05, & P(A_2, B_2) &= 0.03, & P(A_2, B_3) &= 0.09 \\ P(A_3, B_1) &= 0.05, & P(A_3, B_2) &= 0.12, & P(A_3, B_3) &= 0.14 \\ P(A_4, B_1) &= 0.11, & P(A_4, B_2) &= 0.04, & P(A_4, B_3) &= 0.06 \end{aligned}$$

Determine the probabilities $P(A_i)$, $i = 1, 2, 3, 4$, and $P(B_j)$, $j = 1, 2, 3$.

- 2-2 The random variables X_i , $i = 1, 2, \dots, n$, have the joint pdf $p(x_1, x_2, \dots, x_n)$. Prove that

$$\begin{aligned} p(x_1, x_2, x_3, \dots, x_n) \\ = p(x_n | x_{n-1}, \dots, x_1) p(x_{n-1} | x_{n-2}, \dots, x_1) \cdots p(x_3 | x_2, x_1) p(x_2 | x_1) p(x_1) \end{aligned}$$

2-3 The pdf of a random variable X is $p(x)$. A random variable Y is defined as

$$Y = aX + b$$

where $a < 0$. Determine the pdf of Y in terms of the pdf of X .

2-4 Suppose that X is a gaussian random variable with zero mean and unit variance. Let

$$Y = aX^3 + b, \quad a > 0$$

Determine and plot the pdf of Y .

2-5 a Let X_r and X_i be statistically independent zero-mean gaussian random variables with identical variance. Show that a (rotational) transformation of the form

$$Y_r + jY_i = (X_r + jX_i)e^{j\phi}$$

results in another pair (Y_r, Y_i) of gaussian random variables that have the same joint pdf as the pair (X_r, X_i) .

b Note that

$$\begin{bmatrix} Y_r \\ Y_i \end{bmatrix} = \mathbf{A} \begin{bmatrix} X_r \\ X_i \end{bmatrix}$$

where \mathbf{A} is a 2×2 matrix. As a generalization of the two-dimensional transformation of the gaussian random variables considered in (a), what property must the linear transformation \mathbf{A} satisfy if the pdfs for \mathbf{X} and \mathbf{Y} , where $\mathbf{Y} = \mathbf{A}\mathbf{X}$, $\mathbf{X} = (X_1 X_2 \cdots X_n)$ and $\mathbf{Y} = (Y_1 Y_2 \cdots Y_n)$, are identical?

2-6 The random variable Y is defined as

$$Y = \sum_{i=1}^n X_i$$

where the X_i , $i = 1, 2, \dots, n$, are statistically independent random variables with

$$X_i = \begin{cases} 1 & \text{with probability } p \\ 0 & \text{with probability } 1 - p \end{cases}$$

a Determine the characteristic function of Y .

b From the characteristic function, determine the moments $E(Y)$ and $E(Y^2)$.

2-7 The four random variables X_1, X_2, X_3, X_4 are zero-mean jointly gaussian random variables with covariance $\mu_{ij} = E(X_i X_j)$ and characteristic function $\psi(j\nu_1, j\nu_2, j\nu_3, j\nu_4)$. Show that

$$E(X_1 X_2 X_3 X_4) = \mu_{12} \mu_{34} + \mu_{13} \mu_{24} + \mu_{14} \mu_{23}$$

2-8 From the characteristic functions for the central chi-square and noncentral chi-square random variables given by (2-1-109) and (2-1-117), respectively,

determine the corresponding first and second moments given by (2-1-112) and (2-1-125)

2-9 The pdf of a Cauchy distributed random variable X is

$$p(x) = \frac{a/\pi}{x^2 + a^2}, \quad -\infty < x < \infty$$

- a Determine the mean and variance of X .
- b Determine the characteristic function of X .

2-10 The random variable Y is defined as

$$Y = \frac{1}{n} \sum_{i=1}^n X_i$$

where $X_i, i = 1, 2, \dots, n$, are statistically independent and identically distributed random variables each of which has the Cauchy pdf given in Problem 2-9

- a Determine the characteristic function of Y .
- b Determine the pdf of Y .
- c Consider the pdf of Y in the limit as $n \rightarrow \infty$. Does the central limit hold? Explain your answer.

2-11 Assume that random processes $x(t)$ and $y(t)$ are individually and jointly stationary.

- a Determine the autocorrelation function of $z(t) = x(t) + y(t)$.
- b Determine the autocorrelation function of $z(t)$ when $x(t)$ and $y(t)$ are uncorrelated.
- c Determine the autocorrelation function of $z(t)$ when $x(t)$ and $y(t)$ are uncorrelated and have zero means.

2-12 The autocorrelation function of a stochastic process $X(t)$ is

$$\phi_{xx}(\tau) = \frac{1}{2} N_0 \delta(\tau)$$

Such a process is called *white noise*. Suppose $x(t)$ is the input to an ideal bandpass filter having the frequency response characteristic shown in Fig. P2-12. Determine the total noise power at the output of the filter.

2-13 The covariance matrix of three random variables X_1, X_2 and X_3 is

$$\begin{bmatrix} \mu_{11} & 0 & \mu_{13} \\ 0 & \mu_{22} & 0 \\ \mu_{31} & 0 & \mu_{33} \end{bmatrix}$$

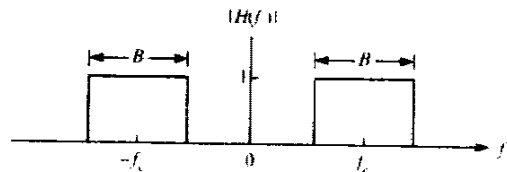


FIGURE P.2-12

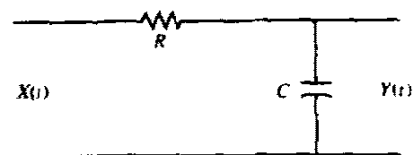


FIGURE P2-16

The linear transformation $\mathbf{Y} = \mathbf{A}\mathbf{X}$ is made where

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Determine the covariance matrix of \mathbf{Y} .

- 2-14** Let $X(t)$ be a stationary real normal process with zero mean. Let a new process $Y(t)$ be defined by

$$Y(t) = X^2(t)$$

Determine the autocorrelation function of $Y(t)$ in terms of the autocorrelation function of $X(t)$. *Hint:* Use the result on gaussian variables derived in Problem 2-7.

- 2-15** For the Nakagami pdf, given by (2-1-147), define the normalized random variable $X = R/\sqrt{\Omega}$. Determine the pdf of X .
- 2-16** The input $X(t)$ in the circuit shown in Fig. P2-16 is a stochastic process with $E[X(t)] = 0$ and $\phi_{xx}(\tau) = \sigma^2\delta(\tau)$, i.e., $X(t)$ is a white noise process.
- Determine the spectral density $\Phi_{yy}(f)$.
 - Determine $\phi_{yy}(\tau)$ and $E\{Y^2(t)\}$.
- 2-17** Demonstrate the validity of (2-2-38).
- 2-18** Use the Chernoff bound to show that $Q(x) \leq e^{-x^2/2}$ where $Q(x)$ is defined by (2-1-97).
- 2-19** Determine the mean, the autocorrelation sequence, and the power density spectrum of the output of a system with unit sample response

$$h(n) = \begin{cases} 1 & (n = 0) \\ -2 & (n = 1) \\ 1 & (n = 2) \\ 0 & (\text{otherwise}) \end{cases}$$

when the input $x(n)$ is a white-noise process with variance σ^2 .

- 2-20** The autocorrelation sequence of a discrete-time stochastic process is $\phi(k) = (\frac{1}{2})^{|k|}$. Determine its power density spectrum.
- 2-21** A discrete-time stochastic process $X(n) \equiv X(nT)$ is obtained by periodic sampling of a continuous-time zero-mean stationary process $X(t)$ where T is the sampling interval, i.e., $f_s = 1/T$ is the sampling rate.
- Determine the relationship between the autocorrelation function of $X(t)$ and the autocorrelation sequence of $X(n)$.
 - Express the power density spectrum of $X(n)$ in terms of the power density spectrum of the process $X(t)$.

- c Determine the conditions under which the power density spectrum of $X(n)$ is equal to the power density spectrum of $X(t)$.
- 2-22 Consider a band-limited zero-mean stationary stochastic $X(t)$ with power density spectrum

$$\Phi(f) = \begin{cases} 1 & (|f| \leq W) \\ 0 & (|f| > W) \end{cases}$$

$X(t)$ is sampled at a rate $f_s = 1/T$ to yield a discrete-time process $X(n) = X(nT)$.

- a Determine the expression for the autocorrelation sequence of $X(n)$.
- b Determine the minimum value of T that results in a white (spectrally flat) sequence.
- c Repeat (b) if the power density spectrum of $X(t)$ is

$$\Phi(f) = \begin{cases} 1 - |f|/W & (|f| \leq W) \\ 0 & (|f| > W) \end{cases}$$

- 2-23 Show that the functions

$$f_k(t) = \frac{\sin \left[2\pi W \left(t - \frac{k}{2W} \right) \right]}{2\pi W \left(t - \frac{k}{2W} \right)}, \quad k = 0, \pm 1, \pm 2, \dots$$

are orthogonal over the interval $[-\infty, \infty]$, i.e.,

$$\int_{-\infty}^{\infty} f_k(t)f_j(t) dt = \begin{cases} 1/2W & (k = j) \\ 0 & (k \neq j) \end{cases}$$

Therefore, the sampling theorem reconstruction formula may be viewed as a series expansion of the band-limited signal $s(t)$, where the weights are samples of $s(t)$ and the $\{f_k(t)\}$ are the set of orthogonal functions used in the series expansion.

- 2-24 The noise equivalent bandwidth of a system is defined as

$$B_{eq} = \frac{1}{G} \int_0^{\infty} |H(f)|^2 df$$

where $G = \max |H(f)|^2$. Using this definition, determine the noise equivalent bandwidth of the ideal bandpass filter shown in Fig. P2-12 and the lowpass system shown in Fig. P2-16.

3

SOURCE CODING

Communication systems are designed to transmit the information generated by a source to some destination. Information sources may take a variety of different forms. For example, in radio broadcasting, the source is generally an *audio source* (voice or music). In TV broadcasting, the information source is a *video source* whose output is a *moving image*. The outputs of these sources are analog signals and, hence, the sources are called *analog sources*. In contrast, computers and storage devices, such as magnetic or optical disks, produce discrete outputs (usually binary or ASCII characters) and, hence, they are called *discrete sources*.

Whether a source is analog or discrete, a digital communication system is designed to transmit information in digital form. Consequently, the output of the source must be converted to a format that can be transmitted digitally. This conversion of the source output to a digital form is generally performed by the source encoder, whose output may be assumed to be a sequence of binary digits.

In this chapter, we treat source encoding based on mathematical models of information sources and a quantitative measure of the information emitted by a source. We consider the encoding of discrete sources first and then we discuss the encoding of analog sources. We begin by developing mathematical models for information sources.

3-1 MATHEMATICAL MODELS FOR INFORMATION SOURCES

Any information source produces an output that is random, i.e., the source output is characterized in statistical terms. Otherwise, if the source output

82

were known exactly, there would be no need to transmit it. In this section, we consider both discrete and analog information sources, and we postulate mathematical models for each type of source.

The simplest type of discrete source is one that emits a sequence of letters selected from a finite alphabet. For example, a *binary source* emits a binary sequence of the form 100101110..., where the alphabet consists of the two letters {0, 1}. More generally, a discrete information source with an alphabet of L possible letters, say $\{x_1, x_2, \dots, x_L\}$, emits a sequence of letters selected from the alphabet.

To construct a mathematical model for a discrete source, we assume that each letter in the alphabet $\{x_1, x_2, \dots, x_L\}$ has a given probability p_k of occurrence. That is,

$$p_k = P(X = x_k), \quad 1 \leq k \leq L$$

where

$$\sum_{k=1}^L p_k = 1$$

We consider two mathematical models of discrete sources. In the first, we assume that the output sequence from the source is statistically independent. That is, the current output letter is statistically independent from all past and future outputs. A source whose output satisfies the condition of statistical independence among output letters in the sequence is said to be *memoryless*. Such a source is called a *discrete memoryless source (DMS)*.

If the discrete source output is statistically dependent, as, for example, English text, we may construct a mathematical model based on statistical stationarity. By definition, a discrete source is said to be *stationary* if the joint probabilities of two sequences of length n , say a_1, a_2, \dots, a_n and $a_{1+m}, a_{2+m}, \dots, a_{n+m}$, are identical for all $n \geq 1$ and for all shifts m . In other words, the joint probabilities for any arbitrary length sequence of source outputs are invariant under a shift in the time origin.

An *analog* source has an output waveform $x(t)$ that is a sample function of a stochastic process $X(t)$. We assume that $X(t)$ is a stationary stochastic process with autocorrelation function $\phi_{xx}(\tau)$ and power spectral density $\Phi_{xx}(f)$. When $X(t)$ is a bandlimited stochastic process, i.e., $\Phi_{xx}(f) = 0$ for $|f| \geq W$, the sampling theorem may be used to represent $X(t)$ as

$$X(t) = \sum_{n=-\infty}^{\infty} X\left(\frac{n}{2W}\right) \frac{\sin\left[2\pi W\left(t - \frac{n}{2W}\right)\right]}{2\pi W\left(t - \frac{n}{2W}\right)} \quad (3-1-1)$$

where $\{X(n/2W)\}$ denote the samples of the process $X(t)$ taken at the sampling (Nyquist) rate of $f_s = 2W$ samples/s. Thus, by applying the sampling theorem, we may convert the output of an analog source into an equivalent

discrete-time source. Then, the source output is characterized statistically by the joint pdf $p(x_1, x_2, \dots, x_m)$ for all $m \geq 1$, where $X_n = X(n/2W)$, $1 \leq n \leq m$, are the random variables corresponding to the samples of $X(t)$.

We note that the output samples $\{X(n/2W)\}$ from the stationary sources are generally continuous, and, hence, they cannot be represented in digital form without some loss in precision. For example, we may quantize each sample to a set of discrete values, but the quantization process results in loss of precision, and, consequently, the original signal cannot be reconstructed exactly from the quantized sample values. Later in this chapter, we shall consider the distortion resulting from quantization of the samples from an analog source.

3-2 A LOGARITHMIC MEASURE OF INFORMATION

To develop an appropriate measure of information, let us consider two discrete random variables with possible outcomes x_i , $i = 1, 2, \dots, n$, and y_j , $j = 1, 2, \dots, m$, respectively. Suppose we observe some outcome $Y = y_j$ and we wish to determine, quantitatively, the amount of information that the occurrence of the event $Y = y_j$ provides about the event $X = x_i$, $i = 1, 2, \dots, n$. We observe that when X and Y are statistically independent, the occurrence of $Y = y_j$ provides no information about the occurrence of the event $X = x_i$. On the other hand, when X and Y are fully dependent such that the occurrence of $Y = y_j$ determines the occurrence of $X = x_i$, the information content is simply that provided by the event $X = x_i$. A suitable measure that satisfies these conditions is the logarithm of the ratio of the conditional probability

$$P(X = x_i | Y = y_j) = P(x_i | y_j)$$

divided by the probability

$$P(X = x_i) = P(x_i)$$

That is, the information content provided by the occurrence of the event $Y = y_j$ about the event $X = x_i$ is defined as

$$I(x_i; y_j) = \log \frac{P(x_i | y_j)}{P(x_i)} \quad (3-2-1)$$

$I(x_i; y_j)$ is called the *mutual information* between x_i and y_j .

The units of $I(x_i; y_j)$ are determined by the base of the logarithm, which is usually selected as either 2 or e . When the base of the logarithm is 2, the units of $I(x_i; y_j)$ are bits, and when the base is e , the units of $I(x_i; y_j)$ are called *nats* (natural units). (The standard abbreviation for \log_e is \ln .) Since

$$\ln a = \ln 2 \log_2 a = 0.69315 \log_2 a$$

the information measured in nats is equal to $\ln 2$ times the information measured in bits.

When the random variables X and Y are statistically independent,

$P(x_i | y_j) = P(x_i)$ and, hence, $I(x_i; y_j) = 0$. On the other hand, when the occurrence of the event $Y = y_j$ uniquely determines the occurrence of the event $X = x_i$, the conditional probability in the numerator of (3-2-1) is unity and, hence,

$$I(x_i; y_j) = \log \frac{1}{P(x_i)} = -\log P(x_i) \quad (3-2-2)$$

But (3-2-2) is just the information of the event $X = x_i$. For this reason, it is called the *self-information* of the event $X = x_i$ and it is denoted as

$$I(x_i) = \log \frac{1}{P(x_i)} = -\log P(x_i) \quad (3-2-3)$$

We note that a high-probability event conveys less information than a low-probability event. In fact, if there is only a single event x with probability $P(x) = 1$ then $I(x) = 0$. To demonstrate further that the logarithmic measure of information content is the appropriate one for digital communications, let us consider the following example.

Example 3-2-1

Suppose we have a discrete information source that emits a binary digit, either 0 or 1, with equal probability every τ_s seconds. The information content of each output from source is

$$\begin{aligned} I(x_i) &= -\log_2 P(x_i), \quad x_i = 0, 1 \\ &= -\log_2 \frac{1}{2} = 1 \text{ bit} \end{aligned}$$

Now suppose that successive outputs from the source are statistically independent, i.e., the source is memoryless. Let us consider a block of k binary digits from the source that occurs in a time interval $k\tau_s$. There are $M = 2^k$ possible k -bit blocks, each of which is equally probable with probability $1/M = 2^{-k}$. The self-information of a k -bit block is

$$I(x'_i) = -\log_2 2^{-k} = k \text{ bits}$$

emitted in a time interval $k\tau_s$. Thus the logarithmic measure of information content possesses the desired additivity property when a number of source outputs is considered as a block.

Now let us return to the definition of mutual information given in (3-2-1) and multiply the numerator and denominator of the ratio of probabilities by $P(y_j)$. Since

$$\frac{P(x_i | y_j)}{P(x_i)} = \frac{P(x_i | y_j)P(y_j)}{P(x_i)P(y_j)} = \frac{P(x_i, y_j)}{P(x_i)P(y_j)} = \frac{P(y_j | x_i)}{P(y_j)}$$

we conclude that

$$I(x_i; y_j) = I(y_j; x_i) \quad (3-2-4)$$

Therefore the information provided by the occurrence of the event $Y = y_j$ about the event $X = x_i$ is identical to the information provided by the occurrence of the event $X = x_i$ about the event $Y = y_j$.

Example 3-2-2

Suppose that X and Y are binary-valued $\{0, 1\}$ random variables that represent the input and output of a binary-input, binary-output channel. The input symbols are equally likely and the output symbols depend on the input according to the conditional probabilities

$$P(Y = 0 | X = 0) = 1 - p_0$$

$$P(Y = 1 | X = 0) = p_0$$

$$P(Y = 1 | X = 1) = 1 - p_1$$

$$P(Y = 0 | X = 1) = p_1$$

Let us determine the mutual information about the occurrence of the events $X = 0$ and $X = 1$, given that $Y = 0$.

From the probabilities given above, we obtain

$$\begin{aligned} P(Y = 0) &= P(Y = 0 | X = 0)P(X = 0) + P(Y = 0 | X = 1)P(X = 1) \\ &= \frac{1}{2}(1 - p_0 + p_1) \end{aligned}$$

$$\begin{aligned} P(Y = 1) &= P(Y = 1 | X = 0)P(X = 0) + P(Y = 1 | X = 1)P(X = 1) \\ &= \frac{1}{2}(1 - p_1 + p_0) \end{aligned}$$

Then, the mutual information about the occurrence of the event $X = 0$, given that $Y = 0$ is observed, is

$$I(x_1; y_1) = I(0; 0) = \log_2 \frac{P(Y = 0 | X = 0)}{P(Y = 0)} = \log_2 \frac{2(1 - p_0)}{1 - p_0 + p_1}$$

Similarly, given that $Y = 0$ is observed, the mutual information about the occurrence of the event $X = 1$ is

$$I(x_2; y_1) = I(1; 0) = \log_2 \frac{2p_1}{1 - p_0 + p_1}$$

Let us consider some special cases: First, if $p_0 = p_1 = 0$, the channel is called *noiseless* and

$$I(0; 0) = \log_2 2 = 1 \text{ bit}$$

Hence, the output specifies the input with certainty. On the other hand, if $p_0 = p_1 = \frac{1}{2}$, the channel is *useless* because

$$I(0; 0) = \log_2 1 = 0$$

However, if $p_0 = p_1 = \frac{1}{4}$, then

$$I(0; 0) = \log_2 \frac{3}{2} = 0.587$$

$$I(0; 1) = \log_2 \frac{1}{2} = -1 \text{ bit}$$

In addition to the definition of mutual information and self-information, it is useful to define the *conditional self-information* as

$$I(x_i | y_j) = \log \frac{1}{P(x_i | y_j)} = -\log P(x_i | y_j) \quad (3-2-5)$$

Then, by combining (3-2-1), (3-2-3), and (3-2-5), we obtain the relationship

$$I(x_i; y_j) = I(x_i) - I(x_i | y_j) \quad (3-2-6)$$

We interpret $I(x_i | y_j)$ as the self-information about the event $X = x_i$ after having observed the event $Y = y_j$. Since both $I(x_i) \geq 0$ and $I(x_i | y_j) \geq 0$, it follows that $I(x_i; y_j) < 0$ when $I(x_i | y_j) > I(x_i)$, and $I(x_i; y_j) > 0$ when $I(x_i | y_j) < I(x_i)$. Hence, the mutual information between a pair of events can be either positive, or negative, or zero.

3-2-1 Average Mutual Information and Entropy

Having defined the mutual information associated with the pair of events (x_i, y_j) , which are possible outcomes of the two random variables X and Y , we can obtain the average value of the mutual information by simply weighting $I(x_i; y_j)$ by the probability of occurrence of the joint event and summing over all possible joint events. Thus, we obtain

$$\begin{aligned} I(X; Y) &= \sum_{i=1}^n \sum_{j=1}^m P(x_i, y_j) I(x_i; y_j) \\ &= \sum_{i=1}^n \sum_{j=1}^m P(x_i, y_j) \log \frac{P(x_i, y_j)}{P(x_i)P(y_j)} \end{aligned} \quad (3-2-7)$$

as the average mutual information between X and Y . We observe that

$I(X; Y) = 0$ when X and Y are statistically independent. An important characteristic of the average mutual information is that $I(X; Y) \geq 0$ (see Problem 3-4).

Similarly, we define the average self-information, denoted by $H(X)$, as

$$\begin{aligned} H(X) &= \sum_{i=1}^n P(x_i) I(x_i) \\ &= - \sum_{i=1}^n P(x_i) \log P(x_i) \end{aligned} \quad (3-2-8)$$

When X represents the alphabet of possible output letters from a source, $H(X)$ represents the average self-information per source letter, and it is called the *entropy*† of the source. In the special case in which the letters from the source are equally probable, $P(x_i) = 1/n$ for all i , and, hence,

$$\begin{aligned} H(X) &= - \sum_{i=1}^n \frac{1}{n} \log \frac{1}{n} \\ &= \log n \end{aligned} \quad (3-2-9)$$

In general, $H(X) \leq \log n$ (see Problem 3-5) for any given set of source letter probabilities. In other words, *the entropy of a discrete source is a maximum when the output letters are equally probable.*

Example 3-2-3

Consider a source that emits a sequence of statistically independent letters, where each output letter is either 0 with probability q or 1 with probability $1 - q$. The entropy of this source is

$$H(X) = H(q) = -q \log q - (1 - q) \log (1 - q) \quad (3-2-10)$$

The binary entropy function $H(q)$ is illustrated in Fig. 3-2-1. We observe that the maximum value of the entropy function occurs at $q = \frac{1}{2}$ where $H(\frac{1}{2}) = 1$.

The average conditional self-information is called the *conditional entropy* and is defined

$$H(X | Y) = \sum_{i=1}^n \sum_{j=1}^m P(x_i, y_j) \log \frac{1}{P(x_i | y_j)} \quad (3-2-11)$$

We interpret $H(X | Y)$ as the information or uncertainty in X after Y is

†The term *entropy* is taken from statistical mechanics (thermodynamics), where a function similar to (3-2-8) is called (thermodynamic) entropy.

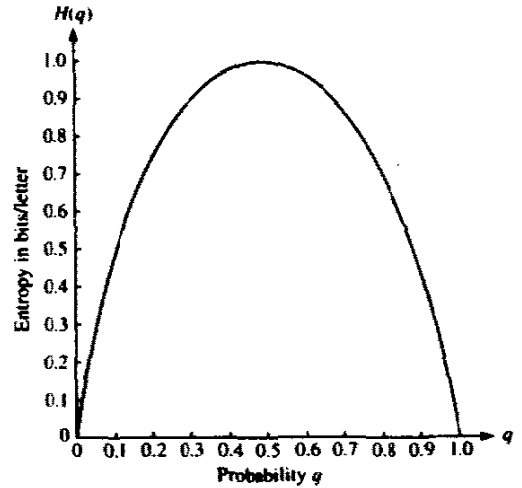


FIGURE 3-2-1 Binary entropy function.

observed. By combining (3-2-7), (3-2-8), and (3-2-11) we obtain the relationship

$$I(X; Y) = H(X) - H(X | Y) \quad (3-2-12)$$

Since $I(X; Y) \geq 0$, it follows that $H(X) \geq H(X | Y)$, with equality if and only if X and Y are statistically independent. If we interpret $H(X | Y)$ as the average amount of (conditional self-information) uncertainty in X after we observe Y , and $H(X)$ as the average amount of uncertainty (self-information) prior to the observation, then $I(X; Y)$ is the average amount of (mutual information) uncertainty provided about the set X by the observation of the set Y . Since $H(X) \geq H(X | Y)$, it is clear that conditioning on the observation Y does not increase the entropy.

Example 3-2-4

Let us evaluate the $H(X | Y)$ and $I(X; Y)$ for the binary-input, binary-output channel treated previously in Example 3-2-2 for the case where $p_0 = p_1 = p$. Let the probabilities of the input symbols be $P(X = 0) = q$ and $P(X = 1) = 1 - q$. Then the entropy is

$$H(X) = H(q) = -q \log q - (1 - q) \log (1 - q)$$

where $H(q)$ is the binary entropy function and the conditional entropy $H(X | Y)$ is defined by (3-2-11). A plot of $H(X | Y)$ as a function of q with

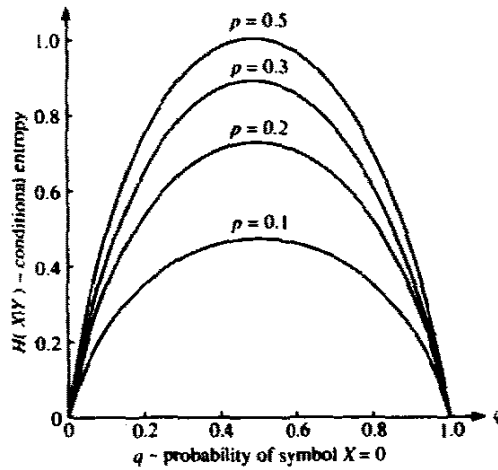


FIGURE 3-2-2 Conditional entropy for binary-input, binary-output symmetric channel.

p as a parameter is shown in Fig. 3-2-2. The average mutual information $I(X; Y)$ is plotted in Fig. 3-2-3.

As in the preceding example, when the conditional entropy $H(X|Y)$ is viewed in terms of a channel whose input is X and whose output is Y , $H(X|Y)$ is called the *equivocation* and is interpreted as the amount of average uncertainty remaining in X after observation of Y .

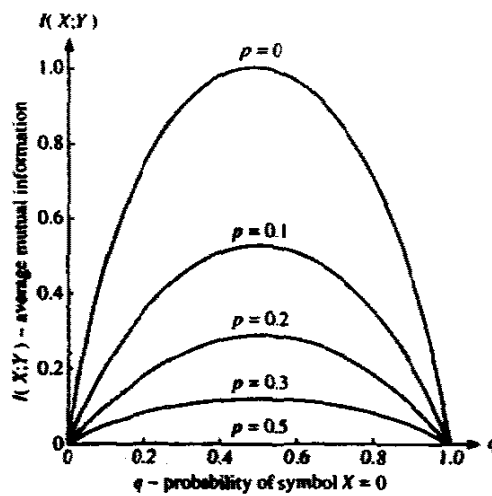


FIGURE 3-2-3 Average mutual information for binary-input, binary-output symmetric channel.

The results given above can be generalized to more than two random variables. In particular, suppose we have a block of k random variables $X_1 X_2 \cdots X_k$, with joint probability $P(x_1 x_2 \cdots x_k) \equiv P(X_1 = x_1, X_2 = x_2, \dots, X_k = x_k)$. Then, the entropy for the block is defined as

$$H(X_1 X_2 \cdots X_k) = - \sum_{j_1=1}^{n_1} \sum_{j_2=1}^{n_2} \cdots \sum_{j_k=1}^{n_k} P(x_{j_1} x_{j_2} \cdots x_{j_k}) \log P(x_{j_1} x_{j_2} \cdots x_{j_k}) \quad (3-2-13)$$

Since the joint probability $P(x_1 x_2 \cdots x_k)$ can be factored as

$$P(x_1 x_2 \cdots x_k) = P(x_1) P(x_2 | x_1) P(x_3 | x_1 x_2) \cdots P(x_k | x_1 x_2 \cdots x_{k-1}) \quad (3-2-14)$$

it follows that

$$\begin{aligned} H(X_1 X_2 X_3 \cdots X_k) &= H(X_1) + H(X_2 | X_1) + H(X_3 | X_1 X_2) \\ &\quad + \dots + H(X_k | X_1 \cdots X_{k-1}) \\ &= \sum_{i=1}^k H(X_i | X_1 X_2 \cdots X_{i-1}) \end{aligned} \quad (3-2-15)$$

By applying the result $H(X) \geq H(X | Y)$, where $X = X_m$ and $Y = X_1 X_2 \cdots X_{m-1}$, in (3-2-15) we obtain

$$H(X_1 X_2 \cdots X_k) \leq \sum_{m=1}^k H(X_m) \quad (3-2-16)$$

with equality if and only if the random variables X_1, X_2, \dots, X_k are statistically independent.

3-2-2 Information Measures for Continuous Random Variables

The definition of mutual information given above for discrete random variables may be extended in a straightforward manner to continuous random variables. In particular, if X and Y are random variables with joint pdf $p(x, y)$ and marginal pdfs $p(x)$ and $p(y)$, the average mutual information between X and Y is defined as

$$I(X; Y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(x) p(y | x) \log \frac{p(y | x) p(x)}{p(x) p(y)} dx dy \quad (3-2-17)$$

Although the definition of the average mutual information carries over to

continuous random variables, the concept of self-information does not. The problem is that a continuous random variable requires an infinite number of binary digits to represent it exactly. Hence, its self-information is infinite and, therefore, its entropy is also infinite. Nevertheless, we shall define a quantity that we call the *differential entropy* of the continuous random variable X as

$$H(X) = - \int_{-\infty}^{\infty} p(x) \log p(x) dx \quad (3-2-18)$$

We emphasize that this quantity does *not* have the physical meaning of self-information, although it may appear to be a natural extension of the definition of entropy for a discrete random variable (see Problem 3-6).

By defining the average conditional entropy of X given Y as

$$H(X | Y) = - \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(x, y) \log p(x | y) dx dy \quad (3-2-19)$$

the average mutual information may be expressed as

$$I(X; Y) = H(X) - H(X | Y)$$

or, alternatively, as

$$I(X; Y) = H(Y) - H(Y | X)$$

In some cases of practical interest, the random variable X is discrete and Y is continuous. To be specific, suppose that X has possible outcomes x_i , $i = 1, 2, \dots, n$, and Y is described by its marginal pdf $p(y)$. When X and Y are statistically dependent, we may express $p(y)$ as

$$p(y) = \sum_{i=1}^n p(y | x_i) P(x_i)$$

The mutual information provided about the event $X = x_i$ by the occurrence of the event $Y = y$ is

$$\begin{aligned} I(x_i; y) &= \log \frac{p(y | x_i) P(x_i)}{p(y) P(x_i)} \\ &= \log \frac{p(y | x_i)}{p(y)} \end{aligned} \quad (3-2-20)$$

Then, the average mutual information between X and Y is

$$I(X; Y) = \sum_{i=1}^n \int_{-\infty}^{\infty} p(y | x_i) P(x_i) \log \frac{p(y | x_i)}{p(y)} dy \quad (3-2-21)$$

Example 3-2-5

Suppose that X is a discrete random variable with two equally probable outcomes $x_1 = A$ and $x_2 = -A$. Let the conditional pdfs $p(y | x_i)$, $i = 1, 2$, be gaussian with mean x_i and variance σ^2 . That is,

$$p(y | A) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(y-A)^2/2\sigma^2} \quad (3-2-22)$$

$$p(y | -A) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(y+A)^2/2\sigma^2} \quad (3-2-22)$$

The average mutual information obtained from (3-2-21) becomes

$$I(X; Y) = \frac{1}{2} \int_{-\infty}^{\infty} \left[p(y | A) \log \frac{p(y | A)}{p(y)} + p(y | -A) \log \frac{p(y | -A)}{p(y)} \right] dy \quad (3-2-23)$$

$$p(y) = \frac{1}{2} [p(y | A) + p(y | -A)] \quad (3-2-24)$$

In Chapter 7, it will be shown that the average mutual information $I(X; Y)$ given by (3-2-23) represents the channel capacity of a binary-input additive white gaussian noise channel.

3-3 CODING FOR DISCRETE SOURCES

In Section 3-2 we introduced a measure for the information content associated with a discrete random variable X . When X is the output of a discrete source, the entropy $H(X)$ of the source represents the average amount of information emitted by the source. In this section, we consider the process of encoding the output of a source, i.e., the process of representing the source output by a sequence of binary digits. A measure of the efficiency of a source-encoding method can be obtained by comparing the average number of binary digits per output letter from the source to the entropy $H(X)$.

The encoding of a discrete source having a finite alphabet size may appear, at first glance, to be a relatively simple problem. However, this is true only when the source is memoryless, i.e., when successive symbols from the source are statistically independent and each symbol is encoded separately. The discrete memoryless source (DMS) is by far the simplest model that can be devised for a physical source. Few physical sources, however, closely fit this idealized mathematical model. For example, successive output letters from a machine printing English text are expected to be statistically dependent. On the other hand, if the machine output is a computer program coded in Fortran, the sequence of output letters is expected to exhibit a much smaller dependence. In any case, we shall demonstrate that it is always more efficient to encode blocks of symbol instead of encoding each symbol separately. By making the block size sufficiently large, the average number of binary digits

per output letter from the source can be made arbitrarily close to the entropy of the source.

3-3-1 Coding for Discrete Memoryless Sources

Suppose that a DMS produces an output letter or symbol every τ_s seconds. Each symbol is selected from a finite alphabet of symbols x_i , $i = 1, 2, \dots, L$, occurring with probabilities $P(x_i)$, $i = 1, 2, \dots, L$. The entropy of the DMS in bits per source symbol is

$$H(X) = - \sum_{i=1}^L P(x_i) \log_2 P(x_i) \leq \log_2 L \quad (3-3-1)$$

where equality holds when the symbols are equally probable. The average number of bits per source symbol is $H(X)$ and the source rate in bits/s is defined as $H(X)/\tau_s$.

Fixed-Length Code Words First we consider a block encoding scheme that assigns a unique set of R binary digits to each symbol. Since there are L possible symbols, the number of binary digits per symbol required for unique encoding when L is a power of 2 is

$$R = \log_2 L \quad (3-3-2)$$

and, when L is not a power of 2, it is

$$R = \lfloor \log_2 L \rfloor + 1 \quad (3-3-3)$$

where $\lfloor x \rfloor$ denotes the largest integer less than x . The code rate R in bits per symbol is now R and, since $H(X) \leq \log_2 L$, it follows that $R \geq H(X)$.

The efficiency of the encoding for the DMS is defined as the ratio $H(X)/R$. We observe that when L is a power of 2 and the source letters are equally probable, $R = H(X)$. Hence, a fixed-length code of R bits per symbol attains 100% efficiency. However, if L is not a power of 2 but the source symbols are still equally probable, R differs from $H(X)$ by at most 1 bit per symbol. When $\log_2 L \gg 1$, the efficiency of this encoding scheme is high. On the other hand, when L is small, the efficiency of the fixed-length code can be increased by encoding a sequence of J symbols at a time. To accomplish the desired encoding, we require L^J unique code words. By using sequences of N binary digits, we can accommodate 2^N possible code words. N must be selected such that

$$N \geq J \log_2 L$$

Hence, the minimum integer value of N required is

$$N = \lfloor J \log_2 L \rfloor + 1 \quad (3-3-4)$$

Now the average number of bits per source symbol is $N/J = R$, and, thus, the

inefficiency has been reduced by approximately a factor of $1/J$ relative to the symbol-by-symbol encoding described above. By making J sufficiently large, the efficiency of the encoding procedure, measured by the ratio $JH(X)/N$, can be made as close to unity as desired.

The encoding methods described above introduce no distortion since the encoding of source symbols or blocks of symbols into code words is unique. This type of encoding is called *noiseless*.

Now, suppose we attempt to reduce the code rate R by relaxing the condition that the encoding process be unique. For example, suppose that only a fraction of the L^J blocks of symbols is encoded uniquely. To be specific, let us select the $2^N - 1$ most probable J -symbol blocks and encode each of them uniquely, while the remaining $L^J - (2^N - 1)$ J -symbol blocks are represented by the single remaining code word. This procedure results in a decoding failure or (distortion) probability of error every time a low probability block is mapped into this single code word. Let P_e denote this probability of error. Based on this block encoding procedure, Shannon (1948a) proved the following source coding theorem.

Source Coding Theorem I

Let X be the ensemble of letters from a DMS with finite entropy $H(X)$. Blocks of J symbols from the source are encoded into code words of length N from a binary alphabet. For any $\epsilon > 0$, the probability P_e of a block decoding failure can be made arbitrarily small if

$$R \equiv \frac{N}{J} \geq H(X) + \epsilon \quad (3-3-5)$$

and J is sufficiently large. Conversely, if

$$R \leq H(X) - \epsilon \quad (3-3-6)$$

then P_e becomes arbitrarily close to 1 as J is made sufficiently large.

From this theorem, we observe that the average number of bits per symbol required to encode the output of a DMS with arbitrarily small probability of decoding failure is lower bounded by the source entropy $H(X)$. On the other hand, if $R < H(X)$, the decoding failure rate approaches 100% as J is arbitrarily increased.

Variable-Length Code Words When the source symbols are not equally probable, a more efficient encoding method is to use variable-length code

TABLE 3-3-1 VARIABLE-LENGTH CODES

Letter	$P(a_i)$	Code I	Code II	Code III
a_1	$\frac{1}{2}$	1	0	0
a_2	$\frac{1}{4}$	00	10	01
a_3	$\frac{1}{8}$	01	110	011
a_4	$\frac{1}{8}$	10	111	111

words. An example of such encoding is the Morse code, which dates back to the nineteenth century. In the Morse code, the letters that occur more frequently are assigned short code words and those that occur infrequently are assigned long code words. Following this general philosophy, we may use the probabilities of occurrence of the different source letters in the selection of the code words. The problem is to devise a method for selecting and assigning the code words to source letters. This type of encoding is called *entropy coding*.

For example, suppose that a DMS with output letters a_1, a_2, a_3, a_4 and corresponding probabilities $P(a_1) = \frac{1}{2}, P(a_2) = \frac{1}{4},$ and $P(a_3) = P(a_4) = \frac{1}{8}$ is encoded as shown in Table 3-3-1. Code I is a variable-length code that has a basic flaw. To see the flaw, suppose we are presented with the sequence 001001 Clearly, the first symbol corresponding to 00 is a_2 . However, the next four bits are ambiguous (not uniquely decodable). They may be decoded either as a_4a_3 or as $a_1a_2a_1$. Perhaps, the ambiguity can be resolved by waiting for additional bits, but such a decoding delay is highly undesirable. We shall only consider codes that are decodable *instantaneously*, that is, without any decoding delay.

Code II in Table 3-3-1 is *uniquely decodable* and *instantaneously decodable*. It is convenient to represent the code words in this code graphically as terminal nodes of a tree, as shown in Fig. 3-3-1. We observe that the digit 0 indicates the end of a code word for the first three code words. This characteristic plus the fact that no code word is longer than three binary digits makes this code instantaneously decodable. Note that no code word in this code is a prefix of any other code word. In general, the *prefix condition* requires that for a given code word C_k of length k having elements (b_1, b_2, \dots, b_k) , there is no other code word of length $l < k$ with elements (b_1, b_2, \dots, b_l) for $1 \leq l \leq k - 1$. In

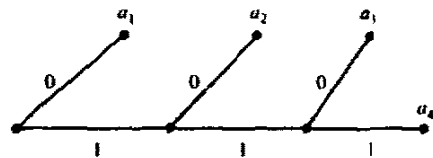


FIGURE 3-3-1 Code tree for code II in Table 3-3-1.

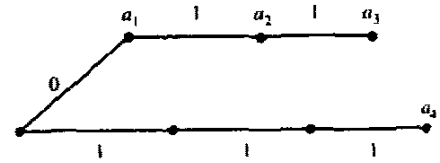


FIGURE 3-3-2 Code tree for code III in Table 3-3-1.

other words, there is no code word of length $l < k$ that is identical to the first l binary digits of another code word of length $k > l$. This property makes the code words instantaneously decodable.

Code III given in Table 3-3-1 has the tree structure shown in Fig. 3-3-2. We note that in this case the code is uniquely decodable but *not* instantaneously decodable. Clearly, this code does *not* satisfy the prefix condition.

Our main objective is to devise a systematic procedure for constructing uniquely decodable variable-length codes that are efficient in the sense that the average number of bits per source letter, defined as the quantity

$$\bar{R} = \sum_{k=1}^L n_k P(a_k) \tag{3-3-7}$$

is minimized. The conditions for the existence of a code that satisfies the prefix condition are given by the Kraft inequality.

Kraft Inequality A necessary and sufficient condition for the existence of a binary code with code words having lengths $n_1 \leq n_2 \leq \dots \leq n_L$ that satisfy the prefix condition is

$$\sum_{k=1}^L 2^{-n_k} \leq 1 \tag{3-3-8}$$

First, we prove that (3-3-8) is a sufficient condition for the existence of a code that satisfies the prefix condition. To construct such a code, we begin with a full binary tree of order $n = n_L$ that has 2^n terminal nodes and two nodes of order k stemming from each node of order $k - 1$, for each $k, 1 \leq k \leq n$. Let us select any node of order n_1 as the first code word C_1 . This choice eliminates 2^{n-n_1} terminal nodes (or the fraction 2^{-n_1} of the 2^n terminal nodes). From the remaining available nodes of order n_2 , we select one node for the second code word C_2 . This choice eliminates 2^{n-n_2} terminal nodes (or the fraction 2^{-n_2} of the 2^n terminal nodes). This process continues until the last code word is assigned at terminal node $n = n_L$. Since, at the node of order $j < L$, the fraction of the number of terminal nodes eliminated is

$$\sum_{k=1}^j 2^{-n_k} < \sum_{k=1}^L 2^{-n_k} \leq 1$$

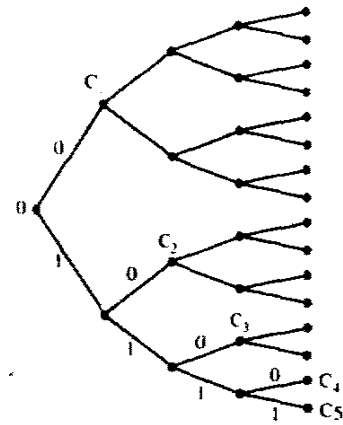


FIGURE 3-3-3 Construction of a binary tree code embedded in a full tree.

there is always a node of order $k > j$ available to be assigned to the next code word. Thus, we have constructed a code tree that is embedded in the full tree of 2^n nodes as illustrated in Fig. 3-3-3, for a tree having 16 terminal nodes and a source output consisting of five letters with $n_1 = 1$, $n_2 = 2$, $n_3 = 3$, and $n_4 = n_5 = 4$.

To prove that (3-3-8) is a necessary condition, we observe that in the code tree of order $n = n_L$, the number of terminal nodes eliminated from the total number of 2^n terminal nodes is

$$\sum_{k=1}^L 2^{n-n_k} \leq 2^n$$

Hence,

$$\sum_{k=1}^L 2^{-n_k} \leq 1$$

and the proof of (3-3-8) is complete.

The Kraft inequality may be used to prove the following (noiseless) source coding theorem, which applies to codes that satisfy the prefix condition.

Source Coding Theorem II

Let X be the ensemble of letters from a DMS with finite entropy $H(X)$, and output letters x_k , $1 \leq k \leq L$, with corresponding probabilities of occurrence p_k , $1 \leq k \leq L$. It is possible to construct a code that satisfies the prefix condition and has an average length \bar{R} that satisfies the inequalities

$$H(X) \leq \bar{R} < H(X) + 1 \tag{3-3-9}$$

To establish the lower bound in (3-3-9), we note that for code words that have length n_k , $1 \leq k \leq L$, the difference $H(X) - \bar{R}$ may be expressed as

$$\begin{aligned} H(X) - \bar{R} &= \sum_{k=1}^L p_k \log_2 \frac{1}{p_k} - \sum_{k=1}^L p_k n_k \\ &= \sum_{k=1}^L p_k \log_2 \frac{2^{-n_k}}{p_k} \end{aligned} \quad (3-3-10)$$

Use of the inequality $\ln x \leq x - 1$ in (3-3-10) yields

$$\begin{aligned} H(X) - \bar{R} &\leq (\log_2 e) \sum_{k=1}^L p_k \left(\frac{2^{-n_k}}{p_k} - 1 \right) \\ &\leq (\log_2 e) \left(\sum_{k=1}^L 2^{-n_k} - 1 \right) \leq 0 \end{aligned}$$

where the last inequality follows from the Kraft inequality. Equality holds if and only if $p_k = 2^{-n_k}$ for $1 \leq k \leq L$.

The upper bound in (3-3-9) may be established under the constraint that n_k , $1 \leq k \leq L$, are integers, by selecting the $\{n_k\}$ such that $2^{-n_k} \leq p_k < 2^{-n_k+1}$. But if the terms $p_k \geq 2^{-n_k}$ are summed over $1 \leq k \leq L$, we obtain the Kraft inequality, for which we have demonstrated that there exists a code that satisfies the prefix condition. On the other hand, if we take the logarithm of $p_k < 2^{-n_k+1}$, we obtain

$$\log p_k < -n_k + 1$$

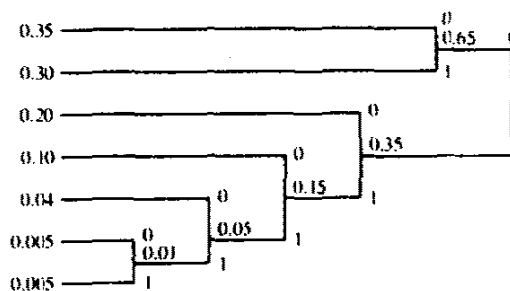
or, equivalently,

$$n_k < 1 - \log p_k \quad (3-3-11)$$

If we multiply both sides of (3-3-11) by p_k and sum over $1 \leq k \leq L$, we obtain the desired upper bound given in (3-3-9). This completes the proof of (3-3-9).

We have now established that variable length codes that satisfy the prefix condition are efficient source codes for any DMS with source symbols that are not equally probable. Let us now describe an algorithm for constructing such codes.

Huffman Coding Algorithm Huffman (1952) devised a variable-length encoding algorithm, based on the source letter probabilities $P(x_i)$, $i = 1, 2, \dots, L$. This algorithm is optimum in the sense that the average number of binary digits required to represent the source symbols is a minimum, subject to the constraint that the code words satisfy the prefix condition, as defined above, which allows the received sequence to be uniquely and instantaneously decodable. We illustrate this encoding algorithm by means of two examples.



Letter	Probability	Self-information	Code
x_1	0.35	1.5146	00
x_2	0.30	1.7370	01
x_3	0.20	2.3219	10
x_4	0.10	3.3219	110
x_5	0.04	4.6439	1110
x_6	0.005	7.6439	11110
x_7	0.005	7.6439	11111

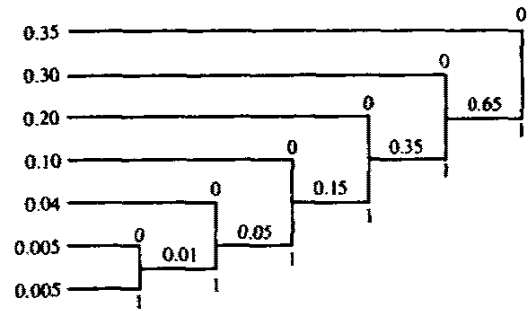
$H(X) = 2.11$ $\bar{R} = 2.21$

FIGURE 3-3-4 An example of variable-length-source encoding for a DMS.

Example 3-3-1

Consider a DMS with seven possible symbols x_1, x_2, \dots, x_7 having the probabilities of occurrence illustrated in Fig. 3-3-4. We have ordered the source symbols in decreasing order of the probabilities, i.e., $P(x_1) > P(x_2) > \dots > P(x_7)$. We begin the encoding process with the two least probable symbols x_6 and x_7 . These two symbols are tied together as shown in Fig. 3-3-4, with the upper branch assigned a 0 and the lower branch assigned a 1. The probabilities of these two branches are added together at the node where the two branches meet to yield the probability 0.01. Now we have the source symbols x_1, \dots, x_5 plus a new symbol, say x'_6 , obtained by combining x_6 and x_7 . The next step is to join the two least probable symbols from the set $x_1, x_2, x_3, x_4, x_5, x'_6$. These are x_5 and x'_6 , which have a combined probability of 0.05. The branch from x_5 is assigned a 0 and the branch from x'_6 is assigned a 1. This procedure continues until we exhaust the set of possible source letters. The result is a code tree with branches that contain the desired code words. The code words are obtained by beginning at the rightmost node in the tree and proceeding to the left. The resulting code words are listed in Fig. 3-3-4. The average number of binary digits per symbol for this code is $\bar{R} = 2.21$ bits/symbol. The entropy of the source is 2.11 bits/symbol.

We make the observation that the code is not necessarily unique. For example, at the next to the last step in the encoding procedure, we have a tie between x_1 and x'_3 , since these symbols are equally probable. At this point, we chose to pair x_1 with x_2 . An alternative is to pair x_2 with x'_3 . If we choose this



Letter	Code
x_1	0
x_2	10
x_3	110
x_4	1110
x_5	11110
x_6	111110
x_7	111111

$\bar{R} = 2.21$

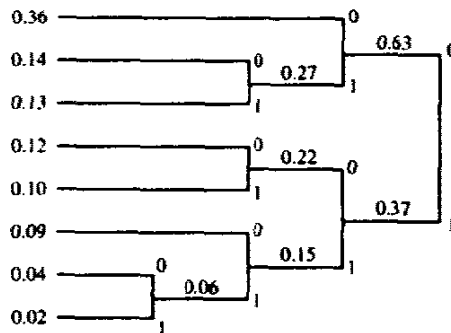
FIGURE 3-3-5 An alternative code for the DMS in Example 3-3-1.

pairing, the resulting code is illustrated in Fig. 3-3-5. The average number of bits per source symbol for this code is also 2.21. Hence, the resulting codes are equally efficient. Secondly, the assignment of a 0 to the upper branch and a 1 to the lower (less probable) branch is arbitrary. We may simply reverse the assignment of a 0 and 1 and still obtain an efficient code satisfying the prefix condition.

Example 3-3-2

As a second example, let us determine the Huffman code for the output of a DMS illustrated in Fig. 3-3-6. The entropy of this source is $H(X) = 2.63$ bits/symbol. The Huffman code as illustrated in Fig. 3-3-6 has an average length of $\bar{R} = 2.70$ bits/symbol. Hence, its efficiency is 0.97.

FIGURE 3-3-6 Huffman code for Example 3-3-2.



Letter	Code
x_1	00
x_2	010
x_3	011
x_4	100
x_5	101
x_6	110
x_7	1110
x_8	1111

$H(X) = 2.63 \quad \bar{R} = 2.70$

The variable-length encoding (Huffman) algorithm described in the above examples generates a prefix code having an \bar{R} that satisfies (3-3-9). However, instead of encoding on a symbol-by-symbol basis, a more efficient procedure is to encode blocks of J symbols at a time. In such a case, the bounds in (3-3-9) of source coding theorem II become

$$JH(X) \leq \bar{R}_J < JH(X) + 1, \quad (3-3-12)$$

since the entropy of a J -symbol block from a DMS is $JH(X)$, and \bar{R}_J is the average number of bits per J -symbol blocks. If we divide (3-3-12) by J , we obtain

$$H(X) \leq \frac{\bar{R}_J}{J} < H(X) + \frac{1}{J} \quad (3-3-13)$$

where $\bar{R}_J/J = \bar{R}$ is the average number of bits per source symbol. Hence \bar{R} can be made as close to $H(X)$ as desired by selecting J sufficiently large.

Example 3-3-3

The output of a DMS consists of letters x_1 , x_2 , and x_3 with probabilities 0.45, 0.35, and 0.20, respectively. The entropy of this source is $H(X) = 1.518$ bits/symbol. The Huffman code for this source, given in Table 3-3-2, requires $\bar{R}_1 = 1.55$ bits/symbol and results in an efficiency of 97.9%. If pairs of symbols are encoded by means of the Huffman algorithm, the resulting code is as given in Table 3-3-3. The entropy of the source output for pairs of letters is $2H(X) = 3.036$ bits/symbol pair. On the other hand, the Huffman code requires $\bar{R}_2 = 3.0675$ bits/symbol pair. Thus, the efficiency of the encoding increases to $2H(X)/\bar{R}_2 = 0.990$ or, equivalently, to 99.0%.

In summary, we have demonstrated that efficient encoding for a DMS may be done on a symbol-by-symbol basis using a variable-length code based on

TABLE 3-3-2 HUFFMAN CODE FOR EXAMPLE 3-3-3

Letter	Probability	Self-information	Code
x_1	0.45	1.156	1
x_2	0.35	1.520	00
x_3	0.20	2.330	01
$H(X) = 1.518$ bits/letter $\bar{R}_1 = 1.55$ bits/letter Efficiency = 97.9%			

TABLE 3-3-3 HUFFMAN CODE FOR ENCODING PAIRS OF LETTERS

Letter pair	Probability	Self-information	Code
x_1x_1	0.2025	2.312	10
x_1x_2	0.1575	2.676	001
x_2x_1	0.1575	2.676	010
x_2x_2	0.1225	3.039	011
x_1x_3	0.09	3.486	111
x_3x_1	0.09	3.486	0000
x_2x_3	0.07	3.850	0001
x_3x_2	0.07	3.850	1100
x_3x_3	0.04	4.660	1101
$2H(X) = 3.036$ bits/letter pair $\bar{R}_2 = 3.0675$ bits/letter pair $\frac{1}{2}\bar{R}_2 = 1.534$ bits/letter Efficiency = 99.0%			

the Huffman algorithm. Furthermore, the efficiency of the encoding procedure is increased by encoding blocks of J symbols at a time. Thus, the output of a DMS with entropy $H(X)$ may be encoded by a variable-length code with an average number of bits per source letter that approaches $H(X)$ as closely as desired.

3-3-2 Discrete Stationary Sources

In the previous section, we described the efficient encoding of the output of a DMS. In this section, we consider discrete sources for which the sequence of output letters is statistically dependent. We limit our treatment to sources that are statistically stationary.

Let us evaluate the entropy of any sequence of letters from a stationary source. From the definition in (3-2-13) and the result given in (3-2-15), the entropy of a block of random variables $X_1X_2 \cdots X_k$ is

$$H(X_1X_2 \cdots X_k) = \sum_{i=1}^k H(X_i | X_1X_2 \cdots X_{i-1}) \quad (3-3-14)$$

where $H(X_i | X_1X_2 \cdots X_{i-1})$ is the conditional entropy of the i th symbol from the source given the previous $i-1$ symbols. The entropy per letter for the k -symbol block is defined as

$$H_k(X) = \frac{1}{k} H(X_1X_2 \cdots X_k) \quad (3-3-15)$$

We define the information content of a stationary source as the entropy per letter in (3-3-15) in the limit as $k \rightarrow \infty$. That is,

$$H_\infty(X) = \lim_{k \rightarrow \infty} H_k(X) = \lim_{k \rightarrow \infty} \frac{1}{k} H(X_1X_2 \cdots X_k) \quad (3-3-16)$$

The existence of this limit is established below.

As an alternative, we may define the entropy per letter from the source in terms of the conditional entropy $H(X_k | X_1 X_2 \cdots X_{k-1})$ in the limit as k approaches infinity. Fortunately, this limit also exists and is identical to the limit in (3-3-16). That is,

$$H_z(X) = \lim_{k \rightarrow \infty} H(X_k | X_1 X_2 \cdots X_{k-1}) \quad (3-3-17)$$

This result is also established below. Our development follows the approach in Gallager (1968).

First, we show that

$$H(X_k | X_1 X_2 \cdots X_{k-1}) \leq H(X_{k-1} | X_1 X_2 \cdots X_{k-2}) \quad (3-3-18)$$

for $k \geq 2$. From our previous result that conditioning on a random variable cannot increase entropy, we have

$$H(X_k | X_1 X_2 \cdots X_{k-1}) \leq H(X_k | X_2 X_3 \cdots X_{k-1}) \quad (3-3-19)$$

From the stationarity of the source, we have

$$H(X_k | X_2 X_3 \cdots X_{k-1}) = H(X_{k-1} | X_1 X_2 \cdots X_{k-2}) \quad (3-3-20)$$

Hence, (3-3-18) follows immediately. This result demonstrates that $H(X_k | X_1 X_2 \cdots X_{k-1})$ is a nonincreasing sequence in k .

Second, we have the result

$$H_k(X) \geq H(X_k | X_1 X_2 \cdots X_{k-1}) \quad (3-3-21)$$

which follows immediately from (3-3-14) and (3-3-15) and the fact that the last term in the sum of (3-3-14) is a lower bound on each of the other $k - 1$ terms.

Third, from the definition of $H_k(X)$, we may write

$$\begin{aligned} H_k(X) &= \frac{1}{k} [H(X_1 X_2 \cdots X_{k-1}) + H(X_k | X_1 \cdots X_{k-1})] \\ &= \frac{1}{k} [(k-1)H_{k-1}(X) + H(X_k | X_1 \cdots X_{k-1})] \\ &\leq \frac{k-1}{k} H_{k-1}(X) + \frac{1}{k} H_k(X) \end{aligned}$$

which reduces to

$$H_k(X) \leq H_{k-1}(X) \quad (3-3-22)$$

Hence, $H_k(X)$ is a nonincreasing sequence in k .

Since $H_k(X)$ and the conditional entropy $H(X_k | X_1 \cdots X_{k-1})$ are both

nonnegative and nonincreasing with k , both limits must exist. Their limiting forms can be established by using (3-3-14) and (3-3-15) to express $H_{k+j}(X)$ as

$$\begin{aligned} H_{k+j}(X) &= \frac{1}{k+j} H(X_1 X_2 \cdots X_{k-1}) \\ &\quad + \frac{1}{k+j} [H(X_k | X_1 \cdots X_{k-1}) + H(X_{k+1} | X_1 \cdots X_k) \\ &\quad + \cdots + H(X_{k+j} | X_1 \cdots X_{k+j-1})] \end{aligned}$$

Since the conditional entropy is nonincreasing, the first term in the square brackets serves as an upper bound on the other terms. Hence,

$$H_{k+j}(X) \leq \frac{1}{k+j} H(X_1 X_2 \cdots X_{k-1}) + \frac{j+1}{k+j} H(X_k | X_1 X_2 \cdots X_{k-1}) \quad (3-3-23)$$

For a fixed k , the limit of (3-3-23) as $j \rightarrow \infty$ yields

$$H_\infty(X) \leq H(X_k | X_1 X_2 \cdots X_{k-1}) \quad (3-3-24)$$

But (3-3-24) is valid for all k ; hence, it is valid for $k \rightarrow \infty$. Therefore,

$$H_\infty(X) \leq \lim_{k \rightarrow \infty} H(X_k | X_1 X_2 \cdots X_{k-1}) \quad (3-3-25)$$

On the other hand, from (3-3-21), we obtain in the limit as $k \rightarrow \infty$,

$$H_\infty(X) \geq \lim_{k \rightarrow \infty} H(X_k | X_1 X_2 \cdots X_{k-1}) \quad (3-3-26)$$

which establishes (3-3-17).

Now suppose we have a discrete stationary source that emits J letters with $H_j(X)$ as the entropy per letter. We can encode the sequence of J letters with a variable-length Huffman code that satisfies the prefix condition by following the procedure described in the previous section. The resulting code has an average number of bits for the J -letter block that satisfies the condition

$$H(X_1 \cdots X_J) \leq \bar{R}_J < H(X_1 \cdots X_J) + 1 \quad (3-3-27)$$

By dividing each term of (3-3-27) by J , we obtain the bounds on the average number $\bar{R} = \bar{R}_J/J$ of bits per source letter as

$$H_j(X) \leq \bar{R} < H_j(X) + \frac{1}{J} \quad (3-3-28)$$

By increasing the block size J , we can approach $H_j(X)$ arbitrarily closely, and in the limit as $J \rightarrow \infty$, \bar{R} satisfies

$$H_\infty(X) \leq \bar{R} < H_\infty(X) + \epsilon \quad (3-3-29)$$

where ϵ approaches zero as $1/J$. Thus, efficient encoding of stationary sources is accomplished by encoding large blocks of symbols into code words. We should emphasize, however, that the design of the Huffman code requires knowledge of the joint pdf for the J -symbol blocks.

ie Lempel–Ziv Algorithm

From our preceding discussion, we have observed that the Huffman coding algorithm yields optimal source codes in the sense that the code words satisfy the prefix condition and the average block length is a minimum. To design a Huffman code for a DMS, we need to know the probabilities of occurrence of all the source letters. In the case of a discrete source with memory, we must know the joint probabilities of blocks of length $n \geq 2$. However, in practice, the statistics of a source output are often unknown. In principle, it is possible to estimate the probabilities of the discrete source output by simply observing a long information sequence emitted by the source and obtaining the probabilities empirically. Except for the estimation of the marginal probabilities $\{p_k\}$, corresponding to the frequency of occurrence of the individual source output letters, the computational complexity involved in estimating joint probabilities is extremely high. Consequently, the application of the Huffman coding method to source coding for many real sources with memory is generally impractical.

In contrast to the Huffman coding algorithm, the Lempel–Ziv source coding algorithm is designed to be independent of the source statistics. Hence, the Lempel–Ziv algorithm belongs to the class of *universal source coding algorithms*. It is a variable-to-fixed-length algorithm, where the encoding is performed as described below.

In the Lempel–Ziv algorithm, the sequence at the output of the discrete source is parsed into variable-length blocks, which are called *phrases*. A new phrase is introduced every time a block of letters from the source differs from some previous phrase in the last letter. The phrases are listed in a dictionary, which stores the location of the existing phrases. In encoding a new phrase, we simply specify the location of the existing phrase in the dictionary and append the new letter.

As an example, consider the binary sequence

10101101001001110101000011001110101100011011

Parsing the sequence as described above produces the following phrases:

1, 0, 10, 11, 01, 00, 100, 111, 010, 1000, 011, 001, 110, 101, 10001, 1011

We observe that each phrase in the sequence is a concatenation of a previous phrase with a new output letter from the source. To encode the phrases, we

TABLE 3-3-4 DICTIONARY FOR LEMPEL-ZIV ALGORITHM

	Dictionary location	Dictionary contents	Code word
1	0001	1	00001
2	0010	0	00000
3	0011	10	00010
4	0100	11	00011
5	0101	01	00101
6	0110	00	00100
7	0111	100	00110
8	1000	111	01001
9	1001	010	01010
10	1010	1000	01110
11	1011	011	01011
12	1100	001	01101
13	1101	110	01000
14	1110	101	00111
15	1111	10001	10101
16		1011	11101

construct a dictionary as shown in Table 3-3-4. The dictionary locations are numbered consecutively, beginning with 1 and counting up, in this case to 16, which is the number of phrases in the sequence. The different phrases corresponding to each location are also listed, as shown. The codewords are determined by listing the dictionary location (in binary form) of the previous phrase that matches the new phrase in all but the last location. Then, the new output letter is appended to the dictionary location of the previous phrase. Initially, the location 0000 is used to encode a phrase that has not appeared previously.

The source decoder for the code constructs an identical table at the receiving end of the communication system and decodes the received sequence accordingly.

It should be observed that the table encoded 44 source bits into 16 code words of five bits each, resulting in 80 coded bits. Hence, the algorithm provided no data compression at all. However, the inefficiency is due to the fact that the sequence we have considered is very short. As the sequence is increased in length, the encoding procedure becomes more efficient and results in a compressed sequence at the output of the source.

How do we select the overall length of the table? In general, no matter how large the table is, it will eventually overflow. To solve the overflow problem, the source encoder and source decoder must agree to remove phrases from the respective dictionaries that are not useful and substitute new phrases in their place.

The Lempel–Ziv algorithm is widely used in the compression of computer files. The “compress” and “uncompress” utilities under the UNIX[®] operating system and numerous algorithms under the MS-DOS operating system are implementations of various versions of this algorithm.

3-4 CODING FOR ANALOG SOURCES—OPTIMUM QUANTIZATION

As indicated in Section 3-1, an analog source emits a message waveform $x(t)$ that is a sample function of a stochastic process $X(t)$. When $X(t)$ is a bandlimited, stationary stochastic process, the sampling theorem allows us to represent $X(t)$ by a sequence of uniform samples taken at the Nyquist rate.

By applying the sampling theorem, the output of an analog source is converted to an equivalent discrete-time sequence of samples. The samples are then quantized in amplitude and encoded. One type of simple encoding is to represent each discrete amplitude level by a sequence of binary digits. Hence, if we have L levels, we need $R = \log_2 L$ bits per sample if L is a power of 2, or $R = \lfloor \log_2 L \rfloor + 1$ if L is not a power of 2. On the other hand, if the levels are not equally probable, and the probabilities of the output levels are known, we may use Huffman coding (also called *entropy coding*) to improve the efficiency of the encoding process.

Quantization of the amplitudes of the sampled signal results in data compression but it also introduces some distortion of the waveform or a loss of signal fidelity. The minimization of this distortion is considered in this section. Many of the results given in this section apply directly to a discrete-time, continuous amplitude, memoryless gaussian source. Such a source serves as a good model for the residual error in a number of source coding methods described in Section 3-5.

3-4-1 Rate-Distortion Function

Let us begin the discussion of signal quantization by considering the distortion introduced when the samples from the information source are quantized to a fixed number of bits. By the term “distortion,” we mean some measure of the difference between the actual source samples $\{x_k\}$ and the corresponding quantized values \tilde{x}_k , which we denote by $d\{x_k, \tilde{x}_k\}$. For example, a commonly used distortion measure is the *squared-error distortion*, defined as

$$d(x_k, \tilde{x}_k) = (x_k - \tilde{x}_k)^2 \quad (3-4-1)$$

which is used to characterize the quantization error in PCM in Section 3-5-1. Other distortion measures may take the general form

$$d(x_k, \tilde{x}_k) = |x_k - \tilde{x}_k|^p \quad (3-4-2)$$

where p takes values from the set of positive integers. The case $p = 2$ has the advantage of being mathematically tractable.

If $d(x_k, \tilde{x}_k)$ is the distortion measure per letter, the distortion between a sequence of n samples \mathbf{X}_n and the corresponding n quantized values $\tilde{\mathbf{X}}_n$ is the average over the n source output samples, i.e.,

$$d(\mathbf{X}_n, \tilde{\mathbf{X}}_n) = \frac{1}{n} \sum_{k=1}^n d(x_k, \tilde{x}_k) \quad (3-4-3)$$

The source output is a random process, and, hence, the n samples in \mathbf{X}_n are random variables. Therefore, $d(\mathbf{X}_n, \tilde{\mathbf{X}}_n)$ is a random variable. Its expected value is defined as the distortion D , i.e.,

$$D = E[d(\mathbf{X}_n, \tilde{\mathbf{X}}_n)] = \frac{1}{n} \sum_{k=1}^n E[d(x_k, \tilde{x}_k)] = E[d(x, \tilde{x})] \quad (3-4-4)$$

where the last step follows from the assumption that the source output process is stationary.

Now suppose we have a memoryless source with a continuous-amplitude output \mathbf{X} that has a pdf $p(x)$, a quantized amplitude output alphabet $\tilde{\mathbf{X}}$, and a per letter distortion measure $d(x, \tilde{x})$, where $x \in \mathbf{X}$ and $\tilde{x} \in \tilde{\mathbf{X}}$. Then, the minimum rate in bits per source output that is required to represent the output \mathbf{X} of the memoryless source with a distortion less than or equal to D is called the *rate-distortion function* $R(D)$ and is defined as

$$R(D) = \min_{p(\tilde{x}|x): E[d(\mathbf{X}, \tilde{\mathbf{X}})] \leq D} I(\mathbf{X}, \tilde{\mathbf{X}}) \quad (3-4-5)$$

where $I(\mathbf{X}; \tilde{\mathbf{X}})$ is the average mutual information between \mathbf{X} and $\tilde{\mathbf{X}}$. In general, the rate $R(D)$ decreases as D increases or, conversely, $R(D)$ increases as D decreases.

One interesting model of a continuous-amplitude, memoryless information source is the gaussian source model. In this case, Shannon proved the following fundamental theorem on the rate-distortion function.

Theorem: Rate-Distortion Function for a Memoryless Gaussian Source (Shannon, 1959a)

The minimum information rate necessary to represent the output of a discrete-time, continuous-amplitude memoryless gaussian source based on a mean-square-error distortion measure per symbol (single letter distortion measure) is

$$R_s(D) = \begin{cases} \frac{1}{2} \log_2 (\sigma_x^2 / D) & (0 \leq D \leq \sigma_x^2) \\ 0 & (D > \sigma_x^2) \end{cases} \quad (3-4-6)$$

where σ_x^2 is the variance of the gaussian source output.

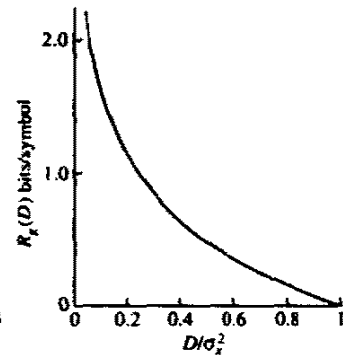


FIGURE 3-4-1 Rate distortion function for a continuous-amplitude memoryless gaussian source.

We should note that (3-4-6) implies that no information need be transmitted when the distortion $D \geq \sigma_x^2$. Specifically, $D = \sigma_x^2$ can be obtained by using zeros in the reconstruction of the signal. For $D > \sigma_x^2$, we can use statistically independent, zero-mean gaussian noise samples with a variance of $D - \sigma_x^2$ for the reconstruction. $R_g(D)$ is plotted in Fig. 3-4-1.

The rate distortion function $R(D)$ of a source is associated with the following basic source coding theorem in information theory.

Theorem: Source Coding with a Distortion Measure (Shannon, 1959a)

There exists an encoding scheme that maps the source output into code words such that for any given distortion D , the minimum rate $R(D)$ bits per symbol (sample) is sufficient to reconstruct the source output with an average distortion that is arbitrarily close to D .

It is clear, therefore, that the rate distortion function $R(D)$ for any source represents a lower bound on the source rate that is possible for a given level of distortion.

Let us return to the result in (3-4-6) for the rate distortion function of a memoryless gaussian source. If we reverse the functional dependence between D and R , we may express D in terms of R as

$$D_g(R) = 2^{-2R} \sigma_x^2 \quad (3-4-7)$$

This function is called the *distortion-rate function* for the discrete-time, memoryless gaussian source.

When we express the distortion in (3-4-7) in dB, we obtain

$$10 \log_{10} D_g(R) = -6R + 10 \log_{10} \sigma_x^2 \quad (3-4-8)$$

Note that the mean square distortion decreases at a rate of 6 dB/bit.

Explicit results on the rate distortion functions for memoryless non-gaussian sources are not available. However, there are useful upper and lower bounds

on the rate distortion function for any discrete-time, continuous-amplitude, memoryless source. An upper bound is given by the following theorem.

Theorem: Upper Bound on $R(D)$

The rate-distortion function of a memoryless, continuous-amplitude source with zero mean and finite variance σ_x^2 with respect to the mean-square-error distortion measure is upper bounded as

$$R(D) \leq \frac{1}{2} \log_2 \frac{\sigma_x^2}{D} \quad (0 \leq D \leq \sigma_x^2) \quad (3-4-9)$$

A proof of this theorem is given by Berger (1971). It implies that the gaussian source requires the maximum rate among all other sources for a specified level of mean square distortion. Thus, the rate distortion $R(D)$ of any continuous-amplitude, memoryless source with zero mean and finite variance σ_x^2 satisfies the condition $R(D) \leq R_g(D)$. Similarly, the distortion-rate function of the same source satisfies the condition

$$D(R) \leq D_g(R) = 2^{-2R} \sigma_x^2 \quad (3-4-10)$$

A lower bound on the rate-distortion function also exists. This is called the *Shannon lower bound* for a mean-square-error distortion measure, and is given as

$$R^*(D) = H(X) - \frac{1}{2} \log_2 2\pi e D \quad (3-4-11)$$

where $H(X)$ is the differential entropy of the continuous-amplitude, memoryless source. The distortion-rate function corresponding to (3-4-11) is

$$D^*(R) = \frac{1}{2\pi e} 2^{-2[R - H(X)]} \quad (3-4-12)$$

Therefore, the rate-distortion function for any continuous-amplitude, memoryless source is bounded from above and below as

$$R^*(D) \leq R(D) \leq R_g(D) \quad (3-4-13)$$

and the corresponding distortion-rate function is bounded as

$$D^*(R) \leq D(R) \leq D_g(R) \quad (3-4-14)$$

The differential entropy of the memoryless gaussian source is

$$H_g(X) = \frac{1}{2} \log_2 2\pi e \sigma_x^2 \quad (3-4-15)$$

so that the lower bound $R^*(D)$ in (3-4-11) reduces to $R_g(D)$. Now, if we

express $D^*(R)$ in terms of decibels and normalize it by setting $\sigma_x^2 = 1$ [or dividing $D^*(R)$ by σ_x^2], we obtain from (3-4-12)

$$10 \log_{10} D^*(R) = -6R - 6[H_g(X) - H(X)] \quad (3-4-16)$$

or, equivalently,

$$\begin{aligned} 10 \log_{10} \frac{D_g(R)}{D^*(R)} &= 6[H_g(X) - H(X)] \text{ dB} \\ &= 6[R_g(D) - R^*(D)] \text{ dB} \end{aligned} \quad (3-4-17)$$

The relations in (3-4-16) and (3-4-17) allow us to compare the lower bound in the distortion with the upper bound which is the distortion for the gaussian source. We note that $D^*(R)$ also decreases at -6 dB/bit. We should also mention that the differential entropy $H(X)$ is upper-bounded by $H_g(X)$, as shown by Shannon (1948b).

Table 3-4-1 lists four pdfs that are models commonly used for source signal distributions. The table shows the differential entropies, the differences in rates in bits/sample, and the difference in distortion between the upper and lower bounds. Note that the gamma pdf shows the greatest deviation from the gaussian. The Laplacian pdf is the most similar to the gaussian, and the uniform pdf ranks second of the pdfs shown in the table. These results provide some benchmarks on the difference between the upper and lower bounds on distortion and rate.

Before concluding this section, let us consider a band-limited gaussian source with spectral density

$$\Phi(f) = \begin{cases} \sigma_x^2/2W & (|f| \leq W) \\ 0 & (|f| > W) \end{cases} \quad (3-4-18)$$

When the output of this source is sampled at the Nyquist rate, the samples are uncorrelated and, since the source is gaussian, they are also statistically

TABLE 3-4-2

TABLE 3-4-1 DIFFERENTIAL ENTROPIES AND RATE DISTORTION COMPARISONS OF FOUR COMMON PDFs FOR SIGNAL MODELS

pdf	$p(x)$	$H(X)$	$R_g(D) - R^*(D)$ (bits/sample)	$D_g(R) - D^*(R)$ (dB)
Gaussian	$\frac{1}{\sqrt{2\pi}\sigma_x} e^{-x^2/2\sigma_x^2}$	$\frac{1}{2} \log_2(2\pi e\sigma_x^2)$	0	0
Uniform	$\frac{1}{2\sqrt{3}\sigma_x}, x \leq \sqrt{3}\sigma_x$	$\frac{1}{2} \log_2(12\sigma_x^2)$	0.255	1.53
Laplacian	$\frac{1}{\sqrt{2}\sigma_x} e^{-\sqrt{2} x /\sigma_x}$	$\frac{1}{2} \log_2(2e^2\sigma_x^2)$	0.104	0.62
Gamma	$\frac{\sqrt[4]{3}}{\sqrt{8\pi\sigma_x} x } e^{-\sqrt{3} x /2\sigma_x}$	$\frac{1}{2} \log_2(4\pi e^{0.423}\sigma_x^2/3)$	0.709	4.25

independent. Hence, the equivalent discrete-time gaussian source is memoryless. The rate-distortion function for each sample is given by (3-4-6). Therefore, the rate-distortion function for the band-limited white gaussian source in bits/s is

$$R_g(D) = W \log_2 \frac{\sigma_x^2}{D} \quad (0 \leq D \leq \sigma_x^2) \quad (3-4-19)$$

The corresponding distortion-rate function is

$$D_g(R) = 2^{-R/W} \sigma_x^2 \quad (3-4-20)$$

which, when expressed in decibels and normalized by σ_x^2 , becomes

$$10 \log D_g(R)/\sigma_x^2 = -3R/W \quad (3-4-21)$$

The more general case in which the gaussian process is neither white nor band-limited has been treated by Gallager (1968) and Goblick and Holsinger (1967).

3-4-2 Scalar Quantization

In source encoding, the quantizer can be optimized if we know the probability density function of the signal amplitude at the input to the quantizer. For example, suppose that the sequence $\{x_n\}$ at the input to the quantizer has a pdf $p(x)$ and let $L = 2^R$ be the desired number of levels. We wish to design the optimum scalar quantizer that minimizes some function of the quantization error $q = \bar{x} - x$, where \bar{x} is the quantized value of x . To elaborate, suppose that $f(\bar{x} - x)$ denotes the desired function of the error. Then, the distortion resulting from quantization of the signal amplitude is

$$D = \int_{-\infty}^{\infty} f(\bar{x} - x) p(x) dx \quad (3-4-22)$$

In general, an optimum quantizer is one that minimizes D by optimally selecting the output levels and the corresponding input range of each output level. This optimization problem has been considered by Lloyd (1982) and Max (1960), and the resulting optimum quantizer is usually called the *Lloyd-Max quantizer*.

For a uniform quantizer, the output levels are specified as $\bar{x}_k = \frac{1}{2}(2k - 1)\Delta$, corresponding to an input signal amplitude in the range $(k - 1)\Delta \leq x < k\Delta$, where Δ is the step size. When the uniform quantizer is symmetric with an even number of levels, the average distortion in (3-4-22) may be expressed as

$$D = 2 \sum_{k=1}^{L/2-1} \int_{(k-1)\Delta}^{k\Delta} f(\frac{1}{2}(2k - 1)\Delta - x) p(x) dx + 2 \int_{(L/2-1)\Delta}^{\infty} f(\frac{1}{2}(2k - 1)\Delta - x) p(x) dx \quad (3-4-23)$$

TABLE 3-4-2 OPTIMUM STEP SIZES FOR UNIFORM QUANTIZATION OF A GAUSSIAN RANDOM VARIABLE

Number of output levels	Optimum step size Δ_{opt}	Minimum MSE D_{min}	$10 \log D_{min}$ (dB)
2	1.596	0.3634	-4.4
4	0.9957	0.1188	-9.25
8	0.5860	0.03744	-14.27
16	0.3352	0.01154	-19.38
32	0.1881	0.00349	-24.57

In this case, the minimization of D is carried out with respect to the step-size parameter Δ . By differentiating D with respect to Δ , we obtain

$$\sum_{k=1}^{L/2-1} (2k-1) \int_{(k-1)\Delta}^{k\Delta} f(\frac{1}{2}(2k-1)\Delta - x)p(x) dx + (L-1) \int_{-(L/2-1)\Delta}^{\infty} f'(\frac{1}{2}(L-1)\Delta - x)p(x) dx = 0 \quad (3-4-24)$$

where $f'(x)$ denotes the derivative of $f(x)$.

By selecting the error criterion function $f(x)$, the solution of (3-4-24) for the optimum step size can be obtained numerically on a digital computer for any given pdf $p(x)$. For the mean-square-error criterion, for which $f(x) = x^2$, Max (1960) evaluated the optimum step size Δ_{opt} and the minimum mean square error when the pdf $p(x)$ is zero-mean gaussian with unit variance. Some of these results are given in Table 3-4-2. We observe that the minimum mean square distortion D_{min} decreases by a little more than 5 dB for each doubling of the number of levels L . Hence, each additional bit that is employed in a uniform quantizer with optimum step size Δ_{opt} for a gaussian-distributed signal amplitude reduces the distortion by more than 5 dB.

By relaxing the constraint that the quantizer be uniform, the distortion can be reduced further. In this case, we let the output level be $\bar{x} = \bar{x}_k$ when the input signal amplitude is in the range $x_{k-1} \leq x < x_k$. For an L -level quantizer, the end points are $x_0 = -\infty$ and $x_L = \infty$. The resulting distortion is

$$D = \sum_{k=1}^L \int_{x_{k-1}}^{x_k} f(\bar{x}_k - x)p(x) dx \quad (3-4-25)$$

which is now minimized by optimally selecting the $\{\bar{x}_k\}$ and $\{x_k\}$.

The necessary conditions for a minimum distortion are obtained by differentiating D with respect to the $\{x_k\}$ and $\{\bar{x}_k\}$. The result of this minimization is the pair of equations

$$f(\bar{x}_k - x_k) = f(\bar{x}_{k+1} - x_k), \quad k = 1, 2, \dots, L-1 \quad (3-4-26)$$

$$\int_{x_{k-1}}^{x_k} f'(\bar{x}_k - x)p(x) dx = 0, \quad k = 1, 2, \dots, L \quad (3-4-27)$$

TABLE 3-4-3 OPTIMUM FOUR-LEVEL
QUANTIZER FOR A GAUSSIAN
RANDOM VARIABLE

Level k	x_k	\bar{x}_k
1	-0.9816	-1.510
2	0.0	-0.4528
3	0.9816	0.4528
4	∞	1.510

$D_{\min} = 0.1175$
 $10 \log D_{\min} = -9.3 \text{ dB}$

As a special case, we again consider minimizing the mean square value of the distortion. In this case, $f(x) = x^2$ and, hence, (3-4-26) becomes

$$x_k = \frac{1}{2}(\bar{x}_k + \bar{x}_{k+1}), \quad k = 1, 2, \dots, L-1 \quad (3-4-28)$$

which is the midpoint between \bar{x}_k and \bar{x}_{k+1} . The corresponding equations determining $\{\bar{x}_k\}$ are

$$\int_{x_{k-1}}^{x_k} (\bar{x}_k - x)p(x) dx = 0, \quad k = 1, 2, \dots, L \quad (3-4-29)$$

Thus, \bar{x}_k is the centroid of the area of $p(x)$ between x_{k-1} and x_k . These equations may be solved numerically for any given $p(x)$.

Tables 3-4-3 and 3-4-4 give the results of this optimization obtained by Max

TABLE 3-4-4 OPTIMUM EIGHT-LEVEL
QUANTIZER FOR A GAUSSIAN
RANDOM VARIABLE (MAX, 1960)

Level k	x_k	\bar{x}_k
1	-1.748	-2.152
2	-1.050	-1.344
3	-0.5006	-0.7560
4	0	-0.2451
5	0.5006	0.2451
6	1.050	0.7560
7	1.748	1.344
8	∞	2.152

$D_{\min} = 0.03454$
 $10 \log D_{\min} = -14.62 \text{ dB}$

TABLE 3-4-5 COMPARISON OF OPTIMUM UNIFORM AND NONUNIFORM QUANTIZERS FOR A GAUSSIAN RANDOM VARIABLE (MAX, 1960; PAEZ AND GLISSON, 1972)

R (bits/sample)	$10 \log_{10} D_{min}$	
	Uniform (dB)	Nonuniform (dB)
1	-4.4	-4.4
2	-9.25	-9.30
3	-14.27	-14.62
4	-19.38	-20.22
5	-24.57	-26.02
6	-29.83	-31.89
7	-35.13	-37.81

(1960) for the optimum four-level and eight-level quantizers of a gaussian distributed signal amplitude having zero mean and unit variance. In Table 3-4-5, we compare the minimum mean square distortion of a uniform quantizer to that of a nonuniform quantizer for the gaussian-distributed signal amplitude. From the results of this table, we observe that the difference in the performance of the two types of quantizers is relatively small for small values of R (less than 0.5 dB for $R \leq 3$), but it increases as R increases. For example, at $R = 5$, the nonuniform quantizer is approximately 1.5 dB better than the uniform quantizer.

It is instructive to plot the minimum distortion as a function of the bit rate $R = \log_2 L$ bits per source sample (letter) for both the uniform and nonuniform quantizers. These curves are illustrated in Fig. 3-4-2. The functional dependence of the distortion D on the bit rate R may be expressed as $D(R)$, the distortion-rate function. We observe that the distortion-rate function for the optimum nonuniform quantizer falls below that of the optimum uniform quantizer.

Since any quantizer reduces a continuous amplitude source into a discrete amplitude source, we may treat the discrete amplitude as letters, say $\tilde{X} = \{\tilde{x}_k, 1 \leq k \leq L\}$, with associated probabilities $\{p_k\}$. If the signal amplitudes are statistically independent, the discrete source is memoryless and, hence, its entropy is

$$H(\tilde{X}) = - \sum_{k=1}^L p_k \log_2 p_k \quad (3-4-30)$$

For example, the optimum four-level nonuniform quantizer for the gaussian-distributed signal amplitude results in the probabilities $p_1 = p_4 = 0.1635$ for the two outer levels and $p_2 = p_3 = 0.3365$ for the two inner levels. The entropy for the discrete source is $H(\tilde{X}) = 1.911$ bits/letter. Hence, with

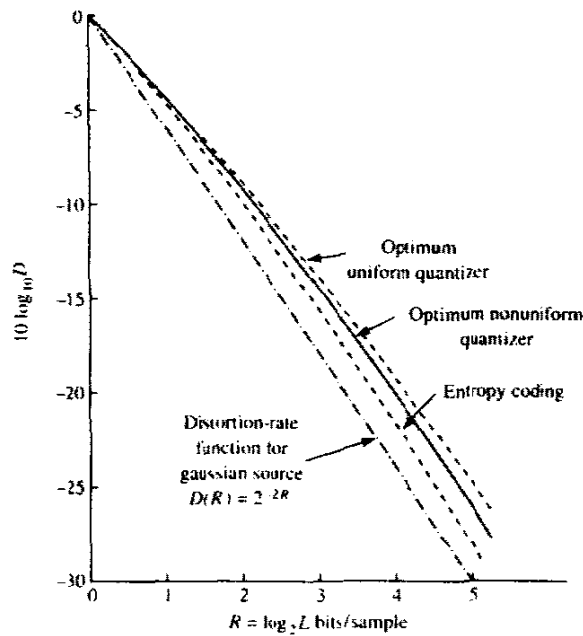


FIGURE 3-4-2 Distortion versus rate curves for discrete-time memoryless gaussian source.

entropy coding (Huffman coding) of blocks of output letters, we can achieve the minimum distortion of -9.30 dB with 1.911 bits/letter instead of 2 bits/letter. Max (1960) has given the entropy for the discrete source letters resulting from quantization. Table 3-4-6 lists the values of the entropy for the nonuniform quantizer. These values are also plotted in Fig. 3-4-2 and labeled *entropy coding*.

From this discussion, we conclude that the quantizer can be optimized when the pdf of the continuous source output is known. The optimum quantizer of $L = 2^R$ levels results in a minimum distortion of $D(R)$, where $R = \log_2 L$

TABLE 3-4-6 ENTROPY OF THE OUTPUT OF AN OPTIMUM NONUNIFORM QUANTIZER FOR A GAUSSIAN RANDOM VARIABLE (MAX, 1960)

\bar{R} (bits/sample)	Entropy (bits/letter)	Distortion $10 \log_{10} D_{min}$
1	1.0	-4.4
2	1.911	-9.30
3	2.825	-14.62
4	3.765	-20.22
5	4.730	-26.02

bits/sample. Thus, this distortion can be achieved by simply representing each quantized sample by R bits. However, more efficient encoding is possible. The discrete source output that results from quantization is characterized by a set of probabilities $\{p_k\}$ that can be used to design efficient variable-length codes for the source output (entropy coding). The efficiency of any encoding method can be compared with the distortion-rate function or, equivalently, the rate-distortion function for the discrete-time, continuous-amplitude source that is characterized by the given pdf.

If we compare the performance of the optimum nonuniform quantizer with the distortion-rate function, we find, for example, that at a distortion of -26 dB, entropy coding is 0.41 bits/sample more than the minimum rate given by (3-4-8), and simple block coding of each letter requires 0.68 bits/sample more than the minimum rate. We also observe that the distortion rate functions for the optimal uniform and nonuniform quantizers for the gaussian source approach the slope of -6 dB/bit asymptotically for large R .

3-4-3 Vector Quantization

In the previous section, we considered the quantization of the output signal from a continuous-amplitude source when the quantization is performed on a sample-by-sample basis, i.e., by scalar quantization. In this section, we consider the joint quantization of a block of signal samples or a block of signal parameters. This type of quantization is called *block* or *vector quantization*. It is widely used in speech coding for digital cellular systems.

A fundamental result of rate-distortion theory is that better performance can be achieved by quantizing vectors instead of scalars, even if the continuous-amplitude source is memoryless. If, in addition, the signal samples or signal parameters are statistically dependent, we can exploit the dependency by jointly quantizing blocks of samples or parameters and, thus, achieve an even greater efficiency (lower bit rate) compared with that which is achieved by scalar quantization.

The vector quantization problem may be formulated as follows. We have an n -dimensional vector $\mathbf{X} = [x_1 \ x_2 \ \cdots \ x_n]$ with real-valued, continuous-amplitude components $\{x_k, 1 \leq k \leq n\}$ that are described by a joint pdf $p(x_1, x_2, \dots, x_n)$. The vector \mathbf{X} is quantized into another n -dimensional vector $\tilde{\mathbf{X}}$ with components $\{\tilde{x}_k, 1 \leq k \leq n\}$. We express the quantization as $Q(\cdot)$, so that

$$\tilde{\mathbf{X}} = Q(\mathbf{X}) \quad (3-4-31)$$

where $\tilde{\mathbf{X}}$ is the output of the vector quantizer when the input vector is \mathbf{X} .

Basically, vector quantization of blocks of data may be viewed as a pattern recognition problem involving the classification of blocks of data into a discrete number of categories or *cells* in a way that optimizes some fidelity criterion, such as mean square distortion. For example, let us consider the quantization

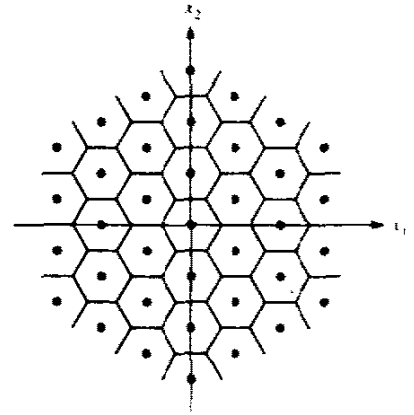


FIGURE 3-4-3 An example of quantization in two-dimensional space.

of two-dimensional vectors $\mathbf{X} = [x_1, x_2]$. The two-dimensional space is partitioned into cells as illustrated in Fig. 3-4-3, where we have arbitrarily selected hexagonal-shaped cells $\{C_k\}$. All input vectors that fall in cell C_k are quantized into the vector $\tilde{\mathbf{X}}_k$, which is shown in Fig. 3-4-3 as the center of the hexagon. In this example, there are $L = 37$ vectors, one for each of the 37 cells into which the two-dimensional space has been partitioned. We denote the set of possible output vectors as $\{\tilde{\mathbf{X}}_k, 1 \leq k \leq L\}$.

In general, quantization of the n -dimensional vector \mathbf{X} into an n -dimensional vector $\tilde{\mathbf{X}}$ introduces a quantization error or a distortion $d(\mathbf{X}, \tilde{\mathbf{X}})$. The average distortion over the set of input vectors \mathbf{X} is

$$\begin{aligned} D &= \sum_{k=1}^L P(\mathbf{X} \in C_k) E[d(\mathbf{X}, \tilde{\mathbf{X}}_k) | \mathbf{X} \in C_k] \\ &= \sum_{k=1}^L P(\mathbf{X} \in C_k) \int_{\mathbf{X} \in C_k} d(\mathbf{X}, \tilde{\mathbf{X}}_k) p(\mathbf{X}) d\mathbf{X} \end{aligned} \quad (3-4-32)$$

where $P(\mathbf{X} \in C_k)$ is the probability that the vector \mathbf{X} falls in the cell C_k and $p(\mathbf{X})$ is the joint pdf of the n random variables. As in the case of scalar quantization, we can minimize D by selecting the cells $\{C_k, 1 \leq k \leq L\}$ for a given pdf $p(\mathbf{X})$.

A commonly used distortion measure is the mean square error (l_2 norm) defined as

$$d_2(\mathbf{X}, \tilde{\mathbf{X}}) = \frac{1}{n} (\mathbf{X} - \tilde{\mathbf{X}})' (\mathbf{X} - \tilde{\mathbf{X}}) = \frac{1}{n} \sum_{k=1}^n (x_k - \tilde{x}_k)^2 \quad (3-4-33)$$

or, more generally, the weighted mean square error

$$d_{2w}(\mathbf{X}, \tilde{\mathbf{X}}) = (\mathbf{X} - \tilde{\mathbf{X}})' \mathbf{W} (\mathbf{X} - \tilde{\mathbf{X}}) \quad (3-4-34)$$

where \mathbf{W} is a positive-definite weighting matrix. Usually, \mathbf{W} is selected to be the inverse of the covariance matrix of the input data vector \mathbf{X} .

Other distortion measures that are sometimes used are special cases of the l_p norm defined as

$$d_p(\mathbf{X}, \tilde{\mathbf{X}}) = \frac{1}{n} \sum_{k=1}^n |x_k - \tilde{x}_k|^p \quad (3-4-35)$$

The special case $p = 1$ is often used as an alternative to $p = 2$.

Vector quantization is not limited to quantizing a block of signal samples of a source waveform. It can also be applied to quantizing a set of parameters extracted from the data. For example, in linear predictive coding (LPC), described in Section 3-5-3, the parameters extracted from the signal are the prediction coefficients, which are the coefficients in the all-pole filter model for the source that generates the observed data. These parameters can be considered as a block and quantized as a block by application of some appropriate distortion measure. In the case of speech encoding, an appropriate distortion measure, proposed by Itakura and Saito (1968, 1975), is the weighted square error where the weighting matrix \mathbf{W} is selected to be the normalized autocorrelation matrix Φ of the observed data.

In speech processing, an alternative set of parameters that may be quantized as a block and transmitted to the receiver is the set of reflection coefficients $\{a_{ii}, 1 \leq i \leq m\}$. Yet another set of parameters that is sometimes used for vector quantization in linear predictive coding of speech comprises the log-area ratios $\{r_k\}$, which are defined in terms of the reflection coefficients as

$$r_k = \log \frac{1 + a_{kk}}{1 - a_{kk}}, \quad 1 \leq k \leq m \quad (3-4-36)$$

Now, let us return to the mathematical formulation of vector quantization and let us consider the partitioning of the n -dimensional space into L cells $\{C_k, 1 \leq k \leq L\}$ so that the average distortion is minimized over all L -level quantizers. There are two conditions for optimality. The first is that the optimal quantizer employs a nearest-neighbor selection rule, which may be expressed mathematically as

$$Q(\mathbf{X}) = \mathbf{X}_k$$

if and only if

$$D(\mathbf{X}, \tilde{\mathbf{X}}_k) \leq D(\mathbf{X}, \tilde{\mathbf{X}}_j), \quad k \neq j, \quad 1 \leq j \leq L \quad (3-4-37)$$

The second condition necessary for optimality is that each output vector $\tilde{\mathbf{X}}_k$ be chosen to minimize the average distortion in cell C_k . In other words, $\tilde{\mathbf{X}}_k$ is the vector in C_k that minimizes

$$D_k = E[d(\mathbf{X}, \tilde{\mathbf{X}}) | \mathbf{X} \in C_k] = \int_{\mathbf{X} \in C_k} d(\mathbf{X}, \tilde{\mathbf{X}}) p(\mathbf{X}) d\mathbf{X} \quad (3-4-38)$$

The vector $\tilde{\mathbf{X}}_k$ that minimizes D_k is called the *centroid* of the cell. Thus, these conditions for optimality can be applied to partition the n -dimensional space

into cells $\{C_k, 1 \leq k \leq L\}$ when the joint pdf $p(\mathbf{X})$ is known. It is clear that these two conditions represent the generalization of the optimum scalar quantization problem to the n -dimensional vector quantization problem. In general, we expect the code vectors to be closer together in regions where the joint pdf is large and farther apart in regions where $p(\mathbf{X})$ is small.

As an upper bound on the distortion of a vector quantizer, we may use the distortion of the optimal scalar quantizer, which can be applied to each component of the vector as described in the previous section. On the other hand, the best performance that can be achieved by optimum vector quantization is given by the rate-distortion function or, equivalently, the distortion-rate function.

The distortion-rate function, which was introduced in the previous section, may be defined in the context of vector quantization as follows. Suppose we form a vector \mathbf{X} of dimension n from n consecutive samples $\{x_m\}$. The vector \mathbf{X} is then quantized to form $\tilde{\mathbf{X}} = Q(\mathbf{X})$, where $\tilde{\mathbf{X}}$ is a vector from the set of $\{\tilde{\mathbf{X}}_k, 1 \leq k \leq L\}$. As described above, the average distortion D resulting from representing \mathbf{X} by $\tilde{\mathbf{X}}$ is $E[d(\mathbf{X}, \tilde{\mathbf{X}})]$, where $d(\mathbf{X}, \tilde{\mathbf{X}})$ is the distortion per dimension, e.g.,

$$d(\mathbf{X}, \tilde{\mathbf{X}}) = \frac{1}{n} \sum_{k=1}^n (x_k - \tilde{x}_k)^2$$

The vectors $\{\tilde{\mathbf{X}}_k, 1 \leq k \leq L\}$ can be transmitted at an average bit rate of

$$R = \frac{H(\tilde{\mathbf{X}})}{n} \text{ bits/sample} \quad (3-4-39)$$

where $H(\tilde{\mathbf{X}})$ is the entropy of the quantized source output defined as

$$H(\tilde{\mathbf{X}}) = - \sum_{i=1}^L p(\tilde{\mathbf{X}}_i) \log_2 P(\tilde{\mathbf{X}}_i) \quad (3-4-40)$$

For a given average rate R , the minimum achievable distortion $D_n(R)$ is

$$D_n(R) = \min_{Q(\mathbf{X})} E[d(\mathbf{X}, \tilde{\mathbf{X}})] \quad (3-4-41)$$

where $R \geq H(\tilde{\mathbf{X}})/n$ and the minimum in (3-4-41) is taken over all possible mappings $Q(\mathbf{X})$. In the limit as the number of dimensions n is allowed to approach infinity, we obtain

$$D(R) = \lim_{n \rightarrow \infty} D_n(R) \quad (3-4-42)$$

where $D(R)$ is the distortion-rate function that was introduced in the previous section. It is apparent from this development that the distortion-rate function can be approached arbitrarily closely by increasing the size n of the vectors.

The development above is predicated on the assumption that the joint pdf $p(\mathbf{X})$ of the data vector is known. However, in practice, the joint pdf $p(\mathbf{X})$ of the data may not be known. In such a case, it is possible to select the

quantized output vectors adaptively from a set of training vectors $\mathbf{X}(m)$. Specifically, suppose that we are given a set of M training vectors where M is much greater than L ($M \gg L$). An iterative clustering algorithm, called the K means algorithm, where in our case $K = L$, can be applied to the training vectors. This algorithm iteratively subdivides the M training vectors into L clusters such that the two necessary conditions for optimality are satisfied. The K means algorithm may be described as follows [Makhoul *et al.* (1985)].

K Means Algorithm

Step 1 Initialize by setting the iteration number $i = 0$. Choose a set of output vectors $\tilde{\mathbf{X}}_k(0)$, $1 \leq k \leq L$.

Step 2 Classify the training vectors $\{\mathbf{X}(m), 1 \leq m \leq M\}$ into the clusters $\{C_k\}$ by applying the nearest-neighbor rule

$$\mathbf{X} \in C_k(i) \text{ iff } D(\mathbf{X}, \tilde{\mathbf{X}}_k(i)) \leq D(\mathbf{X}, \tilde{\mathbf{X}}_j(i)) \text{ for all } k \neq j$$

Step 3 Recompute (set i to $i + 1$) the output vectors of every cluster by computing the centroid

$$\tilde{\mathbf{X}}_k(i) = \frac{1}{M_k} \sum_{\mathbf{X} \in C_k} \mathbf{X}(m), \quad 1 \leq k \leq L$$

of the training vectors that fall in each cluster. Also, compute the resulting distortion $D(i)$ at the i th iteration.

Step 4 Terminate the test if the change $D(i - 1) - D(i)$ in the average distortion is relatively small. Otherwise, go to Step 2.

The K means algorithm converges to a local minimum (see Anderberg, 1973; Linde *et al.*, 1980). By beginning the algorithm with different sets of initial output vectors $\{\mathbf{X}_k(0)\}$ and each time performing the optimization described in the K means algorithm, it is possible to find a global optimum. However, the computational burden of this search procedure may limit the search to a few initializations.

Once we have selected the output vectors $\{\tilde{\mathbf{X}}_k, 1 \leq k \leq L\}$, each signal vector $\mathbf{X}(m)$ is quantized to the output vector that is nearest to it according to the distortion measure that is adopted. If the computation involves evaluating the distance between $\mathbf{X}(m)$ and each of the L possible output vectors $\{\tilde{\mathbf{X}}_k\}$, the procedure constitutes a *full search*. If we assume that each computation requires n multiplications and additions, the computational requirement for a full search is

$$\mathcal{C} = nL \quad (3-4-43)$$

multiplication and additions per input vector.

If we select L to be a power of 2 then $\log_2 L$ is the number of bits required to represent each vector. Now, if R denotes the bit rate per sample [per component or dimension of $\mathbf{X}(m)$], we have $nR = \log_2 L$, and, hence, the computational cost is

$$\mathcal{C} = n2^{nR} \quad (3-4-44)$$

Note that the number of computations grows exponentially with the dimensionality parameter n and the bit rate R per dimension. Because of this exponential increase of the computational cost, vector quantization has been applied to low-bit-source encoding, such as coding the reflection coefficients or log area ratios in LPC.

The computational cost associated with full search can be reduced by slightly suboptimum algorithms (see Chang *et al.*, 1984; Gersho, 1982).

In order to demonstrate the benefits of vector quantization compared with scalar quantization, we present the following example taken from Makhoul *et al.* (1985).

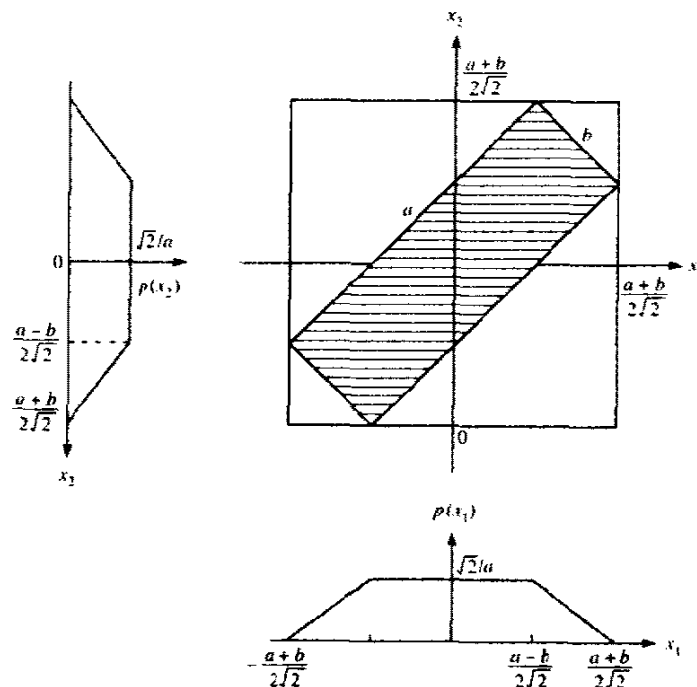
Example 3-4-1

Let x_1 and x_2 be two random variables with a uniform joint pdf

$$p(x_1, x_2) \equiv p(\mathbf{X}) = \begin{cases} \frac{1}{ab} & (\mathbf{X} \in \mathbf{C}) \\ 0 & (\text{otherwise}) \end{cases} \quad (3-4-45)$$

where \mathbf{C} is the rectangular region illustrated in Fig. 3-4-4. Note that the rectangle is rotated by 45° relative to the horizontal axis. Also shown in Fig. 3-4-4 are the marginal densities $p(x_1)$ and $p(x_2)$.

FIGURE 3-4-4 A uniform pdf in two dimensions. (Makhoul *et al.*, 1985.)



If we quantize x_1 and x_2 separately by using uniform intervals of length Δ , the number of levels needed is

$$L_1 = L_2 = \frac{a+b}{\sqrt{2}\Delta} \quad (3-4-46)$$

Hence, the number of bits needed for coding the vector $\mathbf{X} = [x_1 \ x_2]$ is

$$\begin{aligned} R_x &= R_1 + R_2 = \log_2 L_1 + \log_2 L_2 \\ R_x &= \log_2 \frac{(a+b)^2}{2\Delta^2} \end{aligned} \quad (3-4-47)$$

Thus, scalar quantization of each component is equivalent to vector quantization with the total number of levels

$$L_x = L_1 L_2 = \frac{(a+b)^2}{2\Delta^2} \quad (3-4-48)$$

We observe that this approach is equivalent to covering the large square that encloses the rectangle by square cells, where each cell represents one of the L_x quantized regions. Since $p(\mathbf{X}) = 0$ except for $\mathbf{X} \in C$, this encoding is wasteful and results in an increase of the bit rate.

If we were to cover only the region for which $p(\mathbf{X}) \neq 0$ with squares having area Δ^2 , the total number of levels that will result is the area of the rectangle divided by Δ^2 , i.e.,

$$L'_x = \frac{ab}{\Delta^2} \quad (3-4-49)$$

Therefore, the difference in bit rate between the scalar and vector quantization methods is

$$R_x - R'_x = \log_2 \frac{(a+b)^2}{2ab} \quad (3-4-50)$$

For instance, if $a = 4b$, the difference in bit rate is

$$R_x - R'_x = 1.64 \text{ bits/vector}$$

Thus, vector quantization is 0.82 bits/sample better for the same distortion.

It is interesting to note that a linear transformation (rotation by 45°) will decorrelate x_1 and x_2 and render the two random variables statistically independent. Then scalar quantization and vector quantization achieve the same efficiency. Although a linear transformation can decorrelate a vector of random variables, it does not result in statistically independent random variables, in general. Consequently, vector quantization will always equal or exceed the performance of scalar quantization (see Problem 3-40).

Vector quantization has been applied to several types of speech encoding

methods including both waveform and model-based methods which are treated in Section 3-5. In model-based methods such as LPC, vector quantization has made possible the coding of speech at rates below 1000 bits/s (see Buzo *et al.*, 1980; Roucos *et al.*, 1982; Paul 1983). When applied to waveform encoding methods, it is possible to obtain good quality speech at 16 000 bits/s, or, equivalently, at $R = 2$ bits/sample. With additional computational complexity, it may be possible in the future to implement waveform encoders producing good quality speech at a rate of $R = 1$ bit/sample.

3-5 CODING TECHNIQUES FOR ANALOG SOURCES

A number of coding techniques for analog sources have been developed over the past 40 years. Most of these have been applied to the encoding of speech and images. In this section, we briefly describe several of these methods and use speech encoding as an example in assessing their performance.

It is convenient to subdivide analog source encoding methods into three types. One type is called *temporal waveform coding*. In this type of encoding, the source encoder is designed to represent digitally the temporal characteristics of the source waveform. A second type of source encoding is *spectral waveform coding*. The signal waveform is usually subdivided into different frequency bands, and either the time waveform in each band or its spectral characteristics are encoded for transmission. The third type of source encoding is based on a mathematical model of the source and is called *model-based coding*.

3-5-1 Temporal Waveform Coding

There are several analog source coding techniques that are designed to represent the time-domain characteristics of the signal. The most commonly used methods are described in this section.

Pulse Code Modulation† (PCM) Let $x(t)$ denote a sample function emitted by a source and let x_n denote the samples taken at a sampling rate $f_s \geq 2W$, where W is the highest frequency in the spectrum of $x(t)$. In PCM, each sample of the signal is quantized to one of 2^R amplitude levels, where R is the number of binary digits used to represent each sample. Thus the rate from the source is Rf_s bits/s.

The quantization process may be modeled mathematically as

$$\tilde{x}_n = x_n + q_n \quad (3-5-1)$$

where \tilde{x}_n represents the quantized value of x_n and q_n represents the quantization error, which we treat as an additive noise. Assuming that a

† PCM, DPCM, and ADPCM are source coding techniques. They are not digital modulation methods.

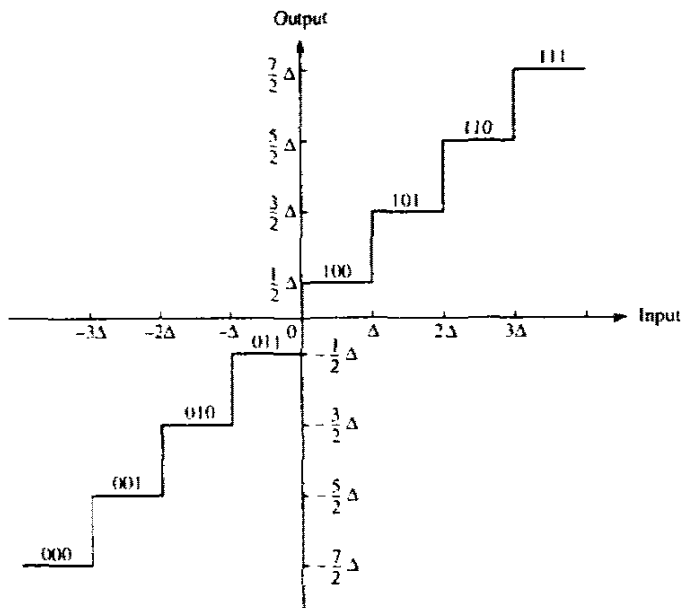


FIGURE 3-5-1 Input-output characteristic for a uniform quantizer.

uniform quantizer is used, having the input-output characteristic illustrated in Fig. 3-5-1, the quantization noise is well characterized statistically by the uniform pdf

$$p(q) = \frac{1}{\Delta}, \quad -\frac{1}{2}\Delta \leq q \leq \frac{1}{2}\Delta \quad (3-5-2)$$

where the step size of the quantizer is $\Delta = 2^{-R}$. The mean square value of the quantization error is

$$E(q^2) = \frac{1}{12}\Delta^2 = \frac{1}{12} \times 2^{-2R} \quad (3-5-3)$$

Measured in decibels, the mean square value of the noise is

$$10 \log \frac{1}{12}\Delta^2 = 10 \log \left(\frac{1}{12} \times 2^{-2R} \right) = -6R - 10.8 \text{ dB} \quad (3-5-4)$$

We observe that the quantization noise decreases by 6 dB/bit used in the quantizer. For example, a 7 bit quantizer results in a quantization noise power of -52.8 dB.

Many source signals such as speech waveforms have the characteristic that small signal amplitudes occur more frequently than large ones. However, a uniform quantizer provides the same spacing between successive levels throughout the entire dynamic range of the signal. A better approach is to employ a nonuniform quantizer. A nonuniform quantizer characteristic is usually obtained by passing the signal through a nonlinear device that compresses the signal amplitude, followed by a uniform quantizer. For

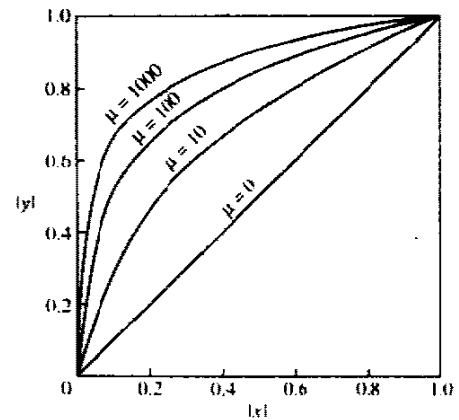


FIGURE 3-5-2 Input-output magnitude characteristic for a logarithmic compressor.

example, a logarithmic compressor has an input-output magnitude characteristics of the form

$$|y| = \frac{\log(1 + \mu|x|)}{\log(1 + \mu)} \quad (3-5-5)$$

where $|x| \leq 1$ is the magnitude of the input, $|y|$ is the magnitude of the output, and μ is a parameter that is selected to give the desired compression characteristic. Figure 3-5-2 illustrates this compression relationship for several values of μ . The value $\mu = 0$ corresponds to no compression.

In the encoding of speech waveforms, for example, the value of $\mu = 255$ has been adopted as a standard in the USA and Canada. This value results in about a 24 dB reduction in the quantization noise power relative to uniform quantization, as shown by Jayant (1974). Consequently, a 7 bit quantizer used in conjunction with a $\mu = 255$ logarithmic compressor produces a quantization noise power of approximately -77 dB compared with the -53 dB for uniform quantization.

In the reconstruction of the signal from the quantized values, the inverse logarithmic relation is used to expand the signal amplitude. The combined compressor-expander pair is termed a *comparator*.

Differential Pulse Code Modulation (DPCM) In PCM, each sample of the waveform is encoded independently of all the others. However, most source signals sampled at the Nyquist rate or faster exhibit significant correlation between successive samples. In other words, the average change in amplitude between successive samples is relatively small. Consequently, an encoding scheme that exploits the redundancy in the samples will result in a lower bit rate for the source output.

A relatively simple solution is to encode the differences between successive samples rather than the samples themselves. Since differences between samples are expected to be smaller than the actual sampled amplitudes, fewer bits are required to represent the differences. A refinement of this general approach is

to predict the current sample based on the previous p samples. To be specific, let x_n denote the current sample from the source and let \hat{x}_n denote the predicted value of x_n , defined as

$$\hat{x}_n = \sum_{i=1}^p a_i x_{n-i} \quad (3-5-6)$$

Thus \hat{x}_n is a weighted linear combination of the past p samples and the $\{a_i\}$ are the predictor coefficients. The $\{a_i\}$ are selected to minimize some function of the error between x_n and \hat{x}_n .

A mathematically and practically convenient error function is the mean square error (MSE). With the MSE as the performance index for the predictor, we select the $\{a_i\}$ to minimize

$$\begin{aligned} \mathcal{E}_p &= E(e_n^2) = E\left[\left(x_n - \sum_{i=1}^p a_i x_{n-i}\right)^2\right] \\ &= E(x_n^2) - 2 \sum_{i=1}^p a_i E(x_n x_{n-i}) + \sum_{i=1}^p \sum_{j=1}^p a_i a_j E(x_{n-i} x_{n-j}) \end{aligned} \quad (3-5-7)$$

Assuming that the source output is (wide-sense) stationary, we may express (3-5-7) as

$$\mathcal{E}_p = \phi(0) - 2 \sum_{i=1}^p a_i \phi(i) + \sum_{i=1}^p \sum_{j=1}^p a_i a_j \phi(i-j) \quad (3-5-8)$$

where $\phi(m)$ is the autocorrelation function of the sampled signal sequence x_n . Minimization of \mathcal{E}_p with respect to the predictor coefficients $\{a_i\}$ results in the set of linear equations

$$\sum_{i=1}^p a_i \phi(i-j) = \phi(j), \quad j = 1, 2, \dots, p \quad (3-5-9)$$

Thus, the values of the predictor coefficients are established. When the autocorrelation function $\phi(n)$ is not known *a priori*, it may be estimated from the samples $\{x_n\}$ using the relation†

$$\hat{\phi}(n) = \frac{1}{N} \sum_{i=1}^{N-n} x_i x_{i+n}, \quad n = 0, 1, 2, \dots, p \quad (3-5-10)$$

and the estimate $\hat{\phi}(n)$ is used in (3-5-9) to solve for the coefficients $\{a_i\}$. Note that the normalization factor of $1/N$ in (3-5-10) drops out when $\hat{\phi}(n)$ is substituted in (3-5-9).

The linear equations in (3-5-9) for the predictor coefficients are called the *normal equations* or the *Yule-Walker equations*. There is an algorithm developed by Levinson (1947) and Durbin (1959) for solving these equations efficiently. It is described in Appendix A. We shall deal with the solution in greater detail in the subsequent discussion on linear predictive coding.

† The estimation of the autocorrelation function from a finite number of observations $\{x_i\}$ is a separate issue, which is beyond the scope of this discussion. The estimate in (3-5-10) is one that is frequently used in practice.

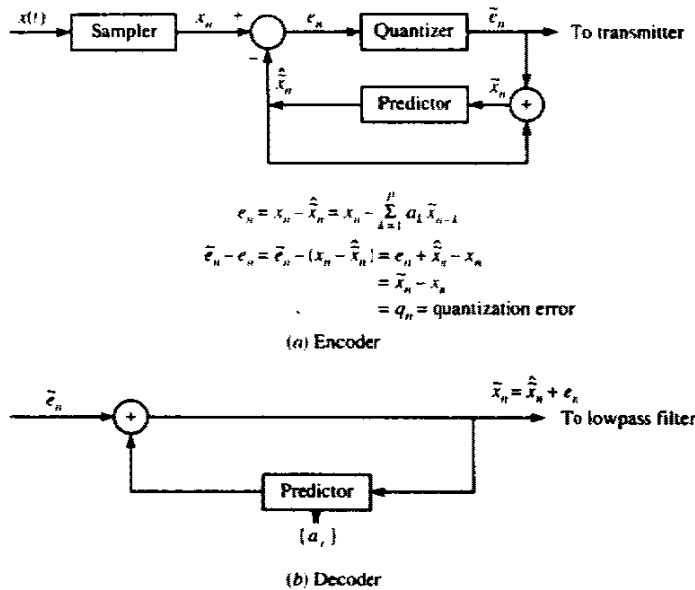


FIGURE 3-5-3 (a) Block diagram of a DPCM encoder. (b) DPCM decoder at the receiver.

Having described the method for determining the predictor coefficients, let us now consider the block diagram of a practical DPCM system, shown in Fig. 3-5-3(a). In this configuration, the predictor is implemented with the feedback loop around the quantizer. The input to the predictor is denoted by \tilde{x}_n , which represents the signal sample x_n modified by the quantization process, and the output of the predictor is

$$\hat{x}_n = \sum_{i=1}^p a_i \tilde{x}_{n-i} \tag{3-5-11}$$

The difference

$$e_n = x_n - \hat{x}_n \tag{3-5-12}$$

is the input to the quantizer and \tilde{e}_n denotes the output. Each value of the quantized prediction error \tilde{e}_n is encoded into a sequence of binary digits and transmitted over the channel to the destination. The quantized error \tilde{e}_n is also added to the predicted value \hat{x}_n to yield \tilde{x}_n .

At the destination, the same predictor that was used at the transmitting end is synthesized and its output \hat{x}_n is added to \tilde{e}_n to yield \tilde{x}_n . The signal \tilde{x}_n is the desired excitation for the predictor and also the desired output sequence from which the reconstructed signal $\hat{x}(t)$ is obtained by filtering, as shown in Fig. 3-5-3(b).

The use of feedback around the quantizer, as described above, ensures that the error in \tilde{x}_n is simply the quantization error $q_n = \tilde{e}_n - e_n$ and that there is no

accumulation of previous quantization errors in the implementation of the decoder. That is,

$$\begin{aligned} q_n &= \tilde{e}_n - e_n \\ &= \tilde{e}_n - (x_n - \hat{x}_n) \\ &= \bar{x}_n - x_n \end{aligned} \tag{3-5-13}$$

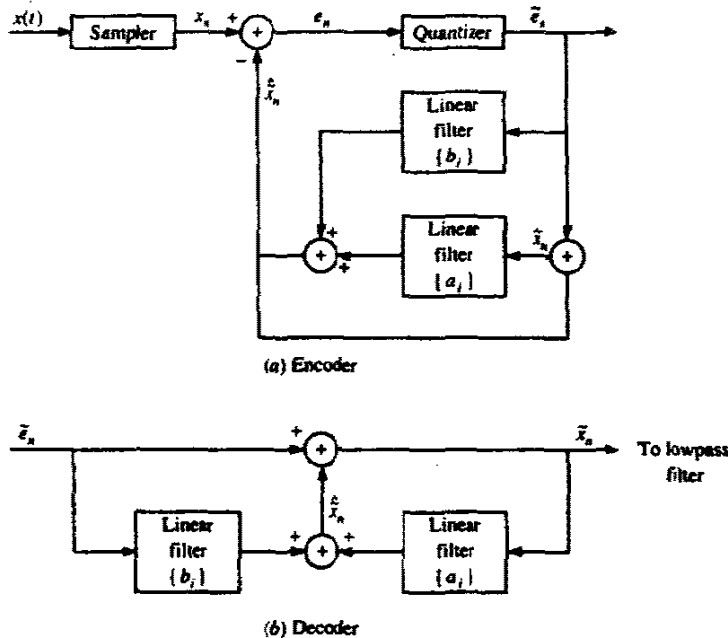
Hence $\bar{x}_n = x_n + q_n$. This means that the quantized sample \bar{x}_n differs from the input x_n by the quantization error q_n independent of the predictor used. Therefore, the quantization errors do not accumulate.

In the DPCM system illustrated in Fig. 3-5-3, the estimate or predicted value \hat{x}_n of the signal sample x_n is obtained by taking a linear combination of past values \bar{x}_{n-k} , $k = 1, 2, \dots, p$, as indicated by (3-5-11). An improvement in the quality of the estimate is obtained by including linearly filtered past values of the quantized error. Specifically, the \hat{x}_n estimate may be expressed as

$$\hat{x}_n = \sum_{i=1}^p a_i \bar{x}_{n-i} + \sum_{i=1}^m b_i \tilde{e}_{n-i} \tag{3-5-14}$$

where $\{b_i\}$ are the coefficients of the filter for the quantized error sequence \tilde{e}_n . The block diagrams of the encoder at the transmitter and the decoder at the receiver are shown in Fig. 3-5-4. The two sets of coefficients $\{a_i\}$ and $\{b_i\}$ are selected to minimize some function of the error $e_n = x_n - \hat{x}_n$, such as the mean square error.

FIGURE 3-5-4 DPCM modified by the addition of linearly filtered error sequence.



Adaptive PCM and DPCM Many real sources are quasistationary in nature. One aspect of the quasistationary characteristic is that the variance and the autocorrelation function of the source output vary slowly with time. PCM and DPCM encoders, however, are designed on the basis that the source output is stationary. The efficiency and performance of these encoders can be improved by having them adapt to the slowly time-variant statistics of the source.

In both PCM and DPCM, the quantization error q_n resulting from a uniform quantizer operating on a quasistationary input signal will have a time-variant variance (quantization noise power). One improvement that reduces the dynamic range of the quantization noise is the use of an adaptive quantizer. Although the quantizer can be made adaptive in different ways, a relatively simple method is to use a uniform quantizer that varies its step size in accordance with the variance of the past signal samples. For example, a short-term running estimate of the variance of x_n can be computed from the input sequence $\{x_n\}$ and the step size can be adjusted on the basis of such an estimate. In its simplest form, the algorithm for the step-size adjustment employs only the previous signal sample. Such an algorithm has been successfully used by Jayant (1974) in the encoding of speech signals. Figure 3-5-5 illustrates such a (3 bit) quantizer in which the step size is adjusted recursively according to the relation

$$\Delta_{n+1} = \Delta_n M(n) \tag{3-5-15}$$

FIGURE 3-5-5 Example of a quantizer with an adaptive step size. (Jayant, 1974.)

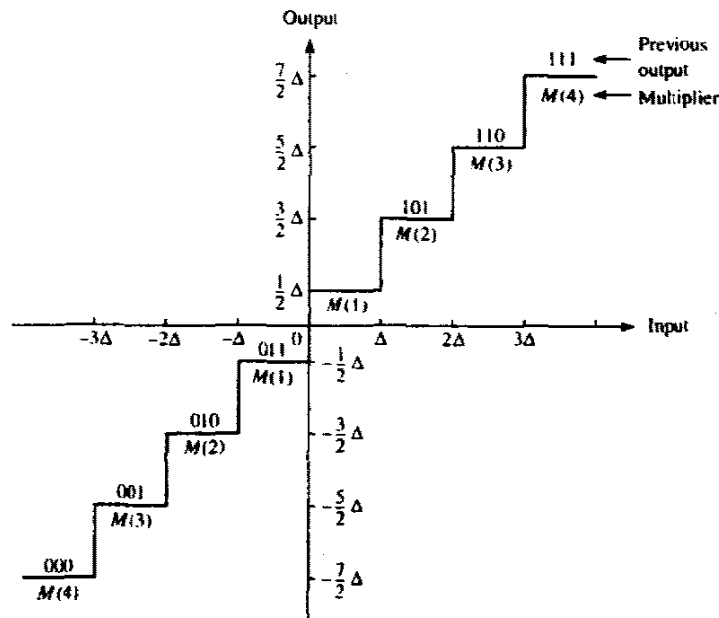


TABLE 3-5-1 MULTIPLICATION FACTORS FOR ADAPTIVE STEP SIZE ADJUSTMENT (JAYANT, 1974)

	PCM			DPCM		
	2	3	4	2	3	4
$M(1)$	0.60	0.85	0.80	0.80	0.90	0.90
$M(2)$	2.20	1.00	0.80	1.60	0.90	0.90
$M(3)$		1.00	0.80		1.25	0.90
$M(4)$		1.50	0.80		1.70	0.90
$M(5)$			1.20			1.20
$M(6)$			1.60			1.60
$M(7)$			2.00			2.00
$M(8)$			2.40			2.40

where $M(n)$ is a factor, whose value depends on the quantizer level for the sample x_n , and Δ_n is the step size of the quantizer for processing x_n . Values of the multiplication factors optimized for speech encoding have been given by Jayant (1974). These values are displayed in Table 3-5-1 for 2, 3, and 4 bit adaptive quantization.

In DPCM, the predictor can also be made adaptive when the source output is quasistationary. The coefficients of the predictor can be changed periodically to reflect the changing signal statistics of the source. The linear equations given by (3-5-9) still apply, with the short-term estimate of the autocorrelation function of x_n substituted in place of the ensemble correlation function. The predictor coefficients thus determined may be transmitted along with the quantized error $\bar{e}(n)$ to the receiver, which implements the same predictor. Unfortunately, the transmission of the predictor coefficients results in a higher bit rate over the channel, offsetting, in part, the lower data rate achieved by having a quantizer with fewer bits (fewer levels) to handle the reduced dynamic range in the error e_n resulting from adaptive prediction.

As an alternative, the predictor at the receiver may compute its own prediction coefficients from \bar{e}_n and \bar{x}_n , where

$$\bar{x}_n = \bar{e}_n + \sum_{i=1}^p a_i \bar{x}_{n-i} \quad (3-5-16)$$

If we neglect the quantization noise, \bar{x}_n is equivalent to x_n . Hence, \bar{x}_n may be used to estimate the autocorrelation function $\phi(n)$ at the receiver, and the resulting estimates can be used in (3-5-9) in place of $\phi(n)$ to solve for the predictor coefficients. For sufficiently fine quantization, the difference between x_n and \bar{x}_n is very small. Hence, the estimate of $\phi(n)$ obtained from \bar{x}_n is usually adequate for determining the predictor coefficients. Implemented in this manner, the adaptive predictor results in a lower source data rate.

Instead of using the block processing approach for determining the

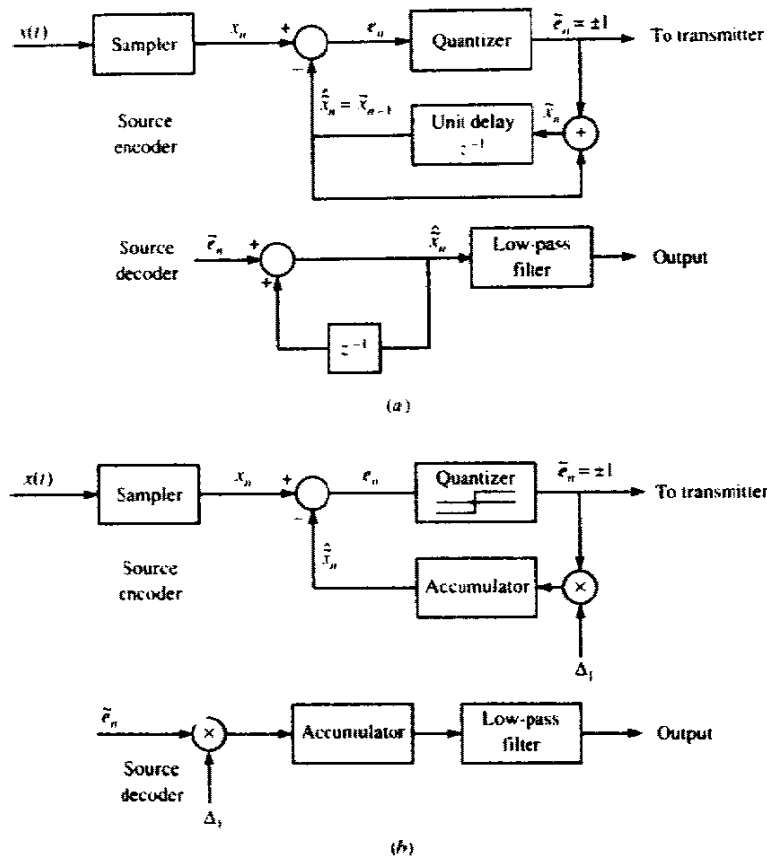


FIGURE 3-5-6 (a) Block diagram of a delta modulation system. (b) An equivalent realization of a delta modulation system.

predictor coefficients $\{a_i\}$ as described above, we may adapt the predictor coefficients on a sample-by-sample basis by using a gradient-type algorithm, similar in form to the adaptive gradient equalization algorithm that is described in Chapter 11. Similar gradient-type algorithms have also been devised for adapting the filter coefficients $\{a_i\}$ and $\{b_i\}$ of the DPCM system shown in Fig. 3-5-4. For details on such algorithms, the reader may refer to the book by Jayant and Noll (1984).

Delta Modulation (DM) Delta modulation may be viewed as a simplified form of DPCM in which a two-level (1 bit) quantizer is used in conjunction with a fixed first-order predictor. The block diagram of a DM encoder–decoder is shown in Fig. 3-5-6(a). We note that

$$\hat{x}_n = \tilde{x}_{n-1} = \hat{x}_{n-1} + \tilde{e}_{n-1} \quad (3-5-17)$$

Since

$$\begin{aligned} q_n &= \bar{e}_n - e_n \\ &= \bar{e}_n - (x_n - \hat{x}_n) \end{aligned}$$

It follows that

$$\hat{x}_n = x_{n-1} + q_{n-1}$$

Thus the estimated (predicted) value of x_n is really the previous sample x_{n-1} modified by the quantization noise q_{n-1} . We also note that the difference equation (3-5-17) represents an integrator with an input \bar{e}_n . Hence, an equivalent realization of the one-step predictor is an accumulator with an input equal to the quantized error signal \bar{e}_n . In general, the quantized error signal is scaled by some value, say Δ_1 , which is called the *step size*. This equivalent realization is illustrated in Fig. 3-5-6(b). In effect, the encoder shown in Fig. 3-5-6 approximates a waveform $x(t)$ by a linear staircase function. In order for the approximation to be relatively good, the waveform $x(t)$ must change slowly relative to the sampling rate. This requirement implies that the sampling rate must be several (a factor of at least 5) times the Nyquist rate.

At any given sampling rate, the performance of the DM encoder is limited by two types of distortion, as illustrated in Fig. 3-5-7. One is called *slope-overload distortion*. It is due to the use of a step size Δ_1 that is too small to follow portions of the waveform that have a steep slope. The second type of distortion, called *granular noise*, results from using a step size that is too large in parts of the waveform having a small slope. The need to minimize both of these two types of distortion results in conflicting requirements in the selection of the step size Δ_1 . One solution is to select Δ_1 to minimize the sum of the mean square values of these two distortions.

Even when Δ_1 is optimized to minimize the total mean square value of the slope-overload distortion and the granular noise, the performance of the DM encoder may still be less than satisfactory. An alternative solution is to employ a variable step size that adapts itself to the short-term characteristics of the source signal. That is, the step size is increased when the waveform has a steep

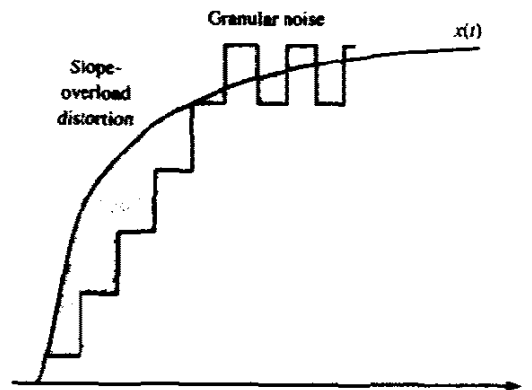


FIGURE 3-5-7 An example of slope overload distortion and granular noise in a delta modulation encoder.

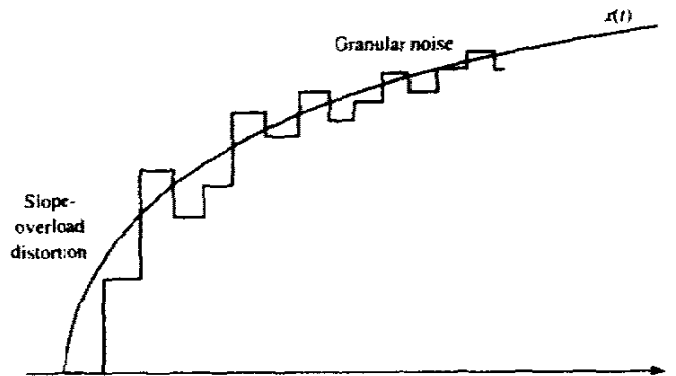


FIGURE 3-5-8 An example of variable-step-size delta modulation encoding.

slope and decreased when the waveform has a relatively small slope. This adaptive characteristic is illustrated in Fig. 3-5-8.

A variety of methods can be used to adaptively set the step size in every iteration. The quantized error sequence \tilde{e}_n provides a good indication of the slope characteristics of the waveform being encoded. When the quantized error \tilde{e}_n is changing signs between successive iterations, this is an indication that the slope of the waveform in that locality is relatively small. On the other hand, when the waveform has a steep slope, successive values of the error \tilde{e}_n are expected to have identical signs. From these observations, it is possible to devise algorithms that decrease or increase the step size depending on successive values of \tilde{e}_n . A relatively simple rule devised by Jayant (1970) is to adaptively vary the step size according to the relation

$$\Delta_n = \Delta_{n-1} K^{\tilde{e}_n \tilde{e}_{n-1}}, \quad n = 1, 2, \dots$$

where $K \geq 1$ is a constant that is selected to minimize the total distortion. A block diagram of a DM encoder-decoder that incorporates this adaptive algorithm is illustrated in Fig. 3-5-9.

Several other variations of adaptive DM encoding have been investigated and described in the technical literature. A particularly effective and popular technique first proposed by Greefkes (1970) is called *continuously variable slope delta modulation* (CVSD). In CVSD the adaptive step-size parameter may be expressed as

$$\Delta_n = \alpha \Delta_{n-1} + k_1$$

if \tilde{e}_n , \tilde{e}_{n-1} , and \tilde{e}_{n-2} have the same sign; otherwise,

$$\Delta_n = \alpha \Delta_{n-1} + k_2$$

The parameters α , k_1 , and k_2 are selected such that $0 < \alpha < 1$ and $k_1 \gg k_2 > 0$. For more discussion on this and other variations of adaptive DM, the interested reader is referred to the papers by Jayant (1974) and Flanagan *et al.* (1979), which contain extensive references.

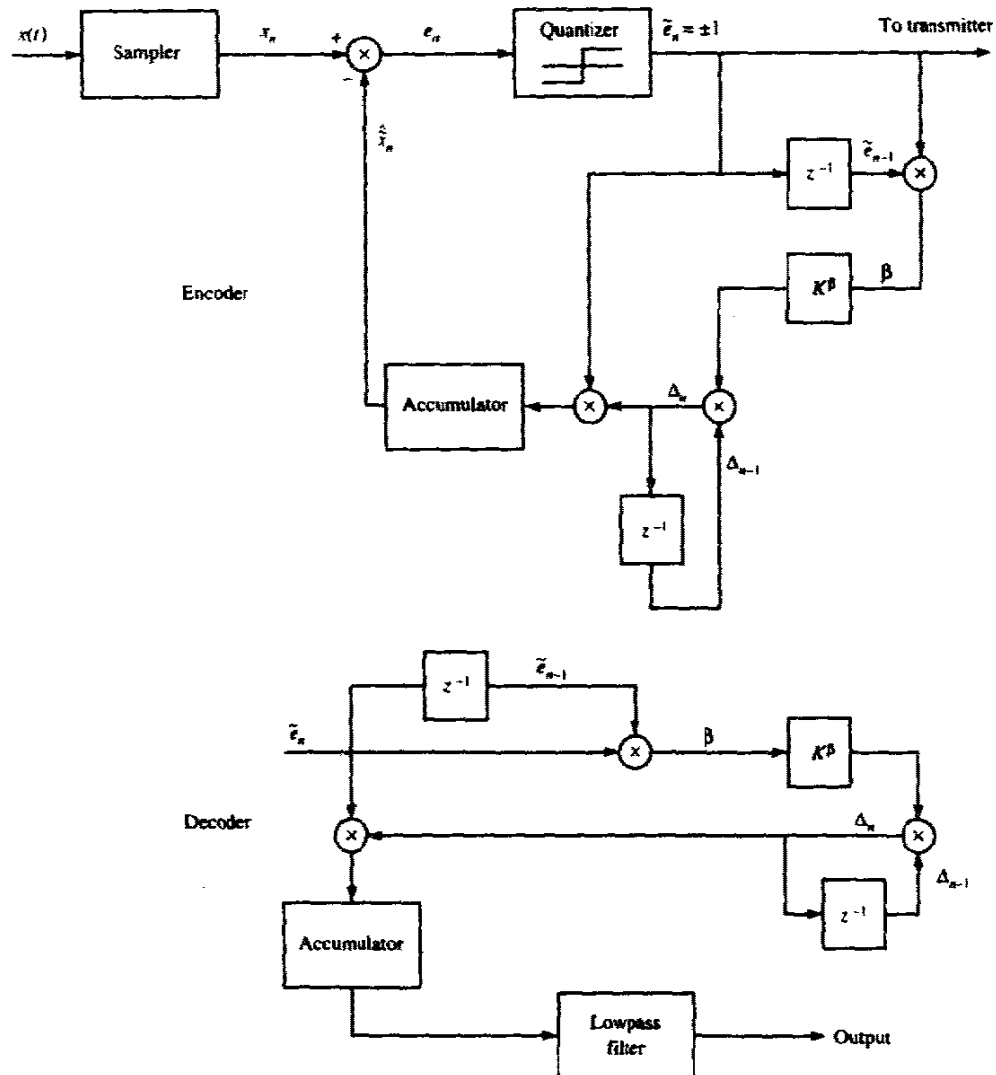


FIGURE 3-5-9 An example of a delta modulation system with adaptive step size.

PCM, DPCM, adaptive PCM, and adaptive DPCM and DM are all source encoding techniques that attempt to faithfully represent the output waveform from the source. The following class of waveform encoding methods is based on a spectral decomposition of the source signal.

3-5-2 Spectral Waveform Coding

In this section, we briefly describe waveform coding methods that filter the source output signal into a number of frequency bands or subbands and separately encode the signal in each subband. The waveform encoding may be

performed either on the time-domain waveforms in each subband or on the frequency-domain representation of the corresponding time-domain waveform in each subband.

Subband Coding In subband coding (SBC) of speech and image signals, the signal is divided into a small number of subbands and the time waveform in each subband is encoded separately. In speech coding, for example, the lower-frequency bands contain most of the spectral energy in voiced speech. In addition, quantization noise is more noticeable to the ear in the lower-frequency bands. Consequently, more bits are used for the lower-band signals and fewer are used for the higher-frequency bands.

Filter design is particularly important in achieving good performance in SBC. In practice, quadrature-mirror filters (QMFs) are generally used because they yield an alias-free response due to their perfect reconstruction property (see Vaidyanathan, 1993). By using QMFs in subband coding, the lower-frequency band is repeatedly subdivided by factors of two, thus creating octave-band filters. The output of each QMF filter is decimated by a factor of two, in order to reduce the sampling rate. For example, suppose that the bandwidth of a speech signal extends to 3200 Hz. The first pair of QMFs divides the spectrum into the low (0–1600 Hz) and high (1600–3200 Hz) bands. Then, the low band is split into low (0–800 Hz) and high (800–1600 Hz) bands by the use of another pair of QMFs. A third subdivision by another pair of QMFs can split the 0–800 Hz band into low (0–400 Hz) and high (400–800 Hz) bands. Thus, with three pairs of QMFs, we have obtained signals in the frequency bands 0–400, 400–800, 800–1600 and 1600–3200 Hz. The time-domain signal in each subband may now be encoded with different precision. In practice, adaptive PCM has been used for waveform encoding of the signal in each subband.

Adaptive Transform Coding In adaptive transform coding (ATC), the source signal is sampled and subdivided into frames of N_f samples, and the data in each frame is transformed into the spectral domain for coding and transmission. At the source decoder, each frame of spectral samples is transformed back into the time domain and the signal is synthesized from the time-domain samples and passed through a D/A converter. To achieve coding efficiency, we assign more bits to the more important spectral coefficients and fewer bits to the less important spectral coefficients. In addition, by designing an adaptive allocation in the assignment of the total number of bits to the spectral coefficients, we can adapt to possibly changing statistics of the source signal.

An objective in selecting the transformation from the time domain to the frequency domain is to achieve uncorrelated spectral samples. In this sense, the Karhunen-Loève transform (KLT) is optimal in that it yields spectral values that are uncorrelated, but the KLT is generally difficult to compute (see

Wintz, 1972). The DFT and the *discrete cosine transform* (DCT) are viable alternatives, although they are suboptimum. Of these two, the DCT yields good performance compared with the KLT, and is generally used in practice (see Campanella and Robinson, 1971; Zelinsky and Noll, 1977).

In speech coding using ATC, it is possible to attain communication-quality speech at a rate of about 9600 bits/s.

3-5-3 Model-Based Source Coding

In contrast to the waveform encoding methods described above, model-based source coding represents a completely different approach. In this, the source is modeled as a linear system (filter) that, when excited by an appropriate input signal, results in the observed source output. Instead of transmitting the samples of the source waveform to the receiver, the parameters of the linear system are transmitted along with an appropriate excitation signal. If the number of parameters is sufficiently small, the model-based methods provide a large compression of the data.

The most widely used model-based coding method is called *linear predictive coding* (LPC). In this, the sampled sequence, denoted by x_n , $n = 0, 1, \dots, N - 1$, is assumed to have been generated by an all-pole (discrete-time) filter having the transfer function

$$H(z) = \frac{G}{1 - \sum_{k=1}^p a_k z^{-k}} \quad (3-5-18)$$

Appropriate excitation functions are an impulse, a sequence of impulses, or a sequence of white noise with unit variance. In any case, suppose that the input sequence is denoted by v_n , $n = 0, 1, 2, \dots$. Then the output sequence of the all-pole model satisfies the difference equation

$$x_n = \sum_{k=1}^p a_k x_{n-k} + Gv_n, \quad n = 0, 1, 2, \dots \quad (3-5-19)$$

In general, the observed source output x_n , $n = 0, 1, 2, \dots, N - 1$, does not satisfy the difference equation (3-5-19), but only its model does. If the input is a white-noise sequence or an impulse, we may form an estimate (or prediction) of x_n by the weighted linear combination

$$\hat{x}_n = \sum_{k=1}^p a_k x_{n-k}, \quad n > 0 \quad (3-5-20)$$

The difference between x_n and \hat{x}_n , namely,

$$\begin{aligned} e_n &= x_n - \hat{x}_n \\ &= x_n - \sum_{k=1}^p a_k x_{n-k} \end{aligned} \quad (3-5-21)$$

represents the error between the observed value x_n and the estimated (predicted) value \hat{x}_n . The filter coefficients $\{a_k\}$ can be selected to minimize the mean square value of this error.

Suppose for the moment that the input $\{v_n\}$ is a white-noise sequence. Then, the filter output x_n is a random sequence and so is the difference $e_n = x_n - \hat{x}_n$. The ensemble average of the squared error is

$$\begin{aligned}\xi_p &= E(e_n^2) \\ &= E\left[\left(x_n - \sum_{k=1}^p a_k x_{n-k}\right)^2\right] \\ &= \phi(0) - 2 \sum_{k=1}^p a_k \phi(k) + \sum_{k=1}^p \sum_{m=1}^p a_k a_m \phi(k-m)\end{aligned}\quad (3-5-22)$$

where $\phi(m)$ is the autocorrelation function of the sequence x_n , $n = 0, 1, \dots, N-1$. But ξ_p is identical to the MSE given by (3-5-8) for a predictor used in DPCM. Consequently, minimization of ξ_p in (3-5-22) yields the set of normal equations given previously by (3-5-9). To completely specify the filter $H(z)$, we must also determine the filter gain G . From (3-5-19), we have

$$E[(Gv_n)^2] = G^2 E(v_n^2) = G^2 = E\left[\left(x_n - \sum_{k=1}^p a_k x_{n-k}\right)^2\right] = \xi_p \quad (3-5-23)$$

where ξ_p is the residual MSE obtained from (3-5-22) by substituting the optimum prediction coefficients, which result from the solution of (3-5-9). With this substitution, the expression for ξ_p and, hence, G^2 simplifies to

$$\xi_p = G^2 = \phi(0) - \sum_{k=1}^p a_k \phi(k) \quad (3-5-24)$$

In practice, we do not usually know *a priori* the true autocorrelation function of the source output. Hence, in place of $\phi(n)$, we substitute an estimate $\hat{\phi}(n)$ as given by (3-5-10), which is obtained from the set of samples x_n , $n = 0, 1, \dots, N-1$, emitted by the source.

As indicated previously, the Levinson-Durbin algorithm derived in Appendix A may be used to solve for the predictor coefficients $\{a_k\}$ recursively, beginning with a first-order predictor and iterating the order of the predictor up to order p . The recursive equations for the $\{a_k\}$ may be expressed as

$$\begin{aligned}a_i &= \frac{\hat{\phi}(i) - \sum_{k=1}^{i-1} a_{i-k} \hat{\phi}(i-k)}{\hat{\xi}_{i-1}} \quad i = 2, 3, \dots, p \\ a_{i,k} &= a_{i-1,k} - a_i a_{i-1-i+k}, \quad 1 \leq k \leq i-1 \\ \hat{\xi}_i &= (1 - a_i) \hat{\xi}_{i-1} \\ a_{11} &= \frac{\hat{\phi}(1)}{\hat{\phi}(0)}, \quad \hat{\xi}_0 = \hat{\phi}(0)\end{aligned}\quad (3-5-25)$$

where a_{ik} , $k = 1, 2, \dots, i$, are the coefficients of the i th-order predictor. The desired coefficients for the predictor of order p are

$$a_k \cong a_{pk}, \quad k = 1, 2, \dots, p \tag{3-5-26}$$

and the residual MSE is

$$\begin{aligned} \hat{\mathcal{E}} &= G^2 = \hat{\phi}(0) - \sum_{k=1}^p a_k \hat{\phi}(k) \\ &= \hat{\phi}(0) \prod_{i=1}^p (1 - a_{ii}^2) \end{aligned} \tag{3-5-27}$$

We observe that the recursive relations in (3-5-25) give us not only the coefficients of the predictor for order p , but also the predictor coefficients of all orders less than p .

The residual MSE $\hat{\mathcal{E}}_i$, $i = 1, 2, \dots, p$, forms a monotone decreasing sequence, i.e. $\hat{\mathcal{E}}_p \leq \hat{\mathcal{E}}_{p-1} \leq \dots \leq \hat{\mathcal{E}}_1 \leq \hat{\mathcal{E}}_0$, and the prediction coefficients a_{ii} satisfy the condition

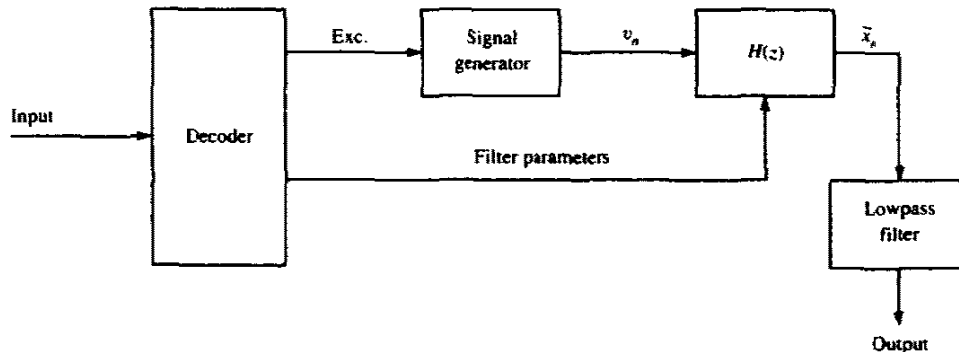
$$|a_{ii}| < 1, \quad i = 1, 2, \dots, p \tag{3-5-28}$$

This condition is necessary and sufficient for all the poles of $H(z)$ to be inside the unit circle. Thus (3-5-28) ensures that the model is stable.

LPC has been successfully used in the modeling of a speech source. In this case, the coefficients a_{ii} , $i = 1, 2, \dots, p$, are called *reflection coefficients* as a consequence of their correspondence to the reflection coefficients in the acoustic tube model of the vocal tract (see Rabiner and Schafer, 1978; Deller *et al.*, 1993).

Once the predictor coefficients and the gain G have been estimated from the source output $\{x_n\}$, each parameter is coded into a sequence of binary digits and transmitted to the receiver. Source decoding or waveform synthesis may be accomplished at the receiver as illustrated in Fig. 3-5-10. The signal generator is used to produce the excitation function $\{v_n\}$, which is scaled by G

FIGURE 3-5-10 Block diagram of a waveform synthesizer (source decoder) for an LPC system.



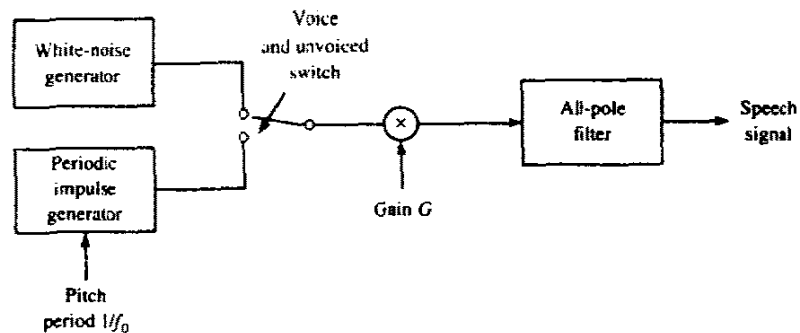


FIGURE 3-5-11 Block diagram model of the generation of a speech signal.

to produce the desired input to the all-pole filter model $H(z)$ synthesized from the received prediction coefficients. The analog signal may be reconstructed by passing the output sequence from $H(z)$ through an analog filter that basically performs the function of interpolating the signal between sample points. In this realization of the waveform synthesizer, the excitation function and the gain parameter must be transmitted along with the prediction coefficients to the receiver.

When the source output is stationary, the filter parameters need to be determined only once. However, the statistics of most sources encountered in practice are at best quasistationary. Under these circumstances, it is necessary to periodically obtain new estimates of the filter coefficients, the gain G , and the type of excitation function, and to transmit these estimates to the receiver.

Example 3-5-1

The block diagram shown in Fig. 3-5-11 illustrates a model for a speech source. There are two mutually exclusive excitation functions to model voiced and unvoiced speech sounds. On a short-time basis, voiced speech is periodic with a fundamental frequency f_0 or a pitch period $1/f_0$ that depends on the speaker. Thus voiced speech is generated by exciting an all-pole filter model of the vocal tract by a periodic impulse train with a period equal to the desired pitch period. Unvoiced speech sounds are generated by exciting the all-pole filter model by the output of a random-noise generator. The speech encoder at the transmitter must determine the proper excitation function, the pitch period for voiced speech, the gain parameter G , and the prediction coefficients. These parameters are encoded into binary digits and transmitted to the receiver. Typically, the voiced and unvoiced information requires 1 bit, the pitch period is adequately represented by 6 bits, and the gain parameter may be represented by 5 bits after its dynamic range is compressed logarithmically. The prediction coefficients require 8–10 bits/coefficient for adequate representation (see Rabiner and Schafer, 1978). The reason for such high accuracy is that relatively small changes in

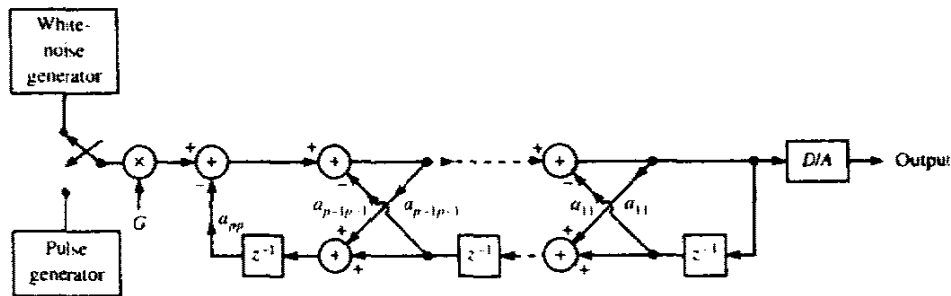


FIGURE 3-5-12 All-pole lattice filter for synthesizing the speech signal.

the prediction coefficients result in a large change in the pole positions of the filter model $H(z)$. The accuracy requirements may be lessened by transmitting the reflection coefficients a_{ii} , which have a smaller dynamic range. These are adequately represented by 6 bits. Thus, for a predictor of order $p = 10$ [five poles in $H(z)$], the total number of bits is 72. Due to the quasistationary nature of the speech signal, the linear system model must be changed periodically, typically once every 15–30 ms. Consequently, the bit rate from the source encoder is in the range 4800–2400 bit/s.

When the reflection coefficients are transmitted to the decoder, it is not necessary to recompute the prediction coefficients in order to realize the speech synthesizer. Instead, the synthesis is performed by realizing a lattice filter, shown in Fig. 3-5-12, which utilizes the reflection coefficients directly and which is equivalent to the linear prediction filter.

The linear all-pole filter model, for which the filter coefficients are estimated via linear prediction, is by far the simplest linear model for a source. A more general source model is a linear filter that contains both poles and zeros. In a pole-zero model, the source output x_n satisfies the difference equation

$$x_n = \sum_{k=1}^p a_k x_{n-k} + \sum_{k=0}^q b_k v_{n-k}$$

where v_n is the input excitation sequence. The problem now is to estimate the filter parameters $\{a_k\}$ and $\{b_k\}$ from the data x_i , $i = 0, 1, \dots, N-1$, emitted by the source. However, the MSE criterion applied to the minimization of the error $e_n = x_n - \hat{x}_n$, where \hat{x}_n is an estimate of x_n , results in a set of nonlinear equations for the parameters $\{a_k\}$ and $\{b_k\}$. Consequently, the evaluation of the $\{a_k\}$ and $\{b_k\}$ becomes tedious and difficult mathematically. To avoid having to solve the nonlinear equations, a number of suboptimum methods have been devised for pole-zero modeling. A discussion of these techniques would lead us too far afield, however.

LPC as described above forms the basis for more complex model-based source encoding methods. When applied to speech coding, the model-based

methods are generally called *vocoders* (for voice coders). In addition to the conventional LPC vocoder described above, other types of vocoders that have been implemented include the residual excited LPC (RELPC) vocoder, the multipulse LPC vocoder, the code-excited LPC (CELP) vocoder, and the vector-sum-excited LPC (VSELP) vocoder. The CELP and VSELP vocoders employ vector-quantized excitation codebooks to achieve communication quality speech at low bit rates.

Before concluding this section, we consider the application of waveform encoding and LPC to the encoding of speech signals and compare the bit rates of these coding techniques.

Encoding Methods Applied to Speech Signals The transmission of speech signals over telephone lines, radio channels, and satellite channels constitutes by far the largest part of our daily communications. It is understandable, therefore, that over the past three decades more research has been performed on speech encoding than on any other type of information-bearing signal. In fact, all the encoding techniques described in this section have been applied to the encoding of speech signals. It is appropriate, therefore, to compare the efficiency of these methods in terms of the bit rate required to transmit the speech signal.

The speech signal is assumed to be band-limited to the frequency range 200–3200 Hz and sampled at a nominal rate of 8000 samples/s for all encoders except DM, where the sampling rate is f_s , identical to the bit rate. For an LPC encoder, the parameters given in Example 3-5-1 are assumed.

Table 3-5-2 summarizes the main characteristics of the encoding methods described in this section and the required bit rate. In terms of the quality of the speech signal synthesized at the receiver from the (error-free) binary sequence, all the waveform encoding methods (PCM, DPCM, ADPCM, DM, ADM) provide telephone (toll) quality speech. In other words, a listener would have difficulty discerning the difference between the digitized speech and the analog speech waveform. ADPCM and ADM are particularly efficient waveform encoding techniques. With CVSD, it is possible to operate down to 9600 bits/s

TABLE 3-5-2 ENCODING TECHNIQUES APPLIED TO SPEECH SIGNALS

Encoding method	Quantizer	Coder	Transmission rate (bits/s)
PCM	Linear	12 bits	96 000
Log PCM	Logarithmic	7–8 bits	56 000–64 000
DPCM	Logarithmic	4–6 bits	32 000–48 000
ADPCM	Adaptive	3–4 bits	24 000–32 000
DM	Binary	1 bit	32 000–64 000
ADM	Adaptive binary	1 bit	16 000–32 000
LPC			2400–4800

with some noticeable waveform distortion. In fact, at rates below 16 000 bits/s, the distortion produced by waveform encoders increases significantly. Consequently, these techniques are not used below 9600 bits/s.

For rates below 9600 bits/s, encoding techniques, such as LPC, that are based on linear models of the source are usually employed. The synthesized speech obtained from this class of encoding techniques is intelligible. However, the speech signal has a synthetic quality and there is noticeable distortion.

3-6 BIBLIOGRAPHICAL NOTES AND REFERENCES

Source coding has been an area of intense research activity since the publication of Shannon's classic papers in 1948 and the paper by Huffman (1952). Over the years, major advances have been made in the development of highly efficient source data compression algorithms. Of particular significance is the research on universal source coding and universal quantization published by Ziv (1985), Ziv and Lempel (1977, 1978), Davisson (1973), Gray (1975), and Davisson *et al.* (1981).

Treatments of rate distortion theory are found in the books by Gallager (1968), Berger (1971), Viterbi and Omura (1979), Blahut (1987) and Gray (1990).

Much work has been done over the past several decades on speech encoding methods. Our treatment provides an overview of this important topic. A more comprehensive treatment is given in the books by Rabiner and Schafer (1978), Jayant and Noll (1984), and Deller *et al.* (1993). In addition to these texts, there have been special issues of the *IEEE Transactions on Communications* (April 1979 and April 1982) and, more recently, the *IEEE Journal on Selected Areas in Communications* (February 1988) devoted to speech encoding. We should also mention the publication by IEEE Press of a book containing reprints of published papers on waveform quantization and coding, edited by Jayant (1976).

Over the past decade, we have also seen a number of important developments in vector quantization. Our treatment of this topic was based on the tutorial paper by Makhoul *et al.* (1985). A comprehensive treatment of vector quantization and signal compression is provided in the book by Gersho and Gray (1992).

PROBLEMS

- 3-1 Consider the joint experiment described in Problem 2-1 with the given joint probabilities $P(A_i, B_j)$. Suppose we observe the outcomes A_i , $i = 1, 2, 3, 4$ of experiment A .
- Determine the mutual information $I(B_j; A_i)$ for $j = 1, 2, 3$ and $i = 1, 2, 3, 4$, in bits.
 - Determine the average mutual information $I(B; A)$.

- 3-2 Suppose the outcomes B_j , $j = 1, 2, 3$, in Problem 3-1 represent the three possible output letters from the DMS. Determine the entropy of the source.
- 3-3 Prove that $\ln u \leq u - 1$ and also demonstrate the validity of this inequality by plotting $\ln u$ and $u - 1$ on the same graph.
- 3-4 X and Y are two discrete random variables with probabilities

$$P(X = x, Y = y) \equiv P(x, y)$$

Show that $I(X; Y) \geq 0$, with equality if and only if X and Y are statistically independent.

[Hint: Use the inequality $\ln u < u - 1$, for $0 < u < 1$, to show that $-I(X; Y) \leq 0$.]

- 3-5 The output of a DMS consists of the possible letters x_1, x_2, \dots, x_n , which occur with probabilities p_1, p_2, \dots, p_n , respectively. Prove that the entropy $H(X)$ of the source is at most $\log n$.
- 3-6 Determine the differential entropy $H(X)$ of the uniformly distributed random variable X with pdf

$$p(x) = \begin{cases} a^{-1} & (0 \leq x \leq a) \\ 0 & (\text{otherwise}) \end{cases}$$

for the following three cases:

- a $a = 1$;
 b $a = 4$;
 c $a = \frac{1}{4}$.

Observe from these results that $H(X)$ is not an absolute measure, but only a relative measure of randomness.

- 3-7 A DMS has an alphabet of eight letters, x_i , $i = 1, 2, \dots, 8$, with probabilities 0.25, 0.20, 0.15, 0.12, 0.10, 0.08, 0.05, and 0.05.
- a Use the Huffman encoding procedure to determine a binary code for the source output.
- b Determine the average number \bar{R} of binary digits per source letter.
- c Determine the entropy of the source and compare it with \bar{R} .
- 3-8 A DMS has an alphabet of five letters, x_i , $i = 1, 2, \dots, 5$, each occurring with probability $\frac{1}{5}$. Evaluate the efficiency of a fixed-length binary code in which
- a each letter is encoded separately into a binary sequence;
- b two letters at a time are encoded into a binary sequence;
- c three letters at a time are encoded into a binary sequence.
- 3-9 Recall (3-2-6):

$$I(x_i; y_j) = I(x_i) - I(x_i | y_j)$$

Prove that

- a $I(x_i; y_j) = I(y_j) - I(y_j | x_i)$;
- b $I(x_i; y_j) = I(x_i) + I(y_j) - I(x_i, y_j)$, where $I(x_i, y_j) = -\log P(x_i, y_j)$.
- 3-10 Let X be a geometrically distributed random variable; that is,

$$p(X = k) = p(1 - p)^{k-1}, \quad k = 1, 2, 3, \dots$$

- a Find the entropy of X .
- b Knowing that $X > K$, where K is a positive integer, what is the entropy of X ?

- 3-11** Let X and Y denote two jointly distributed discrete valued random variables.
 a Show that

$$H(X) = - \sum_{x,y} P(x, y) \log P(x)$$

$$H(Y) = - \sum_{x,y} P(x, y) \log P(y)$$

- b Use the above result to show that

$$H(X, Y) \leq H(X) + H(Y)$$

When does equality hold?

- c Show that

$$H(X | Y) \leq H(X)$$

with equality if and only if X and Y are independent.

- 3-12** Two binary random variables X and Y are distributed according to the joint distributions $p(X = Y = 0) = p(X = 0, Y = 1) = p(X = Y = 1) = \frac{1}{3}$. Compute $H(X)$, $H(Y)$, $H(X | Y)$, $H(Y | X)$, and $H(X, Y)$.
3-13 A Markov process is a process with one-step memory, i.e., a process such that

$$p(x_n | x_{n-1}, x_{n-2}, x_{n-3}, \dots) = p(x_n | x_{n-1})$$

for all n . Show that, for a stationary Markov process, the entropy rate is given by $H(X_n | X_{n-1})$.

- 3-14** Let $Y = g(X)$, where g denotes a deterministic function. Show that, in general, $H(Y) \leq H(X)$. When does equality hold?
3-15 Show that $I(X; Y) = H(X) + H(Y) - H(XY)$.
3-16 Show that, for statistically independent events,

$$H(X_1 X_2 \dots X_n) = \sum_{i=1}^n H(X_i)$$

- 3-17** For a noiseless channel, show that $H(X | Y) = 0$.
3-18 Show that

$$I(X_3; X_2 | X_1) = H(X_3 | X_1) - H(X_3 | X_1 X_2)$$

and that

$$H(X_3 | X_1) \geq H(X_3 | X_1 X_2)$$

- 3-19** Let X be a random variable with pdf $p_X(x)$ and let $Y = aX + b$ be a linear transformation of X , where a and b are two constants. Determine the differential entropy $H(Y)$ in terms of $H(X)$.
3-20 The outputs x_1 , x_2 , and x_3 of a DMS with corresponding probabilities $p_1 = 0.45$, $p_2 = 0.35$, and $p_3 = 0.20$ are transformed by the linear transformation $Y = aX + b$, where a and b are constants. Determine the entropy $H(Y)$ and comment on what effect the transformation has had on the entropy of X .
3-21 The optimum four-level nonuniform quantizer for a gaussian-distributed signal amplitude results in the four levels a_1 , a_2 , a_3 , and a_4 , with corresponding probabilities of occurrence $p_1 = p_2 = 0.3365$ and $p_3 = p_4 = 0.1635$.

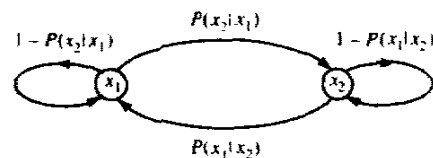


FIGURE P3-22

- a Design a Huffman code that encodes a single level at a time and determine the average bit rate.
- b Design a Huffman code that encodes two output levels at a time and determine the average bit rate.
- c What is the minimum rate obtained by encoding J output levels at a time as $J \rightarrow \infty$?
- 3-22 A first-order Markov source is characterized by the state probabilities $P(x_i)$, $i = 1, 2, \dots, L$, and the transition probabilities $P(x_k | x_i)$, $k = 1, 2, \dots, L$, and $k \neq i$. The entropy of the Markov source is

$$H(X) = \sum_{k=1}^L P(x_k) H(X | x_k)$$

where $H(X | x_k)$ is the entropy conditioned on the source being in state x_k .

- Determine the entropy of the binary, first-order Markov source shown in Fig. P3-22, which has the transition probabilities $P(x_2 | x_1) = 0.2$ and $P(x_1 | x_2) = 0.3$. [Note that the conditional entropies $H(X | x_1)$ and $H(X | x_2)$ are given by the binary entropy functions $H[P(x_2 | x_1)]$ and $H[P(x_1 | x_2)]$, respectively.] How does the entropy of the Markov source compare with the entropy of a binary DMS with the same output letter probabilities $P(x_1)$ and $P(x_2)$?
- 3-23 A memoryless source has the alphabet $\mathcal{X} = \{-5, -3, -1, 0, 1, 3, 5\}$, with corresponding probabilities $\{0.05, 0.1, 0.1, 0.15, 0.05, 0.25, 0.3\}$.
- a Find the entropy of the source.
- b Assuming that the source is quantized according to the quantization rule

$$\begin{aligned} q(-5) &= q(-3) = 4 \\ q(-1) &= q(0) = q(1) = 0 \\ q(3) &= q(5) = 4 \end{aligned}$$

find the entropy of the quantized source.

- 3-24 Design a *ternary* Huffman code, using 0, 1, and 2 as letters, for a source with output alphabet probabilities given by $\{0.05, 0.1, 0.15, 0.17, 0.18, 0.22, 0.13\}$. What is the resulting average codeword length? Compare the average codeword length with the entropy of the source. (In what base would you compute the logarithms in the expression for the entropy for a meaningful comparison?)
- 3-25 Find the Lempel–Ziv source code for the binary source sequence

000100100000011000010000000100000010100001000000110100000001100

Recover the original sequence back from the Lempel–Ziv source code.

[Hint: You require two passes of the binary sequence to decide on the size of the dictionary.]

- 3-26 Find the differential entropy of the continuous random variable X in the following cases:

- a X is an exponential random variable with parameter $\lambda > 0$, i.e.,

$$f_X(x) = \begin{cases} \lambda^{-1} e^{-x/\lambda} & (x > 0) \\ 0 & (\text{otherwise}) \end{cases}$$

- b X is a Laplacian random variable with parameter $\lambda > 0$, i.e.,

$$f_X(x) = \frac{1}{2\lambda} e^{-|x|/\lambda}$$

- c X is a triangular random variable with parameter $\lambda > 0$, i.e.,

$$f_X(x) = \begin{cases} (x + \lambda)/\lambda^2 & (-\lambda \leq x \leq 0) \\ (-x + \lambda)/\lambda^2 & (0 < x \leq \lambda) \\ 0 & (\text{otherwise}) \end{cases}$$

- 3-27 It can be shown that the rate-distortion function for a Laplacian source, $f_X(x) = (2\lambda)^{-1} e^{-|x|/\lambda}$ with an absolute value of error-distortion measure $d(x, \hat{x}) = |x - \hat{x}|$ is given by

$$R(D) = \begin{cases} \log(\lambda/D) & (0 \leq D \leq \lambda) \\ 0 & (D > \lambda) \end{cases}$$

(see Berger, 1971).

- a How many bits per sample are required to represent the outputs of this source with an average distortion not exceeding $\frac{1}{2}\lambda$?
 b Plot $R(D)$ for three different values of λ and discuss the effect of changes in λ on these plots.
- 3-28 It can be shown that if X is a zero-mean continuous random variable with variance σ^2 , its rate distortion function, subject to squared error distortion measure, satisfies the lower and upper bounds given by the inequalities

$$h(X) - \frac{1}{2} \log 2\pi e D \leq R(D) \leq \frac{1}{2} \log \frac{1}{2} \sigma^2$$

where $h(X)$ denotes the differential entropy of the random variable X (see Cover and Thomas, 1991).

- a Show that, for a Gaussian random variable, the lower and upper bounds coincide.
 b Plot the lower and upper bounds for a Laplacian source with $\sigma = 1$.
 c Plot the lower and upper bounds for a triangular source with $\sigma = 1$.
- 3-29 A stationary random process has an autocorrelation function given by $R_X = \frac{1}{2} A^2 e^{-|\tau|} \cos 2\pi f_0 \tau$ and it is known that the random process never exceeds 6 in magnitude. Assuming $A = 6$, how many quantization levels are required to guarantee a signal-to-quantization noise ratio of at least 60 dB?
- 3-30 An additive white gaussian noise channel has the output $Y = X + G$, where X is the channel input and G is the noise with probability density function

$$p(n) = \frac{1}{\sqrt{2\pi} \sigma_n} e^{-n^2/2\sigma_n^2}$$

If X is a white gaussian input with $E(X) = 0$ and $E(X^2) = \sigma_x^2$, determine

- a the conditional differential entropy $H(X|G)$;
 b the average mutual information $I(X; Y)$.
- 3-31 A DMS has an alphabet of eight letters, x_i , $i = 1, 2, \dots, 8$, with probabilities

given in Problem 3-7. Use the Huffman encoding procedure to determine a ternary code (using symbols 0, 1, and 2) for encoding the source output.

[Hint: Add a symbol x_0 with probability $p_0 = 0$, and group three symbols at a time.]

- 3-32** Determine whether there exists a binary code with code word lengths $(n_1, n_2, n_3, n_4) = (1, 2, 2, 3)$ that satisfy the prefix condition.
- 3-33** Consider a binary block code with 2^n code words of the same length n . Show that the Kraft inequality is satisfied for such a code.
- 3-34** Show that the entropy of an n -dimensional gaussian vector $\mathbf{X} = [x_1 \ x_2 \ \dots \ x_n]$ with zero mean and covariance matrix \mathbf{M} is

$$H(\mathbf{X}) = \frac{1}{2} \log_2 (2\pi e)^n |\mathbf{M}|$$

- 3-35** Consider a DMS with output bits (0, 1) that are equiprobable. Define the distortion measure as $D = P_e$, where P_e is the probability of error in transmitting the binary symbols to the user over a BSC. Then the rate distortion function is (Berger, 1971)

$$R(D) = 1 + D \log_2 D + (1 - D) \log_2 (1 - D), \quad 0 \leq D = P_e \leq \frac{1}{2}$$

Plot $R(D)$ for $0 \leq D \leq \frac{1}{2}$.

- 3-36** Evaluate the rate distortion function for an M -ary symmetric channel where $D = P_M$ and

$$R(D) = \log_2 M + D \log_2 D + (1 - D) \log_2 \frac{1 - D}{M - 1}$$

for $M = 2, 4, 8$, and 16 . P_M is the probability of error.

- 3-37** Consider the use of the weighted mean-square-error (MSE) distortion measure defined as

$$d_w(\mathbf{X}, \tilde{\mathbf{X}}) = (\mathbf{X} - \tilde{\mathbf{X}})' \mathbf{W} (\mathbf{X} - \tilde{\mathbf{X}})$$

where \mathbf{W} is a symmetric, positive-definite weighting matrix. By factorizing \mathbf{W} as $\mathbf{W} = \mathbf{P}'\mathbf{P}$, show that $d_w(\mathbf{X}, \tilde{\mathbf{X}})$ is equivalent to an unweighted MSE distortion measure $d_2(\mathbf{X}', \tilde{\mathbf{X}}')$ involving transformed vectors \mathbf{X}' and $\tilde{\mathbf{X}}'$.

- 3-38** Consider a stationary stochastic signal sequence $\{X(n)\}$ with zero mean and autocorrelation sequence

$$\phi(n) = \begin{cases} 1 & (n = 0) \\ \frac{1}{2} & (n = \pm 1) \\ 0 & (\text{otherwise}) \end{cases}$$

- a** Determine the prediction coefficient of the first-order minimum MSE predictor for $\{X(n)\}$ given by

$$\hat{x}(n) = a_1 x(n - 1)$$

and the corresponding minimum mean square error \mathcal{E}_1 .

- b** Repeat (a) for the second-order predictor

$$\hat{x}(n) = a_1 x(n - 1) + a_2 x(n - 2)$$

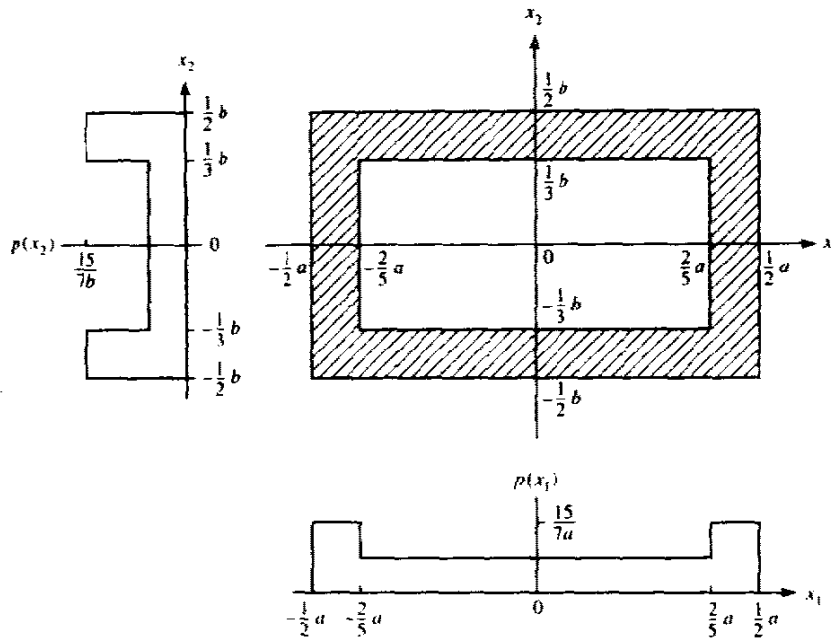


FIGURE P3-39

3-39 Consider the encoding of the random variables x_1 and x_2 that are characterized by the joint pdf $p(x_1, x_2)$ given by

$$p(x_1, x_2) = \begin{cases} 15/7ab & (x_1, x_2 \in C) \\ 0 & (\text{otherwise}) \end{cases}$$

as shown in Fig. P3-39. Evaluate the bit rates required for uniform quantization of x_1 and x_2 separately (scalar quantization) and combined (vector) quantization of (x_1, x_2) . Determine the difference in bit rate when $a = 4b$.

3-40 Consider the encoding of two random variables X and Y that are uniformly distributed on the region between two squares as shown in Fig. P3-40.

- a Find $f_X(x)$ and $f_Y(y)$.
- b Assume that each of the random variables X and Y are quantized using four level uniform quantizers. What is the resulting distortion? What is the resulting number of bits per (X, Y) pair?

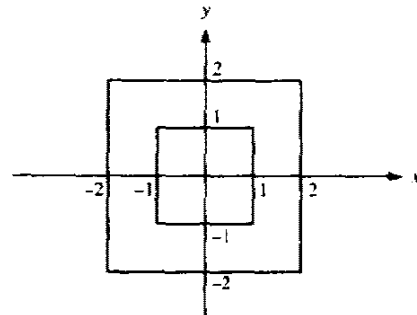


FIGURE P3-40

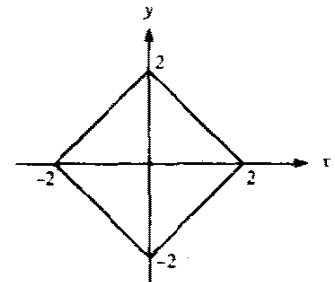


FIGURE P3-41

- c Now assume that instead of scalar quantizers for X and Y , we employ a vector quantizer to achieve the same level of distortion as in (b). What is the resulting number of bits per source output pair (X, Y) ?
- 3-41** Two random variables X and Y are uniformly distributed on the square shown in Fig. P3-41.
- Find $f_X(x)$ and $f_Y(y)$.
 - Assume that each of the random variables X and Y are quantized using four level uniform quantizers. What is the resulting distortion? What is the resulting number of bits per (X, Y) pair?
 - Now assume that, instead of scalar quantizers for X and Y , we employ a vector quantizer with the same number of bits per source output pair (X, Y) as in (b). What is the resulting distortion for this vector quantizer?

4

CHARACTERIZATION OF COMMUNICATION SIGNALS AND SYSTEMS

Signals can be categorized in a number of different ways, such as random versus deterministic, discrete time versus continuous time, discrete amplitude versus continuous amplitude, lowpass versus bandpass, finite energy versus infinite energy, finite average power versus infinite average power, etc. In this chapter, we treat the characterization of signals and systems that are usually encountered in the transmission of digital information over a communication channel. In particular, we introduce the representation of various forms of digitally modulated signals and describe their spectral characteristics.

We begin with the characterization of bandpass signals and systems, including the mathematical representation of bandpass stationary stochastic processes. Then, we present a vector space representation of signals. We conclude with the representation of digitally modulated signals and their spectral characteristics.

4-1 REPRESENTATION OF BANDPASS SIGNALS AND SYSTEMS

Many digital information-bearing signals are transmitted by some type of carrier modulation. The channel over which the signal is transmitted is limited in bandwidth to an interval of frequencies centered about the carrier, as in double-sideband modulation, or adjacent to the carrier, as in single-sideband modulation. Signals and channels (systems) that satisfy the condition that their bandwidth is much smaller than the carrier frequency are termed *narrowband bandpass signals and channels (systems)*. The modulation performed at the

152

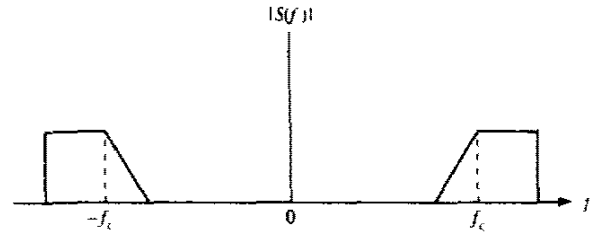


FIGURE 4-1-1 Spectrum of a bandpass signal.

transmitting end of the communication system to generate the bandpass signal and the demodulation performed at the receiving end to recover the digital information involve frequency translations. With no loss of generality and for mathematical convenience, it is desirable to reduce all bandpass signals and channels to equivalent lowpass signals and channels. As a consequence, the results of the performance of the various modulation and demodulation techniques presented in the subsequent chapters are independent of carrier frequencies and channel frequency bands. The representation of bandpass signals and systems in terms of equivalent lowpass waveforms and the characterization of bandpass stationary stochastic processes are the main topics of this section.

4-1-1 Representation of Bandpass Signals

Suppose that a real-valued signal $s(t)$ has a frequency content concentrated in a narrow band of frequencies in the vicinity of a frequency f_c , as shown in Fig. 4-1-1. Our objective is to develop a mathematical representation of such signals. First, we construct a signal that contains only the positive frequencies in $s(t)$. Such a signal may be expressed as

$$S_+(f) = 2u(f)S(f) \quad (4-1-1)$$

where $S(f)$ is the Fourier transform of $s(t)$ and $u(f)$ is the unit step function. The equivalent time-domain expression for (4-1-1) is

$$\begin{aligned} s_+(t) &= \int_{-\infty}^{\infty} S_+(f)e^{j2\pi ft} df \\ &= F^{-1}[2u(f)] \star F^{-1}[S(f)] \end{aligned} \quad (4-1-2)$$

The signal $s_+(t)$ is called the *analytic signal* or the *pre-envelope* of $s(t)$. We note that $F^{-1}[S(f)] = s(t)$ and

$$F^{-1}[2u(f)] = \delta(t) + \frac{j}{\pi t} \quad (4-1-3)$$

Hence,

$$\begin{aligned} s_+(t) &= \left[\delta(t) + \frac{j}{\pi t} \right] \star s(t) \\ &= s(t) + j \frac{1}{\pi t} \star s(t) \end{aligned} \quad (4-1-4)$$

We define $\hat{s}(t)$ as

$$\begin{aligned} \hat{s}(t) &= \frac{1}{\pi t} \star s(t) \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{s(\tau)}{t - \tau} d\tau \end{aligned} \quad (4-1-5)$$

The signal $\hat{s}(t)$ may be viewed as the output of the filter with impulse response

$$h(t) = \frac{1}{\pi t}, \quad -\infty < t < \infty \quad (4-1-6)$$

when excited by the input signal $s(t)$. Such a filter is called a *Hilbert transformer*. The frequency response of this filter is simply

$$\begin{aligned} H(f) &= \int_{-\infty}^{\infty} h(t) e^{-j2\pi f t} dt \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{t} e^{-j2\pi f t} dt \\ &= \begin{cases} -j & (f > 0) \\ 0 & (f = 0) \\ j & (f < 0) \end{cases} \end{aligned} \quad (4-1-7)$$

We observe that $|H(f)| = 1$ and that the phase response $\Theta(f) = -\frac{1}{2}\pi$ for $f > 0$ and $\Theta(f) = \frac{1}{2}\pi$ for $f < 0$. Therefore, this filter is basically a 90° phase shifter for all frequencies in the input signal.

The analytic signal $s_+(t)$ is a bandpass signal. We may obtain an equivalent lowpass representation by performing a frequency translation of $S_+(f)$. Thus, we define $S_I(f)$ as

$$S_I(f) = S_+(f + f_c) \quad (4-1-8)$$

The equivalent time-domain relation is

$$\begin{aligned} s_I(t) &= s_+(t) e^{-j2\pi f_c t} \\ &= [s(t) + j\hat{s}(t)] e^{-j2\pi f_c t} \end{aligned} \quad (4-1-9)$$

or, equivalently,

$$s(t) + j\hat{s}(t) = s_I(t) e^{j2\pi f_c t} \quad (4-1-10)$$

In general, the signal $s_I(t)$ is complex-valued (see Problem 4-5), and may be expressed as

$$s_I(t) = x(t) + jy(t) \quad (4-1-11)$$

If we substitute for $s_i(t)$ in (4-1-11) and equate real and imaginary parts on each side, we obtain the relations

$$s(t) = x(t) \cos 2\pi f_c t - y(t) \sin 2\pi f_c t \quad (4-1-12)$$

$$\hat{s}(t) = x(t) \sin 2\pi f_c t + y(t) \cos 2\pi f_c t \quad (4-1-13)$$

The expression (4-1-12) is the desired form for the representation of a bandpass signal. The low-frequency signal components $x(t)$ and $y(t)$ may be viewed as amplitude modulations impressed on the carrier components $\cos 2\pi f_c t$ and $\sin 2\pi f_c t$, respectively. Since these carrier components are in phase quadrature, $x(t)$ and $y(t)$ are called the *quadrature components* of the bandpass signal $s(t)$.

Another representation of the signal in (4-1-12) is

$$\begin{aligned} s(t) &= \text{Re} \{ [x(t) + jy(t)] e^{j2\pi f_c t} \} \\ &= \text{Re} [s_i(t) e^{j2\pi f_c t}] \end{aligned} \quad (4-1-14)$$

where Re denotes the real part of the complex-valued quantity in the brackets following. The lowpass signal $s_i(t)$ is usually called the *complex envelope* of the real signal $s(t)$, and is basically the *equivalent lowpass signal*.

Finally, a third possible representation of a bandpass signal is obtained by expressing $s_i(t)$ as

$$s_i(t) = a(t) e^{j\theta(t)} \quad (4-1-15)$$

where

$$a(t) = \sqrt{x^2(t) + y^2(t)} \quad (4-1-16)$$

$$\theta(t) = \tan^{-1} \frac{y(t)}{x(t)} \quad (4-1-17)$$

Then

$$\begin{aligned} s(t) &= \text{Re} [s_i(t) e^{j2\pi f_c t}] \\ &= \text{Re} [a(t) e^{j[2\pi f_c t + \theta(t)]}] \\ &= a(t) \cos [2\pi f_c t + \theta(t)] \end{aligned} \quad (4-1-18)$$

The signal $a(t)$ is called the *envelope* of $s(t)$, and $\theta(t)$ is called the *phase* of $s(t)$. Therefore, (4-1-12), (4-1-14), and (4-1-18) are equivalent representations of bandpass signals.

The Fourier transform of $s(t)$ is

$$\begin{aligned} S(f) &= \int_{-\infty}^{\infty} s(t) e^{-j2\pi f t} dt \\ &= \int_{-\infty}^{\infty} \{ \text{Re} [s_i(t) e^{j2\pi f_c t}] \} e^{-j2\pi f t} dt \end{aligned} \quad (4-1-19)$$

Use of the identity

$$\text{Re}(\xi) = \frac{1}{2}(\xi + \xi^*) \quad (4-1-20)$$

in (4-1-19) yields the result

$$\begin{aligned} S(f) &= \frac{1}{2} \int_{-\infty}^{\infty} [s_i(t)e^{j2\pi f_c t} + s_i^*(t)e^{-j2\pi f_c t}] e^{-j2\pi f t} dt \\ &= \frac{1}{2} [S_i(f - f_c) + S_i^*(-f - f_c)] \end{aligned} \quad (4-1-21)$$

where $S_i(f)$ is the Fourier transform of $s_i(t)$. This is the basic relationship between the spectrum of the real bandpass signal $S(f)$ and the spectrum of the equivalent lowpass signal $S_i(f)$.

The energy in the signal $s(t)$ is defined as

$$\begin{aligned} \mathcal{E} &= \int_{-\infty}^{\infty} s^2(t) dt \\ &= \int_{-\infty}^{\infty} \{\text{Re} [s_i(t)e^{j2\pi f_c t}]\}^2 dt \end{aligned} \quad (4-1-22)$$

When the identity in (4-1-20) is used in (4-1-22), we obtain the following result:

$$\begin{aligned} \mathcal{E} &= \frac{1}{2} \int_{-\infty}^{\infty} |s_i(t)|^2 dt \\ &\quad + \frac{1}{2} \int_{-\infty}^{\infty} |s_i(t)|^2 \cos [4\pi f_c t + 2\theta(t)] dt \end{aligned} \quad (4-1-23)$$

Consider the second integral in (4-1-23). Since the signal $s(t)$ is narrowband, the real envelope $a(t) \equiv |s_i(t)|$ or, equivalently, $a^2(t)$ varies slowly relative to the rapid variations exhibited by the cosine function. A graphical illustration of the integrand in the second integral of (4-1-23) is shown in Fig. 4-1-2. The value of the integral is just the net area under the cosine function modulated by $a^2(t)$. Since the modulating waveform $a^2(t)$ varies slowly relative to the cosine function, the net area contributed by the second integral is very small relative to the value of the first integral in (4-1-23) and, hence, it can be

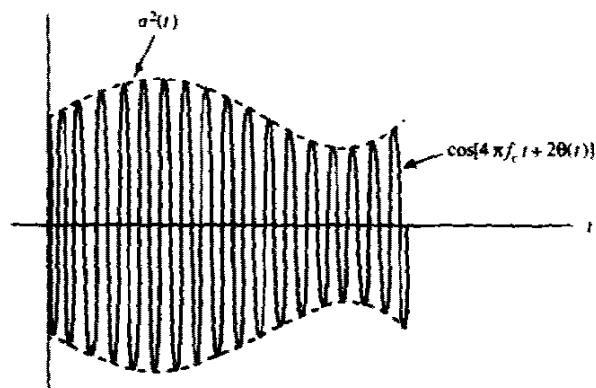


FIGURE 4-1-2 The signal $a^2(t) \cos [4\pi f_c t + 2\theta(t)]$.

neglected. Thus, for all practical purposes, the energy in the bandpass signal $s(t)$, expressed in terms of the equivalent lowpass signal $s_l(t)$, is

$$\mathcal{E} = \frac{1}{2} \int_{-\infty}^{\infty} |s_l(t)|^2 dt \quad (4-1-24)$$

where $|s_l(t)|$ is just the envelope $a(t)$ of $s(t)$.

4-1-2 Representation of Linear Bandpass Systems

A linear filter or system may be described either by its impulse response $h(t)$ or by its frequency response $H(f)$, which is the Fourier transform of $h(t)$. Since $h(t)$ is real,

$$H^*(-f) = H(f) \quad (4-1-25)$$

Let us define $H_l(f - f_c)$ as

$$H_l(f - f_c) = \begin{cases} H(f) & (f > 0) \\ 0 & (f < 0) \end{cases} \quad (4-1-26)$$

Then

$$H_l^*(-f - f_c) = \begin{cases} 0 & (f > 0) \\ H^*(-f) & (f < 0) \end{cases} \quad (4-1-27)$$

Using (4-1-25), we have

$$H(f) = H_l(f - f_c) + H_l^*(-f - f_c) \quad (4-1-28)$$

which resembles (4-1-21) except for the factor $\frac{1}{2}$. The inverse transform of $H(f)$ in (4-1-28) yields $h(t)$ in the form

$$\begin{aligned} h(t) &= h_l(t)e^{j2\pi f_c t} + h_l^*(t)e^{-j2\pi f_c t} \\ &= 2 \operatorname{Re} [h_l(t)e^{j2\pi f_c t}] \end{aligned} \quad (4-1-29)$$

where $h_l(t)$ is the inverse Fourier transform of $H_l(f)$. In general, the impulse response $h_l(t)$ of the equivalent lowpass system is complex-valued.

4-1-3 Response of a Bandpass System to a Bandpass Signal

In Sections 4-1-1 and 4-1-2, we have shown that narrowband bandpass signals and systems can be represented by equivalent lowpass signals and systems. In this section, we demonstrate that the output of a bandpass system to a

bandpass input signal is simply obtained from the equivalent lowpass input signal and the equivalent lowpass impulse response of the system.

Suppose that $s(t)$ is a narrowband bandpass signal and $s_l(t)$ is the equivalent lowpass signal. This signal excites a narrowband bandpass system characterized by its bandpass impulse response $h(t)$ or by its equivalent lowpass impulse response $h_l(t)$. The output of the bandpass system is also a bandpass signal, and, therefore, it can be expressed in the form

$$r(t) = \text{Re} [r_l(t)e^{j2\pi f_c t}] \quad (4-1-30)$$

where $r(t)$ is related to the input signal $s(t)$ and the impulse response $h(t)$ by the convolution integral

$$r(t) = \int_{-\infty}^{\infty} s(\tau)h(t - \tau) d\tau \quad (4-1-31)$$

Equivalently, the output of the system, expressed in the frequency domain, is

$$R(f) = S(f)H(f) \quad (4-1-32)$$

Substituting from (4-1-21) for $S(f)$ and from (4-1-28) for $H(f)$, we obtain the result

$$R(f) = \frac{1}{2}\{S_l(f - f_c) + S_l^*(-f - f_c)\}[H_l(f - f_c) + H_l^*(-f - f_c)] \quad (4-1-33)$$

When $s(t)$ is a narrowband signal and $h(t)$ is the impulse response of a narrowband system, $S_l(f - f_c) \approx 0$ and $H_l(f - f_c) = 0$ for $f < 0$. It follows from this narrowband condition that

$$S_l(f - f_c)H_l^*(-f - f_c) = 0, \quad S_l^*(-f - f_c)H_l(f - f_c) = 0$$

Therefore, (4-1-33) simplifies to

$$\begin{aligned} R(f) &= \frac{1}{2}\{S_l(f - f_c)H_l(f - f_c) + S_l^*(-f - f_c)H_l^*(-f - f_c)\} \\ &= \frac{1}{2}\{R_l(f - f_c) + R_l^*(-f - f_c)\} \end{aligned} \quad (4-1-34)$$

where

$$R_l(f) = S_l(f)H_l(f) \quad (4-1-35)$$

is the output spectrum of the equivalent lowpass system excited by the equivalent lowpass signal. It is clear that the time domain relation for the output $r_l(t)$ is given by the convolution of $s_l(t)$ with $h_l(t)$. That is,

$$r_l(t) = \int_{-\infty}^{\infty} s_l(\tau)h_l(t - \tau) d\tau \quad (4-1-36)$$

The combination of (4-1-36) with (4-1-30) gives the relationship between the bandpass output signal $r(t)$ and the equivalent lowpass time functions $s_l(t)$ and $h_l(t)$. This simple relationship allows us to ignore any linear frequency translations encountered in the modulation of a signal for purposes of matching its spectral content to the frequency allocation of a particular channel. Thus, for mathematical convenience, we shall deal only with the transmission of equivalent lowpass signals through equivalent lowpass channels.

4-1-4 Representation of Bandpass Stationary Stochastic Processes

The representation of bandpass signals presented in Section 4-1-1 applied to deterministic signals. In this section, we extend the representation to sample functions of a bandpass stationary stochastic process. In particular, we derive the important relations between the correlation functions and power spectra of the bandpass signal and the correlation functions and power spectra of the equivalent lowpass signal.

Suppose that $n(t)$ is a sample function of a wide-sense stationary stochastic process with zero mean and power spectral density $\Phi_{nn}(f)$. The power spectral density is assumed to be zero outside of an interval of frequencies centered around $\pm f_c$, where f_c is termed the *carrier frequency*. The stochastic process $n(t)$ is said to be a *narrowband bandpass process* if the width of the spectral density is much smaller than f_c . Under this condition, a sample function of the process $n(t)$ can be represented by any of the three equivalent forms given in Section 4-1-1, namely,

$$n(t) = a(t) \cos [2\pi f_c t + \theta(t)] \quad (4-1-37)$$

$$= x(t) \cos 2\pi f_c t - y(t) \sin 2\pi f_c t \quad (4-1-38)$$

$$= \text{Re} [z(t)e^{j2\pi f_c t}] \quad (4-1-39)$$

where $a(t)$ is the envelope and $\theta(t)$ is the phase of the real-valued signal, $x(t)$ and $y(t)$ are the quadrature components of $n(t)$, and $z(t)$ is called the *complex envelope* of $n(t)$.

Let us consider the form given by (4-1-38) in more detail. First, we observe that if $n(t)$ is zero mean, then $x(t)$ and $y(t)$ must also have zero mean values. In addition, the stationarity of $n(t)$ implies that the autocorrelation and cross-correlation functions of $x(t)$ and $y(t)$ satisfy the following properties:

$$\phi_{xx}(\tau) = \phi_{yy}(\tau) \quad (4-1-40)$$

$$\phi_{xy}(\tau) = -\phi_{yx}(\tau) \quad (4-1-41)$$

That these two properties follow from the stationarity of $n(t)$ is now demonstrated. The autocorrelation function $\phi_{nn}(\tau)$ of $n(t)$ is

$$\begin{aligned}
 E\{n(t)n(t+\tau)\} &= E\{[x(t)\cos 2\pi f_c t - y(t)\sin 2\pi f_c t] \\
 &\quad \times [x(t+\tau)\cos 2\pi f_c(t+\tau) \\
 &\quad - y(t+\tau)\sin 2\pi f_c(t+\tau)]\} \\
 &= \phi_{xx}(\tau)\cos 2\pi f_c t \cos 2\pi f_c(t+\tau) \\
 &\quad + \phi_{yy}(\tau)\sin 2\pi f_c t \sin 2\pi f_c(t+\tau) \\
 &\quad - \phi_{xy}(\tau)\sin 2\pi f_c t \cos 2\pi f_c(t+\tau) \\
 &\quad - \phi_{yx}(\tau)\cos 2\pi f_c t \sin 2\pi f_c(t+\tau) \quad (4-1-42)
 \end{aligned}$$

Use of the trigonometric identities

$$\begin{aligned}
 \cos A \cos B &= \frac{1}{2}[\cos(A-B) + \cos(A+B)] \\
 \sin A \sin B &= \frac{1}{2}[\cos(A-B) - \cos(A+B)] \quad (4-1-43) \\
 \sin A \cos B &= \frac{1}{2}[\sin(A-B) + \sin(A+B)]
 \end{aligned}$$

in (4-1-42) yields the result

$$\begin{aligned}
 E\{n(t)n(t+\tau)\} &= \frac{1}{2}[\phi_{xx}(\tau) + \phi_{yy}(\tau)]\cos 2\pi f_c \tau \\
 &\quad + \frac{1}{2}[\phi_{xx}(\tau) - \phi_{yy}(\tau)]\cos 2\pi f_c(2t+\tau) \\
 &\quad - \frac{1}{2}[\phi_{yx}(\tau) - \phi_{xy}(\tau)]\sin 2\pi f_c \tau \\
 &\quad - \frac{1}{2}[\phi_{yx}(\tau) + \phi_{xy}(\tau)]\sin 2\pi f_c(2t+\tau) \quad (4-1-44)
 \end{aligned}$$

Since $n(t)$ is stationary, the right-hand side of (4-1-44) must be independent of t . But this condition can only be satisfied if (4-1-40) and (4-1-41) hold. As a consequence, (4-1-44) reduces to

$$\phi_{nn}(\tau) = \phi_{xx}(\tau)\cos 2\pi f_c \tau - \phi_{yx}(\tau)\sin 2\pi f_c \tau \quad (4-1-45)$$

We note that the relation between the autocorrelation function $\phi_{nn}(\tau)$ of the bandpass process and the autocorrelation and cross-correlation functions $\phi_{xx}(\tau)$ and $\phi_{yx}(\tau)$ of the quadrature components is identical in form to (4-1-38), which expresses the bandpass process in terms of the quadrature components.

The autocorrelation function of the equivalent lowpass process

$$z(t) = x(t) + jy(t) \quad (4-1-46)$$

is defined as

$$\phi_{zz}(\tau) = \frac{1}{2}E[z^*(t)z(t+\tau)] \quad (4-1-47)$$

Substituting (4-1-46) into (4-1-47) and performing the expectation operation, we obtain

$$\phi_{zz}(\tau) = \frac{1}{2}[\phi_{xx}(\tau) + \phi_{yy}(\tau) - j\phi_{xy}(\tau) + j\phi_{yx}(\tau)] \quad (4-1-48)$$

Now if the symmetry properties given in (4-1-40) and (4-1-41) are used in (4-1-48), we obtain

$$\phi_{zz}(\tau) = \phi_{xx}(\tau) + j\phi_{yx}(\tau) \quad (4-1-49)$$

which relates the autocorrelation function of the complex envelope to the autocorrelation and cross-correlation functions of the quadrature components. Finally, we incorporate the result given by (4-1-49) into (4-1-45), and we have

$$\phi_{nn}(\tau) = \text{Re}[\phi_{zz}(\tau)e^{j2\pi f_c \tau}] \quad (4-1-50)$$

Thus, the autocorrelation function $\phi_{nn}(\tau)$ of the bandpass stochastic process is uniquely determined from the autocorrelation function $\phi_{zz}(\tau)$ of the equivalent lowpass process $z(t)$ and the carrier frequency f_c .

The power density spectrum $\Phi_{nn}(f)$ of the stochastic process $n(t)$ is the Fourier transform of $\phi_{nn}(\tau)$. Hence,

$$\begin{aligned} \Phi_{nn}(f) &= \int_{-\infty}^{\infty} \{\text{Re}[\phi_{zz}(\tau)e^{j2\pi f_c \tau}]\} e^{-j2\pi f \tau} d\tau \\ &= \frac{1}{2}[\Phi_{zz}(f - f_c) + \Phi_{zz}(-f - f_c)] \end{aligned} \quad (4-1-51)$$

where $\Phi_{zz}(f)$ is the power density spectrum of the equivalent lowpass process $z(t)$. Since the autocorrelation function of $z(t)$ satisfies the property $\phi_{zz}(\tau) = \phi_{zz}^*(-\tau)$, it follows that $\Phi_{zz}(f)$ is a real-valued function of frequency.

Properties of the Quadrature Components It was just demonstrated above that the cross-correlation function of the quadrature components $x(t)$ and $y(t)$ of the bandpass stationary stochastic process $n(t)$ satisfies the symmetry condition in (4-1-41). Furthermore, any cross-correlation function satisfies the condition

$$\phi_{yx}(\tau) = \phi_{xy}(-\tau) \quad (4-1-52)$$

From these two conditions, we conclude that

$$\phi_{xy}(\tau) = -\phi_{yx}(-\tau) \quad (4-1-53)$$

That is, $\phi_{xy}(\tau)$ is an odd function of τ . Consequently, $\phi_{xy}(0) = 0$, and, hence, $x(t)$ and $y(t)$ are uncorrelated (for $\tau = 0$, only). Of course, this does not mean that the processes $x(t)$ and $y(t + \tau)$ are uncorrelated for all τ , since that would imply that $\phi_{xy}(\tau) = 0$ for all τ . If, indeed, $\phi_{xy}(\tau) = 0$ for all τ , then $\phi_{zz}(\tau)$ is real and the power spectral density $\Phi_{zz}(f)$ satisfies the condition

$$\Phi_{zz}(f) = \Phi_{zz}(-f) \quad (4-1-54)$$

and vice versa. That is, $\Phi_{zz}(f)$ is symmetric about $f = 0$.

In the special case in which the stationary stochastic process $n(t)$ is gaussian, the quadrature components $x(t)$ and $y(t + \tau)$ are jointly gaussian. Moreover, for $\tau = 0$, they are statistically independent, and, hence, their joint probability density function is

$$p(x, y) = \frac{1}{2\pi\sigma^2} e^{-(x^2+y^2)/2\sigma^2} \tag{4-1-55}$$

where the variance σ^2 is defined as $\sigma^2 = \phi_{xx}(0) = \phi_{yy}(0) = \phi_{nn}(0)$.

Representation of White Noise White noise is a stochastic process that is defined to have a flat (constant) power spectral density over the entire frequency range. This type of noise cannot be expressed in terms of quadrature components, as a result of its wideband character.

In problems concerned with the demodulation of narrowband signals in noise, it is mathematically convenient to model the additive noise process as white and to represent the noise in terms of quadrature components. This can be accomplished by postulating that the signals and noise at the receiving terminal have passed through an ideal bandpass filter, having a passband that includes the spectrum of the signals but is much wider. Such a filter will introduce negligible, if any, distortion on the signal but it does eliminate the noise frequency components outside of the passband.

The noise resulting from passing the white noise process through a spectrally flat (ideal) bandpass filter is termed *bandpass white noise* and has the power spectral density depicted in Fig. 4-1-3. Bandpass white noise can be represented by any of the forms given in (4-1-37), (4-1-38), and (4-1-39). The equivalent lowpass noise $z(t)$ has a power spectral density

$$\Phi_{zz}(f) = \begin{cases} N_0 & (|f| \leq \frac{1}{2}B) \\ 0 & (|f| > \frac{1}{2}B) \end{cases} \tag{4-1-56}$$

and its autocorrelation function is

$$\phi_{zz}(\tau) = N_0 \frac{\sin \pi B \tau}{\pi \tau} \tag{4-1-57}$$

The limiting form of $\phi_{zz}(\tau)$ as B approaches infinity is

$$\phi_{zz}(\tau) = N_0 \delta(\tau) \tag{4-1-58}$$

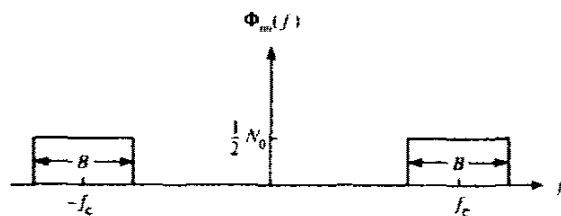


FIGURE 4-1-3 Bandpass noise with a flat spectrum.

The power spectral density for white noise and bandpass white noise is symmetric about $f = 0$, so $\phi_{zz}(\tau) = 0$ for all τ . Therefore,

$$\phi_{z,z}(\tau) = \phi_{x,x}(\tau) = \phi_{y,y}(\tau) \quad (4-1-59)$$

That is, the quadrature components $x(t)$ and $y(t)$ are uncorrelated for all time shifts τ and the autocorrelation functions of $z(t)$, $x(t)$, and $y(t)$ are all equal.

4-2 SIGNAL SPACE REPRESENTATIONS

In this section, we demonstrate that signals have characteristics that are similar to vectors and develop a vector representation for signal waveforms. We begin with some basic definitions and concepts involving vectors.

4-2-1 Vector Space Concepts

A vector \mathbf{v} in an n -dimensional space is characterized by its n components $[v_1 \ v_2 \ \dots \ v_n]$. It may also be represented as a linear combination of *unit vectors* or *basis vectors* \mathbf{e}_i , $1 \leq i \leq n$, i.e.,

$$\mathbf{v} = \sum_{i=1}^n v_i \mathbf{e}_i \quad (4-2-1)$$

where, by definition, a unit vector has length unity and v_i is the projection of the vector \mathbf{v} onto the unit vector \mathbf{e}_i .

The *inner product* of two n -dimensional vectors $\mathbf{v}_1 = [v_{11} \ v_{12} \ \dots \ v_{1n}]$ and $\mathbf{v}_2 = [v_{21} \ v_{22} \ \dots \ v_{2n}]$ is defined as

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = \sum_{i=1}^n v_{1i} v_{2i} \quad (4-2-2)$$

Two vectors \mathbf{v}_1 and \mathbf{v}_2 are orthogonal if $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$. More generally, a set of m vectors \mathbf{v}_k , $1 \leq k \leq m$, are orthogonal if

$$\mathbf{v}_i \cdot \mathbf{v}_j = 0 \quad (4-2-3)$$

for all $1 \leq i, j \leq m$ and $i \neq j$.

The *norm* of a vector \mathbf{v} is denoted by $\|\mathbf{v}\|$ and is defined as

$$\|\mathbf{v}\| = (\mathbf{v} \cdot \mathbf{v})^{1/2} = \sqrt{\sum_{i=1}^n v_i^2} \quad (4-2-4)$$

which is simply its length. A set of m vectors is said to be *orthonormal* if the vectors are orthogonal and each vector has a unit norm. A set of m vectors is said to be *linearly independent* if no one vector can be represented as a linear combination of the remaining vectors.

Two n -dimensional vectors \mathbf{v}_1 and \mathbf{v}_2 satisfy the *triangle inequality*

$$\|\mathbf{v}_1 + \mathbf{v}_2\| \leq \|\mathbf{v}_1\| + \|\mathbf{v}_2\| \quad (4-2-5)$$

with equality if \mathbf{v}_1 and \mathbf{v}_2 are in the same direction, i.e., $\mathbf{v}_1 = a\mathbf{v}_2$ where a is a

positive real scalar. From the triangle inequality there follows the *Cauchy-Schwartz inequality*

$$|\mathbf{v}_1 \cdot \mathbf{v}_2| \leq \|\mathbf{v}_1\| \|\mathbf{v}_2\| \quad (4-2-6)$$

with equality if $\mathbf{v}_1 = a\mathbf{v}_2$. The norm square of the sum of two vectors may be expressed as

$$\|\mathbf{v}_1 + \mathbf{v}_2\|^2 = \|\mathbf{v}_1\|^2 + \|\mathbf{v}_2\|^2 + 2\mathbf{v}_1 \cdot \mathbf{v}_2 \quad (4-2-7)$$

If \mathbf{v}_1 and \mathbf{v}_2 are orthogonal then $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$ and, hence,

$$\|\mathbf{v}_1 + \mathbf{v}_2\|^2 = \|\mathbf{v}_1\|^2 + \|\mathbf{v}_2\|^2 \quad (4-2-8)$$

This is the Pythagorean relation for two orthogonal n -dimensional vectors.

From matrix algebra, we recall that a linear transformation in an n -dimensional vector space is a matrix transformation of the form

$$\mathbf{v}' = \mathbf{A}\mathbf{v} \quad (4-2-9)$$

where the matrix \mathbf{A} transforms the vector \mathbf{v} into some vector \mathbf{v}' . In the special case where $\mathbf{v}' = \lambda\mathbf{v}$, i.e.,

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v} \quad (4-2-10)$$

where λ is some (positive or negative) scalar, the vector \mathbf{v} is called an *eigenvector* of the transformation and λ is the corresponding *eigenvalue*.

Finally, let us review the Gram-Schmidt procedure for constructing a set of orthonormal vectors from a set of n -dimensional vectors \mathbf{v}_i , $1 \leq i \leq m$. We begin by arbitrarily selecting a vector from the set, say \mathbf{v}_1 . By normalizing its length, we obtain the first vector, say

$$\mathbf{u}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} \quad (4-2-11)$$

Next, we may select \mathbf{v}_2 and, first, subtract the projection of \mathbf{v}_2 onto \mathbf{u}_1 . Thus, we obtain

$$\mathbf{u}'_2 = \mathbf{v}_2 - (\mathbf{v}_2 \cdot \mathbf{u}_1)\mathbf{u}_1 \quad (4-2-12)$$

Then, we normalize the vector \mathbf{u}'_2 to unit length. This yields

$$\mathbf{u}_2 = \frac{\mathbf{u}'_2}{\|\mathbf{u}'_2\|} \quad (4-2-13)$$

The procedure continues by selecting \mathbf{v}_3 and subtracting the projections of \mathbf{v}_3 into \mathbf{u}_1 and \mathbf{u}_2 . Thus, we have

$$\mathbf{u}'_3 = \mathbf{v}_3 - (\mathbf{v}_3 \cdot \mathbf{u}_1)\mathbf{u}_1 - (\mathbf{v}_3 \cdot \mathbf{u}_2)\mathbf{u}_2 \quad (4-2-14)$$

Then, the orthonormal vector \mathbf{u}_3 is

$$\mathbf{u}_3 = \frac{\mathbf{u}'_3}{\|\mathbf{u}'_3\|} \quad (4-2-15)$$

By continuing this procedure, we shall construct a set of n_1 orthonormal vectors, where $n_1 \leq n$, in general. If $m < n$ then $n_1 \leq m$, and if $m \geq n$ then $n_1 \leq n$.

4-2-2 Signal Space Concepts

As in the case of vectors, we may develop a parallel treatment for a set of signals defined on some interval $[a, b]$. The *inner product* of two generally complex-valued signals $x_1(t)$ and $x_2(t)$ is denoted by $\langle x_1(t), x_2(t) \rangle$ and defined as

$$\langle x_1(t), x_2(t) \rangle = \int_a^b x_1(t)x_2^*(t) dt \quad (4-2-16)$$

The signals are *orthogonal* if their inner product is zero.

The *norm* of a signal is defined as

$$\|x(t)\| = \left(\int_a^b |x(t)|^2 dt \right)^{1/2} \quad (4-2-17)$$

A set of m signals are *orthonormal* if they are orthogonal and their norms are all unity. A set of m signals is *linearly independent*, if no signal can be represented as a linear combination of the remaining signals.

The *triangle inequality* for two signals is simply

$$\|x_1(t) + x_2(t)\| \leq \|x_1(t)\| + \|x_2(t)\| \quad (4-2-18)$$

and the *Cauchy-Schwartz inequality* is

$$\left| \int_a^b x_1(t)x_2^*(t) dt \right| \leq \left| \int_a^b |x_1(t)|^2 dt \right|^{1/2} \left| \int_a^b |x_2(t)|^2 dt \right|^{1/2} \quad (4-2-19)$$

with equality when $x_2(t) = ax_1(t)$, where a is any complex number.

4-2-3 Orthogonal Expansions of Signals

In this section, we develop a vector representation for signal waveforms, and, thus, we demonstrate an equivalence between a signal waveform and its vector representation.

Suppose that $s(t)$ is a deterministic, real-valued signal with finite energy

$$\mathcal{E}_s = \int_{-\infty}^{\infty} [s(t)]^2 dt \quad (4-2-20)$$

Furthermore, suppose that there exists a set of functions $\{f_n(t), n = 1, 2, \dots, N\}$ that are orthonormal in the sense that

$$\int_{-\infty}^{\infty} f_n(t)f_m(t) dt = \begin{cases} 0 & (m \neq n) \\ 1 & (m = n) \end{cases} \quad (4-2-21)$$

We may approximate the signal $s(t)$ by a weighted linear combination of these functions, i.e.,

$$\hat{s}(t) = \sum_{k=1}^K s_k f_k(t) \quad (4-2-22)$$

where $\{s_k, 1 \leq k \leq K\}$ are the coefficients in the approximation of $s(t)$. The approximation error incurred is

$$e(t) = s(t) - \hat{s}(t) \quad (4-2-23)$$

Let us select the coefficients $\{s_k\}$ so as to minimize the energy \mathcal{E}_e of the approximation error. Thus,

$$\begin{aligned} \mathcal{E}_e &= \int_{-\infty}^{\infty} [s(t) - \hat{s}(t)]^2 dt \\ &= \int_{-\infty}^{\infty} \left[s(t) - \sum_{k=1}^K s_k f_k(t) \right]^2 dt \end{aligned} \quad (4-2-24)$$

The optimum coefficients in the series expansion of $s(t)$ may be found by differentiating (4-2-24) with respect to each of the coefficients $\{s_k\}$ and setting the first derivatives to zero. Alternatively, we may use a well-known result from estimation theory based on the mean-square-error criterion, which, simply stated, is that the minimum of \mathcal{E}_e with respect to the $\{s_k\}$ is obtained when the error is orthogonal to each of the functions in the series expansion. Thus,

$$\int_{-\infty}^{\infty} \left[s(t) - \sum_{k=1}^K s_k f_k(t) \right] f_n(t) dt = 0, \quad n = 1, 2, \dots, K \quad (4-2-25)$$

Since the functions $\{f_n(t)\}$ are orthonormal, (4-2-25) reduces to

$$s_n = \int_{-\infty}^{\infty} s(t) f_n(t) dt, \quad n = 1, 2, \dots, K \quad (4-2-26)$$

Thus, the coefficients are obtained by projecting the signal $s(t)$ onto each of the functions $\{f_n(t)\}$. Consequently, $\hat{s}(t)$ is the projection of $s(t)$ onto the K -dimensional signal space spanned by the functions $\{f_n(t)\}$. The minimum mean square approximation error is

$$\begin{aligned} \mathcal{E}_{\min} &= \int_{-\infty}^{\infty} e(t) s(t) dt \\ &= \int_{-\infty}^{\infty} [s(t)]^2 dt - \int_{-\infty}^{\infty} \sum_{k=1}^K s_k f_k(t) s(t) dt \\ &= \mathcal{E}_s - \sum_{k=1}^K s_k^2 \end{aligned} \quad (4-2-27)$$

which is nonnegative, by definition.

When the minimum mean square approximation error $\mathcal{E}_{\min} = 0$,

$$\mathcal{E}_k = \sum_{k=1}^K s_k^2 = \int_{-\infty}^{\infty} [s(t)]^2 dt \quad (4-2-28)$$

Under the condition that $\mathcal{E}_{\min} = 0$, we may express $s(t)$ as

$$s(t) = \sum_{k=1}^K s_k f_k(t) \quad (4-2-29)$$

where it is understood that equality of $s(t)$ to its series expansion holds in the sense that the approximation error has zero energy.

When every finite energy signal can be represented by a series expansion of the form in (4-2-29) for which $\mathcal{E}_{\min} = 0$, the set of orthonormal functions $\{f_n(t)\}$ is said to be *complete*.

Example 4-2-1: Trigonometric Fourier Series

A finite energy signal $s(t)$ that is zero everywhere except in the range $0 \leq t \leq T$ and has a finite number of discontinuities in this interval, can be represented in a Fourier series as

$$s(t) = \sum_{k=0}^{\infty} \left(a_k \cos \frac{2\pi kt}{T} + b_k \sin \frac{2\pi kt}{T} \right) \quad (4-2-30)$$

where the coefficients $\{a_k, b_k\}$ that minimize the mean square error are given by

$$\begin{aligned} a_k &= \frac{1}{\sqrt{T}} \int_0^T s(t) \cos \frac{2\pi kt}{T} dt \\ b_k &= \frac{1}{\sqrt{T}} \int_0^T s(t) \sin \frac{2\pi kt}{T} dt \end{aligned} \quad (4-2-31)$$

The set of trigonometric functions $\{\sqrt{2/T} \cos 2\pi kt/T, \sqrt{2/T} \sin 2\pi kt/T\}$ is complete, and, hence, the series expansion results in zero mean square error. These properties are easily established from the development given above.

Gram-Schmidt Procedure Now suppose that we have a set of finite energy signal waveforms $\{s_i(t), i = 1, 2, \dots, M\}$ and we wish to construct a set of orthonormal waveforms. The Gram-Schmidt orthogonalization procedure allows us to construct such a set. We begin with the first waveform $s_1(t)$, which is assumed to have energy \mathcal{E}_1 . The first waveform is simply constructed as

$$f_1(t) = \frac{s_1(t)}{\sqrt{\mathcal{E}_1}} \quad (4-2-32)$$

Thus, $f_1(t)$ is simply $s_1(t)$ normalized to unit energy.

The second waveform is constructed from $s_2(t)$ by first computing the projection of $f_1(t)$ onto $s_2(t)$, which is

$$c_{12} = \int_{-\infty}^{\infty} s_2(t) f_1(t) dt \quad (4-2-33)$$

Then, $c_{12} f_1(t)$ is subtracted from $s_2(t)$ to yield

$$f'_2(t) = s_2(t) - c_{12} f_1(t) \quad (4-2-34)$$

This waveform is orthogonal to $f_1(t)$ but it does not have unit energy. If \mathcal{E}_2 denotes the energy of $f'_2(t)$, the normalized waveform that is orthogonal to $f_1(t)$ is

$$f_2(t) = \frac{f'_2(t)}{\sqrt{\mathcal{E}_2}} \quad (4-2-35)$$

In general, the orthogonalization of the k th function leads to

$$f_k(t) = \frac{f'_k(t)}{\sqrt{\mathcal{E}_k}} \quad (4-2-36)$$

where

$$f'_k(t) = s_k(t) - \sum_{i=1}^{k-1} c_{ik} f_i(t) \quad (4-2-37)$$

and

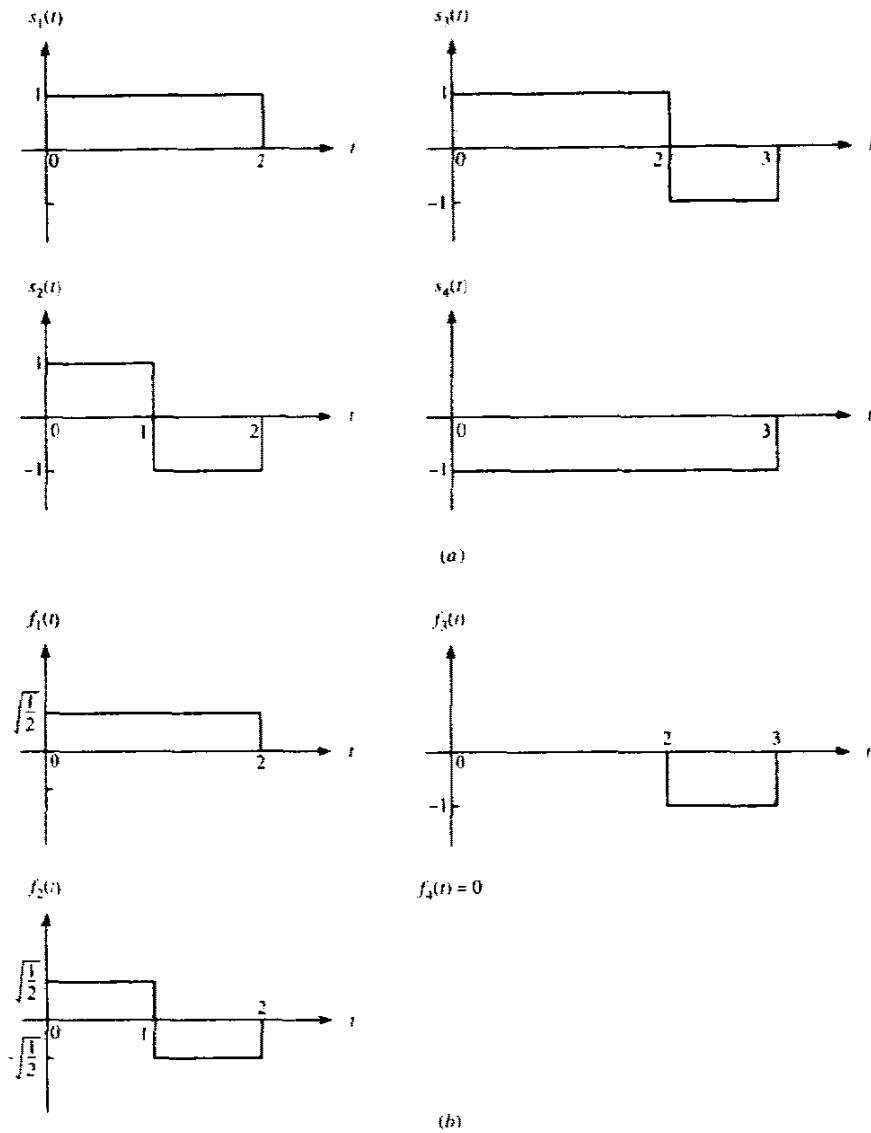
$$c_{ik} = \int_{-\infty}^{\infty} s_k(t) f_i(t) dt, \quad i = 1, 2, \dots, k-1 \quad (4-2-38)$$

Thus, the orthogonalization process is continued until all the M signal waveforms $\{s_i(t)\}$ have been exhausted and $N \leq M$ orthonormal waveforms have been constructed. The dimensionality N of the signal space will be equal to M if all the signal waveforms are linearly independent, i.e., none of the signals waveforms is a linear combination of the other signal waveforms.

Example 4-2-2

Let us apply the Gram-Schmidt procedure to the set of four waveforms illustrated in Fig. 4-2-1(a). The waveform $s_1(t)$ has energy $\mathcal{E}_1 = 2$, so that $f_1(t) = \sqrt{\frac{1}{2}} s_1(t)$. Next, we observe that $c_{12} = 0$; hence, $s_2(t)$ and $f_1(t)$ are orthogonal. Therefore, $f_2(t) = s_2(t) / \sqrt{\mathcal{E}_2} = \sqrt{\frac{1}{2}} s_2(t)$. To obtain $f_3(t)$, we compute c_{13} and c_{23} , which are $c_{13} = \sqrt{2}$ and $c_{23} = 0$. Thus,

$$f'_3(t) = s_3(t) - \sqrt{2} f_1(t) = \begin{cases} -1 & (2 \leq t \leq 3) \\ 0 & (\text{otherwise}) \end{cases}$$



Gram-Schmidt orthogonalization of the signals $\{s_i(t), i = 1, 2, 3, 4\}$ and the corresponding orthogonal signals.

Since $f_3'(t)$ has unit energy, it follows that $f_3(t) = f_3'(t)$. In determining $f_4(t)$, we find that $c_{14} = -\sqrt{2}$, $c_{24} = 0$, and $c_{34} = 1$. Hence,

$$f_4'(t) = s_4(t) + \sqrt{2}f_1(t) - f_3(t) = 0$$

Consequently, $s_4(t)$ is a linear combination of $f_1(t)$ and $f_3(t)$ and, hence, $f_4(t) = 0$. The three orthonormal functions are illustrated in Fig. 4-2-1(b).

Once we have constructed the set of orthonormal waveforms $\{f_n(t)\}$, we can express the M signals $\{s_k(t)\}$ as linear combinations of the $\{f_n(t)\}$. Thus, we may write

$$s_k(t) = \sum_{n=1}^N s_{kn} f_n(t), \quad k = 1, 2, \dots, M \quad (4-2-39)$$

and

$$\mathcal{E}_k = \int_{t_1}^{t_2} [s_k(t)]^2 dt = \sum_{n=1}^N s_{kn}^2 = \|\mathbf{s}_k\|^2 \quad (4-2-40)$$

Based on the expression in (4-2-39), each signal may be represented by the vector

$$\mathbf{s}_k = [s_{k1} \ s_{k2} \ \dots \ s_{kN}] \quad (4-2-41)$$

or, equivalently, as a point in the N -dimensional signal space with coordinates $\{s_{ki}, i = 1, 2, \dots, N\}$. The energy in the k th signal is simply the square of the length of the vector or, equivalently, the square of the Euclidean distance from the origin to the point in the N -dimensional space. Thus, any signal can be represented geometrically as a point in the signal space spanned by the orthonormal functions $\{f_n(t)\}$.

Example 4-2-3

Let us obtain the vector representation of the four signals shown in Fig. 4-2-1(a) by using the orthonormal set of functions in Fig. 4-2-1(b). Since the dimensionality of the signal space is $N = 3$, each signal is described by three components. The signal $s_1(t)$ is characterized by the vector $\mathbf{s}_1 = (\sqrt{2}, 0, 0)$. Similarly, the signals $s_2(t)$, $s_3(t)$, and $s_4(t)$ are characterized by the vectors $\mathbf{s}_2 = (0, \sqrt{2}, 0)$, $\mathbf{s}_3 = (\sqrt{2}, 0, 1)$, and $\mathbf{s}_4 = (-\sqrt{2}, 0, 1)$, respectively. These vectors are shown in Fig. 4-2-2. Their lengths are $|\mathbf{s}_1| = \sqrt{2}$, $|\mathbf{s}_2| = \sqrt{2}$,

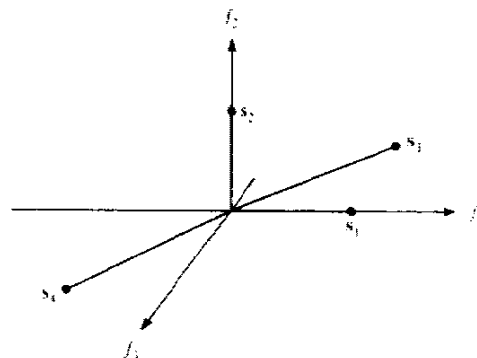


FIGURE 4-2-2 The four signal vectors represented as points in three dimensional function space.

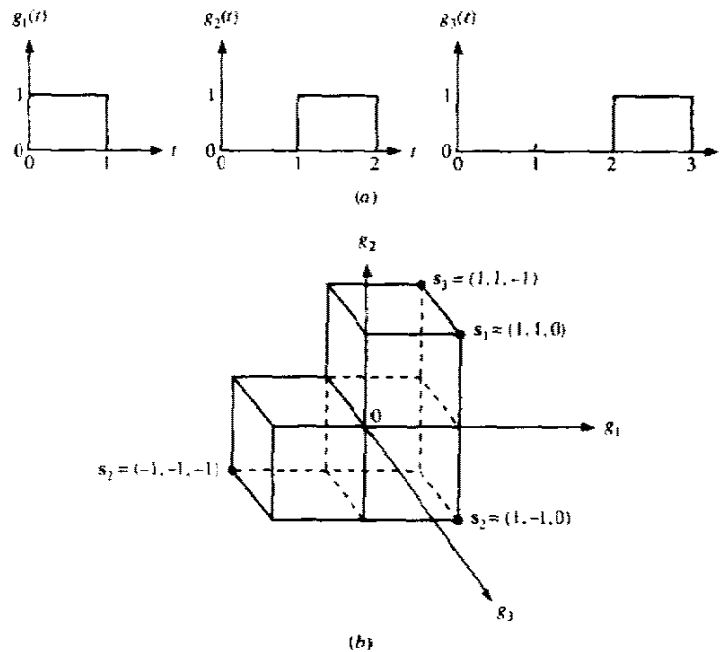
$|s_3| = \sqrt{3}$, and $|s_4| = \sqrt{3}$, and the corresponding signal energies are $\mathcal{E}_k = |s_k|^2$, $k = 1, 2, 3, 4$.

We have demonstrated that a set of M finite energy waveforms $\{s_n(t)\}$ can be represented by a weighted linear combination of orthonormal functions $\{f_n(t)\}$ of dimensionality $N \leq M$. The functions $\{f_n(t)\}$ are obtained by applying the Gram-Schmidt orthogonalization procedure on $\{s_n(t)\}$. It should be emphasized, however, that the functions $\{f_n(t)\}$ obtained from the Gram-Schmidt procedure are not unique. If we alter the order in which the orthogonalization of the signals $\{s_n(t)\}$ is performed, the orthonormal waveforms will be different and the corresponding vector representation of the signals $\{s_n(t)\}$ will depend on the choice of the orthonormal functions $\{f_n(t)\}$. Nevertheless, the vectors $\{s_n\}$ will retain their geometrical configuration and their lengths will be invariant to the choice of orthonormal functions $\{f_n(t)\}$.

Example 4-2-4

An alternative set of orthonormal functions for the four signals in Fig. 4-2-1 is illustrated in Fig. 4-2-3(a). By using these functions to expand $\{s_n(t)\}$, we

FIGURE 4-2-3 An alternative set of orthonormal functions for the four signals in Fig. 4-2-1(a) and the corresponding signal points.



obtain the corresponding vectors $\mathbf{s}_1 = (1, 1, 0)$, $\mathbf{s}_2 = (1, -1, 0)$, $\mathbf{s}_3 = (1, 1, -1)$, and $\mathbf{s}_4 = (-1, -1, -1)$, which are shown in Fig. 4-2-3(b). Note that the vector lengths are identical to those obtained from the orthonormal functions $\{f_n(t)\}$.

The orthogonal expansions described above were developed for real-valued signal waveforms. The extension to complex-valued signal waveforms is left as an exercise for the reader (see Problems 4-6 and 4-7).

Finally, let us consider the case in which the signal waveforms are bandpass and represented as

$$s_m(t) = \text{Re} [s_{lm}(t)e^{j2\pi f_c t}], \quad m = 1, 2, \dots, M \quad (4-2-42)$$

where $\{s_{lm}(t)\}$ denote the equivalent lowpass signals. Recall that the signal energies may be expressed either in terms of $s_m(t)$ or $s_{lm}(t)$, as

$$\begin{aligned} \mathcal{E}_m &= \int_{-\infty}^{\infty} s_m^2(t) dt \\ &= \frac{1}{2} \int_{-\infty}^{\infty} |s_{lm}(t)|^2 dt \end{aligned} \quad (4-2-43)$$

The similarity between any pair of signal waveforms, say $s_m(t)$ and $s_k(t)$, is measured by the normalized cross-correlation

$$\frac{1}{\sqrt{\mathcal{E}_m \mathcal{E}_k}} \int_{-\infty}^{\infty} s_m(t)s_k(t) dt = \text{Re} \left\{ \frac{1}{2\sqrt{\mathcal{E}_m \mathcal{E}_k}} \int_{-\infty}^{\infty} s_{lm}(t)s_{lk}^*(t) dt \right\} \quad (4-2-44)$$

We define the complex-valued cross-correlation coefficient ρ_{km} as

$$\rho_{km} = \frac{1}{2\sqrt{\mathcal{E}_m \mathcal{E}_k}} \int_{-\infty}^{\infty} s_{lm}^*(t)s_{lk}(t) dt \quad (4-2-45)$$

Then,

$$\text{Re}(\rho_{km}) = \frac{1}{\sqrt{\mathcal{E}_m \mathcal{E}_k}} \int_{-\infty}^{\infty} s_m(t)s_k(t) dt \quad (4-2-46)$$

or, equivalently,

$$\text{Re}(\rho_{km}) = \frac{\mathbf{s}_m \cdot \mathbf{s}_k}{\|\mathbf{s}_m\| \|\mathbf{s}_k\|} = \frac{\mathbf{s}_m \cdot \mathbf{s}_k}{\sqrt{\mathcal{E}_m \mathcal{E}_k}} \quad (4-2-47)$$

The cross-correlation coefficients between pairs of signal waveforms or signal vectors comprise one set of parameters that characterize the similarity

of a set of signals. Another related parameter is the Euclidean distance $d_{km}^{(e)}$ between a pair of signals, defined as

$$\begin{aligned} d_{km}^{(e)} &= \|s_m - s_k\| \\ &= \left\{ \int_{-\infty}^{\infty} [s_m(t) - s_k(t)]^2 dt \right\}^{1/2} \\ &= \{ \mathcal{E}_m + \mathcal{E}_k - 2\sqrt{\mathcal{E}_m \mathcal{E}_k} \operatorname{Re}(\rho_{km}) \}^{1/2} \end{aligned} \quad (4-2-48)$$

When $\mathcal{E}_m = \mathcal{E}_k = \mathcal{E}$ for all m and k , this expression simplifies to

$$d_{km}^{(e)} = \{2\mathcal{E}[1 - \operatorname{Re}(\rho_{km})]\}^{1/2} \quad (4-2-49)$$

Thus, the Euclidean distance is an alternative measure of the similarity (or dissimilarity) of the set of signal waveforms or the corresponding signal vectors.

In the following section, we describe digitally modulated signals and make use of the signal space representation for such signals. We shall observe that digitally modulated signals, which are classified as linear, are conveniently expanded in terms of two orthonormal basis functions of the form

$$\begin{aligned} f_1(t) &= \sqrt{\frac{2}{T}} \cos 2\pi f_c t \\ f_2(t) &= -\sqrt{\frac{2}{T}} \sin 2\pi f_c t \end{aligned} \quad (4-2-50)$$

Hence, if $s_{im}(t)$ is expressed as $s_{im}(t) = x_i(t) + jy_i(t)$, it follows that $s_m(t)$ in (4-2-42) may be expressed as

$$s_m(t) = x_i(t)f_1(t) + y_i(t)f_2(t) \quad (4-2-51)$$

where $x_i(t)$ and $y_i(t)$ represent the signal modulations.

4-3 REPRESENTATION OF DIGITALLY MODULATED SIGNALS

In the transmission of digital information over a communications channel, the modulator is the interface device that maps the digital information into analog waveforms that match the characteristics of the channel. The mapping is generally performed by taking blocks of $k = \log_2 M$ binary digits at a time from the information sequence $\{a_n\}$ and selecting one of $M = 2^k$ deterministic, finite energy waveforms $\{s_m(t), m = 1, 2, \dots, M\}$ for transmission over the channel.

When the mapping from the digital sequence $\{a_n\}$ to waveforms is performed under the constraint that a waveform transmitted in any time interval depends on one or more previously transmitted waveforms, the modulator is said to have *memory*. On the other hand, when the mapping

from the sequence $\{a_n\}$ to the waveforms $\{s_m(t)\}$ is performed without any constraint on previously transmitted waveforms, the modulator is called *memoryless*.

In addition to classifying the modulator as either memoryless or having memory, we may classify it as either *linear* or *nonlinear*. Linearity of a modulation method requires that the principle of superposition applies in the mapping of the digital sequence into successive waveforms. In nonlinear modulation, the superposition principle does not apply to signals transmitted in successive time intervals. We shall begin by describing memoryless modulation methods.

4-3-1 Memoryless Modulation Methods

As indicated above, the modulator in a digital communication system maps a sequence of binary digits into a set of corresponding signal waveforms. These waveforms may differ in either amplitude or in phase or in frequency, or some combination of two or more signal parameters. We consider each of these signal types separately, beginning with digital pulse amplitude modulation (PAM). In all cases, we assume that the sequence of binary digits at the input to the modulator occurs at a rate of R bits/s.

Pulse Amplitude Modulated (PAM) Signals In digital PAM, the signal waveforms may be represented as

$$\begin{aligned} s_m(t) &= \text{Re} [A_m g(t) e^{j2\pi f_c t}] \\ &= A_m g(t) \cos 2\pi f_c t, \quad m = 1, 2, \dots, M, \quad 0 \leq t \leq T \end{aligned} \quad (4-3-1)$$

where $\{A_m, 1 \leq m \leq M\}$ denote the set of M possible amplitudes corresponding to $M = 2^k$ possible k -bit blocks or *symbols*. The signal amplitudes A_m take the discrete values (levels)

$$A_m = (2m - 1 - M)d, \quad m = 1, 2, \dots, M \quad (4-3-2)$$

where $2d$ is the distance between adjacent signal amplitudes. The waveform $g(t)$ is a real-valued signal pulse whose shape influences the spectrum of the transmitted signal, as we shall observe later. The symbol rate for the PAM signal is R/k . This is the rate at which changes occur in the amplitude of the carrier to reflect the transmission of new information. The time interval $T_b = 1/R$ is called the *bit interval* and the time interval $T = k/R = kT_b$ is called the *symbol interval*.

The M PAM signals have energies

$$\begin{aligned} \mathcal{E}_m &= \int_0^T s_m^2(t) dt \\ &= \frac{1}{2} A_m^2 \int_0^T g^2(t) dt \\ &= \frac{1}{2} A_m^2 \mathcal{E}_g \end{aligned} \quad (4-3-3)$$

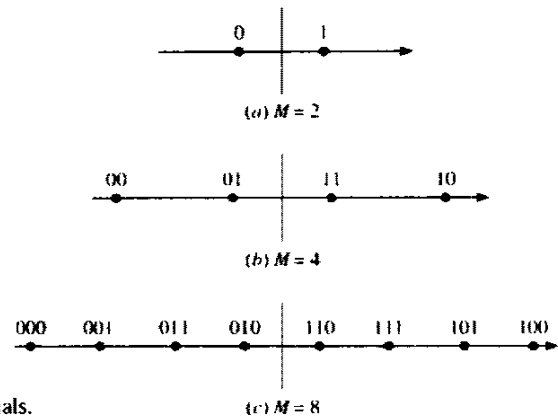


FIGURE 4-3-1 Signal space diagram for digital PAM signals.

where \mathcal{E}_g denotes the energy in the pulse $g(t)$. Clearly, these signals are one-dimensional ($N = 1$), and, hence, are represented by the general form

$$s_m(t) = s_m f(t) \quad (4-3-4)$$

where $f(t)$ is defined as the unit-energy signal waveform given as

$$f(t) = \sqrt{\frac{2}{\mathcal{E}_g}} g(t) \cos 2\pi f_c t \quad (4-3-5)$$

and

$$s_m = A_m \sqrt{\frac{1}{2} \mathcal{E}_g}, \quad m = 1, 2, \dots, M \quad (4-3-6)$$

The corresponding signal space diagrams for $M = 2$, $M = 4$ and $M = 8$ are shown in Fig. 4-3-1. Digital PAM is also called *amplitude-shift keying* (ASK).

The mapping or assignment of k information bits to the $M = 2^k$ possible signal amplitudes may be done in a number of ways. The preferred assignment is one in which the adjacent signals amplitudes differ by one binary digit as illustrated in Fig. 4-3-1. This mapping is called *Gray encoding*. It is important in the demodulation of the signal because the most likely errors caused by noise involve the erroneous selection of an adjacent amplitude to the transmitted signal amplitude. In such a case, only a single bit error occurs in the k -bit sequence.

We note that the Euclidean distance between any pair of signal points is

$$\begin{aligned} d_{mn}^{(e)} &= \sqrt{(s_m - s_n)^2} \\ &= \sqrt{\frac{1}{2} \mathcal{E}_g} |A_m - A_n| \\ &= d \sqrt{2 \mathcal{E}_g} |m - n| \end{aligned} \quad (4-3-7)$$

Hence, the distance between a pair of adjacent signal points, i.e., the minimum Euclidean distance, is

$$d_{\min}^{(e)} = d \sqrt{2 \mathcal{E}_g} \quad (4-3-8)$$

The carrier-modulated PAM signal represented by (4-3-1) is a double-sideband (DSB) signal and requires twice the channel bandwidth of the equivalent lowpass signal for transmission. Alternatively, we may use single-sideband (SSB) PAM, which has the representation (lower or upper sideband).

$$s_m(t) = \text{Re} \{ A_m [g(t) \pm j\hat{g}(t)] e^{j2\pi f_c t} \}, \quad m = 1, 2, \dots, M \quad (4-3-9)$$

where $\hat{g}(t)$ is the Hilbert transform of $g(t)$. Thus, the bandwidth of the SSB signal is half that of the DSB signal.

The digital PAM signal is also appropriate for transmission over a channel that does not require carrier modulation. In this case, the signal waveform may be simply represented as

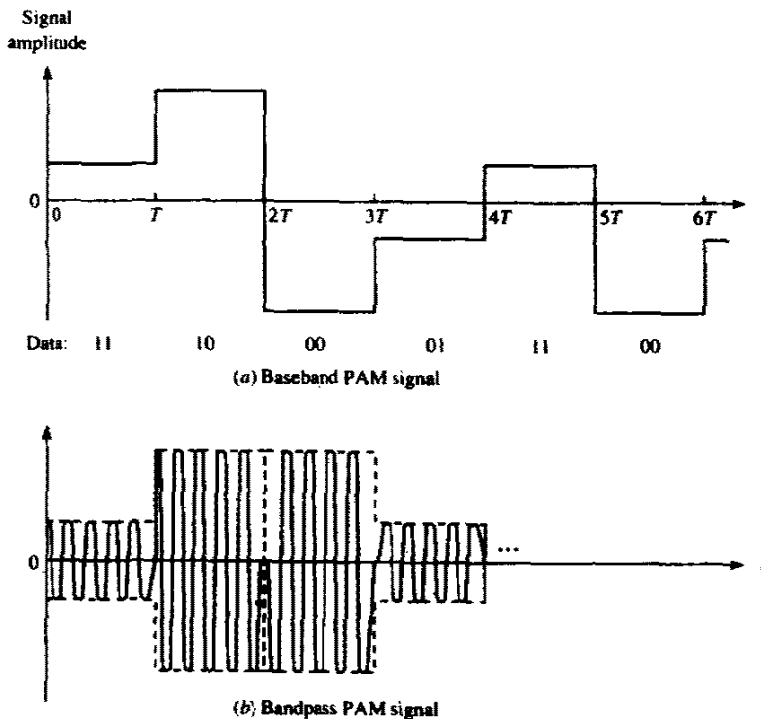
$$s_m(t) = A_m g(t), \quad m = 1, 2, \dots, M \quad (4-3-10)$$

This is now called a *baseband* signal. For example a four-amplitude level baseband PAM signal is illustrated in Fig. 4-3-2(a). The carrier-modulated version of the signal is shown in Fig. 4-3-2(b).

In the special case of $M = 2$ signals, the binary PAM waveforms have the special property that

$$s_1(t) = -s_2(t)$$

FIGURE 4-3-2 Baseband and bandpass PAM signals.



Hence, these two signals have the same energy and a cross-correlation coefficient of -1 . Such signals are called *antipodal*.

Phase-Modulated Signals In digital phase modulation, the M signal waveforms are represented as

$$\begin{aligned} s_m(t) &= \operatorname{Re} [g(t)e^{j2\pi(m-1)/M}e^{j2\pi f_c t}], \quad m = 1, 2, \dots, M, \quad 0 \leq t \leq T \\ &= g(t) \cos \left[2\pi f_c t + \frac{2\pi}{M}(m-1) \right] \\ &= g(t) \cos \frac{2\pi}{M}(m-1) \cos 2\pi f_c t - g(t) \sin \frac{2\pi}{M}(m-1) \sin 2\pi f_c t \end{aligned} \quad (4-3-11)$$

where $g(t)$ is the signal pulse shape and $\theta_m = 2\pi(m-1)/M$, $m = 1, 2, \dots, M$, are the M possible phases of the carrier that convey the transmitted information. Digital phase modulation is usually called *phase-shift keying* (PSK).

We note that these signal waveforms have equal energy, i.e.,

$$\begin{aligned} \mathcal{E} &= \int_0^T s_m^2(t) dt \\ &= \frac{1}{2} \int_0^T g^2(t) dt = \frac{1}{2} \mathcal{E}_g \end{aligned} \quad (4-3-12)$$

Furthermore, the signal waveforms may be represented as a linear combination of two-orthonormal signal waveforms, $f_1(t)$ and $f_2(t)$, i.e.,

$$s_m(t) = s_{m1}f_1(t) + s_{m2}f_2(t) \quad (4-3-13)$$

where

$$f_1(t) = \sqrt{\frac{2}{\mathcal{E}_g}} g(t) \cos 2\pi f_c t \quad (4-3-14)$$

$$f_2(t) = -\sqrt{\frac{2}{\mathcal{E}_g}} g(t) \sin 2\pi f_c t \quad (4-3-15)$$

and the two-dimensional vectors $\mathbf{s}_m = [s_{m1} \ s_{m2}]$ are given by

$$\mathbf{s}_m = \left[\sqrt{\frac{\mathcal{E}_g}{2}} \cos \frac{2\pi}{M}(m-1) \quad \sqrt{\frac{\mathcal{E}_g}{2}} \sin \frac{2\pi}{M}(m-1) \right], \quad m = 1, 2, \dots, M \quad (4-3-16)$$

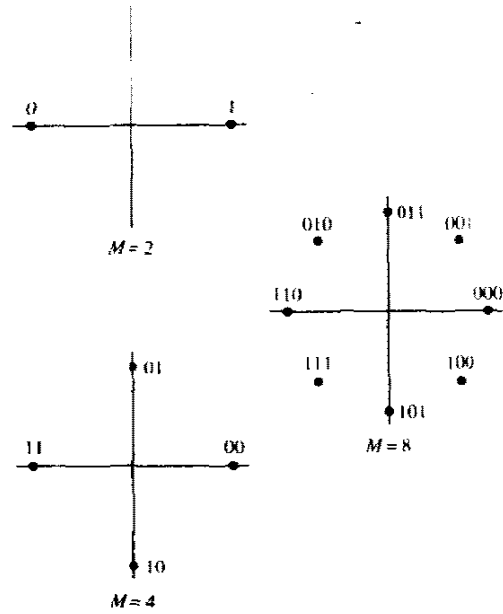


FIGURE 4-3-3 Signal space diagrams for PSK signals.

Signal space diagrams for $M = 2, 4,$ and 8 are shown in Fig. 4-3-3. We note that $M = 2$ corresponds to one-dimensional signals, which are identical to binary PAM signals.

As is the case of PAM, the mapping or assignment of k information bits to the $M = 2^k$ possible phases may be done in a number of ways. The preferred assignment is Gray encoding, so that the most likely errors caused by noise will result in a single bit error in the k -bit symbol.

The Euclidean distance between signal points is

$$d_{mn}^{(e)} = |s_m - s_n|$$

$$= \left\{ \mathcal{E}_s \left[1 - \cos \frac{2\pi}{M} (m - n) \right] \right\}^{1/2} \quad (4-3-17)$$

The minimum Euclidean distance corresponds to the case in which $|m - n| = 1$, i.e., adjacent signal phases. In this case,

$$d_{\min}^{(e)} = \sqrt{\mathcal{E}_s \left(1 - \cos \frac{2\pi}{M} \right)} \quad (4-3-18)$$

Quadrature Amplitude Modulation The bandwidth efficiency of PAM/SSB can also be obtained by simultaneously impressing two separate k -bit symbols from the information sequence $\{a_n\}$ on two quadrature carriers

$\cos 2\pi f_c t$ and $\sin 2\pi f_c t$. The resulting modulation technique is called quadrature PAM or QAM, and the corresponding signal waveforms may be expressed as

$$\begin{aligned} s_m(t) &= \text{Re} [(A_{m_c} + jA_{m_s})g(t)e^{j2\pi f_c t}], \quad m = 1, 2, \dots, M, \quad 0 \leq t \leq T \\ &= A_{m_c}g(t) \cos 2\pi f_c t - A_{m_s}g(t) \sin 2\pi f_c t \end{aligned} \quad (4-3-19)$$

where A_{m_c} and A_{m_s} are the information-bearing signal amplitudes of the quadrature carriers and $g(t)$ is the signal pulse.

Alternatively, the QAM signal waveforms may be expressed as

$$\begin{aligned} s_m(t) &= \text{Re} [V_m e^{j\theta_m} g(t) e^{j2\pi f_c t}] \\ &= V_m g(t) \cos(2\pi f_c t + \theta_m) \end{aligned} \quad (4-3-20)$$

where $V_m = \sqrt{A_{m_c}^2 + A_{m_s}^2}$ and $\theta_m = \tan^{-1}(A_{m_s}/A_{m_c})$. From this expression, it is apparent that the QAM signal waveforms may be viewed as combined amplitude and phase modulation.

In fact, we may select any combination of M_1 -level PAM and M_2 -phase PSK to construct an $M = M_1 M_2$ combined PAM-PSK signal constellation. If $M_1 = 2^n$ and $M_2 = 2^m$, the combined PAM-PSK signal constellation results in the simultaneous transmission of $m + n = \log M_1 M_2$ binary digits occurring at a symbol rate $R/(m + n)$. Examples of signal space diagrams for combined PAM-PSK are shown in Fig. 4-3-4, for $M = 8$ and $M = 16$.

As in the case of PSK signals, the QAM signal waveforms may be represented as a linear combination of two orthonormal signal waveforms, $f_1(t)$ and $f_2(t)$, i.e.,

$$s_m(t) = s_{m1}f_1(t) + s_{m2}f_2(t) \quad (4-3-21)$$

where

$$\begin{aligned} f_1(t) &= \sqrt{\frac{2}{\mathcal{E}_x}} g(t) \cos 2\pi f_c t \\ f_2(t) &= -\sqrt{\frac{2}{\mathcal{E}_x}} g(t) \sin 2\pi f_c t \end{aligned} \quad (4-3-22)$$

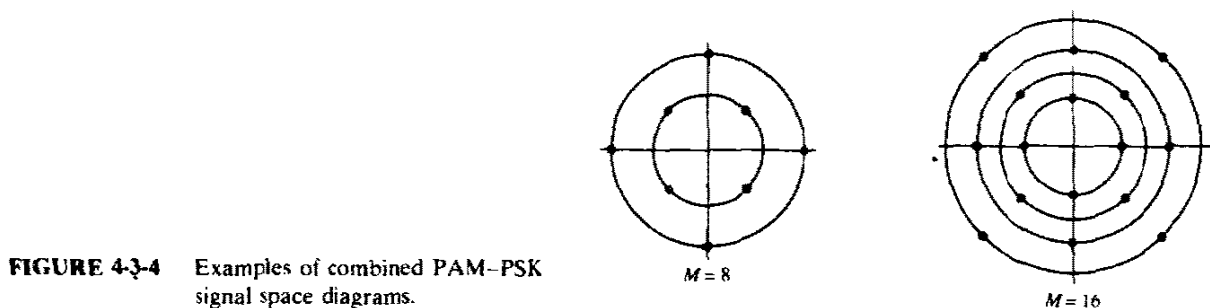


FIGURE 4-3-4 Examples of combined PAM-PSK signal space diagrams.

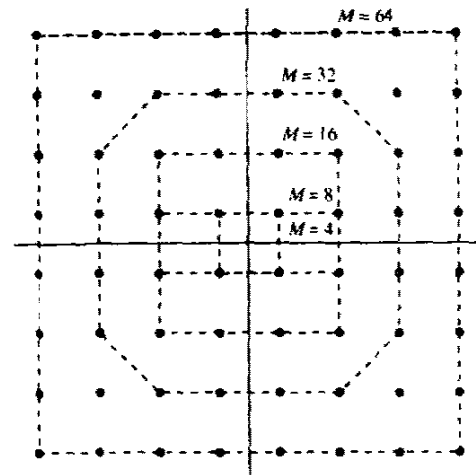


FIGURE 4-3-5 Several signal space diagrams for rectangular QAM.

and

$$\begin{aligned} \mathbf{s}_m &= [s_{m1} \quad s_{m2}] \\ &= [A_{mc} \sqrt{\frac{1}{2} \mathcal{E}_g} \quad A_{ms} \sqrt{\frac{1}{2} \mathcal{E}_g}] \end{aligned} \tag{4-3-23}$$

\mathcal{E}_g is the energy of the signal pulse $g(t)$.

The Euclidean distance between any pair of signal vectors is

$$\begin{aligned} d_{mn}^{(c)} &= |\mathbf{s}_m - \mathbf{s}_n| \\ &= \sqrt{\frac{1}{2} \mathcal{E}_g [(A_{mc} - A_{nc})^2 + (A_{ms} - A_{ns})^2]} \end{aligned} \tag{4-3-24}$$

In the special case where the signal amplitudes takes the set of discrete values $\{(2m - 1 - M)d, m = 1, 2, \dots, M\}$, the signal space diagram is rectangular, as shown in Fig. 4-3-5. In this case, the Euclidean distance between adjacent points, i.e., the minimum distance, is

$$d_{\min}^{(c)} = d\sqrt{2} \sqrt{\mathcal{E}_g} \tag{4-3-25}$$

which is the same result as for PAM.

Multidimensional Signals It is apparent from the discussion above that the digital modulation of the carrier amplitude and phase allows us to construct signal waveforms that correspond to two-dimensional vectors and signal space diagrams. If we wish to construct signal waveforms corresponding to higher-dimensional vectors, we may use either the time domain or the frequency domain or both in order to increase the number of dimensions.

Suppose we have N -dimensional signal vectors. For any N , we may subdivide a time interval of length $T_1 = NT$ into N subintervals of length $T = T_1/N$. In each subinterval of length T , we may use binary PAM (a one-dimensional signal) to transmit an element of the N -dimensional signal

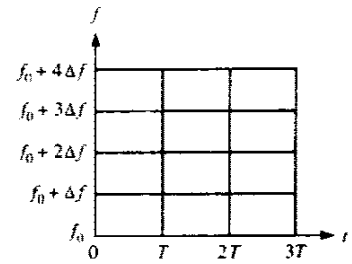


FIGURE 4-3-6 Subdivision of time and frequency axes into distinct slots.

vector. Thus, the N time slots are used to transmit the N -dimensional signal vector. If N is even, a time slot of length T may be used to simultaneously transmit two components of the N -dimensional vector by modulating the amplitude of quadrature carriers independently by the corresponding components. In this manner, the N -dimensional signal vector is transmitted in $\frac{1}{2}NT$ seconds ($\frac{1}{2}N$ time slots).

Alternatively, a frequency band of width $N\Delta f$ may be subdivided into N frequency slots each of width Δf . An N -dimensional signal vector can be transmitted over the channel by simultaneously modulating the amplitude of N carriers, one in each of the N frequency slots. Care must be taken to provide sufficient frequency separation Δf between successive carriers so that there is no cross talk interference among the signals on the N carriers. If quadrature carriers are used in each frequency slot, the N -dimensional vector (even N) may be transmitted in $\frac{1}{2}N$ frequency slots, thus reducing the channel bandwidth utilization by a factor of 2.

More generally, we may use both the time and frequency domains jointly to transmit an N -dimensional signal vector. For example, Fig. 4-3-6 illustrates a subdivision of the time and frequency axes into 12 slots. Thus, an $N = 12$ -dimensional signal vector may be transmitted by PAM or an $N = 24$ -dimensional signal vector may be transmitted by use of two quadrature carriers (QAM) in each slot.

Orthogonal Multidimensional Signals As a special case of the construction of multidimensional signals, let us consider the construction of M equal-energy orthogonal signal waveforms that differ in frequency, and are represented as

$$\begin{aligned} s_m(t) &= \operatorname{Re} [s_{lm}(t)e^{j2\pi f_l t}], \quad m = 1, 2, \dots, M, \quad 0 \leq t \leq T \\ &= \sqrt{\frac{2\mathcal{E}}{T}} \cos [2\pi f_l t + 2\pi m \Delta f t] \end{aligned} \quad (4-3-26)$$

where the equivalent lowpass signal waveforms are defined as

$$s_{lm}(t) = \sqrt{\frac{2\mathcal{E}}{T}} e^{j2\pi m \Delta f t}, \quad m = 1, 2, \dots, M, \quad 0 \leq t \leq T \quad (4-3-27)$$

This type of frequency modulation is called *frequency-shift keying* (FSK).

These waveforms are characterized as having equal energy and cross-correlation coefficients

$$\begin{aligned} \rho_{km} &= \frac{2\mathcal{E}/T}{2\mathcal{E}} \int_0^T e^{j2\pi(m-k)\Delta f t} dt \\ &= \frac{\sin \pi T(m-k)\Delta f}{\pi T(m-k)\Delta f} e^{j\pi T(m-k)\Delta f} \end{aligned} \quad (4-3-28)$$

The real part of ρ_{km} is

$$\begin{aligned} \rho_r &\equiv \text{Re}(\rho_{km}) = \frac{\sin [\pi T(m-k)\Delta f]}{\pi T(m-k)\Delta f} \cos [\pi T(m-k)\Delta f] \\ &= \frac{\sin [2\pi T(m-k)\Delta f]}{2\pi T(m-k)\Delta f} \end{aligned} \quad (4-3-29)$$

First, we observe that $\text{Re}(\rho_{km}) = 0$ when $\Delta f = 1/2T$ and $m \neq k$. Since $|m - k| = 1$ corresponds to adjacent frequency slots, $\Delta f = 1/2T$ represents the minimum frequency separation between adjacent signals for orthogonality of the M signals. Plots of $\text{Re}(\rho_{km})$ versus Δf and $|\rho_{km}|$ versus Δf are shown in Fig. 4-3-7. Note that $|\rho_{km}| = 0$ for multiples of $1/T$ whereas $\text{Re}(\rho_{km}) = 0$ for multiples of $1/2T$.

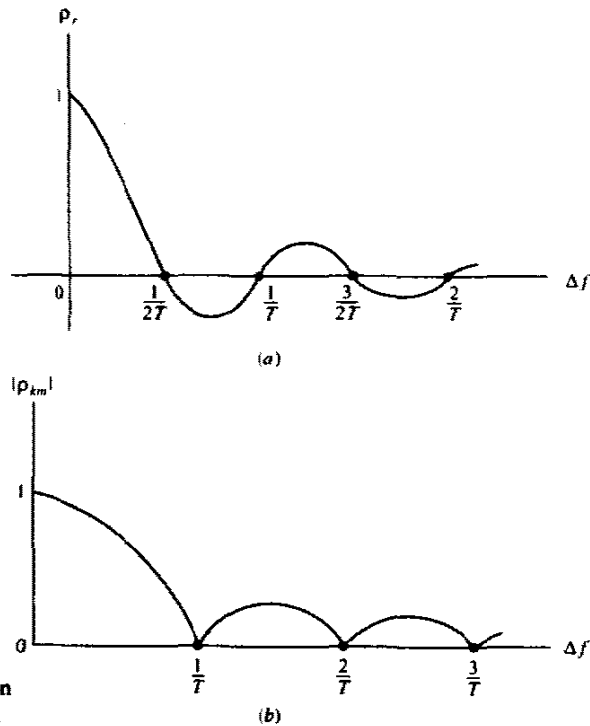


FIGURE 4-3-7 Cross-correlation coefficient as a function of frequency separation for FSK signals.

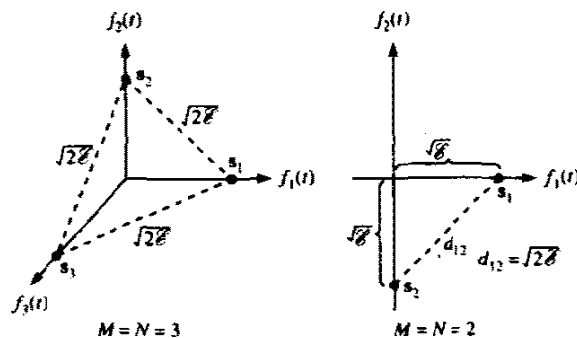


FIGURE 4-3-8 Orthogonal signals for $M = N = 3$ and $M = N = 2$.

For the case in which $\Delta f = 1/2T$, the M FSK signals are equivalent to the N -dimensional vectors

$$\begin{aligned}
 \mathbf{s}_1 &= [\sqrt{E} \ 0 \ 0 \ \dots \ 0 \ 0] \\
 \mathbf{s}_2 &= [0 \ \sqrt{E} \ 0 \ \dots \ 0 \ 0] \\
 &\vdots \\
 \mathbf{s}_N &= [0 \ 0 \ 0 \ \dots \ 0, \ \sqrt{E}]
 \end{aligned}
 \tag{4-3-30}$$

where $N = M$. The distance between pairs of signals is

$$d_{km}^{(e)} = \sqrt{2E} \quad \text{for all } m, k \tag{4-3-31}$$

which is also the minimum distance. Figure 4-3-8 illustrates the signal space diagram for $M = N = 2$ and $M = N = 3$.

Biorthogonal Signals A set of M biorthogonal signals can be constructed from $\frac{1}{2}M$ orthogonal signals by simply including the negatives of the orthogonal signals. Thus, we require $N = \frac{1}{2}M$ dimensions for the construction of a set of M biorthogonal signals. Figure 4-3-9 illustrates the biorthogonal signals for $M = 4$ and 6.

We note that the correlation between any pair of waveforms is either $\rho_r = -1$ or 0. The corresponding distances are $d = 2\sqrt{E}$ or $\sqrt{2E}$, with the latter being the minimum distance.

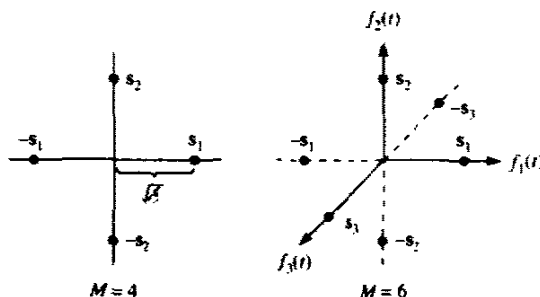


FIGURE 4-3-9 Signal space diagrams for $M = 4$ and $M = 6$ biorthogonal signals.

Simplex Signals Suppose we have a set of M orthogonal waveforms $\{s_m(t)\}$ or, equivalently, their vector representation $\{\mathbf{s}_m\}$. Their mean is

$$\bar{\mathbf{s}} = \frac{1}{M} \sum_{m=1}^M \mathbf{s}_m \quad (4-3-32)$$

Now, let us construct another set of M signals by subtracting the mean from each of the M orthogonal signals. Thus,

$$\mathbf{s}'_m = \mathbf{s}_m - \bar{\mathbf{s}}, \quad m = 1, 2, \dots, M \quad (4-3-33)$$

The effect of the subtraction is to translate the origin of the m orthogonal signals to the point $\bar{\mathbf{s}}$.

The resulting signal waveforms are called *simplex signals* and have the following properties. First, the energy per waveform is

$$\begin{aligned} |\mathbf{s}'_m|^2 &= |\mathbf{s}_m - \bar{\mathbf{s}}|^2 \\ &= \mathcal{E} - \frac{2}{M} \mathcal{E} + \frac{1}{M} \mathcal{E} \\ &= \mathcal{E} \left(1 - \frac{1}{M}\right) \end{aligned} \quad (4-3-34)$$

Second, the cross-correlation of any pair of signals is

$$\begin{aligned} \text{Re}(\rho_{mn}) &= \frac{\mathbf{s}'_m \cdot \mathbf{s}'_n}{|\mathbf{s}'_m| |\mathbf{s}'_n|} \\ &= \frac{-1/M}{1 - 1/M} = -\frac{1}{M-1} \end{aligned} \quad (4-3-35)$$

for all m, n . Hence, the set of simplex waveforms is *equally correlated* and requires less energy, by the factor $1 - 1/M$, than the set of orthogonal waveforms. Since only the origin was translated, the distance between any pair of signal points is maintained at $d = \sqrt{2\mathcal{E}}$, which is the same as the distance between any pair of orthogonal signals.

Figure 4-3-10 illustrates the simplex signals for $M = 2, 3$, and 4. Note that the signal dimensionality is $N = M - 1$.

Signal Waveforms from Binary Codes A set of M signaling waveforms can be generated from a set of M binary code words of the form

$$\mathbf{C}_m = [c_{m1} \ c_{m2} \ \dots \ c_{mN}], \quad m = 1, 2, \dots, M \quad (4-3-36)$$

where $c_{mj} = 0$ or 1 for all m and j . Each component of a code word is mapped into an elementary binary PSK waveform as follows:

$$\begin{aligned} c_{mj} = 1 &\Rightarrow s_{mj}(t) = \sqrt{\frac{2\mathcal{E}_c}{T_c}} \cos 2\pi f_c t \quad (0 \leq t \leq T_c) \\ c_{mj} = 0 &\Rightarrow s_{mj}(t) = -\sqrt{\frac{2\mathcal{E}_c}{T_c}} \cos 2\pi f_c t \quad (0 \leq t \leq T_c) \end{aligned} \quad (4-3-37)$$

where $T_c = T/N$ and $\mathcal{E}_c = \mathcal{E}/N$. Thus, the M code words $\{\mathbf{C}_m\}$ are mapped into a set of M waveforms $\{s_m(t)\}$.

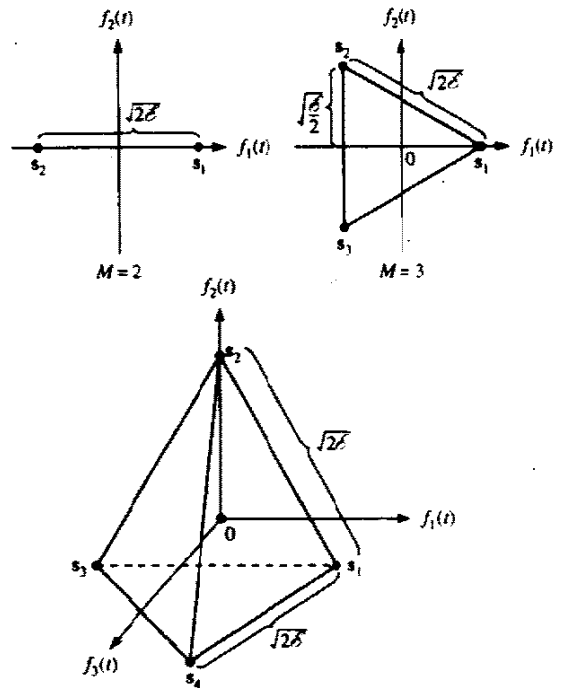


FIGURE 4-3-10 Signal space diagrams for M -ary simplex signals.

The waveforms can be represented in vector form as

$$\mathbf{s}_m = [s_{m1} \ s_{m2} \ \dots \ s_{mN}], \quad m = 1, 2, \dots, M \quad (4-3-38)$$

where $s_{mj} = \pm\sqrt{E/N}$ for all m and j . N is called the block length of the code, and it is also the dimension of the M waveforms.

We note that there are 2^N possible waveforms that can be constructed from the 2^N possible binary code words. We may select a subset of $M < 2^N$ signal waveforms for transmission of the information. We also observe that the 2^N possible signal points correspond to the vertices of an N -dimensional hypercube with its center at the origin. Figure 4-3-11 illustrates the signal points in $N = 2$ and 3 dimensions.

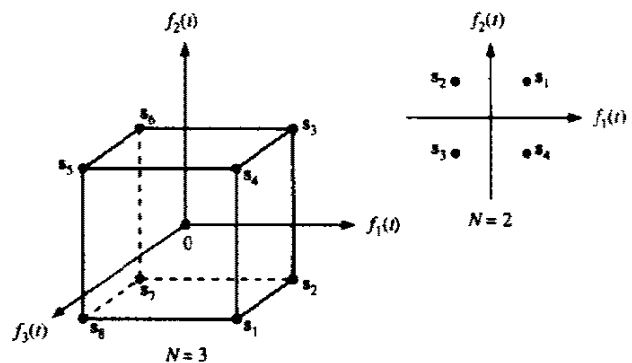


FIGURE 4-3-11 Signal space diagrams for signals generated from binary codes.

Each of the M waveforms has energy \mathcal{E} . The cross-correlation between any pair of waveforms depends on how we select the M waveforms from the 2^N possible waveforms. This topic is treated in Chapter 7. Clearly, any adjacent signal points have a cross-correlation coefficient

$$\rho_r = \frac{\mathcal{E}(1 - 2/N)}{\mathcal{E}} = \frac{N - 2}{N} \quad (4-3-39)$$

and a corresponding distance of

$$\begin{aligned} d^{(e)} &= \sqrt{2\mathcal{E}(1 - \rho_r)} \\ &= \sqrt{4\mathcal{E}/N} \end{aligned} \quad (4-3-40)$$

This concludes our discussion of memoryless modulation signals.

4-3-2 Linear Modulation with Memory

The modulation signals introduced in the previous section were classified as memoryless, because there was no dependence between signals transmitted in non-overlapping symbol intervals. In this section, we present some modulation signals in which there is dependence between the signals transmitted in successive symbol intervals. This signal dependence is usually introduced for the purpose of shaping the spectrum of the transmitted signal so that it matches the spectral characteristics of the channel. Signal dependence between signals transmitted in different signal intervals is generally accomplished by encoding the data sequence at the input to the modulator by means of a *modulation code*, as described in Chapter 9.

In this section, we shall present examples of modulation signals with memory and characterize their memory in terms of Markov chains. We shall confine our treatment to baseband signals. The generalization to bandpass signals is relatively straightforward.

Figure 4-3-12 illustrates three different baseband signals and the corresponding data sequence. The first signal, called NRZ, is the simplest. The binary information digit 1 is represented by a rectangular pulse of polarity A and the binary digit zero is represented by a rectangular pulse of polarity $-A$.

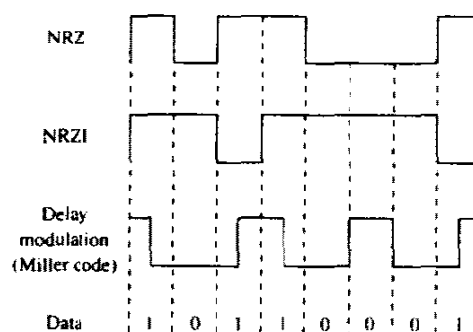


FIGURE 4-3-12 Baseband signals.

Hence, the NRZ modulation is memoryless and is equivalent to a binary PAM or a binary PSK signal in a carrier-modulated system.

The NRZI signal is different from the NRZ signal in that transitions from one amplitude level to another occur only when a 1 is transmitted. The amplitude level remains unchanged when a zero is transmitted. This type of signal encoding is called *differential encoding*. The encoding operation is described mathematically by the relation

$$b_k = a_k \oplus b_{k-1} \tag{4-3-41}$$

where $\{a_k\}$ is the binary information sequence into the encoder, $\{b_k\}$ is the output sequence of the encoder, and \oplus denotes addition modulo 2. When $b_k = 1$, the transmitted waveform is a rectangular pulse of amplitude A , and when $b_k = 0$, the transmitted waveform is a rectangular pulse of amplitude $-A$. Hence, the output of the encoder is mapped into one of two waveforms in exactly the same manner as for the NRZ signal.

The differential encoding operation introduces memory in the signal. The combination of the encoder and the modulator operations may be represented by a state diagram (a Markov chain) as shown in Fig. 4-3-13. The state diagram may be described by two transition matrices corresponding to the two possible input bits $\{0, 1\}$. We note that when $a_k = 0$, the encoder stays in the same state. Hence, the state transition matrix for a zero is simply

$$\mathbf{T}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{4-3-42}$$

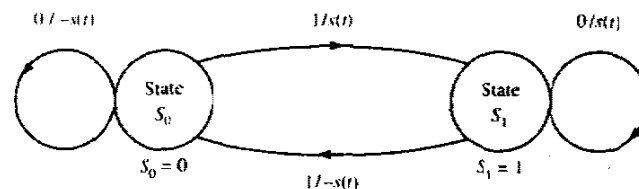
where $t_{ij} = 1$ if a_k results in a transition from state i to state j , $i = 1, 2$, and $j = 1, 2$; otherwise, $t_{ij} = 0$. Similarly, the state transition matrix for $a_k = 1$ is

$$\mathbf{T}_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{4-3-43}$$

Thus, these two state transition matrices characterize the NRZI signal.

Another way to display the memory introduced by the precoding operation is by means of a trellis diagram. The trellis diagram for the NRZI signal is

FIGURE 4-3-13 State diagram for the NRZI signal.



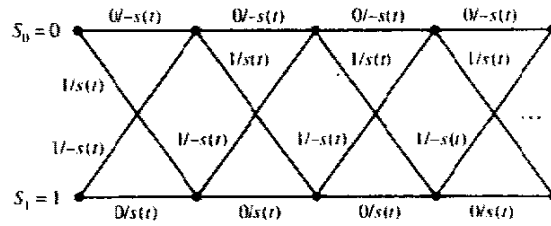


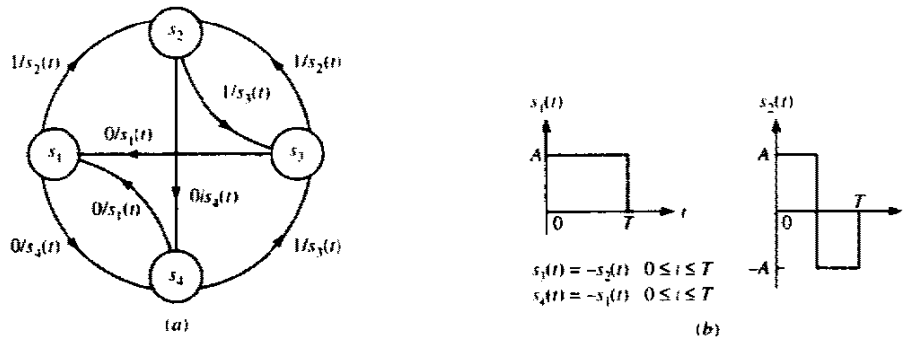
FIGURE 4-3-14 The trellis diagram for the NRZI signal.

illustrated in Fig. 4-3-14. The trellis provides exactly the same information concerning the signal dependence as the state diagram, but also depicts a time evolution of the state transitions.

The signal generated by delay modulation also has memory. As shown in Chapter 9, delay modulation is equivalent to encoding the data sequence by a run-length-limited code called a *Miller code* and using NRZI to transmit the encoded data. This type of digital modulation has been used extensively for digital magnetic recording and in carrier modulation systems employing binary PSK. The signal may be described by a state diagram that has four states as shown in Fig. 4-3-15(a). There are two elementary waveforms $s_1(t)$ and $s_2(t)$ and their negatives $-s_1(t)$ and $-s_2(t)$, which are used for transmitting the binary information. These waveforms are illustrated in Fig. 4-3-15(b). The mapping from bits to corresponding waveforms is illustrated in the state diagram. The state transition matrices that characterize the memory of this encoding and modulation method are easily obtained from the state diagram in Fig. 4-3-15. When $a_k = 0$, we have

$$\mathbf{T}_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad (4-3-44)$$

FIGURE 4-3-15 State diagram (a) and basic waveforms (b) for delay modulated (Miller-encoded) signal.



and when $a_k = 1$, the transition matrix is

$$\mathbf{T}_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad (4-3-45)$$

Thus, these two 4×4 state transition matrices characterize the state diagram for the Miller-encoded signal.

Modulation techniques with memory such as NRZI and Miller coding are generally characterized by a K -state Markov chain with *stationary state probabilities* $\{p_i, i = 1, 2, \dots, K\}$ and *transition probabilities* $\{p_{ij}, i, j = 1, 2, \dots, K\}$. Associated with each transition is a signal waveform $s_j(t)$, $j = 1, 2, \dots, K$. Thus, the transition probability p_{ij} denotes the probability that signal waveform $s_j(t)$ is transmitted in a given signaling interval after the transmission of the signal waveform $s_i(t)$ in the previous signaling interval. The transition probabilities may be arranged in matrix form as

$$\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1K} \\ p_{21} & p_{22} & \cdots & p_{2K} \\ \vdots & \vdots & & \vdots \\ p_{K1} & p_{K2} & \cdots & p_{KK} \end{bmatrix} \quad (4-3-46)$$

where \mathbf{P} is called the *transition probability matrix*.

The transition probability matrix is easily obtained from the transition matrices $\{\mathbf{T}_i\}$ and the corresponding probabilities of occurrence of the input bits (or, equivalently, the stationary state transition probabilities $\{p_i\}$). The general relationship may be expressed as

$$\mathbf{P} = \sum_{i=1}^2 q_i \mathbf{T}_i \quad (4-3-47)$$

where $q_1 = P(a_k = 0)$ and $q_2 = P(a_k = 1)$.

For the NRZI signal with equal state probabilities $p_1 = p_2 = \frac{1}{2}$ and transition matrices given by (4-3-42) and (4-3-43), the transition probability matrix is

$$\mathbf{P} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \quad (4-3-48)$$

Similarly, the transition probability matrix for the Miller-coded signal with equally likely symbols ($q_1 = q_2 = \frac{1}{2}$ or, equivalently, $p_1 = p_2 = p_3 = p_4 = \frac{1}{4}$) is

$$\mathbf{P} = \begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{bmatrix} \quad (4-3-49)$$

The transition probability matrix is useful in the determination of the spectral

characteristics of digital modulation techniques with memory, as we shall observe in Section 4-4.

4-3-3 Nonlinear Modulation Methods with Memory

In this section, we consider a class of digital modulation methods in which the phase of the signal is constrained to be continuous. This constraint results in a phase or frequency modulator that has memory. The modulation method is also nonlinear.

Continuous-Phase FSK (CPFSK) A conventional FSK signal is generated by shifting the carrier by an amount $f_n = \frac{1}{2} \Delta f I_n$, $I_n = \pm 1, \pm 3, \dots, \pm(M-1)$, to reflect the digital information that is being transmitted. This type of FSK signal was described in Section 4-3-1, and it is memoryless. The switching from one frequency to another may be accomplished by having $M = 2^k$ separate oscillators tuned to the desired frequencies and selecting one of the M frequencies according to the particular k -bit symbol that is to be transmitted in a signal interval of duration $T = k/R$ seconds. However, such abrupt switching from one oscillator output to another in successive signaling intervals results in relatively large spectral side lobes outside of the main spectral band of the signal and, consequently, this method requires a large frequency band for transmission of the signal.

To avoid the use of signals having large spectral side lobes, the information-bearing signal frequency modulates a single carrier whose frequency is changed continuously. The resulting frequency-modulated signal is phase-continuous and, hence, it is called *continuous-phase* FSK (CPFSK). This type of FSK signal has memory because the phase of the carrier is constrained to be continuous.

In order to represent a CPFSK signal, we begin with a PAM signal

$$d(t) = \sum_n I_n g(t - nT) \quad (4-3-50)$$

where $\{I_n\}$ denotes the sequence of amplitudes obtained by mapping k -bit blocks of binary digits from the information sequence $\{a_n\}$ into the amplitude levels $\pm 1, \pm 3, \dots, \pm(M-1)$ and $g(t)$ is a rectangular pulse of amplitude $1/2T$ and duration T seconds. The signal $d(t)$ is used to frequency-modulate the carrier. Consequently, the equivalent lowpass waveform $v(t)$ is expressed as

$$v(t) = \sqrt{\frac{2\mathcal{E}}{T}} \exp \left\{ j \left[4\pi T f_d \int_{-\infty}^t d(\tau) d\tau + \phi_0 \right] \right\} \quad (4-3-51)$$

where f_d is the *peak frequency deviation* and ϕ_0 is the initial phase of the carrier.

The carrier-modulated signal corresponding to (4-3-51) may be expressed as

$$s(t) = \sqrt{\frac{2\mathcal{E}}{T}} \cos [2\pi f_c t + \phi(t; \mathbf{I}) + \phi_0] \quad (4-3-52)$$

where $\phi(t; \mathbf{I})$ represents the time-varying phase of the carrier, which is defined as

$$\begin{aligned}\phi(t; \mathbf{I}) &= 4\pi T f_d \int_{-\infty}^t d(\tau) d\tau \\ &= 4\pi T f_d \int_{-\infty}^t \left[\sum_n I_n g(\tau - nT) \right] d\tau\end{aligned}\quad (4-3-53)$$

Note that, although $d(t)$ contains discontinuities, the integral of $d(t)$ is continuous. Hence, we have a continuous-phase signal. The phase of the carrier in the interval $nT \leq t \leq (n+1)T$ is determined by integrating (4-3-53). Thus,

$$\begin{aligned}\phi(t; \mathbf{I}) &= 2\pi f_d T \sum_{k=-\infty}^{n-1} I_k + 2\pi f_d (t - nT) I_n \\ &= \theta_n + 2\pi h I_n q(t - nT)\end{aligned}\quad (4-3-54)$$

where h , θ_n , and $q(t)$ are defined as

$$h = 2f_d T \quad (4-3-55)$$

$$\theta_n = \pi h \sum_{k=-\infty}^{n-1} I_k \quad (4-3-56)$$

$$q(t) = \begin{cases} 0 & (t < 0) \\ t/2T & (0 \leq t \leq T) \\ \frac{1}{2} & (t > T) \end{cases} \quad (4-3-57)$$

We observe that θ_n represents the accumulation (memory) of all symbols up to time $(n-1)T$. The parameter h is called the *modulation index*.

Continuous-Phase Modulation (CPM) When expressed in the form of (4-3-54), CPFSK becomes a special case of a general class of continuous-phase modulated (CPM) signals in which the carrier phase is

$$\phi(t; \mathbf{I}) = 2\pi \sum_{k=-\infty}^n I_k h_k q(t - kT), \quad nT \leq t \leq (n+1)T \quad (4-3-58)$$

where $\{I_k\}$ is the sequence of M -ary information symbols selected from the alphabet $\pm 1, \pm 3, \dots, \pm(M-1)$, $\{h_k\}$ is a sequence of modulation indices, and $q(t)$ is some normalized waveform shape.

When $h_k = h$ for all k , the modulation index is fixed for all symbols. When the modulation index varies from one symbol to another, the CPM signal is called *multi- h* . In such a case, the $\{h_k\}$ are made to vary in a cyclic manner through a set of indices.

The waveform $q(t)$ may be represented in general as the integral of some pulse $g(t)$, i.e.,

$$q(t) = \int_0^t g(\tau) d\tau \quad (4-3-59)$$

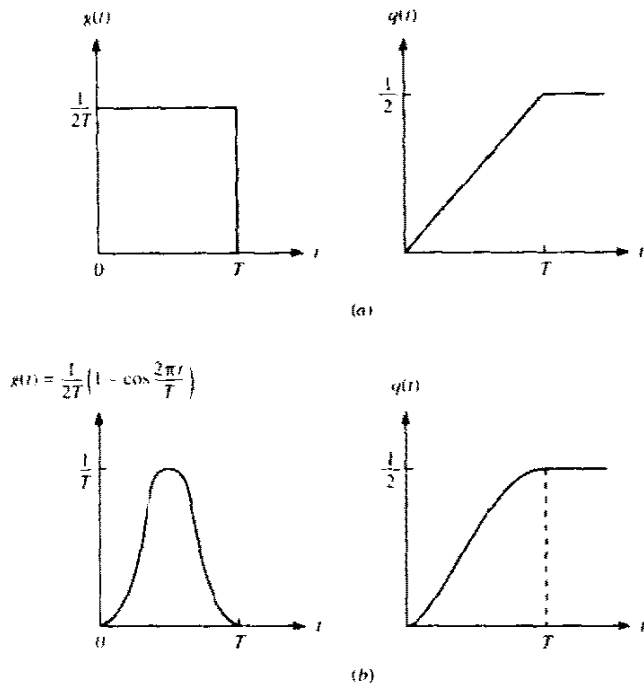
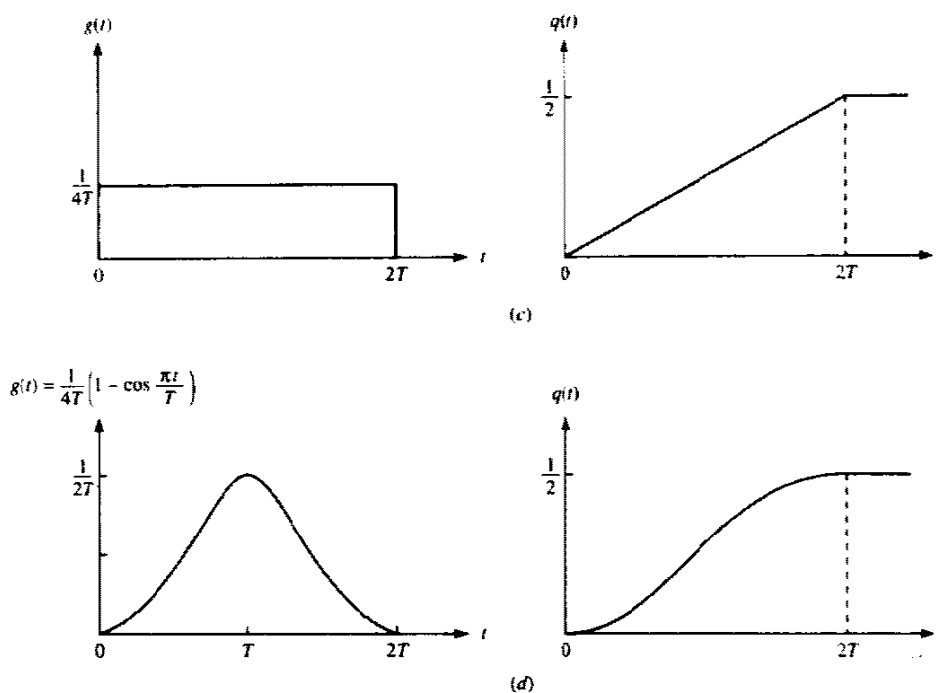


FIGURE 4-3-16 Pulse shapes for full response CPM (a, b) and partial response CPM (c, d).

If $g(t) = 0$ for $t > T$, the CPM signal is called *full response CPM*. If $g(t) \neq 0$ for $t > T$, the modulated signal is called *partial response CPM*. Figure 4-3-16 illustrates several pulse shapes for $g(t)$, and the corresponding $q(t)$. It is apparent that an infinite variety of CPM signals can be generated by choosing different pulse shapes $g(t)$ and by varying the modulation index h and the alphabet size M .

It is instructive to sketch the set of phase trajectories $\phi(t; \mathbf{I})$ generated by all possible values of the information sequence $\{I_n\}$. For example, in the case of CPFSK with binary symbols $I_n = \pm 1$, the set of phase trajectories beginning at time $t = 0$ is shown in Fig. 4-3-17. For comparison, the phase trajectories for quaternary CPFSK are illustrated in Fig. 4-3-18. These phase diagrams are called *phase trees*. We observe that the phase trees for CPFSK are piecewise linear as a consequence of the fact that the pulse $g(t)$ is rectangular. Smoother phase trajectories and phase trees are obtained by using pulses that do not contain discontinuities, such as the class of raised cosine pulses. For example, a phase trajectory generated by the sequence $(1, -1, -1, -1, 1, 1, -1, 1)$ for a partial response, raised cosine pulse of length $3T$ is illustrated in Fig. 4-3-19. For comparison, the corresponding phase trajectory generated by CPFSK is also shown.

The phase trees shown in these figures grow with time. However, the phase



$$g(t) = \frac{1}{4T} \left(1 - \cos \frac{\pi t}{T} \right)$$

FIGURE 4-3-16 (Continued).

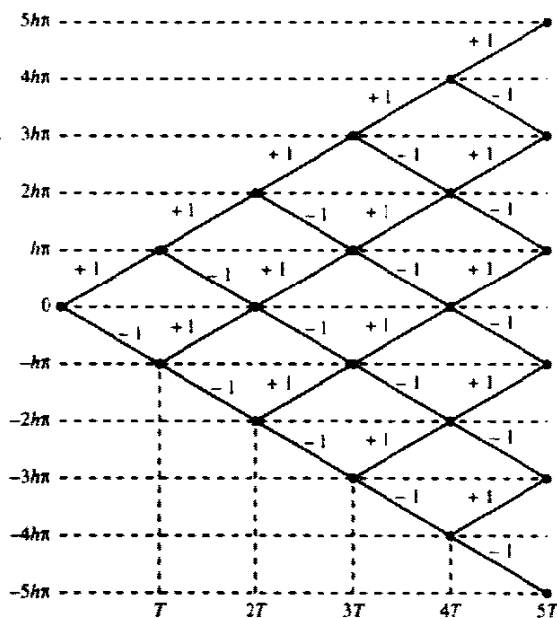


FIGURE 4-3-17 Phase trajectory for binary CPFSK.

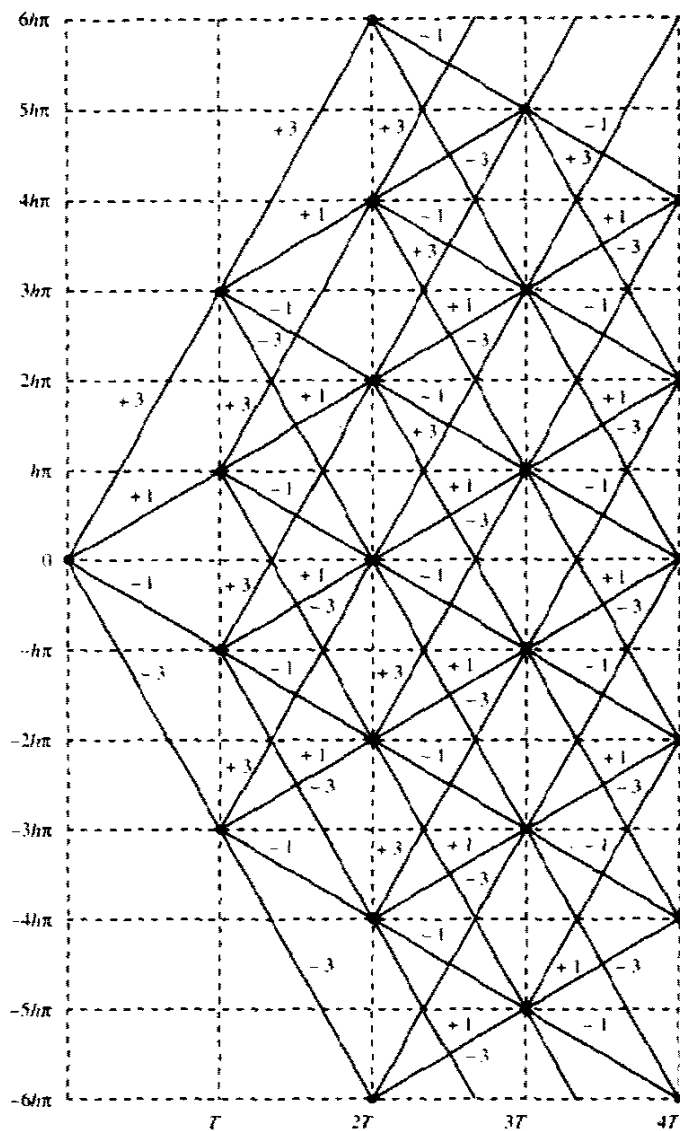


FIGURE 4-3-18 Phase trajectory for quaternary CPFSK.

of the carrier is unique only in the range from $\phi = 0$ to $\phi = 2\pi$ or, equivalently, from $\phi = -\pi$ to $\phi = \pi$. When the phase trajectories are plotted modulo 2π , say in the range $(-\pi, \pi)$, the phase tree collapses into a structure called a *phase trellis*. To properly view the phase trellis diagram, we may plot the two quadrature components $x_c(t; \mathbf{I}) = \cos \phi(t; \mathbf{I})$ and $x_s(t; \mathbf{I}) = \sin \phi(t; \mathbf{I})$ as functions of time. Thus, we generate a three-dimensional plot in which the quadrature components x_c and x_s appear on the surface of a cylinder of unit radius. For example, Fig. 4-3-20 illustrates the phase trellis or phase cylinder

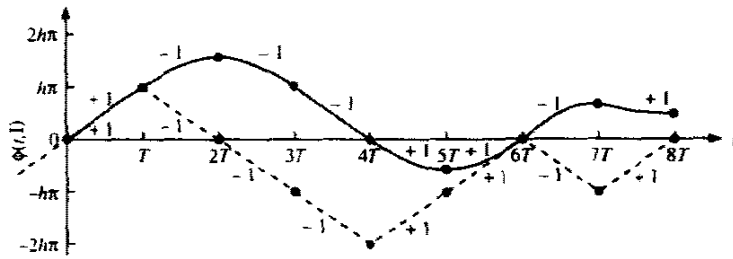


FIGURE 4-3-19 Phase trajectories for binary CPM (dashed) and binary, partial response CPM based on raised cosine pulse of length $3T$. [From Sundberg (1986), © 1986 IEEE.]

obtained with binary modulation, a modulation index $h = \frac{1}{2}$, and a raised cosine pulse of length $3T$.

Simpler representations for the phase trajectories can be obtained by displaying only the terminal values of the signal phase at the time instants $t = nT$. In this case, we restrict the modulation index of the CPM signal to be rational. In particular, let us assume that $h = m/p$, where m and p are relatively prime integers. Then, a full response CPM signal at the time instants $t = nT$ will have the terminal phase states

$$\Theta_s = \left\{ 0, \frac{\pi m}{p}, \frac{2\pi m}{p}, \dots, \frac{(p-1)\pi m}{p} \right\} \quad (4-3-60)$$

when m is even and

$$\Theta_s = \left\{ 0, \frac{\pi m}{p}, \frac{2\pi m}{p}, \dots, \frac{(2p-1)\pi m}{p} \right\} \quad (4-3-61)$$

when m is odd. Hence, there are p terminal phase states when m is even and $2p$ states when m is odd. On the other hand, when the pulse shape extends

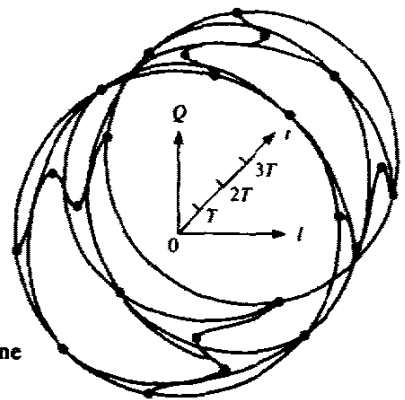


FIGURE 4-3-20 Phase cylinder for binary CPM with $h = \frac{1}{2}$ and a raised cosine pulse of length $3T$. [From Sundberg (1986), © 1986 IEEE.]

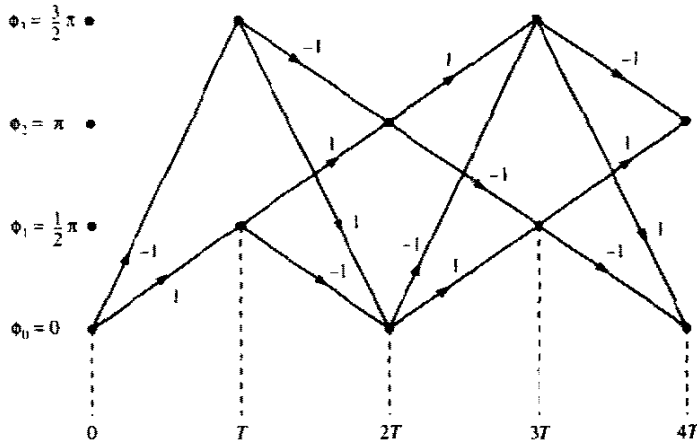


FIGURE 4-3-21 State trellis for binary CPFSK with $h = \frac{1}{2}$.

over L symbol intervals (partial response CPM), the number of phase states may increase up to a maximum of S_t , where

$$S_t = \begin{cases} pM^{L-1} & (\text{even } m) \\ 2pM^{L-1} & (\text{odd } m) \end{cases} \quad (4-3-62)$$

where M is the alphabet size. For example, the binary CPFSK signal (full response, rectangular pulse) with $h = \frac{1}{2}$, has $S_t = 4$ (terminal) phase states. The *state trellis* for this signal is illustrated in Fig. 4-3-21. We emphasize that the phase transitions from one state to another are not true phase trajectories. They represent phase transitions for the (terminal) states at the time instants $t = nT$.

An alternative representation to the state trellis is the state diagram, which also illustrates the state transitions at the time instants $t = nT$. This is an even more compact representation of the CPM signal characteristics. Only the possible (terminal) phase states and their transitions are displayed in the state diagram. Time does not appear explicitly as a variable. For example, the state diagram for the CPFSK signal with $h = \frac{1}{2}$ is shown in Fig. 4-3-22.

Minimum-Shift Keying (MSK) MSK is a special form of binary CPFSK

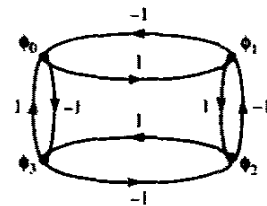


FIGURE 4-3-22 State diagram for binary CPFSK with $h = \frac{1}{2}$.

(and, therefore, CPM) in which the modulation index $h = \frac{1}{2}$. The phase of the carrier in the interval $nT \leq t \leq (n+1)T$ is

$$\begin{aligned}\phi(t; \mathbf{I}) &= \frac{1}{2}\pi \sum_{k=-\infty}^{n-1} I_k + \pi I_n q(t - nT) \\ &= \theta_n + \frac{1}{2}\pi I_n \left(\frac{t - nT}{T} \right), \quad nT \leq t \leq (n+1)T\end{aligned}\quad (4-3-63)$$

and the modulated carrier signal is

$$\begin{aligned}s(t) &= A \cos \left[2\pi f_c t + \theta_n + \frac{1}{2}\pi I_n \left(\frac{t - nT}{T} \right) \right] \\ &= A \cos \left[2\pi \left(f_c + \frac{1}{4T} I_n \right) t - \frac{1}{2}n\pi I_n + \theta_n \right], \quad nT \leq t \leq (n+1)T\end{aligned}\quad (4-3-64)$$

The expression (4-3-64) indicates that the binary CPFSK signal can be expressed as a sinusoid having one of two possible frequencies in the interval $nT \leq t \leq (n+1)T$. If we define these frequencies as

$$\begin{aligned}f_1 &= f_c - \frac{1}{4T} \\ f_2 &= f_c + \frac{1}{4T}\end{aligned}\quad (4-3-65)$$

then the binary CPFSK signal given by (4-3-64) may be written in the form

$$s_i(t) = A \cos [2\pi f_i t + \theta_n + \frac{1}{2}n\pi(-1)^{i-1}], \quad i = 1, 2 \quad (4-3-66)$$

The frequency separation $\Delta f = f_2 - f_1 = 1/2T$. Recall that $\Delta f = 1/2T$ is the minimum frequency separation that is necessary to ensure the orthogonality of the signals $s_1(t)$ and $s_2(t)$ over a signaling interval of length T . This explains why binary CPFSK with $h = \frac{1}{2}$ is called minimum-shift keying (MSK). The phase in the n th signaling interval is the phase state of the signal that results in phase continuity between adjacent intervals.

MSK may also be represented as a form of four-phase PSK. Specifically, we may express the equivalent lowpass digitally modulated signal in the form (see Problem 4-14)

$$v(t) = \sum_{n=-\infty}^{\infty} [I_{2n}g(t - 2nT) - jI_{2n+1}g(t - 2nT - T)] \quad (4-3-67)$$

where $g(t)$ is a sinusoidal pulse defined as

$$g(t) = \begin{cases} \sin \frac{\pi t}{2T} & (0 \leq t \leq 2T) \\ 0 & (\text{otherwise}) \end{cases} \quad (4-3-68)$$

Thus, this type of signal is viewed as a four-phase PSK signal in which the pulse shape is one-half cycle of a sinusoid. The even-numbered binary-valued (± 1) symbols $\{I_{2n}\}$ of the information sequence $\{I_n\}$ are transmitted via the cosine of the carrier, while the odd-numbered symbols $\{I_{2n+1}\}$ are transmitted via the sine of the carrier. The transmission rate on the two orthogonal carrier components is $1/2T$ bits per second so that the combined transmission rate is $1/T$ bits/s. Note that the bit transitions on the sine and cosine carrier components are staggered or offset in time by T seconds. For this reason, the signal

$$s(t) = A \left\{ \left[\sum_{n=-\infty}^{\infty} I_{2n} g(t - 2nT) \right] \cos 2\pi f_c t + \left[\sum_{n=-\infty}^{\infty} I_{2n+1} g(t - 2nT - T) \right] \sin 2\pi f_c t \right\} \quad (4-3-69)$$

is called *offset quadrature PSK (OQPSK)* or *staggered quadrature PSK (SQPSK)*.

Figure 4-3-23 illustrates the representation of the MSK signals as two staggered quadrature-modulated binary PSK signals. The corresponding sum of the two quadrature signals is a constant amplitude, frequency-modulated signal.

It is also interesting to compare the waveforms for MSK with offset QPSK in which the pulse $g(t)$ is rectangular for $0 \leq t \leq 2T$, and with conventional

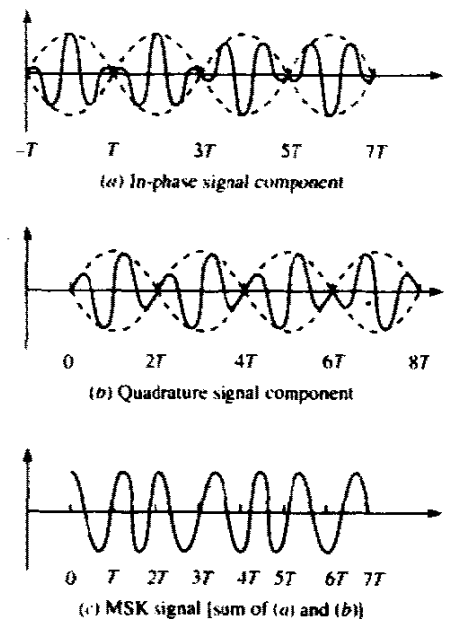


FIGURE 4-3-23 Representation of MSK signal as a form of two staggered binary PSK signals, each with a sinusoidal envelope.

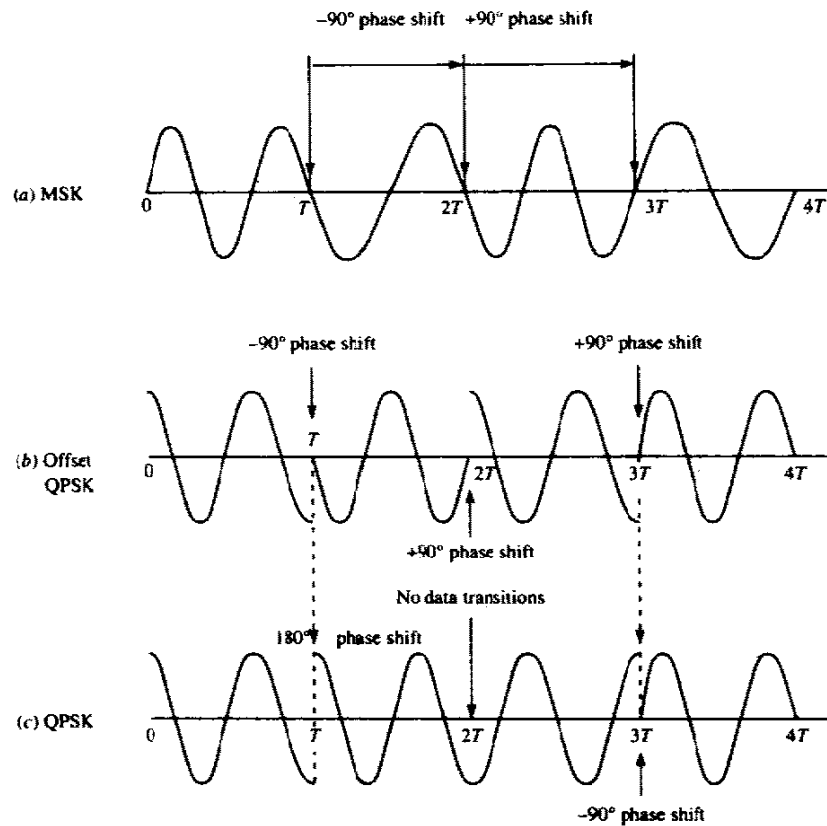


FIGURE 4-3-24 Signal waveforms for (a) MSK, (b) offset QPSK (rectangular pulse), and (c) conventional QPSK (rectangular pulse). [From Gronemeyer and McBride (1976); © 1976 IEEE.]

quadrature (four-phase) PSK (QPSK) in which the pulse $g(t)$ is rectangular for $0 \leq t \leq 2T$. Clearly, all three of the modulation methods result in identical data rates. The MSK signal has continuous phase. The offset QPSK signal with a rectangular pulse is basically two binary PSK signals for which the phase transitions are staggered in time by T seconds. Thus, the signal contains phase jumps of $\pm 90^\circ$ that may occur as often as every T seconds. On the other hand, the conventional four-phase PSK signal with constant amplitude will contain phase jumps of $\pm 180^\circ$ or $\pm 90^\circ$ every $2T$ seconds. An illustration of these three signal types is given in Fig. 4-3-24.

Signal Space Diagrams for CPM In general, continuous-phase signals cannot be represented by discrete points in signal space as in the case of PAM, PSK, and QAM, because the phase of the carrier is time-variant. Instead, a continuous-phase signal is described by the various paths or trajectories from one phase state to another. For a constant-amplitude CPM signal, the various trajectories form a circle.

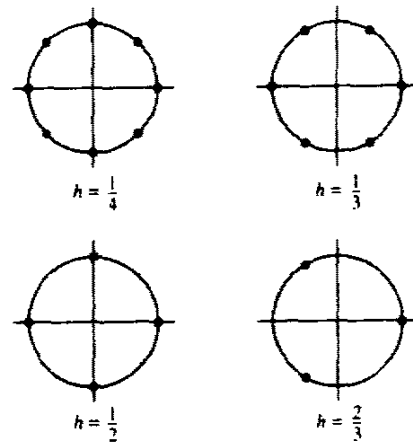


FIGURE 4-3-25 Signal space diagram for CPFSK.

For example, Fig. 4-3-25 illustrates the signal space (phase trajectory) diagram for CPFSK signals with $h = \frac{1}{4}$, $h = \frac{1}{3}$, $h = \frac{1}{2}$, and $h = \frac{2}{3}$. The beginning and ending points of these phase trajectories are marked in the figure by dots. Note that the length of the phase trajectory increases with an increase in h . An increase in h also results in an increase of the signal bandwidth, as demonstrated in the following section.

Multiamplitude CPM Multiamplitude CPM is a generalization of ordinary CPM in which the signal amplitude is allowed to vary over a set of amplitude values while the phase of the signal is constrained to be continuous. For example, let us consider a two-amplitude CPFSK signal, which may be represented as

$$s(t) = 2A \cos [2\pi f_c t + \phi_2(t; \mathbf{I})] + A \cos [2\pi f_c t + \phi_1(t; \mathbf{J})] \quad (4-3-70)$$

where

$$\phi_2(t; \mathbf{I}) = \pi h \sum_{k=-\infty}^{n-1} I_k + \frac{\pi h I_n (t - nT)}{T}, \quad nT \leq t \leq (n+1)T \quad (4-3-71)$$

$$\phi_1(t; \mathbf{J}) = \pi h \sum_{k=-\infty}^{n-1} J_k + \frac{\pi h J_n (t - nT)}{T}, \quad nT \leq t \leq (n+1)T \quad (4-3-72)$$

The information is conveyed by the symbol sequences $\{I_n\}$ and $\{J_n\}$, which are related to two independent binary information sequences $\{a_n\}$ and $\{b_n\}$ that take values $\{0, 1\}$. We observe that the signal in (4-3-70) is a superposition of two CPFSK signals of different amplitude. However, the sequences $\{I_n\}$ and $\{J_n\}$ are not statistically independent, but are constrained in order to achieve phase continuity in the superposition of the two components.

To elaborate, let us consider the case where $h = \frac{1}{2}$, so that we have the superposition of two MSK signals. At the symbol transition points, the two

TABLE 4-3-1

a_n	b_n	I_n	J_n	Amplitude-phase relations
0	0	-1	-1	Amplitude is constant; phase decreases
0	1	-1	1	Amplitude changes; phase decreases
1	0	1	1	Amplitude is constant; phase increases
1	1	1	-1	Amplitude changes; phase increases

amplitude components are either in phase or 180° out of phase. The phase change in the signal is determined by the phase of the larger amplitude component, while the amplitude change is determined by the smaller component. Thus, the smaller component is constrained such that at the start and end of each symbol interval, it is either in phase or 180° out of phase with the larger component, independent of its phase. Under this constraint, the symbol sequences $\{I_n\}$ and $\{J_n\}$ may be expressed as

$$\begin{aligned} I_n &= 2a_n - 1 \\ J_n &= I_n(1 - 2b_n) = I_n\left(1 - \frac{b_n}{h}\right) \end{aligned} \quad (4-3-73)$$

These relationships are summarized in Table 4-3-1.

As a generalization, a multi-amplitude CPFSK signal with n components may be expressed as

$$s(t) = 2^{N-1} \cos[2\pi f_c t + \phi_N(t; \mathbf{I})] + \sum_{m=1}^{N-1} 2^{m-1} \cos[2\pi f_c t + \phi_m(t; \mathbf{J}_m)] \quad (4-3-74)$$

where

$$\phi_N(t; \mathbf{I}) = \pi h I_n \frac{t - nT}{T} + \pi h \sum_{k=-\infty}^{n-1} I_k, \quad nT \leq t \leq (n+1)T \quad (4-3-75)$$

and

$$\begin{aligned} \phi_m(t; \mathbf{J}_m) &= I_n \pi \left[h + \frac{1}{2}(J_{mn} + 1) \right] \frac{t - nT}{T} \\ &+ \sum_{k=-\infty}^{n-1} \pi I_k \left[h + \frac{1}{2}(J_{mk} + 1) \right], \quad nT \leq t \leq (n+1)T \end{aligned} \quad (4-3-76)$$

The sequences $\{I_n\}$ and $\{J_{mn}\}$ are statistically independent, binary-valued sequences that take values from the set $\{1, -1\}$.

From (4-3-75) and (4-3-76), we observe that each component in the sum

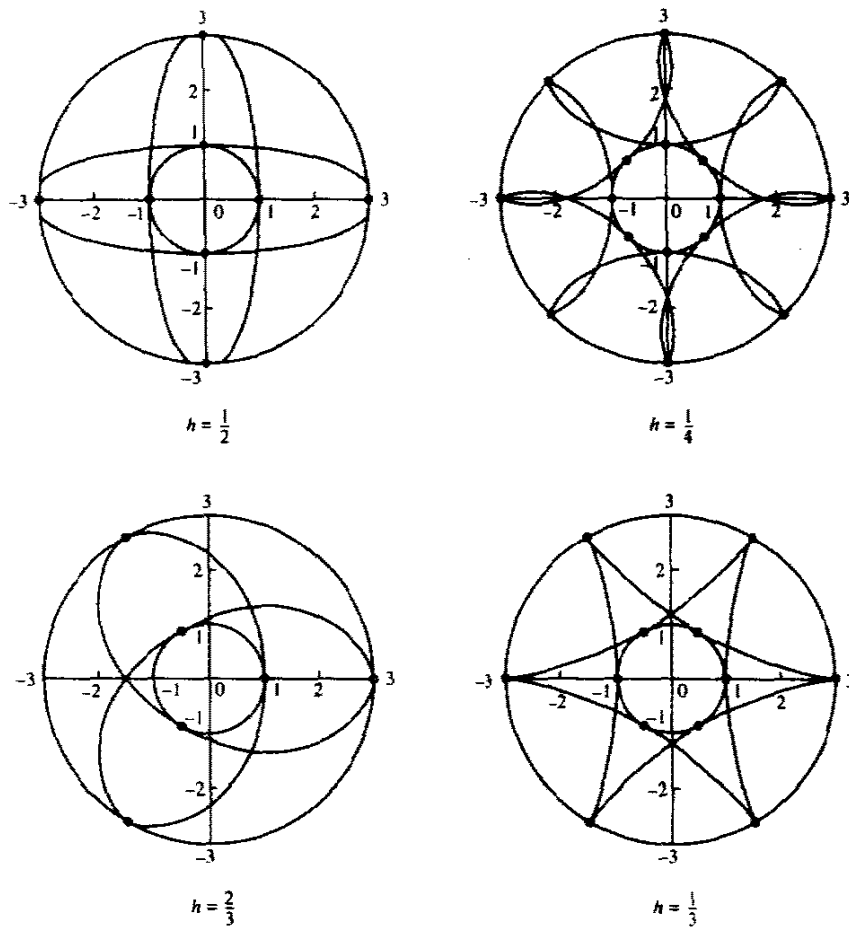


FIGURE 4-3-26 Signal space diagrams for two-component CPFSK.

will be either in phase or 180° out of phase with the largest component at the end of the n th symbol interval, i.e., at $t = (n + 1)T$. Thus, the signal states are specified by an amplitude level from the set of amplitudes $\{1, 3, 5, \dots, 2^N - 1\}$ and a phase level from the set $\{0, \pi\theta, 2\pi\theta, \dots, 2\pi - \pi h\}$. The phase constraint is required to maintain the phase continuity of the CPM signal.

Figure 4-3-26 illustrates the signal space diagrams for two-amplitude ($N = 2$) CPFSK with $h = \frac{1}{4}, \frac{1}{3}, \frac{1}{2},$ and $\frac{2}{3}$. The signal space diagrams for three-component ($N = 3$) CPFSK are shown in Fig. 4-3-27. In this case, there are *four* amplitude levels. The number of states depends on the modulation index h as well as N . Note that the beginning and ending points of the phase trajectories are marked by dots.

Additional multi-amplitude CPM signal formats may be obtained by using

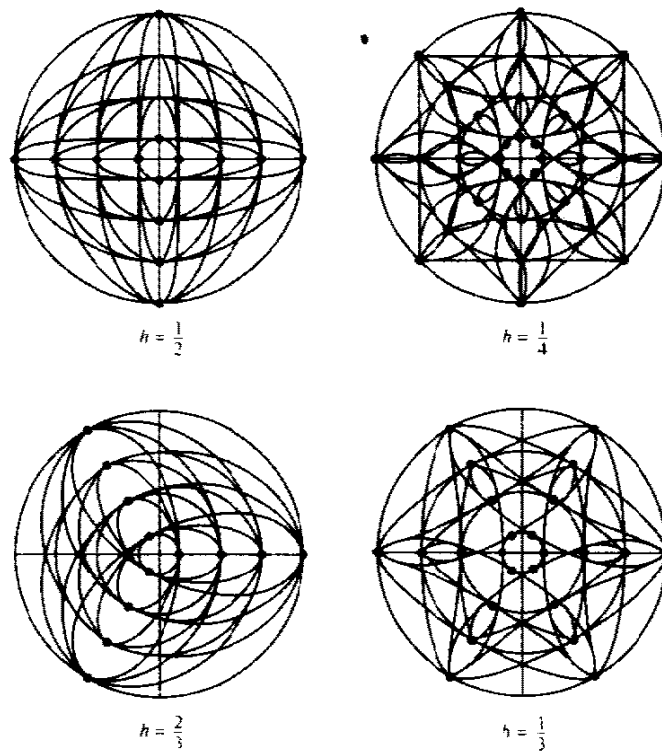


FIGURE 4-3-27 Signal space diagrams for three-component CPFS.

pulse shapes other than rectangular, as well as signal pulses that span more than one symbol (partial response).

4-4 SPECTRAL CHARACTERISTICS OF DIGITALLY MODULATED SIGNALS

In most digital communications systems, the available channel bandwidth is limited. Consequently, the system designer must consider the constraints imposed by the channel bandwidth limitation in the selection of the modulation technique used to transmit the information. For this reason, it is important for us to determine the spectral content of the digitally modulated signals described in Section 4-3.

Since the information sequence is random, a digitally modulated signal is a stochastic process. We are interested in determining the power density spectrum of such a process. From the power density spectrum, we can determine the channel bandwidth required to transmit the information-bearing signal. Below, we first derive the spectral characteristics of the class of linearly

modulated signals. Then, we consider the nonlinear CPFSK, CPM, and baseband modulated signals with memory.

4-4-1 Power Spectra of Linearly Modulated Signals

Beginning with the form

$$s(t) = \text{Re} \{v(t)e^{j2\pi f_c t}\}$$

which relates the bandpass signal $s(t)$ to the equivalent lowpass signal $v(t)$, we can express the autocorrelation function of $s(t)$ as

$$\phi_{ss}(\tau) = \text{Re} \{ \phi_{vv}(\tau)e^{j2\pi f_c \tau} \} \quad (4-4-1)$$

where $\phi_{vv}(\tau)$ is the autocorrelation function of the equivalent lowpass signal $v(t)$. The Fourier transform of (4-4-1) yields the desired expression for the power density spectrum $\Phi_{ss}(f)$ in the form

$$\Phi_{ss}(f) = \frac{1}{2} [\Phi_{vv}(f - f_c) + \Phi_{vv}(-f - f_c)] \quad (4-4-2)$$

where $\Phi_{vv}(f)$ is the power density spectrum of $v(t)$. It suffices to determine the autocorrelation function and the power density spectrum of the equivalent lowpass signal $v(t)$.

First we consider the linear digital modulation methods for which $v(t)$ is represented in the general form

$$v(t) = \sum_{n=-\infty}^{\infty} I_n g(t - nT) \quad (4-4-3)$$

where the transmission rate is $1/T = R/k$ symbols/s and $\{I_n\}$ represents the sequence of symbols that results from mapping k -bit blocks into corresponding signal points selected from the appropriate signal space diagram. Observe that in PAM, the sequence $\{I_n\}$ is real and corresponds to the amplitude values of the transmitted signal, but in PSK, QAM, and combined PAM-PSK, the sequence $\{I_n\}$ is complex-valued, since the signal points have a two-dimensional representation.

The autocorrelation function of $v(t)$ is

$$\begin{aligned} \phi_{vv}(t + \tau; t) &= \frac{1}{2} E \{ v^*(t) v(t + \tau) \} \\ &= \frac{1}{2} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} E \{ I_n^* I_m \} g^*(t - nT) g(t + \tau - mT) \end{aligned} \quad (4-4-4)$$

We assume that the sequence of information symbols $\{I_n\}$ is wide-sense stationary with mean μ_i and autocorrelation function

$$\phi_{ii}(m) = \frac{1}{2} E \{ I_n^* I_{n+m} \} \quad (4-4-5)$$

Hence (4-4-4) can be expressed as

$$\begin{aligned}\phi_{vv}(t + \tau; t) &= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \phi_{ii}(m - n) g^*(t - nT) g(t + \tau - mT) \\ &= \sum_{m=-\infty}^{\infty} \phi_{ii}(m) \sum_{n=-\infty}^{\infty} g^*(t - nT) g(t + \tau - nT - mT) \quad (4-4-6)\end{aligned}$$

The second summation in (4-4-6), namely,

$$\sum_{n=-\infty}^{\infty} g^*(t - nT) g(t + \tau - nT - mT)$$

is periodic in the t variable with period T . Consequently, $\phi_{vv}(t + \tau; t)$ is also periodic in the t variable with period T . That is,

$$\phi_{vv}(t + T + \tau; t + T) = \phi_{vv}(t + \tau; t) \quad (4-4-7)$$

In addition, the mean value of $v(t)$, which is

$$E[v(t)] = \mu_i \sum_{n=-\infty}^{\infty} g(t - nT) \quad (4-4-8)$$

is periodic with period T . Therefore $v(t)$ is a stochastic process having a periodic mean and autocorrelation function. Such a process is called a *cyclostationary process* or a *periodically stationary process in the wide sense*, as described in Section 2-2-6.

In order to compute the power density spectrum of a cyclostationary process, the dependence of $\phi_{vv}(t + \tau; t)$ on the t variable must be eliminated. This can be accomplished simply by averaging $\phi_{vv}(t + \tau; t)$ over a single period. Thus,

$$\begin{aligned}\bar{\phi}_{vv}(\tau) &= \frac{1}{T} \int_{-T/2}^{T/2} \phi_{vv}(t + \tau; t) dt \\ &= \sum_{m=-\infty}^{\infty} \phi_{ii}(m) \sum_{n=-\infty}^{\infty} \frac{1}{T} \int_{-T/2}^{T/2} g^*(t - nT) g(t + \tau - nT - mT) dt \\ &= \sum_{m=-\infty}^{\infty} \phi_{ii}(m) \sum_{n=-\infty}^{\infty} \frac{1}{T} \int_{-T/2 - nT}^{T/2 - nT} g^*(t) g(t + \tau - mT) dt \quad (4-4-9)\end{aligned}$$

We interpret the integral in (4-4-9) as the time-autocorrelation function of $g(t)$ and define it as

$$\phi_{gg}(\tau) = \int_{-\infty}^{\infty} g^*(t) g(t + \tau) dt \quad (4-4-10)$$

Consequently (4-4-9) can be expressed as

$$\bar{\phi}_{vv}(\tau) = \frac{1}{T} \sum_{m=-\infty}^{\infty} \phi_{ii}(m) \phi_{gg}(\tau - mT) \quad (4-4-11)$$

The Fourier transform of the relation in (4-4-11) yields the (average) power density spectrum of $v(t)$ in the form

$$\Phi_{vv}(f) = \frac{1}{T} |G(f)|^2 \Phi_{ii}(f) \quad (4-4-12)$$

where $G(f)$ is the Fourier transform of $g(t)$, and $\Phi_{ii}(f)$ denotes the power density spectrum of the information sequence, defined as

$$\Phi_{ii}(f) = \sum_{m=-\infty}^{\infty} \phi_{ii}(m) e^{-j2\pi f m T} \quad (4-4-13)$$

The result (4-4-12) illustrates the dependence of the power density spectrum of $v(t)$ on the spectral characteristics of the pulse $g(t)$ and the information sequence $\{I_n\}$. That is, the spectral characteristics of $v(t)$ can be controlled by design of the pulse shape $g(t)$ and by design of the correlation characteristics of the information sequence.

Whereas the dependence of $\Phi_{vv}(f)$ on $G(f)$ is easily understood upon observation of (4-4-12), the effect of the correlation properties of the information sequence is more subtle. First of all, we note that for an arbitrary autocorrelation $\phi_{ii}(m)$ the corresponding power density spectrum $\Phi_{ii}(f)$ is periodic in frequency with period $1/T$. In fact, the expression (4-4-13) relating the spectrum $\Phi_{ii}(f)$ to the autocorrelation $\phi_{ii}(m)$ is in the form of an exponential Fourier series with the $\{\phi_{ii}(m)\}$ as the Fourier coefficients. As a consequence, the autocorrelation sequence $\phi_{ii}(m)$ is given by

$$\phi_{ii}(m) = T \int_{-1/2T}^{1/2T} \Phi_{ii}(f) e^{j2\pi f m T} df \quad (4-4-14)$$

Second, let us consider the case in which the information symbols in the sequence are real and mutually uncorrelated. In this case, the autocorrelation function $\phi_{ii}(m)$ can be expressed as

$$\phi_{ii}(m) = \begin{cases} \sigma_i^2 + \mu_i^2 & (m = 0) \\ \mu_i^2 & (m \neq 0) \end{cases} \quad (4-4-15)$$

where σ_i^2 denotes the variance of an information symbol. When (4-4-15) is used to substitute for $\phi_{ii}(m)$ in (4-4-13), we obtain

$$\Phi_{ii}(f) = \sigma_i^2 + \mu_i^2 \sum_{m=-\infty}^{\infty} e^{-j2\pi f m T} \quad (4-4-16)$$

The summation in (4-4-16) is periodic with period $1/T$. It may be viewed as

the exponential Fourier series of a periodic train of impulses with each impulse having an area $1/T$. Therefore (4-4-16) can also be expressed in the form

$$\Phi_{ii}(f) = \sigma_i^2 + \frac{\mu_i^2}{T} \sum_{m=-\infty}^{\infty} \delta\left(f - \frac{m}{T}\right) \quad (4-4-17)$$

Substitution of (4-4-17) into (4-4-12) yields the desired result for the power density spectrum of $v(t)$ when the sequence of information symbols is uncorrelated. That is,

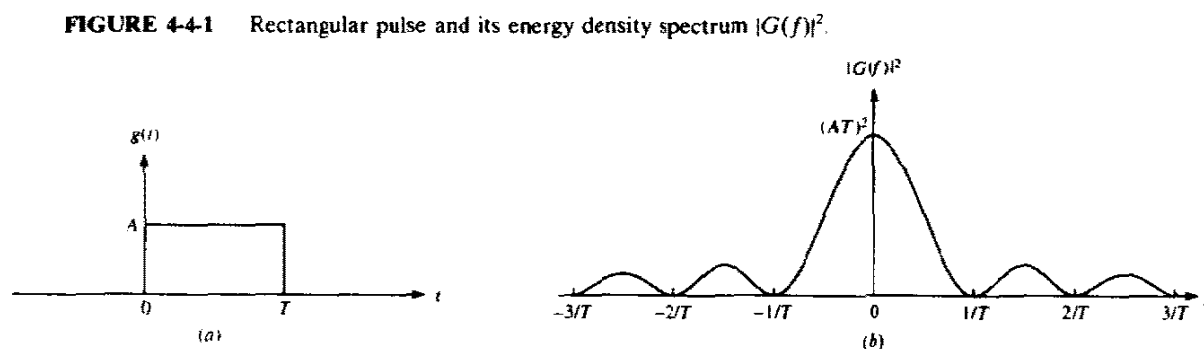
$$\Phi_{vv}(f) = \frac{\sigma_i^2}{T} |G(f)|^2 + \frac{\mu_i^2}{T^2} \sum_{m=-\infty}^{\infty} \left|G\left(\frac{m}{T}\right)\right|^2 \delta\left(f - \frac{m}{T}\right) \quad (4-4-18)$$

The expression (4-4-18) for the power density spectrum is purposely separated into two terms to emphasize the two different types of spectral components. The first term is the continuous spectrum, and its shape depends only on the spectral characteristic of the signal pulse $g(t)$. The second term consists of discrete frequency components spaced $1/T$ apart in frequency. Each spectral line has a power that is proportional to $|G(f)|^2$ evaluated at $f = m/T$. Note that the discrete frequency components vanish when the information symbols have zero mean, i.e., $\mu_i = 0$. This condition is usually desirable for the digital modulation techniques under consideration, and it is satisfied when the information symbols are equally likely and symmetrically positioned in the complex plane. Thus, the system designer can control the spectral characteristics of the digitally modulated signal by proper selection of the characteristics of the information sequence to be transmitted.

Example 4-4-1

To illustrate the spectral shaping resulting from $g(t)$, consider the rectangular pulse shown in Fig. 4-4-1(a). The Fourier transform of $g(t)$ is

$$G(f) = AT \frac{\sin \pi f T}{\pi f T} e^{-j\pi f T}$$



Hence

$$|G(f)|^2 = (AT)^2 \left(\frac{\sin \pi f T}{\pi f T} \right)^2 \tag{4-4-19}$$

This spectrum is illustrated in Fig. 4-4-1(b). Note that it contains zeros at multiples of $1/T$ in frequency and that it decays inversely as the square of the frequency variable. As a consequence of the spectral zeros in $G(f)$, all but one of the discrete spectral components in (4-4-18) vanish. Thus, upon substitution for $|G(f)|^2$ from (4-4-19), (4-4-18) reduces to

$$\Phi_{vv}(f) = \sigma_i^2 A^2 T \left(\frac{\sin \pi f T}{\pi f T} \right)^2 + A^2 \mu_i^2 \delta(f) \tag{4-4-20}$$

Example 4-4-2

As a second illustration of the spectral shaping resulting from $g(t)$, we consider the raised cosine pulse

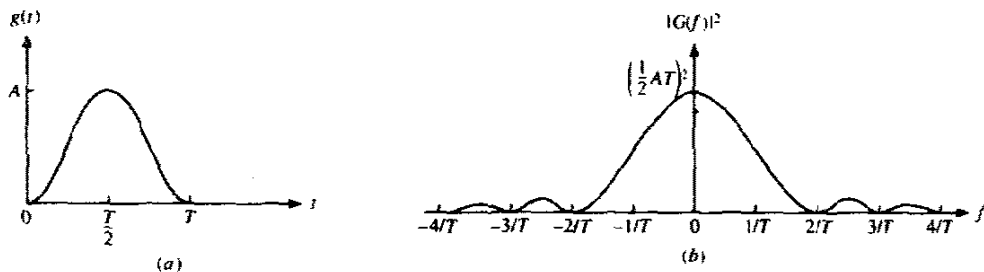
$$g(t) = \frac{A}{2} \left[1 + \cos \frac{2\pi}{T} \left(t - \frac{T}{2} \right) \right], \quad 0 \leq t \leq T \tag{4-2-21}$$

This pulse is graphically illustrated in Fig. 4-4-2(a). Its Fourier transform is easily derived and it may be expressed in the form

$$G(f) = \frac{AT}{2} \frac{\sin \pi f T}{\pi f T (1 - f^2 T^2)} e^{-j\pi f T} \tag{4-4-22}$$

The square of the magnitude of $G(f)$ is shown in Fig. 4-4-2(b). It is interesting to note that the spectrum has zeros at $f = n/T$, $n = \pm 2, \pm 3, \pm 4, \dots$. Consequently, all the discrete spectral components in (4-4-18), except the ones at $f = 0$ and $f = \pm 1/T$, vanish. When compared with the

FIGURE 4-4-2 Raised cosine pulse and its energy density spectrum $|G(f)|^2$.



spectrum of the rectangular pulse, the spectrum of the raised cosine pulse has a broader main lobe but the tails decay inversely as f^6 .

Example 4-4-3

To illustrate that spectral shaping can also be accomplished by operations performed on the input information sequence, we consider a binary sequence $\{b_n\}$ from which we form the symbols

$$I_n = b_n + b_{n-1} \quad (4-4-23)$$

The $\{b_n\}$ are assumed to be uncorrelated random variables, each having zero mean and unit variance. Then the autocorrelation function of the sequence $\{I_n\}$ is

$$\begin{aligned} \phi_{ii}(m) &= E(I_n I_{n-m}) \\ &= \begin{cases} 2 & (m = 0) \\ 1 & (m = \pm 1) \\ 0 & (\text{otherwise}) \end{cases} \end{aligned} \quad (4-4-24)$$

Hence, the power density spectrum of the input sequence is

$$\begin{aligned} \Phi_{ii}(f) &= 2(1 + \cos 2\pi f T) \\ &= 4 \cos^2 \pi f T \end{aligned} \quad (4-4-25)$$

and the corresponding power density spectrum for the (lowpass) modulated signal is

$$\Phi_{vv}(f) = \frac{4}{T} |G(f)|^2 \cos^2 \pi f T \quad (4-4-26)$$

4-4-2 Power Spectra of CPFSK and CPM Signals

In this section, we derive the power density spectrum for the class of constant amplitude CPM signals that were described in Section 4-3-3. We begin by computing the autocorrelation function and its Fourier transform, as was done in the case of linearly modulated signals.

The constant amplitude CPM signal is expressed as

$$s(t; \mathbf{I}) = A \cos [2\pi f_c t + \phi(t; \mathbf{I})] \quad (4-4-27)$$

where

$$\phi(t; \mathbf{I}) = 2\pi h \sum_{k=-\infty}^{\infty} I_k q(t - kT) \quad (4-4-28)$$

Each symbol in the sequence $\{I_n\}$ can take one of the M values $\{\pm 1, \pm 3, \dots, \pm(M-1)\}$. These symbols are statistically independent and identically distributed with prior probabilities

$$P_n = P(I_k = n), \quad n = \pm 1, \pm 3, \dots, \pm(M-1) \quad (4-4-29)$$

where $\sum_n P_n = 1$. The pulse $g(t) = q'(t)$ is zero outside of the interval $[0, LT]$, $q(t) = 0, t < 0$, and $q(t) = \frac{1}{2}$ for $t > LT$.

The autocorrelation function of the equivalent lowpass signal

$$v(t) = e^{j\phi(t;T)}$$

is

$$\phi_{vv}(t + \tau; t) = \frac{1}{2} E \left[\exp \left(j2\pi h \sum_{k=-\infty}^{\infty} I_k [q(t + \tau - kT) - q(t - kT)] \right) \right] \quad (4-4-30)$$

First we express the sum in the exponent as a product of exponents. The result is

$$\phi_{vv}(t + \tau; t) = \frac{1}{2} E \left(\prod_{k=-\infty}^{\infty} \exp \{ j2\pi h I_k [q(t + \tau - kT) - q(t - kT)] \} \right) \quad (4-4-31)$$

Next, we perform the expectation over the data symbols $\{I_k\}$. Since these symbols are statistically independent, we obtain

$$\phi_{vv}(t + \tau; t) = \frac{1}{2} \prod_{k=-\infty}^{\infty} \left(\sum_{\substack{n = \pm 1, \pm 3, \dots, \\ n \text{ odd}}}^{M-1} P_n \exp \{ j2\pi h n [q(t + \tau - kT) - q(t - kT)] \} \right) \quad (4-4-32)$$

Finally, the average autocorrelation function is

$$\bar{\phi}_{vv}(\tau) = \frac{1}{T} \int_0^T \phi_{vv}(t + \tau; t) dt \quad (4-4-33)$$

Although (4-4-32) implies that there are an infinite number of factors in the product, the pulse $g(t) = q'(t) = 0$ for $t < 0$ and $t > LT$, and $q(t) = 0$ for $t < 0$. Consequently only a finite number of terms in the product have nonzero exponents. Thus, (4-4-32) can be simplified considerably. In addition, if we let $\tau = \xi + mT$, where $0 \leq \xi < T$ and $m = 0, 1, \dots$, the average autocorrelation in (4-4-33) reduces to

$$\begin{aligned} & \bar{\phi}_{vv}(\xi + mT) \\ &= \frac{1}{2T} \int_0^T \prod_{k=1-L}^{m-1} \left(\sum_{\substack{n = \pm 1, \pm 3, \dots, \\ n \text{ odd}}}^{M-1} P_n \exp \{ j2\pi h n [q(t + \xi - (k-m)T) - q(t - kT)] \} \right) dt \end{aligned} \quad (4-4-34)$$

Let us focus on $\bar{\phi}_{vv}(\xi + mT)$ for $\xi + mT \geq LT$. In this case, (4-4-34) may be expressed as

$$\bar{\phi}_{vv}(\xi + mT) = [\psi(jh)]^{m-L} \lambda(\xi), \quad m \geq L, \quad 0 \leq \xi < T \quad (4-4-35)$$

where $\psi(jh)$ is the characteristic function of the random sequence $\{I_n\}$, defined as

$$\begin{aligned} \psi(jh) &= E(e^{jnhI_n}) \\ &= \sum_{\substack{n=-(M-1) \\ n \text{ odd}}}^{M-1} P_n e^{jnh} \end{aligned} \quad (4-4-36)$$

and $\lambda(\xi)$ is the remaining part of the average autocorrelation function, which may be expressed as

$$\begin{aligned} \lambda(\xi) &= \frac{1}{2T} \int_0^T \prod_{k=1-L}^0 \left(\sum_{\substack{n=-(M-1) \\ n \text{ odd}}}^{M-1} P_n \exp \{j2\pi hn [\frac{1}{2} - q(t - kT)]\} \right) \\ &\quad \times \prod_{k=1-L}^1 \left(\sum_{\substack{n=-(M-1) \\ n \text{ odd}}}^{M-1} P_n \exp \{j2\pi hn q(t + \xi - kT)\} \right) dt, \quad m \geq L \end{aligned} \quad (4-4-37)$$

Thus, $\bar{\phi}_{vv}(\tau)$ may be separated into a product of $\lambda(\xi)$ and $\psi(jh)$ as indicated in (4-4-35) for $\tau = \xi + mT \geq LT$ and $0 \leq \xi < T$. This property is used below.

The Fourier transform of $\bar{\phi}_{vv}(\tau)$ yields the average power density spectrum as

$$\begin{aligned} \Phi_{vv}(f) &= \int_{-\infty}^{\infty} \bar{\phi}_{vv}(\tau) e^{-j2\pi f\tau} d\tau \\ &= 2 \operatorname{Re} \left[\int_0^{\infty} \bar{\phi}_{vv}(\tau) e^{-j2\pi f\tau} d\tau \right] \end{aligned} \quad (4-4-38)$$

But

$$\begin{aligned} \int_0^{\infty} \bar{\phi}_{vv}(\tau) e^{-j2\pi f\tau} d\tau &= \int_0^{LT} \bar{\phi}_{vv}(\tau) e^{-j2\pi f\tau} d\tau \\ &\quad + \int_{LT}^{\infty} \bar{\phi}_{vv}(\tau) e^{-j2\pi f\tau} d\tau \end{aligned} \quad (4-3-39)$$

With the aid of (4-4-35), the integral in the range $LT \leq \tau < \infty$ may be expressed as

$$\int_{LT}^{\infty} \bar{\phi}_{vv}(\tau) e^{-j2\pi f\tau} d\tau = \sum_{m=L}^{\infty} \int_{mT}^{(m+1)T} \bar{\phi}_{vv}(\tau) e^{-j2\pi f\tau} d\tau \quad (4-4-40)$$

Now, let $\tau = \xi + mT$. Then (4-4-40) becomes

$$\begin{aligned} \int_{LT}^{\infty} \bar{\phi}_{vv}(\tau) e^{-j2\pi f\tau} dt &= \sum_{m=L}^{\infty} \int_0^T \bar{\phi}_{vv}(\xi + mT) e^{-j2\pi f(\xi + mT)} d\xi \\ &= \sum_{m=L}^{\infty} \int_0^T \lambda(\xi) [\psi(jh)]^{m-L} e^{-j2\pi f(\xi + mT)} d\xi \\ &= \sum_{n=0}^{\infty} \psi^n(jh) e^{-j2\pi fnT} \int_0^T \lambda(\xi) e^{-j2\pi f(\xi + LT)} d\xi \end{aligned} \quad (4-4-41)$$

A property of the characteristic function is $|\psi(jh)| \leq 1$. For values of h for which $|\psi(jh)| < 1$, the summation in (4-4-41) converges and yields

$$\sum_{n=0}^{\infty} \psi^n(jh) e^{-j2\pi fnT} = \frac{1}{1 - \psi(jh) e^{-j2\pi fT}} \quad (4-4-42)$$

In this case, (4-4-41) reduces to

$$\int_{LT}^{\infty} \bar{\phi}_{vv}(\tau) e^{-j2\pi f\tau} dt = \frac{1}{1 - \psi(jh) e^{-j2\pi fT}} \int_0^T \bar{\phi}_{vv}(\xi + LT) e^{-j2\pi f(\xi + LT)} d\xi \quad (4-4-43)$$

By combining (4-4-38), (4-4-39), and (4-4-43), we obtain the power density spectrum of the CPM signal in the form

$$\Phi_{vv}(f) = 2 \operatorname{Re} \left[\int_0^{LT} \bar{\phi}_{vv}(\tau) e^{-j2\pi f\tau} d\tau + \frac{1}{1 - \psi(jh) e^{-j2\pi fT}} \int_{LT}^{(L+1)T} \bar{\phi}_{vv}(\tau) e^{-j2\pi f\tau} d\tau \right] \quad (4-4-44)$$

This is the desired result when $|\psi(jh)| < 1$. In general, the power density spectrum is evaluated numerically from (4-4-44). The average autocorrelation function $\bar{\phi}_{vv}(\tau)$ for the range $0 \leq \tau \leq (L+1)T$ may be computed numerically from (4-4-34).

For values of h for which $|\psi(jh)| = 1$, e.g., $h = K$, where K is an integer, we can set

$$\psi(jh) = e^{j2\pi v}, \quad 0 \leq v < 1 \quad (4-4-45)$$

Then, the sum in (4-4-41) becomes

$$\sum_{n=0}^{\infty} e^{-j2\pi T(f - v/T)n} = \frac{1}{2} + \frac{1}{2T} \sum_{n=-\infty}^{\infty} \delta\left(f - \frac{v}{T} - \frac{n}{T}\right) - j \frac{1}{2} \cot \pi T \left(f - \frac{v}{T}\right) \quad (4-4-46)$$

Thus, the power density spectrum now contains impulses located at frequencies

$$f_n = \frac{n + \nu}{T}, \quad 0 \leq \nu < 1, \quad n = 0, 1, 2, \dots \quad (4-4-47)$$

The result (4-4-46) can be combined with (4-4-41) and (4-4-39) to obtain the entire power density spectrum, which includes both a continuous spectrum component and a discrete spectrum component.

Let us return to the case for which $|\psi(jh)| < 1$. When the symbols are equally probable, i.e.,

$$P_n = \frac{1}{M} \quad \text{for all } n$$

the characteristic function simplifies to the form

$$\begin{aligned} \psi(jh) &= \frac{1}{M} \sum_{\substack{n=-(M-1) \\ n \text{ odd}}}^{M-1} e^{j\pi hn} \\ &= \frac{1}{M} \frac{\sin M\pi h}{\sin \pi h} \end{aligned} \quad (4-4-48)$$

Note that in this case $\psi(jh)$ is real. The average autocorrelation function given by (4-4-34) also simplifies in this case to

$$\bar{\phi}_{vv}(\tau) = \frac{1}{2T} \int_0^T \prod_{k=1-L}^{\lfloor \tau/T \rfloor} \frac{1}{M} \frac{\sin 2\pi h M [q(t + \tau - kT) - q(t - kT)]}{\sin 2\pi h [q(t + \tau - kT) - q(t - kT)]} dt \quad (4-4-49)$$

The corresponding expression for the power density spectrum reduces to

$$\begin{aligned} \Phi_{vv}(f) &= 2 \left[\int_0^{LT} \bar{\phi}_{vv}(\tau) \cos 2\pi f\tau d\tau \right. \\ &\quad \left. + \frac{1 - \psi(jh) \cos 2\pi fT}{1 + \psi^2(jh) - 2\psi(jh) \cos 2\pi fT} \int_{LT}^{(L+1)T} \bar{\phi}_{vv}(\tau) \cos 2\pi f\tau d\tau \right] \end{aligned} \quad (4-4-50)$$

Power Density Spectrum of CPFSK A closed-form expression for the power density spectrum can be obtained from (4-4-50) when the pulse shape $g(t)$ is rectangular and zero outside the interval $[0, T]$. In this case, $q(t)$ is linear for $0 \leq t \leq T$. The resulting power spectrum may be expressed as

$$\Phi_{vv}(f) = T \left[\frac{1}{M} \sum_{n=1}^M A_n^2(f) + \frac{2}{M^2} \sum_{n=1}^M \sum_{m=1}^M B_{nm}(f) A_n(f) A_m(f) \right] \quad (4-4-51)$$

where

$$\begin{aligned}
 A_n(f) &= \frac{\sin \pi [fT - \frac{1}{2}(2n-1-M)h]}{\pi [fT - \frac{1}{2}(2n-1-M)h]} \\
 B_{nm}(f) &= \frac{\cos(2\pi fT - \alpha_{nm}) - \psi \cos \alpha_{nm}}{1 + \psi^2 - 2\psi \cos 2\pi fT} \\
 \alpha_{nm} &= \pi h(m+n-1-M) \\
 \psi &\equiv \psi(jh) = \frac{\sin M\pi h}{M \sin \pi h}
 \end{aligned} \tag{4-4-52}$$

The power density spectrum of CPFSK for $M = 2, 4,$ and 8 is plotted in Figs 4-4-3, 4-4-4, and 4-4-5 as a function of the normalized frequency fT , with the modulation index $h = 2f_d T$ as a parameter. Note that only one-half of the bandwidth occupancy is shown in these graphs. The origin corresponds to the carrier f_c . The graphs illustrate that the spectrum of CPFSK is relatively smooth and well confined for $h < 1$. As h approaches unity, the spectra become very peaked and, for $h = 1$ when $|\psi| = 1$, we find that impulses occur at M frequencies. When $h > 1$ the spectrum becomes much broader. In communication systems where CPFSK is used, the modulation index is designed to conserve bandwidth, so that $h < 1$.

The special case of binary CPFSK with $h = \frac{1}{2}$ (or $f_d = 1/4T$) and $\psi = 0$ corresponds to MSK. In this case, the spectrum of the signal is

$$\Phi_{vv}(f) = \frac{16A^2T}{\pi^2} \left(\frac{\cos 2\pi fT}{1 - 16f^2T^2} \right)^2 \tag{4-4-53}$$

where the signal amplitude $A = 1$ in (4-4-52). In contrast the spectrum of four-phase offset (quadrature) PSK (OQPSK) with a rectangular pulse $g(t)$ of duration T is

$$\Phi_{vv}(f) = A^2T \left(\frac{\sin \pi fT}{\pi fT} \right)^2 \tag{4-4-54}$$

If we compare these spectral characteristics, we should normalize the frequency variable by the bit rate or the bit interval T_b . Since MSK is binary FSK, it follows that $T = T_b$ in (4-4-53). On the other hand, in OQPSK, $T = 2T_b$, so that (4-4-54) becomes

$$\Phi_{vv}(f) = 2A^2T_b \left(\frac{\sin 2\pi fT_b}{2\pi fT_b} \right)^2 \tag{4-4-55}$$

The spectra of the MSK and OQPSK signals are illustrated in Fig. 4-4-6. Note that the main lobe of MSK is 50% wider than that for OQPSK. However, the side lobes in MSK fall off considerably faster. For example, if we compare the bandwidth W that contains 99% of the total power, we find that $W = 1.2/T_b$ for MSK and $W \approx 8/T_b$ for OQPSK. Consequently, MSK has a narrower spectral

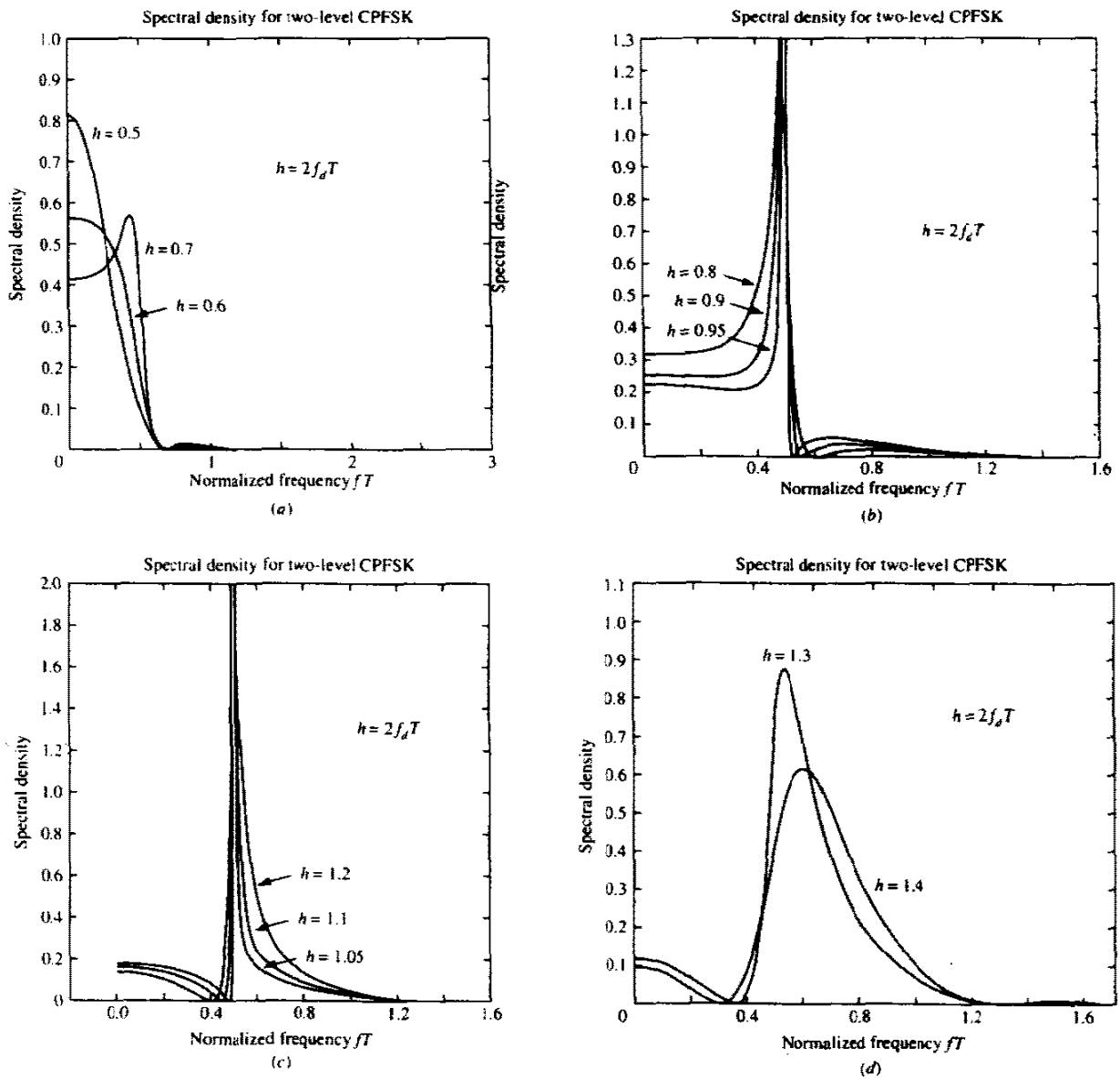


FIGURE 4-4-3 Power density spectrum of binary CPFSK.

occupancy when viewed in terms of fractional out-of-band power above $fT_b = 1$. Graphs for the fractional out-of-band power for OQPSK and MSK are shown in Fig. 4-4-7. Note that MSK is significantly more bandwidth-efficient than QPSK. This efficiency accounts for the popularity of MSK in many digital communications systems.

Even greater bandwidth efficiency than MSK can be achieved by reducing

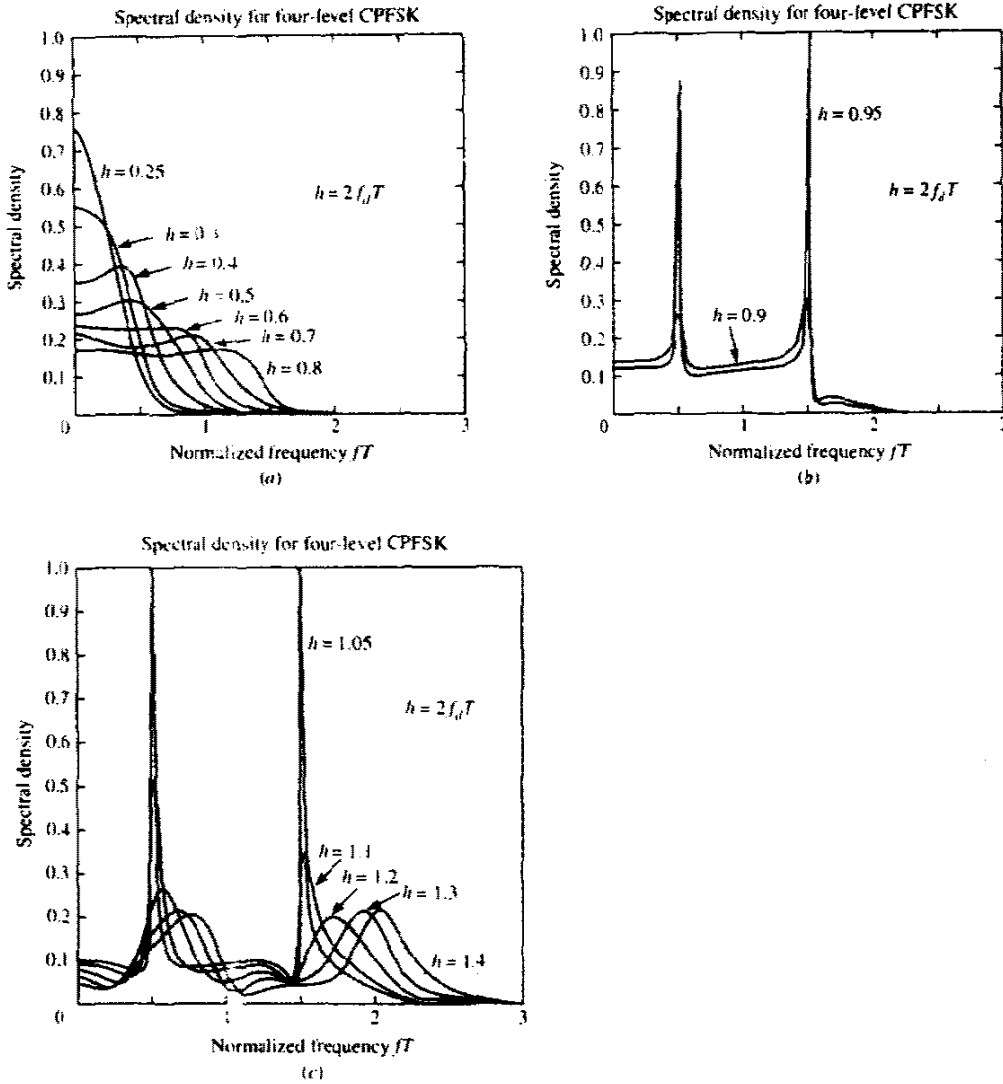


FIGURE 4-44 Power density spectrum of quaternary CPFSK.

the modulation index. However, the FSK signals will no longer be orthogonal and there will be an increase in the error probability.

Spectral Characteristics of CPM In general, the bandwidth occupancy of CPM depends on the choice of the modulation index h , the pulse shape $g(t)$, and the number of signals M . As we have observed for CPFSK, small values of h result in CPM signals with relatively small bandwidth occupancy, while large values of h result in signals with large bandwidth occupancy. This is also the case for the more general CPM signals.

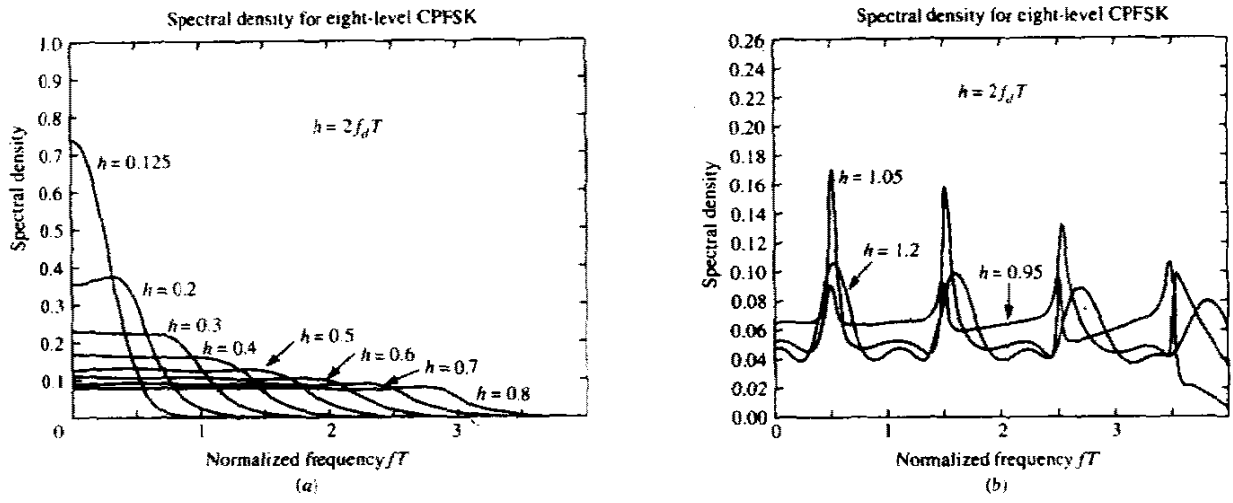
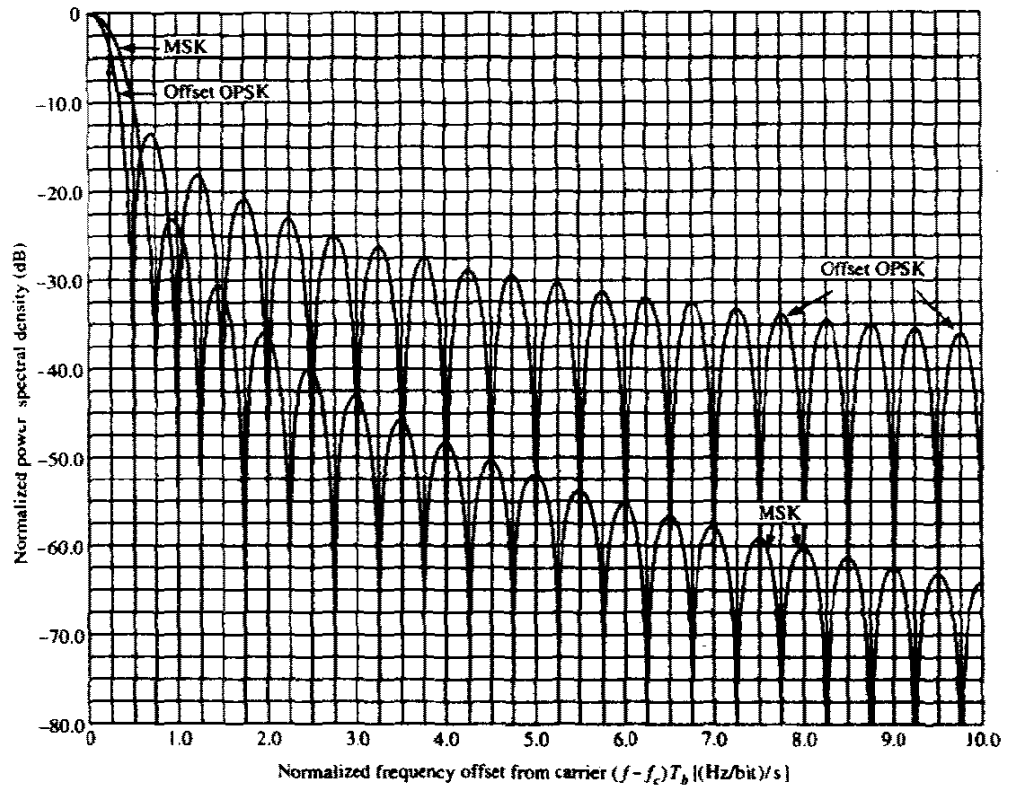


FIGURE 4-4-5 Power density spectrum of octal CPFSK.

FIGURE 4-4-6 Power density spectra of MSK and offset QPSK. [From Gronemeyer and McBride (1976); © IEEE.]



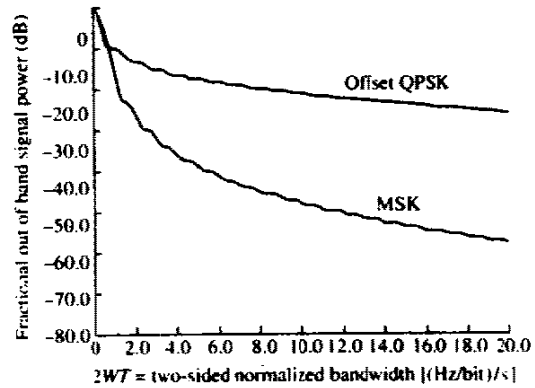


FIGURE 4-4-7 Fractional out-of-band power (normalized two-sided bandwidth = 2 BT). [From Gronemeyer and McBride (1976); © 1976 IEEE.]

The use of smooth pulses such as raised cosine pulses of the form

$$g(t) = \begin{cases} \frac{1}{2LT} \left(1 - \cos \frac{2\pi t}{LT} \right) & (0 \leq t \leq LT) \\ 0 & (\text{otherwise}) \end{cases} \quad (4-4-56)$$

where $L = 1$ for full response and $L > 1$ for partial response, result in smaller bandwidth occupancy and, hence, greater bandwidth efficiency than the use of rectangular pulses. For example, Fig. 4-4-8 illustrates the power density spectrum for binary CPM with different partial response raised cosine (LRC) pulses when $h = \frac{1}{2}$. For comparison, the spectrum of binary CPFSK is also shown. Note that as L increases the pulse $g(t)$ becomes smoother and the corresponding spectral occupancy of the signal is reduced.

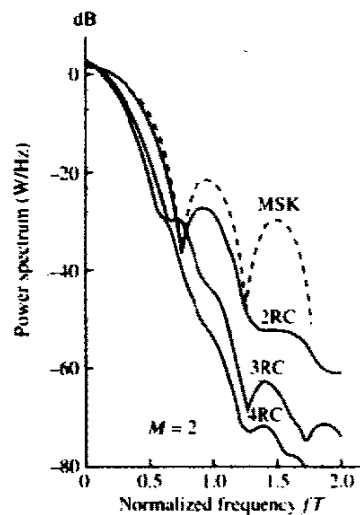


FIGURE 4-4-8 Power density spectrum for binary CPM with $h = \frac{1}{2}$ and different pulse shapes. [From Aulin et al. (1981); © 1981 IEEE.]

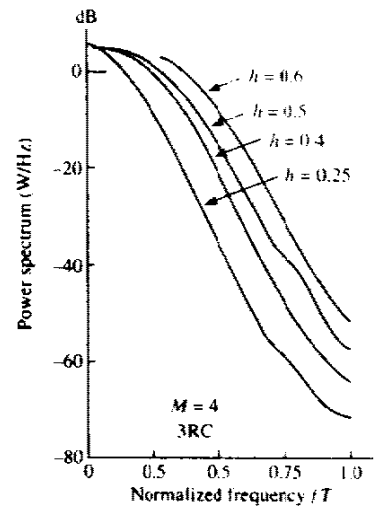
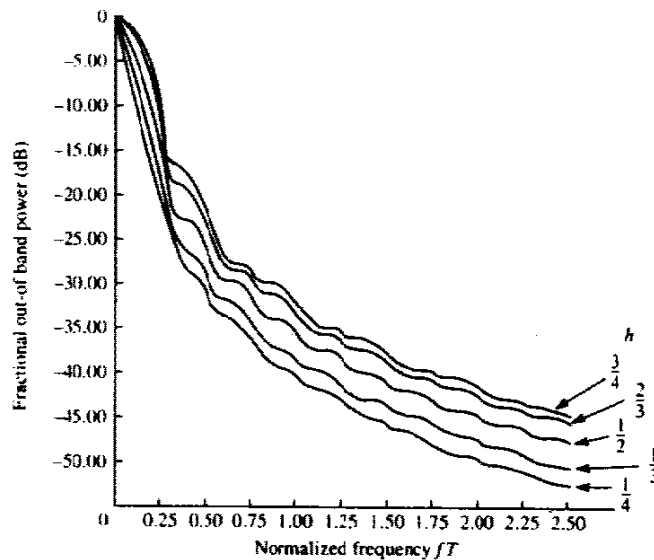


FIGURE 4-4-9 Power density spectrum for $M = 4$ CPM with 3RC and different modulation indices. [From Aulin et al. (1981); © 1981 IEEE.]

The effect of varying the modulation index in a CPM signal is illustrated in Fig. 4-4-9 for the case of $M = 4$ and a raised cosine pulse of the form given in (4-4-56) with $L = 3$. Note that these spectral characteristics are similar to the ones illustrated previously for CPFSK, except that these spectra are narrower due to the use of a smoother pulse shape.

Finally, in Fig. 4-4-10, we illustrate the fractional out-of-band power for two-amplitude CPFSK with several different values of h .

FIGURE 4-4-10 Fractional out-of-band power for two-component CPFSK. (Mulligan, 1988.)



4-4-3 Power Spectra of Modulated Signals with Memory

In the last two sections, we have determined the spectral characteristics for the class of linearly modulated signals without memory and for the class of angle-modulated signals such as CPFSK and CPM, which are nonlinear and possess memory. In this section, we consider the spectral characteristics of linearly modulated signals that have memory that can be modeled by a Markov chain. We have already encountered such signals in Section 4-3-2, where we described several types of baseband signals.

The power density spectrum of a digitally modulated signal that is generated by a Markov chain may be derived by following the basic procedure given in the previous section. Thus, we can determine the autocorrelation function and then evaluate its Fourier transform to obtain the power density spectrum. For signals that are generated by a Markov chain with transition probability matrix \mathbf{P} , the power density spectrum of the modulated signal may be expressed in the general form (see Tiltworth and Welch, 1961)

$$\Phi(f) = \frac{1}{T^2} \sum_{n=-\infty}^{\infty} \left| \sum_{i=1}^K p_i S_i \left(\frac{n}{T} \right) \right|^2 \delta \left(f - \frac{n}{T} \right) + \frac{1}{T} \sum_{i=1}^K p_i |S_i'(f)|^2 + \frac{2}{T} \operatorname{Re} \left[\sum_{i=1}^K \sum_{j=1}^K p_i S_i'^*(f) S_j'(f) P_{ij}(f) \right] \quad (4-4-57)$$

where $S_i(f)$ is the Fourier transform of the signal waveform $s_i(t)$,

$$s_i'(t) = s_i(t) - \sum_{k=1}^K p_k s_k(t)$$

$P_{ij}(f)$ is the Fourier transform of the discrete-time sequence $p_{ij}(n)$, defined as

$$P_{ij}(f) = \sum_{n=1}^{\infty} p_{ij}(n) e^{-j2\pi n f T} \quad (4-4-58)$$

and K is the number of states of the modulator. The term $p_{ij}(n)$ denotes the probability that the signal $s_j(t)$ is transmitted n signaling intervals after the transmission of $s_i(t)$. Hence, $\{p_{ij}(n)\}$ are the transition probabilities in the transition probability matrix \mathbf{P}^n . Note that $p_{ij}(1) = p_{ij}$.

When there is no memory in the modulation method, the signal waveform transmitted on each signaling interval is independent of the waveforms transmitted in previous signaling intervals. The power density spectrum of the resultant signal may still be expressed in the form of (4-4-57), if the transition probability matrix is replaced by

$$\mathbf{P} = \begin{bmatrix} p_1 & p_2 & \dots & p_K \\ p_1 & p_2 & \dots & p_K \\ \vdots & \vdots & & \vdots \\ p_1 & p_2 & \dots & p_K \end{bmatrix} \quad (4-4-59)$$

and we impose the condition that $\mathbf{P}^n = \mathbf{P}$ for all $n \geq 1$. Under these conditions, the expression for the power density spectrum becomes a function of the stationary state probabilities $\{p_i\}$ only, and, hence, it reduces to the simpler form

$$\begin{aligned} \Phi(f) &= \frac{1}{T^2} \sum_{n=-\infty}^{\infty} \left| \sum_{i=1}^K p_i S_i\left(\frac{n}{T}\right) \right|^2 \delta\left(f - \frac{n}{T}\right) \\ &\quad + \frac{1}{T} \sum_{i=1}^K p_i (1 - p_i) |S_i(f)|^2 \\ &\quad - \frac{2}{T} \sum_{i=1}^K \sum_{j=1}^K p_i p_j \operatorname{Re} [S_i(f) S_j^*(f)] \end{aligned} \quad (4-4-60)$$

We observe that our previous result for the power density spectrum of memoryless linear modulation given by (4-4-18) may be viewed as a special case of (4-4-60) in which all waveforms are identical except for a set of scale factors that convey the digital information (Problem 4-30).

We also make the observation that the first term in the expression for the power density spectrum given by either (4-4-57) or (4-4-60) consists of discrete frequency components. This line spectrum vanishes when

$$\sum_{i=1}^K p_i S_i\left(\frac{n}{T}\right) = 0 \quad (4-4-61)$$

The condition (4-4-61) is usually imposed in the design of practical digital communications systems and is easily satisfied by an appropriate choice of signaling waveforms (Problem 4-31).

Now, let us determine the power density spectrum of the baseband-modulated signals described in Section 4-3-2. First, the NRZ signal is characterized by the two waveforms $s_1(t) = g(t)$ and $s_2(t) = -g(t)$, where $g(t)$ is a rectangular pulse of amplitude A . For $K = 2$, (4-4-60) reduces to

$$\Phi(f) = \frac{(2p-1)^2}{T^2} \sum_{n=-\infty}^{\infty} \left| G\left(\frac{n}{T}\right) \right|^2 \delta\left(f - \frac{n}{T}\right) + \frac{4p(1-p)}{T} |G(f)|^2 \quad (4-4-62)$$

where

$$|G(f)|^2 = (AT)^2 \left(\frac{\sin \pi f T}{\pi f T} \right)^2 \quad (4-4-63)$$

Observe that when $p = \frac{1}{2}$, the line spectrum vanishes and $\Phi(f)$ reduces to

$$\Phi(f) = \frac{1}{T} |G(f)|^2 \quad (4-4-64)$$

The NRZI signal is characterized by the transition probability matrix

$$\mathbf{P} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \quad (4-4-65)$$

Notice that in this case $\mathbf{P}^n = \mathbf{P}$ for all $n \geq 1$. Hence, the special form for the power density spectrum given by (4-4-62) applies to this modulation format as well. Consequently, the power density spectrum for the NRZI signal is identical to the spectrum of the NRZ signal.

Delay modulation has a transition probability matrix

$$\mathbf{P} = \begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{bmatrix} \quad (4-4-66)$$

and stationary state probabilities $p_i = \frac{1}{4}$ for $i = 1, 2, 3, 4$. Powers of \mathbf{P} may be obtained by use of the relation

$$\mathbf{P}^k \mathbf{p} = -\frac{1}{4} \mathbf{p} \quad (4-4-67)$$

where \mathbf{p} is the signal correlation matrix with elements

$$\rho_{ij} = \frac{1}{T} \int_0^T s_i(t) s_j(t) dt \quad (4-4-68)$$

and where the four signals $\{s_i(t), i = 1, 2, 3, 4\}$ are shown in Fig. 4-3-15. It is easily seen that

$$\mathbf{p} = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \quad (4-4-69)$$

Consequently, powers of \mathbf{P} can be generated from the relation

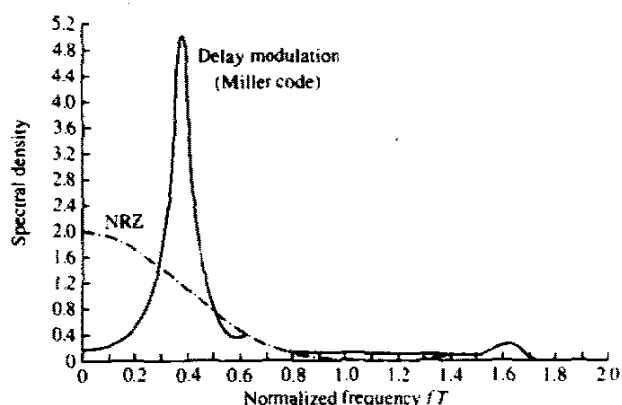
$$\mathbf{P}^{k+4} \mathbf{p} = -\frac{1}{4} \mathbf{P}^k \mathbf{p}, \quad k > 1 \quad (4-4-70)$$

Use of (4-4-66), (4-4-69), and (4-4-70) in (4-4-57) yields the power density spectrum of delay modulation. It may be expressed in the form

$$\Phi(f) = \frac{1}{2\psi^2(17 + 8 \cos 8\psi)} [23 - 2 \cos \psi - 22 \cos 2\psi - 12 \cos 3\psi + 5 \cos 4\psi + 12 \cos 5\psi + 2 \cos 6\psi - 8 \cos 7\psi + 2 \cos 8\psi] \quad (4-4-71)$$

where $\psi = \pi f T$.

FIGURE 4-4-11 Power spectral density (one-sided) of Miller code (delay modulation) and NRZ/NRZI baseband signals. [From Hecht and Guida (1969); © 1969 IEEE.]



The spectra of these baseband signals are illustrated in Fig. 4-4-11. Observe that the spectra of the NRZ and NRZI signals peak at $f = 0$. Delay modulation has a narrower spectrum and a relatively small zero-frequency content. Its bandwidth occupancy is significantly smaller than that of the NRZ signal. These two characteristics make delay modulation an attractive choice for channels that do not pass dc, such as magnetic recording media.

4-5 BIBLIOGRAPHICAL NOTES AND REFERENCES

The characteristics of signals and systems given in this chapter are very useful in the design of optimum modulation/demodulation and coding/decoding techniques for a variety of channel models. In particular, the digital modulation methods introduced in this chapter are widely used in digital communication systems. The next chapter is concerned with optimum demodulation techniques for these signals and their performance in an additive, white gaussian noise channel. A general reference for signal characterization is the book by Franks (1969).

Of particular importance in the design of digital communications systems are the spectral characteristics of the digitally modulated signals, which are presented in this chapter in some depth. Of these modulation techniques, CPM is one of the most important due to its efficient use of bandwidth. For this reason, it has been widely investigated by many researchers, and a large number of papers have been published in the technical literature. The most comprehensive treatment of CPM, including its performance and its spectral characteristics, can be found in the book by Anderson *et al.* (1986). In addition to this text, the tutorial paper by Sundberg (1986) presents the basic concepts and an overview of the performance characteristics of various CPM techniques. This paper also contains over 100 references to published papers on this topic.

There are a large number of references dealing with the spectral characteristics of CPFSK and CPM. As a point of reference, we should mention that MSK was invented by Doelz and Heald in 1961. The early work on the power

spectral density of CPFSK and CPM was done by Bennett and Rice (1963), Anderson and Salz (1965), and Bennett and Davey (1965). The book by Lucky *et al.* (1968) also contains a treatment of the spectral characteristics of CPFSK. Most of the recent work is referenced in the paper by Sundberg (1986). We should also cite the special issue on bandwidth-efficient modulation and coding published by the *IEEE Transactions on Communications* (March 1981), which contains several papers on the spectral characteristics and performance of CPM.

The generalization of MSK to multiple amplitudes was investigated by Weber *et al.* (1978). The combination of multiple amplitudes with general CPM was proposed by Mulligan (1988) who investigated its spectral characteristics and its error probability performance in gaussian noise with and without coding.

4-1 Prove the following properties of Hilbert transforms:

- a If $x(t) = x(-t)$ then $\hat{x}(t) = -\hat{x}(-t)$;
- b If $x(t) = -x(-t)$ then $\hat{x}(t) = \hat{x}(-t)$;
- c If $x(t) = \cos \omega_0 t$ then $\hat{x}(t) = \sin \omega_0 t$;
- d If $x(t) = \sin \omega_0 t$ then $\hat{x}(t) = -\cos \omega_0 t$;
- e $\hat{\hat{x}}(t) = -x(t)$;
- f $\int_{-\infty}^{\infty} x^2(t) dt = \int_{-\infty}^{\infty} \hat{x}^2(t) dt$;
- g $\int_{-\infty}^{\infty} x(t)\hat{x}(t) dt = 0$.

4-2 If $x(t)$ is a stationary random process with autocorrelation function $\phi_{xx}(\tau) = E[x(t)x(t+\tau)]$ and spectral density $\Phi_{xx}(f)$ then show that $\phi_{xx}(\tau) = \phi_{xx}(\tau)$, $\phi_{xx}(\tau) = -\hat{\phi}_{xx}(\tau)$, and $\Phi_{xx}(f) = \Phi_{xx}(f)$.

4-3 Suppose that $n(t)$ is a zero-mean stationary narrowband process represented by either (4-1-37), (4-1-38), or (4-1-39). The autocorrelation function of the equivalent lowpass process $z(t) = x(t) + jy(t)$ is defined as

$$\phi_{zz}(\tau) = \frac{1}{2}E[z^*(t)z(t+\tau)]$$

a Show that

$$E[z(t)z(t+\tau)] = 0$$

b Suppose $\phi_{zz}(\tau) = N_0\delta(\tau)$, and let

$$V = \int_{t_1}^{t_2} z(t) dt$$

Determine $E(V^2)$ and $E(VV^*) = E(|V|^2)$.

4-4 Determine the autocorrelation function of the stochastic process

$$x(t) = A \sin(2\pi f_c t + \theta)$$

where f_c is a constant and θ is a uniformly distributed phase, i.e.,

$$p(\theta) = \frac{1}{2\pi}, \quad 0 \leq \theta \leq 2\pi$$

4-5 Prove that $s_r(t)$ is generally a complex-valued signal and give the condition under which it is real. Assume that $s(t)$ is a real-valued bandpass signal.

4-6 Suppose that $s(t)$ is either a real- or complex-valued signal that is represented as a linear combination of orthonormal functions $\{f_n(t)\}$, i.e.,

$$\hat{s}(t) = \sum_{k=1}^K s_k f_k(t)$$

where

$$\int_{-\infty}^{\infty} f_n(t) f_m^*(t) dt = \begin{cases} 0 & (m \neq n) \\ 1 & (m = n) \end{cases}$$

Determine the expressions for the coefficients $\{s_k\}$ in the expansion $\hat{s}(t)$ that minimize the energy

$$\mathcal{E}_e = \int_{-\infty}^{\infty} |s(t) - \hat{s}(t)|^2 dt$$

and the corresponding residual error \mathcal{E}_e .

4-7 Suppose that a set of M signal waveforms $\{s_m(t)\}$ are complex-valued. Derive the equations for the Gram-Schmidt procedure that will result in a set of $N \leq M$ orthonormal signal waveforms.

4-8 Determine the correlation coefficients ρ_{km} among the four signal waveforms $\{s_i(t)\}$ shown in Fig. 4-2-1, and the corresponding Euclidean distances.

4-9 Consider a set of M orthogonal signal waveforms $s_m(t)$, $1 \leq m \leq M$, $0 \leq t \leq T$, all of which have the same energy \mathcal{E} . Define a new set of M waveforms as

$$s'_m(t) = s_m(t) - \frac{1}{M} \sum_{k=1}^M s_k(t), \quad 1 \leq m \leq M, \quad 0 \leq t \leq T$$

Show that the M signal waveforms $\{s'_m(t)\}$ have equal energy, given by

$$\mathcal{E}' = (M - 1)\mathcal{E}/M$$

and are equally correlated, with correlation coefficient

$$\rho_{mn} = \frac{1}{\mathcal{E}'} \int_0^T s'_m(t) s'_n(t) dt = -\frac{1}{M - 1}$$

4-10 Consider the three waveforms $f_n(t)$ shown in Fig. P4-10.

a Show that these waveforms are orthonormal.

b Express the waveform $x(t)$ as a weighted linear combination of $f_n(t)$, $n = 1, 2, 3$, if

$$x(t) = \begin{cases} -1 & (0 \leq t < 1) \\ 1 & (1 \leq t < 3) \\ -1 & (3 \leq t < 4) \end{cases}$$

and determine the weighting coefficients.

4-11 Consider the four waveforms shown in Fig. P4-11.

a Determine the dimensionality of the waveforms and a set of basis functions.

b Use the basis functions to represent the four waveforms by vectors \mathbf{s}_1 , \mathbf{s}_2 , \mathbf{s}_3 , and \mathbf{s}_4 .

c Determine the minimum distance between any pair of vectors.

4-12 Determine a set of orthonormal functions for the four signals shown in Fig. P4-12.

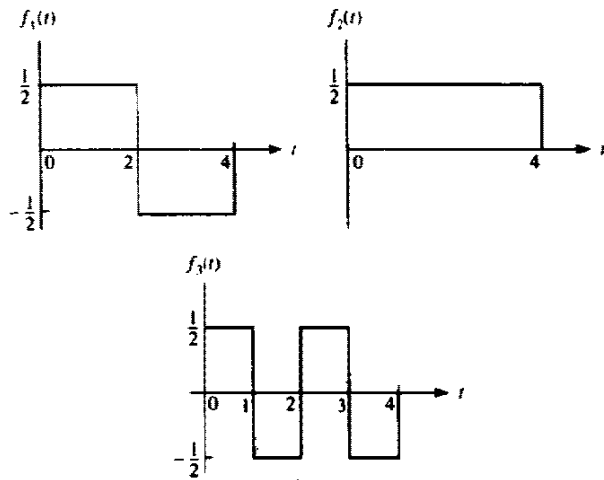


FIGURE P4-10

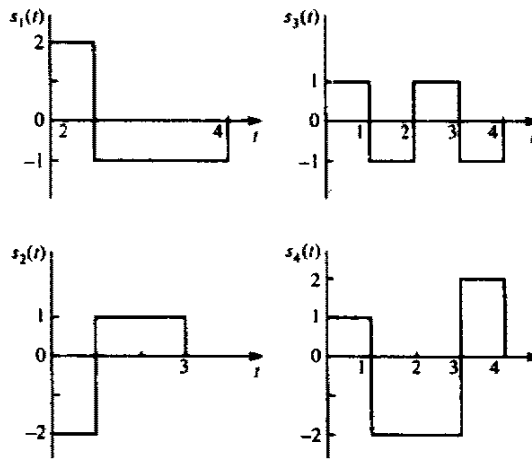


FIGURE P4-11

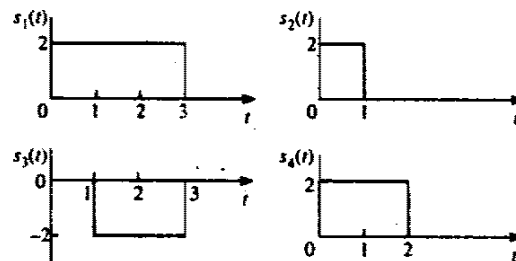


FIGURE P4-12

4-13 A lowpass gaussian stochastic process $x(t)$ has a power spectral density

$$\Phi(f) = \begin{cases} N_0 & (|f| < B) \\ 0 & (|f| > B) \end{cases}$$

Determine the power spectral density and the autocorrelation function of $y(t) = x^2(t)$.

4-14 Consider an equivalent lowpass digitally modulated signal of the form

$$u(t) = \sum_n \{a_n g(t - 2nT) - jb_n g(t - 2nT - T)\}$$

where $\{a_n\}$ and $\{b_n\}$ are two sequences of statistically independent binary digits and $g(t)$ is a sinusoidal pulse defined as

$$g(t) = \begin{cases} \sin(\pi t/2T) & (0 < t < 2T) \\ 0 & (\text{otherwise}) \end{cases}$$

This type of signal is viewed as a four-phase PSK signal in which the pulse shape is one-half cycle of a sinusoid. Each of the information sequences $\{a_n\}$ and $\{b_n\}$ is transmitted at a rate of $1/2T$ bits/s and, hence, the combined transmission rate is $1/T$ bits/s. The two sequences are staggered in time by T seconds in transmission. Consequently, the signal $u(t)$ is called *staggered four-phase PSK*.

- a Show that the envelope $|u(t)|$ is a constant, independent of the information a_n on the in-phase component and information b_n on the quadrature component. In other words, the amplitude of the carrier used in transmitting the signal is constant.
 - b Determine the power density spectrum of $u(t)$.
 - c Compare the power density spectrum obtained from (b) with the power density spectrum of the MSK signal. What conclusion can you draw from this comparison?
- 4-15 Consider a four-phase PSK signal represented by the equivalent lowpass signal

$$u(t) = \sum_n I_n g(t - nT)$$

where I_n takes on one of the four possible values $\sqrt{1/2}(\pm 1 \pm j)$ with equal probability. The sequence of information symbols $\{I_n\}$ is statistically independent.

- a Determine and sketch the power density spectrum of $u(t)$ when

$$g(t) = \begin{cases} A & (0 \leq t \leq T) \\ 0 & (\text{otherwise}) \end{cases}$$

- b Repeat (a) when

$$g(t) = \begin{cases} A \sin(\pi t/T) & (0 \leq t \leq T) \\ 0 & (\text{otherwise}) \end{cases}$$

- c Compare the spectra obtained in (a) and (b) in terms of the 3 dB bandwidth and the bandwidth to the first spectral zero.

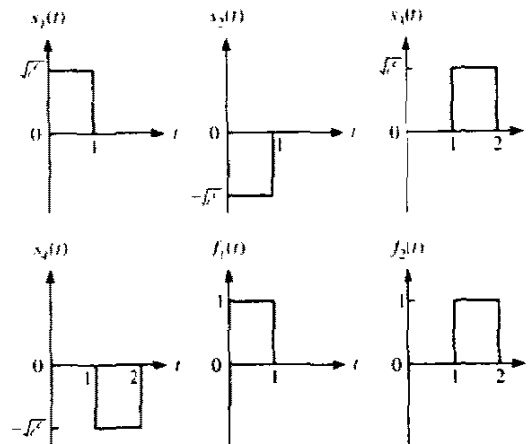


FIGURE P4-18

4-16 The random process $v(t)$ is defined as

$$v(t) = X \cos 2\pi f_c t - Y \sin 2\pi f_c t$$

where X and Y are random variables. Show that $v(t)$ is wide-sense stationary if and only if $E(X) = E(Y) = 0$, $E(X^2) = E(Y^2)$, and $E(XY) = 0$.

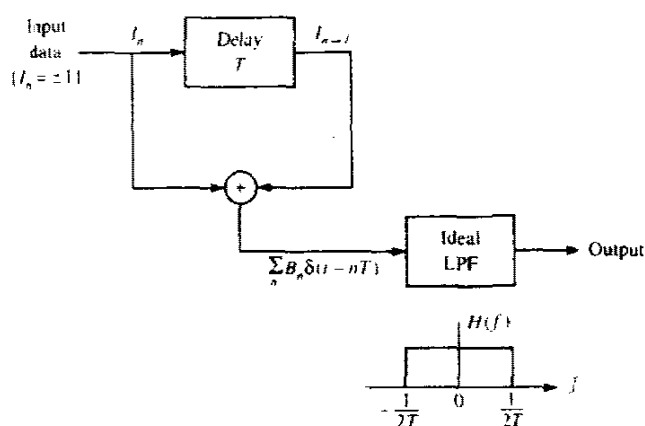
- 4-17 Carry out the Gram-Schmidt orthogonalization of the signals in Fig. 4-2-1(a) in the order $s_4(t)$, $s_3(t)$, $s_1(t)$, and, thus, obtain a set of orthonormal functions $\{f_m(t)\}$. Then, determine the vector representation of the signals $\{s_n(t)\}$ by using the orthonormal functions $\{f_m(t)\}$. Also, determine the signal energies.
- 4-18 Determine the signal space representation of the four signals $s_k(t)$, $k = 1, 2, 3, 4$, shown in Fig. P4-18, by using as basis functions the orthonormal functions $f_1(t)$ and $f_2(t)$. Plot the signal space diagram and show that this signal set is equivalent to that for a four-phase PSK signal.
- 4-19 The power density spectrum of the cyclostationary process

$$v(t) = \sum_n I_n g(t - nT)$$

was derived in Section 4-4-1 by averaging the autocorrelation function $\phi_{vv}(t + \tau, t)$ over the period T of the process and then evaluating the Fourier transform of the average autocorrelation function. An alternative approach is to change the cyclostationary process into a stationary process $v_\Delta(t)$ by adding a random variable Δ , uniformly distributed over $0 \leq \Delta < T$, so that

$$v_\Delta(t) = \sum_n I_n g(t - nT - \Delta)$$

and defining the spectral density of $v(t)$ as the Fourier transform of the autocorrelation function of the stationary process $v_\Delta(t)$. Derive the result in (4-4-11), by evaluating the autocorrelation function of $v_\Delta(t)$ and its Fourier transform.



4-20 A PAM partial response signal (PRS) is generated as shown in Fig. P4-20 by exciting an ideal lowpass filter of bandwidth W by the sequence

$$B_n = I_n + I_{n-1}$$

at a rate $1/T = 2W$ symbols/s. The sequence $\{I_n\}$ consists of binary digits selected independently from the alphabet $\{1, -1\}$ with equal probability. Hence, the filtered signal has the form

$$v(t) = \sum_{n=-\infty}^{\infty} B_n g(t - nT), \quad T = \frac{1}{2W}$$

- Sketch the signal space diagram for $v(t)$ and determine the probability of occurrence of each symbol.
 - Determine the autocorrelation and power density spectrum of the three-level sequence $\{B_n\}$.
 - The signal points of the sequence $\{B_n\}$ form a Markov chain. Sketch this Markov chain and indicate the transition probabilities among the states.
- 4-21** The lowpass equivalent representation of a PAM signal is

$$u(t) = \sum_n I_n g(t - nT)$$

Suppose $g(t)$ is a rectangular pulse and

$$I_n = a_n - a_{n-2}$$

where $\{a_n\}$ is a sequence of uncorrelated binary-valued $(1, -1)$ random variables that occur with equal probability.

- Determine the autocorrelation function of the sequence $\{I_n\}$.
 - Determine the power density spectrum of $u(t)$.
 - Repeat (b) if the possible values of the a_n are $(0, 1)$.
- 4-22** Show that $x(t) = s(t) \cos 2\pi f_c t \pm \hat{s}(t) \sin 2\pi f_c t$ is a single-sideband signal, where $s(t)$ is band-limited to $B \leq f_c$ Hz and $\hat{s}(t)$ is its Hilbert transform.

4-23 Use the results in Section 4-4-3 to determine the power density spectrum of the binary FSK signals in which the waveforms are

$$s_i(t) = \sin \omega_i t, \quad i = 1, 2, \quad 0 \leq t \leq T$$

where $\omega_1 = n\pi/T$ and $\omega_2 = m\pi/T$, $n \neq m$, and m and n are arbitrary positive integers. Assume that $p_1 = p_2 = \frac{1}{2}$. Sketch the spectrum and compare this result with the spectrum of the MSK signal.

4-24 Use the results in Section 4-4-3 to determine the power density spectrum of multitone FSK (MFSK) signals for which the signal waveforms are

$$s_n(t) = \sin \frac{2\pi n t}{T}, \quad n = 1, 2, \dots, M, \quad 0 \leq t \leq T$$

Assume that the probabilities $p_i = 1/M$ for all i . Sketch the power spectral density.

4-25 A quadrature partial response signal (QPRS) is generated by two separate partial response signals of the type described in Problem 4-20 placed in phase quadrature. Hence, the QPRS is represented as

$$s(t) = \text{Re} \{v(t)e^{j2\pi f_c t}\}$$

where

$$\begin{aligned} v(t) &= v_c(t) + jv_s(t) \\ &= \sum_n B_n u(t - nT) + j \sum_n C_n u(t - nT) \end{aligned}$$

and $B_n = I_n + I_{n-1}$ and $C_n = J_n + J_{n-1}$. The sequences $\{B_n\}$ and $\{C_n\}$ are uncorrelated and $I_n = \pm 1$, $J_n = \pm 1$ with equal probability.

a Sketch the signal space diagram for the QPRS signal and determine the probability of occurrence of each symbol.

b Determine the autocorrelations and power spectra density of $v_c(t)$, $v_s(t)$, and $v(t)$.

c Sketch the Markov chain model and indicate the transition probabilities for the QPRS.

4-26 Determine the autocorrelation functions for the MSK and offset QPSK modulated signals based on the assumption that the information sequences for each of the two signals are uncorrelated and zero-mean.

4-27 Sketch the phase tree, the state trellis, and the state diagram for partial response CPM with $h = \frac{1}{2}$ and

$$u(t) = \begin{cases} 1/4T & (0 \leq t \leq 2T) \\ 0 & (\text{otherwise}) \end{cases}$$

4-28 Determine the number of terminal phase states in the state trellis diagram for

a a full response binary CPFSK with either $h = \frac{2}{3}$ or $\frac{1}{4}$;

b a partial response $L = 3$ binary CPFSK with either $h = \frac{2}{3}$ or $\frac{1}{4}$.

4-29 Show that 16 QAM can be represented as a superposition of two four-phase constant envelope signals where each component is amplified separately before summing, i.e.

$$s(t) = G[A_n \cos 2\pi f_c t + B_n \sin 2\pi f_c t] + [C_n \cos 2\pi f_c t + D_n \sin 2\pi f_c t]$$

where $\{A_n\}$, $\{B_n\}$, $\{C_n\}$, and $\{D_n\}$ are statistically independent binary sequences

with elements from the set $\{+1, -1\}$ and G is the amplifier gain. Thus, show that the resulting signal is equivalent to

$$s(t) = I_n \cos 2\pi f_c t + Q_n \sin 2\pi f_c t$$

and determine I_n and Q_n in terms of A_n , B_n , C_n , and D_n .

- 4-30** Use the result in (4-4-60) to derive the expression for the power density spectrum of memoryless linear modulation given by (4-4-18) under the condition that

$$s_k(t) = I_k s(t), \quad k = 1, 2, \dots, K$$

where I_k is one of the K possible transmitted symbols that occur with equal probability.

- 4-31** Show that a sufficient condition for the absence of the line spectrum component in (4-4-60) is

$$\sum_{k=1}^K p_k s_k(t) = 0$$

Is this condition necessary? Justify your answer.

- 4-32** The information sequence $\{a_n\}_{n=-\infty}^{\infty}$ is a sequence of iid random variables, each taking values $+1$ and -1 with equal probability. This sequence is to be transmitted at baseband by a biphas coding scheme, described by

$$s(t) = \sum_{n=-\infty}^{\infty} a_n g(t - nT)$$

where $g(t)$ is shown in Fig. P4-32.

- a** Find the power spectral density of $s(t)$.
 - b** Assume that it is desirable to have a zero in the power spectrum at $f = 1/T$. To this end, we use a precoding scheme by introducing $b_n = a_n + ka_{n-1}$, where k is some constant, and then transmit the $\{b_n\}$ sequence using the same $g(t)$. Is it possible to choose k to produce a frequency null at $f = 1/T$? If yes, what are the appropriate value and the resulting power spectrum?
 - c** Now assume we want to have zeros at all multiples of $f_b = 1/4T$. Is it possible to have these zeros with an appropriate choice of k in the previous part? If not then what kind of precoding do you suggest to result in the desired nulls?
- 4-33** Starting with the definition of the transition probability matrix for delay modulation given in (4-4-66), demonstrate that the relation

$$\mathbf{P}^4 \boldsymbol{\rho} = -\frac{1}{4} \boldsymbol{\rho}$$

holds, and, hence,

$$\mathbf{P}^{4k} \boldsymbol{\rho} = -\frac{1}{4} \mathbf{P}^k \boldsymbol{\rho}, \quad k \geq 1$$

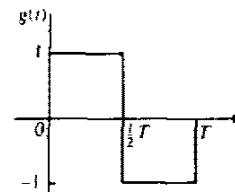


FIGURE P4-32

4-34 The two signal waveforms for binary FSK signal transmission with discontinuous phase are

$$s_0(t) = \sqrt{\frac{2\mathcal{E}_b}{T_b}} \cos \left[2\pi \left(f - \frac{\Delta f}{2} \right) t + \theta_0 \right], \quad 0 \leq t < T$$

$$s_1(t) = \sqrt{\frac{2\mathcal{E}_b}{T_b}} \cos \left[2\pi \left(f + \frac{\Delta f}{2} \right) t + \theta_1 \right], \quad 0 \leq t \leq T$$

where $\Delta f = 1/T \ll f_c$, and θ_0 and θ_1 are uniformly distributed random variables on the interval $(0, 2\pi)$. The signals $s_0(t)$ and $s_1(t)$ are equally probable.

- a** Determine the power spectral density of the FSK signal.
- b** Show that the power spectral density decays as $1/f^2$ for $f \gg f_c$.

OPTIMUM RECEIVERS FOR THE ADDITIVE WHITE GAUSSIAN NOISE CHANNEL

In Chapter 4, we described various types of modulation methods that may be used to transmit digital information through a communication channel. As we have observed, the modulator at the transmitter performs the function of mapping the digital sequence into signal waveforms.

This chapter deals with the design and performance characteristics of optimum receivers for the various modulation methods, when the channel corrupts the transmitted signal by the addition of gaussian noise. In Section 5-1, we first treat memoryless modulation signals, followed by modulation signals with memory. We evaluate the probability of error of the various modulation methods in Section 5-2. We treat the optimum receiver for CPM signals and its performance in Section 5-3. In Section 5-4, we derive the optimum receiver when the carrier phase of the signals is unknown at the receiver and is treated as a random variable. Finally, in Section 5-5, we consider the use of regenerative repeaters in signal transmission and carry out a link budget analysis for radio channels.

5-1 OPTIMUM RECEIVER FOR SIGNALS CORRUPTED BY ADDITIVE WHITE GAUSSIAN NOISE

Let us begin by developing a mathematical model for the signal at the input to the receiver. We assume that the transmitter sends digital information by use of M signal waveforms $\{s_m(t), m = 1, 2, \dots, M\}$. Each waveform is transmitted within the symbol (signaling) interval of duration T . To be specific, we consider the transmission of information over the interval $0 \leq t \leq T$.

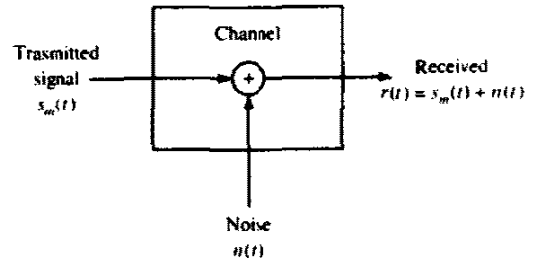


FIGURE 5-1-1 Model for received signal passed through an AWGN channel.

The channel is assumed to corrupt the signal by the addition of white gaussian noise, as illustrated in Fig. 5-1-1. Thus, the received signal in the interval $0 \leq t \leq T$ may be expressed as

$$r(t) = s_m(t) + n(t), \quad 0 \leq t \leq T \tag{5-1-1}$$

where $n(t)$ denotes a sample function of the additive white gaussian noise (AWGN) process with power spectral density $\Phi_{nn}(f) = \frac{1}{2}N_0$ W/Hz. Based on the observation of $r(t)$ over the signal interval, we wish to design a receiver that is optimum in the sense that it minimizes the probability of making an error.

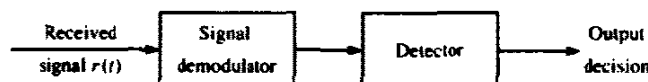
It is convenient to subdivide the receiver into two parts—the signal demodulator and the detector—as shown in Fig. 5-1-2. The function of the signal demodulator is to convert the received waveform $r(t)$ into an N -dimensional vector $\mathbf{r} = [r_1 \ r_2 \ \dots \ r_N]$, where N is the dimension of the transmitted signal waveforms. The function of the detector is to decide which of the M possible signal waveforms was transmitted based on the vector \mathbf{r} .

Two realizations of the signal demodulator are described in the next two sections. One is based on the use of signal correlators. The second is based on the use of matched filters. The optimum detector that follows the signal demodulator is designed to minimize the probability of error.

5-1-1 Correlation Demodulator

In this section, we describe a correlation demodulator that decomposes the received signal and the noise into N -dimensional vectors. In other words, the signal and the noise are expanded into a series of linearly weighted orthonormal basis functions $\{f_n(t)\}$. It is assumed that the N basis functions $\{f_n(t)\}$ span the signal space, so that every one of the possible transmitted

FIGURE 5-1-2 Receiver configuration.



signals of the set $\{s_m(t), 1 \leq m \leq M\}$ can be represented as a weighted linear combination of $\{f_n(t)\}$. In the case of the noise, the functions $\{f_n(t)\}$ do not span the noise space. However, we show below that the noise terms that fall outside the signal space are irrelevant to the detection of the signal.

Suppose the received signal $r(t)$ is passed through a parallel bank of N crosscorrelators which basically compute the projection of $r(t)$ onto the N basis functions $\{f_n(t)\}$, as illustrated in Fig. 5-1-3. Thus, we have

$$\int_0^T r(t) f_k(t) dt = \int_0^T [s_m(t) + n(t)] f_k(t) dt \quad (5-1-2)$$

$$r_k = s_{mk} + n_k, \quad k = 1, 2, \dots, N$$

where

$$s_{mk} = \int_0^T s_m(t) f_k(t) dt, \quad k = 1, 2, \dots, N \quad (5-1-3)$$

$$n_k = \int_0^T n(t) f_k(t) dt, \quad k = 1, 2, \dots, N$$

The signal is now represented by the vector s_m with components s_{mk} , $k = 1, 2, \dots, N$. Their values depend on which of the M signals was transmitted. The components $\{n_k\}$ are random variables that arise from the presence of the additive noise.

In fact, we can express the received signal $r(t)$ in the interval $0 \leq t \leq T$ as

$$r(t) = \sum_{k=1}^N s_{mk} f_k(t) + \sum_{k=1}^N n_k f_k(t) + n'(t)$$

$$= \sum_{k=1}^N r_k f_k(t) + n'(t) \quad (5-1-4)$$

The term $n'(t)$, defined as

$$n'(t) = n(t) - \sum_{k=1}^N n_k f_k(t) \quad (5-1-5)$$

is a zero-mean gaussian noise process that represents the difference between the original noise process $n(t)$ and the part corresponding to the projection of $n(t)$ onto the basis functions $\{f_k(t)\}$. We shall show below that $n'(t)$ is irrelevant to the decision as to which signal was transmitted. Consequently, the decision may be based entirely on the correlator output signal and noise components $r_k = s_{mk} + n_k$, $k = 1, 2, \dots, N$.

Since the signals $\{s_m(t)\}$ are deterministic, the signal components are deterministic. The noise components $\{n_k\}$ are gaussian. Their mean values are

$$E(n_k) = \int_0^T E[n(t)] f_k(t) dt = 0 \quad (5-1-6)$$

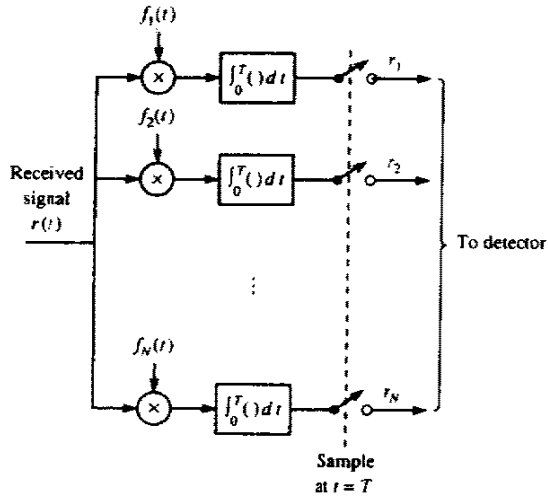


FIGURE 5-1-3 Correlation-type demodulator.

for all n . Their covariances are

$$\begin{aligned}
 E(n_k n_m) &= \int_0^T \int_0^T E[n(t)n(\tau)] f_k(t) f_m(\tau) dt d\tau \\
 &= \frac{1}{2} N_0 \int_0^T \int_0^T \delta(t - \tau) f_k(t) f_m(\tau) dt d\tau \\
 &= \frac{1}{2} N_0 \int_0^T f_k(t) f_m(t) dt \\
 &= \frac{1}{2} N_0 \delta_{mk}
 \end{aligned}
 \tag{5-1-7}$$

where $\delta_{mk} = 1$ when $m = k$ and zero otherwise. Therefore, the N noise components $\{n_k\}$ are zero-mean uncorrelated gaussian random variables with a common variance $\sigma_n^2 = \frac{1}{2} N_0$.

From the above development, it follows that the correlator outputs $\{r_k\}$ conditioned on the m th signal being transmitted are gaussian random variables with mean

$$E(r_k) = E(s_{mk} + n_k) = s_{mk} \tag{5-1-8}$$

and equal variance

$$\sigma_r^2 = \sigma_n^2 = \frac{1}{2} N_0 \tag{5-1-9}$$

Since the noise components $\{n_k\}$ are uncorrelated gaussian random variables, they are also statistically independent. As a consequence, the correlator outputs $\{r_k\}$ conditioned on the m th signal being transmitted are statistically independent gaussian variables. Hence, the conditional probability density functions of the random variables $[r_1 r_2 \dots r_N] = \mathbf{r}$ are simply

$$p(\mathbf{r} | s_m) = \prod_{k=1}^N p(r_k | s_{mk}), \quad m = 1, 2, \dots, M \tag{5-1-10}$$

where

$$p(r_k | s_{mk}) = \frac{1}{\sqrt{\pi N_0}} \exp \left[-\frac{(r_k - s_{mk})^2}{N_0} \right], \quad k = 1, 2, \dots, N \quad (5-1-11)$$

By substituting (5-1-11) into (5-1-10), we obtain the joint conditional pdfs

$$p(\mathbf{r} | \mathbf{s}_m) = \frac{1}{(\pi N_0)^{N/2}} \exp \left[-\sum_{k=1}^N \frac{(r_k - s_{mk})^2}{N_0} \right], \quad m = 1, 2, \dots, M \quad (5-1-12)$$

As a final point we wish to show that the correlator outputs (r_1, r_2, \dots, r_N) are *sufficient statistics* for reaching a decision on which of the M signals was transmitted, i.e., that no additional relevant information can be extracted from the remaining noise process $n'(t)$. Indeed, $n'(t)$ is uncorrelated with the N correlator outputs $\{r_k\}$, i.e.,

$$\begin{aligned} E[n'(t)r_k] &= E[n'(t)s_{mk} + E[n'(t)n_k]] \\ &= E[n'(t)n_k] \\ &= E \left\{ \left[n(t) - \sum_{j=1}^N n_j f_j(t) \right] n_k \right\} \\ &= \int_0^T E[n(t)n(\tau)] f_k(\tau) d\tau - \sum_{j=1}^N E(n_j n_k) f_j(t) \\ &= \frac{1}{2} N_0 f_k(t) - \frac{1}{2} N_0 f_k(t) = 0 \end{aligned} \quad (5-1-13)$$

Since $n'(t)$ and $\{r_k\}$ are gaussian and uncorrelated, they are also statistically independent. Consequently, $n'(t)$ does not contain any information that is relevant to the decision as to which signal waveform was transmitted. All the relevant information is contained in the correlator outputs $\{r_k\}$. Hence, $n'(t)$ may be ignored.

Example 5-1-1

Consider an M -ary baseband PAM signal set in which the basic pulse shape $g(t)$ is rectangular as shown in Fig. 5-1-4. The additive noise is a zero-mean white gaussian noise process. Let us determine the basis function $f(t)$ and the output of the correlation-type demodulator. The energy in the rectangular pulse is

$$\mathcal{E}_g = \int_0^T g^2(t) dt = \int_0^T a^2 dt = a^2 T$$

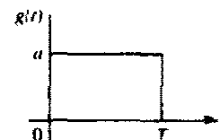


FIGURE 5-1-4 Signal pulse for Example 5-1-1.

Since the PAM signal set has dimension $N=1$, there is only one basis function $f(t)$. This is given as

$$f(t) = \frac{1}{\sqrt{a^2 T}} g(t) = \begin{cases} 1/\sqrt{T} & (0 \leq t \leq T) \\ 0 & (\text{otherwise}) \end{cases}$$

The output of the correlation-type demodulator is

$$r = \int_0^T r(t)f(t) dt = \frac{1}{\sqrt{T}} \int_0^T r(t) dt$$

It is interesting to note that the correlator becomes a simple integrator when $f(t)$ is rectangular. If we substitute for $r(t)$, we obtain

$$\begin{aligned} r &= \frac{1}{\sqrt{T}} \left\{ \int_0^T [s_m(t) + n(t)] dt \right\} \\ &= \frac{1}{\sqrt{T}} \left[\int_0^T s_m(t) dt + \int_0^T n(t) dt \right] \\ r &= s_m + n \end{aligned}$$

where the noise term $E(n) = 0$ and

$$\begin{aligned} \sigma_n^2 &= E \left[\frac{1}{T} \int_0^T \int_0^T n(t)n(\tau) dt d\tau \right] \\ &= \frac{1}{T} \int_0^T \int_0^T E[n(t)n(\tau)] dt d\tau \\ &= \frac{N_0}{2T} \int_0^T \int_0^T \delta(t - \tau) dt d\tau = \frac{1}{2} N_0 \end{aligned}$$

The probability density function for the sampled output is

$$p(r | s_m) = \frac{1}{\sqrt{\pi N_0}} \exp \left[-\frac{(r - s_m)^2}{N_0} \right]$$

5-1-2 Matched-Filter Demodulator

Instead of using a bank of N correlators to generate the variables $\{r_k\}$, we may use a bank of N linear filters. To be specific, let us suppose that the impulse responses of the N filters are

$$h_k(t) = f_k(T - t), \quad 0 \leq t \leq T \quad (5-1-14)$$

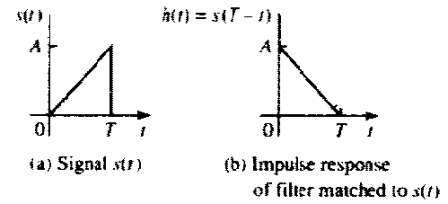


FIGURE 5-1-5 Signal $s(t)$ and filter matched to $s(t)$.

where $\{f_k(t)\}$ are the N basis functions and $h_k(t) = 0$ outside of the interval $0 \leq t \leq T$. The outputs of these filters are

$$\begin{aligned}
 y_k(t) &= \int_0^t r(\tau)h_k(t - \tau) d\tau \\
 &= \int_0^t r(\tau)f_k(T - t + \tau) d\tau, \quad k = 1, 2, \dots, N \quad (5-1-15)
 \end{aligned}$$

Now, if we sample the outputs of the filters at $t = T$, we obtain

$$y_k(T) = \int_0^T r(\tau)f_k(\tau) d\tau = r_k, \quad k = 1, 2, \dots, N \quad (5-1-16)$$

Hence, the sampled outputs of the filters at time $t = T$ are exactly the set of values $\{r_k\}$ obtained from the N linear correlators.

A filter whose impulse response $h(t) = s(T - t)$, where $s(t)$ is assumed to be confined to the time interval $0 \leq t \leq T$, is called the *matched filter* to the signal $s(t)$. An example of a signal and its matched filter are shown in Fig. 5-1-5. The response of $h(t) = s(T - t)$ to the signal $s(t)$ is

$$y(t) = \int_0^t s(\tau)s(T - t + \tau) d\tau \quad (5-1-17)$$

which is basically the time-autocorrelation function of the signal $s(t)$. Figure 5-1-6 illustrates $y(t)$ for the triangular signal pulse shown in Fig. 5-1-5. Note that the autocorrelation function $y(t)$ is an even function of t , which attains a peak at $t = T$.

In the case of the demodulator described above, the N matched filters are

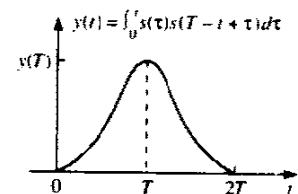


FIGURE 5-1-6 The matched filter output is the autocorrelation function of $s(t)$.

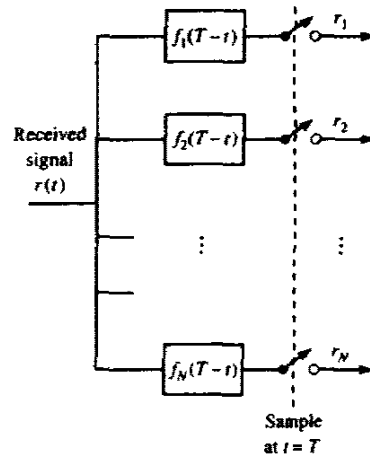


FIGURE 5-1-7 Matched filter demodulator.

matched to the basis functions $\{f_k(t)\}$. Figure 5-1-7 illustrates the matched filter demodulator that generates the observed variables $\{r_k\}$.

Properties of the Matched Filter A matched filter has some interesting properties. Let us prove the most important property, which may be stated as follows: If a signal $s(t)$ is corrupted by AWGN, the filter with impulse response matched to $s(t)$ maximizes the output signal-to-noise ratio (SNR).

To prove this property, let us assume that the received signal $r(t)$ consists of the signal $s(t)$ and AWGN $n(t)$ which has zero-mean and power spectral density $\Phi_{nn}(f) = \frac{1}{2}N_0$ W/Hz. Suppose the signal $r(t)$ is passed through a filter with impulse response $h(t)$, $0 \leq t \leq T$, and its output is sampled at time $t = T$. The filter response to the signal and noise components is

$$\begin{aligned}
 y(t) &= \int_0^t r(\tau)h(t - \tau) d\tau \\
 &= \int_0^t s(\tau)h(t - \tau) d\tau + \int_0^t n(\tau)h(t - \tau) d\tau \quad (5-1-18)
 \end{aligned}$$

At the sampling instant $t = T$, the signal and noise components are

$$\begin{aligned}
 y(T) &= \int_0^T s(\tau)h(T - \tau) d\tau + \int_0^T n(\tau)h(T - \tau) d\tau \\
 &= y_s(T) + y_n(T) \quad (5-1-19)
 \end{aligned}$$

where $y_s(T)$ represents the signal component and $y_n(T)$ the noise component. The problem is to select the filter impulse response that maximizes the output signal-to-noise ratio (SNR₀) defined as

$$\text{SNR}_0 = \frac{y_s^2(T)}{E[y_n^2(T)]} \quad (5-1-20)$$

The denominator in (5-1-20) is simply the variance of the noise term at the output of the filter. Let us evaluate $E[y_n^2(T)]$. We have

$$\begin{aligned} E[y_n^2(T)] &= \int_0^T \int_0^T E[n(\tau)n(t)]h(T-\tau)h(T-t) dt d\tau \\ &= \frac{1}{2}N_0 \int_0^T \int_0^T \delta(t-\tau)h(T-\tau)h(T-t) dt d\tau \\ &= \frac{1}{2}N_0 \int_0^T h^2(T-t) dt \end{aligned} \quad (5-1-21)$$

Note that the variance depends on the power spectral density of the noise and the energy in the impulse response $h(t)$.

By substituting for $y_s(T)$ and $E[y_n^2(T)]$ into (5-1-20), we obtain the expression for the output SNR as

$$\text{SNR}_0 = \frac{[\int_0^T s(\tau)h(T-\tau) d\tau]^2}{\frac{1}{2}N_0 \int_0^T h^2(T-t) dt} = \frac{[\int_0^T h(\tau)s(T-\tau) d\tau]^2}{\frac{1}{2}N_0 \int_0^T h^2(T-t) dt} \quad (5-1-22)$$

Since the denominator of the SNR depends on the energy in $h(t)$, the maximum output SNR over $h(t)$ is obtained by maximizing the numerator subject to the constraint that the denominator is held constant. The maximization of the numerator is most easily performed by use of the Cauchy-Schwarz inequality, which states, in general, that if $g_1(t)$ and $g_2(t)$ are finite-energy signals then

$$\left[\int_{-\infty}^{\infty} g_1(t)g_2(t) dt \right]^2 \leq \int_{-\infty}^{\infty} g_1^2(t) dt \int_{-\infty}^{\infty} g_2^2(t) dt \quad (5-1-23)$$

with equality when $g_1(t) = Cg_2(t)$ for any arbitrary constant C . If we set $g_1(t) = h(t)$ and $g_2(t) = s(T-t)$, it is clear that the SNR is maximized when $h(t) = Cs(T-t)$, i.e., $h(t)$ is matched to the signal $s(t)$. The scale factor C^2 drops out of the expression for the SNR, since it appears in both the numerator and the denominator.

The output (maximum) SNR obtained with the matched filter is

$$\begin{aligned} \text{SNR}_0 &= \frac{2}{N_0} \int_0^T s^2(t) dt \\ &= 2\mathcal{E}/N_0 \end{aligned} \quad (5-1-24)$$

Note that the output SNR from the matched filter depends on the energy of the waveform $s(t)$ but not on the detailed characteristics of $s(t)$. This is another interesting property of the matched filter.

Frequency-Domain Interpretation of the Matched Filter The matched filter has an interesting frequency-domain interpretation. Since $h(t) = s(T-t)$,

the Fourier transform of this relationship is

$$\begin{aligned}
 H(f) &= \int_0^T s(T-t)e^{-j2\pi ft} dt \\
 &= \left[\int_0^T s(\tau)e^{j2\pi f\tau} d\tau \right] e^{-j2\pi fT} \\
 &= S^*(f)e^{-j2\pi fT} \tag{5-1-25}
 \end{aligned}$$

We observe that the matched filter has a frequency response that is the complex conjugate of the transmitted signal spectrum multiplied by the phase factor $e^{-j2\pi fT}$, which represents the sampling delay of T . In other words, $|H(f)| = |S(f)|$, so that the magnitude response of the matched filter is identical to the transmitted signal spectrum. On the other hand, the phase of $H(f)$ is the negative of the phase of $S(f)$.

Now, if the signal $s(t)$ with spectrum $S(f)$ is passed through the matched filter, the filter output has a spectrum $Y(f) = |S(f)|^2 e^{-j2\pi fT}$. Hence, the output waveform is

$$\begin{aligned}
 y_s(t) &= \int_{-\infty}^{\infty} Y(f)e^{j2\pi ft} df \\
 &= \int_{-\infty}^{\infty} |S(f)|^2 e^{-j2\pi fT} e^{j2\pi ft} df \tag{5-1-26}
 \end{aligned}$$

By sampling the output of the matched filter at $t = T$, we obtain

$$y_s(T) = \int_{-\infty}^{\infty} |S(f)|^2 df = \int_0^T s^2(t) dt = \mathcal{E} \tag{5-1-27}$$

where the last step follows from Parseval's relation.

The noise at the output of the matched filter has a power spectral density

$$\Phi_0(f) = \frac{1}{2} |H(f)|^2 N_0 \tag{5-1-28}$$

Hence, the total noise power at the output of the matched filter is

$$\begin{aligned}
 P_n &= \int_{-\infty}^{\infty} \Phi_0(f) df \\
 &= \frac{1}{2} N_0 \int_{-\infty}^{\infty} |H(f)|^2 df = \frac{1}{2} N_0 \int_{-\infty}^{\infty} |S(f)|^2 df = \frac{1}{2} \mathcal{E} N_0 \tag{5-1-29}
 \end{aligned}$$

The output SNR is simply the ratio of the signal power P_s , given by

$$P_s = y_s^2(T) \tag{5-1-30}$$

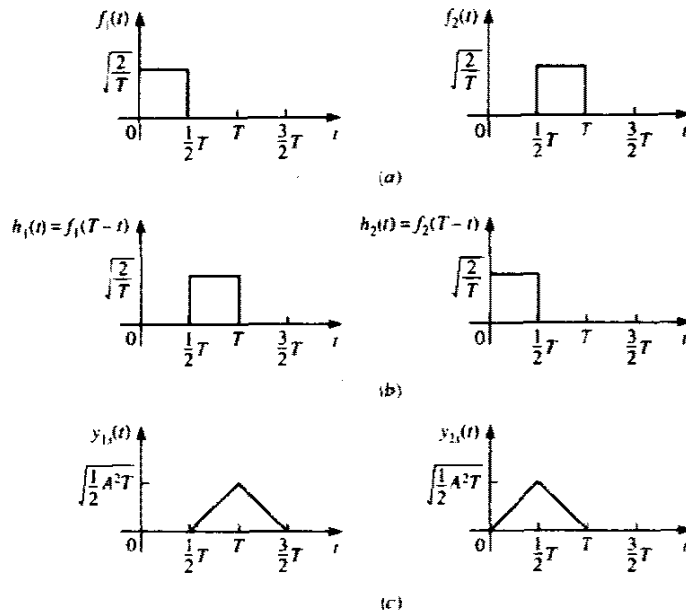


FIGURE 5-1-8 Basis functions and matched filter responses for Example 5-1-2.

to the noise power P_n . Hence,

$$\text{SNR}_0 = \frac{P_s}{P_n} = \frac{\mathcal{E}^2}{\frac{1}{2}\mathcal{E}N_0} = \frac{2\mathcal{E}}{N_0} \quad (5-1-31)$$

which agrees with the result given by (5-1-24).

Example 5-1-2

Consider the $M = 4$ biorthogonal signals shown in Fig. 5-1-8 for transmitting information over an AWGN channel. The noise is assumed to have zero mean and power spectral density $\frac{1}{2}N_0$. Let us determine the basis functions for this signal set, the impulse responses of the matched-filter demodulators, and the output waveforms of the matched-filter demodulators when the transmitted signal is $s_1(t)$.

The $M = 4$ biorthogonal signals have dimension $N = 2$. Hence, two basis functions are needed to represent the signals. From Fig. 5-1-8, we choose $f_1(t)$ and $f_2(t)$ as

$$\begin{aligned} f_1(t) &= \begin{cases} \sqrt{2/T} & (0 \leq t \leq \frac{1}{2}T) \\ 0 & (\text{otherwise}) \end{cases} \\ f_2(t) &= \begin{cases} \sqrt{2/T} & (\frac{1}{2}T \leq t \leq T) \\ 0 & (\text{otherwise}) \end{cases} \end{aligned} \quad (5-1-32)$$

These waveforms are illustrated in Fig. 5-1-8(a). The impulse responses of the two matched filters are

$$\begin{aligned} h_1(t) = f_1(T-t) &= \begin{cases} \sqrt{2/T} & (\frac{1}{2}T \leq t \leq T) \\ 0 & (\text{otherwise}) \end{cases} \\ h_2(t) = f_2(T-t) &= \begin{cases} \sqrt{2/T} & (0 \leq t \leq \frac{1}{2}T) \\ 0 & (\text{otherwise}) \end{cases} \end{aligned} \quad (5-1-33)$$

and are illustrated in Fig. 5-1-8(b).

If $s_1(t)$ is transmitted, the (noise-free) responses of the two matched filters are as shown in Fig. 5-1-8(c). Since $y_1(t)$ and $y_2(t)$ are sampled at $t = T$, we observe that $y_{1s}(T) = \sqrt{\frac{1}{2}A^2T}$ and $y_{2s}(T) = 0$. Note that $\frac{1}{2}A^2T = \mathcal{E}$, the signal energy. Hence, the received vector formed from the two matched filter outputs at the sampling instant $t = T$ is

$$\mathbf{r} = [r_1 \quad r_2] = [\sqrt{\mathcal{E}} + n_1 \quad n_2] \quad (5-1-34)$$

where $n_1 = y_{1n}(T)$ and $n_2 = y_{2n}(T)$ are the noise components at the outputs of the matched filters, given by

$$y_{kn}(T) = \int_0^T n(t)f_k(t) dt, \quad k = 1, 2 \quad (5-1-35)$$

Clearly, $E(n_k) = E[y_{kn}(T)] = 0$. Their variance is

$$\begin{aligned} \sigma_n^2 &= E[y_{kn}^2(T)] = \int_0^T \int_0^T E[n(t)n(\tau)]f_k(t)f_k(\tau) dt d\tau \\ &= \frac{1}{2}N_0 \int_0^T \int_0^T \delta(t-\tau)f_k(\tau)f_k(t) dt d\tau \\ &= \frac{1}{2}N_0 \int_0^T f_k^2(t) dt = \frac{1}{2}N_0 \end{aligned} \quad (5-1-36)$$

Observe that the SNR_0 for the first matched filter is

$$\text{SNR}_0 = \frac{(\sqrt{\mathcal{E}})^2}{\frac{1}{2}N_0} = \frac{2\mathcal{E}}{N_0} \quad (5-1-37)$$

which agrees with our previous result. Also note that the four possible outputs of the two matched filters, corresponding to the four possible transmitted signals in Fig. 5-1-8 are $(r_1, r_2) = (\sqrt{\mathcal{E}} + n_1, n_2)$, $(n_1, \sqrt{\mathcal{E}} + n_2)$, $(-\sqrt{\mathcal{E}} + n_1, n_2)$ and $(n_1, -\sqrt{\mathcal{E}} + n_2)$.

5-1-3 The Optimum Detector

We have demonstrated that, for a signal transmitted over an AWGN channel, either a correlation demodulator or a matched filter demodulator produces the vector $\mathbf{r} = [r_1 \quad r_2 \quad \cdots \quad r_N]$, which contains all the relevant information in the received signal waveform. In this section, we describe the optimum decision

rule based on the observation vector \mathbf{r} . For this development, we assume that there is no memory in signals transmitted in successive signal intervals.

We wish to design a signal detector that makes a decision on the transmitted signal in each signal interval based on the observation of the vector \mathbf{r} in each interval such that the probability of a correct decision is maximized. With this goal in mind, we consider a decision rule based on the computation of the *posterior probabilities* defined as

$$P(\text{signal } \mathbf{s}_m \text{ was transmitted} | \mathbf{r}), \quad m = 1, 2, \dots, M$$

which we abbreviate as $P(\mathbf{s}_m | \mathbf{r})$. The decision criterion is based on selecting the signal corresponding to the maximum of the set of posterior probabilities $\{P(\mathbf{s}_m | \mathbf{r})\}$. Later, we show that this criterion maximizes the probability of a correct decision and, hence, minimizes the probability of error. This decision criterion is called the *maximum a posteriori probability* (MAP) criterion.

Using Bayes' rule, the posterior probabilities may be expressed as

$$P(\mathbf{s}_m | \mathbf{r}) = \frac{p(\mathbf{r} | \mathbf{s}_m)P(\mathbf{s}_m)}{p(\mathbf{r})} \quad (5-1-38)$$

where $p(\mathbf{r} | \mathbf{s}_m)$ is the conditional pdf of the observed vector given \mathbf{s}_m , and $P(\mathbf{s}_m)$ is the *a priori probability* of the m th signal being transmitted. The denominator of (5-1-38) may be expressed as

$$p(\mathbf{r}) = \sum_{m=1}^M p(\mathbf{r} | \mathbf{s}_m)P(\mathbf{s}_m) \quad (5-1-39)$$

From (5-1-38) and (5-1-39), we observe that the computation of the posterior probabilities $P(\mathbf{s}_m | \mathbf{r})$ requires knowledge of the *a priori probabilities* $P(\mathbf{s}_m)$ and the conditional pdfs $p(\mathbf{r} | \mathbf{s}_m)$ for $m = 1, 2, \dots, M$.

Some simplification occurs in the MAP criterion when the M signals are equally probable a priori, i.e., $P(\mathbf{s}_m) = 1/M$ for all M . Furthermore, we note that the denominator in (5-1-38) is independent of which signal is transmitted. Consequently, the decision rule based on finding the signal that maximizes $P(\mathbf{s}_m | \mathbf{r})$ is equivalent to finding the signal that maximizes $p(\mathbf{r} | \mathbf{s}_m)$.

The conditional pdf $p(\mathbf{r} | \mathbf{s}_m)$ or any monotonic function of it is usually called the *likelihood function*. The decision criterion based on the maximum of $p(\mathbf{r} | \mathbf{s}_m)$ over the M signals is called the *maximum-likelihood* (ML) *criterion*. We observe that a detector based on the MAP criterion and one that is based on the ML criterion make the same decisions as long as the a priori probabilities $P(\mathbf{s}_m)$ are all equal, i.e., the signals $\{\mathbf{s}_m\}$ are equiprobable.

In the case of an AWGN channel, the likelihood function $p(\mathbf{r} | \mathbf{s}_m)$ is given by (5-1-12). To simplify the computations, we may work with the natural logarithm of $p(\mathbf{r} | \mathbf{s}_m)$, which is a monotonic function. Thus,

$$\ln p(\mathbf{r} | \mathbf{s}_m) = -\frac{1}{2}N \ln(\pi N_0) - \frac{1}{N_0} \sum_{k=1}^N (r_k - s_{mk})^2 \quad (5-1-40)$$

The maximum of $\ln p(\mathbf{r} | \mathbf{s}_m)$ over \mathbf{s}_m is equivalent to finding the signal \mathbf{s}_m that minimizes the Euclidean distance

$$D(\mathbf{r}, \mathbf{s}_m) = \sum_{k=1}^N (r_k - s_{mk})^2 \quad (5-1-41)$$

We call $D(\mathbf{r}, \mathbf{s}_m)$, $m = 1, 2, \dots, M$, the *distance metrics*. Hence, for the AWGN channel, the decision rule based on the ML criterion reduces to finding the signal \mathbf{s}_m that is closest in distance to the received signal vector \mathbf{r} . We shall refer to this decision rule as *minimum distance detection*.

Another interpretation of the optimum decision rule based on the ML criterion is obtained by expanding the distance metrics in (5-1-41) as

$$\begin{aligned} D(\mathbf{r}, \mathbf{s}_m) &= \sum_{n=1}^N r_n^2 - 2 \sum_{n=1}^N r_n s_{mn} + \sum_{n=1}^N s_{mn}^2 \\ &= |\mathbf{r}|^2 - 2\mathbf{r} \cdot \mathbf{s}_m + |\mathbf{s}_m|^2, \quad m = 1, 2, \dots, M \end{aligned} \quad (5-1-42)$$

The term $|\mathbf{r}|^2$ is common to all decision metrics, and, hence, it may be ignored in the computations of the metrics. The result is a set of *modified distance metrics*

$$D'(\mathbf{r}, \mathbf{s}_m) = -2\mathbf{r} \cdot \mathbf{s}_m + |\mathbf{s}_m|^2 \quad (5-1-43)$$

Note that selecting the signal \mathbf{s}_m that minimizes $D'(\mathbf{r}, \mathbf{s}_m)$ is equivalent to selecting the signal that maximizes the metric $C(\mathbf{r}, \mathbf{s}_m) = -D'(\mathbf{r}, \mathbf{s}_m)$, i.e.,

$$C(\mathbf{r}, \mathbf{s}_m) = 2\mathbf{r} \cdot \mathbf{s}_m - |\mathbf{s}_m|^2 \quad (5-1-44)$$

The term $\mathbf{r} \cdot \mathbf{s}_m$ represents the projection of the received signal vector onto each of the M possible transmitted signal vectors. The value of each of these projections is a measure of the correlation between the received vector and the m th signal. For this reason, we call $C(\mathbf{r}, \mathbf{s}_m)$, $m = 1, 2, \dots, M$, the *correlation metrics* for deciding which of the M signals was transmitted. Finally, the terms $|\mathbf{s}_m|^2 = \mathcal{E}_m$, $m = 1, 2, \dots, M$, may be viewed as bias terms that serve as compensation for signal sets that have unequal energies, such as PAM. If all signals have the same energy, $|\mathbf{s}_m|^2$ may also be ignored in the computation of the correlation metrics $C(\mathbf{r}, \mathbf{s}_m)$ and the distance metrics $D(\mathbf{r}, \mathbf{s}_m)$ or $D'(\mathbf{r}, \mathbf{s}_m)$.

It is easy to show (see Problem 5-5) that the correlation metrics $C(\mathbf{r}, \mathbf{s}_m)$ can also be expressed as

$$C(\mathbf{r}, \mathbf{s}_m) = 2 \int_0^T r(t) s_m(t) dt - \mathcal{E}_m, \quad m = 0, 1, \dots, M \quad (5-1-45)$$

Therefore, these metrics can be generated by a demodulator that cross-correlates the received signal $r(t)$ with each of the M possible transmitted signals and adjusts each correlator output for the bias in the case of unequal signal energies. Equivalently, the received signal may be passed through a bank of M filters matched to the possible transmitted signals $\{s_m(t)\}$ and

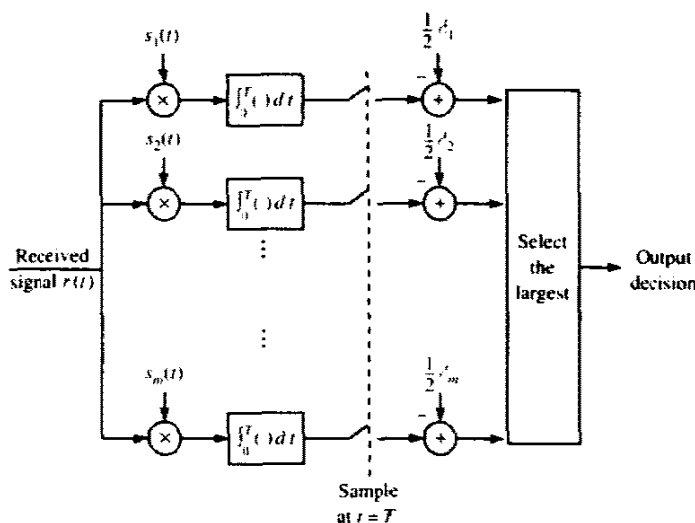


FIGURE 5-1-9 An alternative realization of the optimum AWGN receiver.

sampled at $t = T$, the end of the symbol interval. Consequently, the optimum receiver (demodulator and detector) can be implemented in the alternative configuration illustrated in Fig. 5-1-9.

In summary, we have demonstrated that the optimum ML detector computes a set of M distances $D(\mathbf{r}, \mathbf{s}_m)$ or $D'(\mathbf{r}, \mathbf{s}_m)$ and selects the signal corresponding to the smallest (distance) metric. Equivalently, the optimum ML detector computes a set of M correlation metrics $C(\mathbf{r}, \mathbf{s}_m)$ and selects the signal corresponding to the largest correlation metric.

The above development for the optimum detector treated the important case in which all signals are equally probable. In this case, the MAP criterion is equivalent to the ML criterion. However, when the signals are not equally probable, the optimum MAP detector bases its decision on the probabilities $P(\mathbf{s}_m | \mathbf{r})$, $m = 1, 2, \dots, M$, given by (5-1-38) or, equivalently, on the *metrics*,

$$PM(\mathbf{r}, \mathbf{s}_m) = p(\mathbf{r} | \mathbf{s}_m)P(\mathbf{s}_m)$$

The following example illustrates this computation for binary PAM signals.

Example 5-1-3

Consider the case of binary PAM signals in which the two possible signal points are $s_1 = -s_2 = \sqrt{\mathcal{E}_b}$, where \mathcal{E}_b is the energy per bit. The prior probabilities are $P(s_1) = p$ and $P(s_2) = 1 - p$. Let us determine the metrics for the optimum MAP detector when the transmitted signal is corrupted with AWGN.

The received signal vector (one-dimensional) for binary PAM is

$$r = \pm\sqrt{\mathcal{E}_b} + y_n(T) \quad (5-1-46)$$

where $y_n(T)$ is a zero-mean Gaussian random variable with variance $\sigma_n^2 = \frac{1}{2}N_0$. Consequently, the conditional pdfs $p(r | s_m)$ for the two signals are

$$p(r | s_1) = \frac{1}{\sqrt{2\pi}\sigma_n} \exp\left[-\frac{(r - \sqrt{\mathcal{E}_b})^2}{2\sigma_n^2}\right] \tag{5-1-47}$$

$$p(r | s_2) = \frac{1}{\sqrt{2\pi}\sigma_n} \exp\left[-\frac{(r + \sqrt{\mathcal{E}_b})^2}{2\sigma_n^2}\right] \tag{5-1-48}$$

Then the metrics $PM(r, s_1)$ and $PM(r, s_2)$ are

$$PM(r, s_1) = pp(r | s_1) = \frac{p}{\sqrt{2\pi}\sigma_n} \exp\left[-\frac{(r - \sqrt{\mathcal{E}_b})^2}{2\sigma_n^2}\right] \tag{5-1-49}$$

$$PM(r, s_2) = \frac{1-p}{\sqrt{2\pi}\sigma_n} \exp\left[-\frac{(r + \sqrt{\mathcal{E}_b})^2}{2\sigma_n^2}\right] \tag{5-1-50}$$

If $PM(r, s_1) > PM(r, s_2)$, we select s_1 as the transmitted signal; otherwise, we select s_2 . This decision rule may be expressed as

$$\frac{PM(r, s_1)}{PM(r, s_2)} \underset{s_2}{\overset{s_1}{\geq}} 1 \tag{5-1-51}$$

But

$$\frac{PM(r, s_1)}{PM(r, s_2)} = \frac{p}{1-p} \exp\left[\frac{(r + \sqrt{\mathcal{E}_b})^2 - (r - \sqrt{\mathcal{E}_b})^2}{2\sigma_n^2}\right] \tag{5-1-52}$$

so that (5-1-51) may be expressed as

$$\frac{(r + \sqrt{\mathcal{E}_b})^2 - (r - \sqrt{\mathcal{E}_b})^2}{2\sigma_n^2} \underset{s_2}{\overset{s_1}{\geq}} \ln \frac{1-p}{p} \tag{5-1-53}$$

or equivalently,

$$\sqrt{\mathcal{E}_b} r \underset{s_2}{\overset{s_1}{\geq}} \frac{1}{2}\sigma_n^2 \ln \frac{1-p}{p} = \frac{1}{4}N_0 \ln \frac{1-p}{p} \tag{5-1-54}$$

This is the final form for the optimum detector. It computes the correlation metric $C(r, s_1) = r\sqrt{\mathcal{E}_b}$ and compares it with threshold $\frac{1}{4}N_0 \ln [(1-p)/p]$. Figure 5-1-10 illustrates the two signal points s_1 and s_2 . The threshold, denoted by τ_h , divides the real line into two regions, say R_1 and R_2 , where R_1 consists of the set of points that are greater than τ_h , and

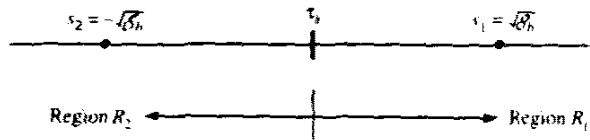


FIGURE 5-1-10 Signal space representation illustrating the operation of the optimum detector for binary (PAM) modulation.

R_2 consists of the set of points that are less than τ_h . If $r\sqrt{\mathcal{E}_b} > \tau_h$, the decision is made that s_1 was transmitted, and if $r\sqrt{\mathcal{E}_b} < \tau_h$, the decision is made that s_2 was transmitted. The threshold τ_h depends on N_0 and p . If $p = \frac{1}{2}$, $\tau_h = 0$. If $p > \frac{1}{2}$, the signal point s_1 is more probable and, hence, $\tau_h < 0$. In this case, the region R_1 is larger than R_2 , so that s_1 is more likely to be selected than s_2 . If $p < \frac{1}{2}$, the opposite is the case. Thus, the average probability of error is minimized.

It is interesting to note that in the case of unequal prior probabilities, it is necessary to know not only the values of the prior probabilities but also the value of the power spectral density N_0 in order to compute the threshold. When $p = \frac{1}{2}$, the threshold is zero, and knowledge of N_0 is not required by the detector.

We conclude this section with the proof that the decision rule based on the maximum-likelihood criterion minimizes the probability of error when the M signals are equally probable a priori. Let us denote by R_m the region in the N -dimensional space for which we decide that signal $s_m(t)$ was transmitted when the vector $\mathbf{r} = [r_1 \ r_2 \ \cdots \ r_N]$ is received. The probability of a decision error given that $s_m(t)$ was transmitted is

$$P(e | \mathbf{s}_m) = \int_{R_m^c} p(\mathbf{r} | \mathbf{s}_m) d\mathbf{r} \quad (5-1-55)$$

where R_m^c is the complement of R_m . The average probability of error is

$$\begin{aligned} P(e) &= \sum_{m=1}^M \frac{1}{M} P(e | \mathbf{s}_m) \\ &= \sum_{m=1}^M \frac{1}{M} \int_{R_m^c} p(\mathbf{r} | \mathbf{s}_m) d\mathbf{r} \\ &= \sum_{m=1}^M \frac{1}{M} \left[1 - \int_{R_m} p(\mathbf{r} | \mathbf{s}_m) d\mathbf{r} \right] \end{aligned} \quad (5-1-56)$$

Note that $P(e)$ is minimized by selecting the signal \mathbf{s}_m if $p(\mathbf{r} | \mathbf{s}_m)$ is larger than $p(\mathbf{r} | \mathbf{s}_k)$ for all $m \neq k$.

When the M signals are not equally probable, the above proof can be generalized to show that the MAP criterion minimizes the average probability of error.

5-1-4 The Maximum-Likelihood Sequence Detector

When the signal has no memory, the symbol-by-symbol detector described in the preceding section is optimum in the sense of minimizing the probability of a symbol error. On the other hand, when the transmitted signal has memory, i.e., the signals transmitted in successive symbol intervals are interdependent, the optimum detector is a detector that bases its decisions on observation of a

sequence of received signals over successive signal intervals. Below, we describe two different types of detection algorithms. In this section, we describe a *maximum-likelihood sequence detection* algorithm that searches for the minimum euclidean distance path through the trellis that characterizes the memory in the transmitted signal. In the following section, we describe a maximum a posteriori probability algorithm that makes decisions on a symbol-by-symbol basis, but each symbol decision is based on an observation of a sequence of received signal vectors.

To develop the maximum likelihood sequence detection algorithm, let us consider, as an example, the NRZI signal described in Section 4-3-2. Its memory is characterized by the trellis shown in Fig. 4-3-14. The signal transmitted in each signal interval is binary PAM. Hence, there are two possible transmitted signals corresponding to the signal points $s_1 = -s_2 = \sqrt{\mathcal{E}_b}$, where \mathcal{E}_b is the energy per bit. The output of the matched-filter or correlation demodulator for binary PAM in the k th signal interval may be expressed as

$$r_k = \pm \sqrt{\mathcal{E}_b} + n_k \quad (5-1-57)$$

where n_k is a zero-mean gaussian random variable with variance $\sigma_n^2 = N_0/2$. Consequently, the conditional pdfs for the two possible transmitted signals are

$$\begin{aligned} p(r_k | s_1) &= \frac{1}{\sqrt{2\pi}\sigma_n} \exp\left[-\frac{(r_k - \sqrt{\mathcal{E}_b})^2}{2\sigma_n^2}\right] \\ p(r_k | s_2) &= \frac{1}{\sqrt{2\pi}\sigma_n} \exp\left[-\frac{(r_k + \sqrt{\mathcal{E}_b})^2}{2\sigma_n^2}\right] \end{aligned} \quad (5-1-58)$$

Now, suppose we observe the sequence of matched-filter outputs r_1, r_2, \dots, r_K . Since the channel noise is assumed to be white and gaussian, and $f(t-iT), f(t-jT)$ for $i \neq j$ are orthogonal, it follows that $E(n_k n_j) = 0, k \neq j$. Hence, the noise sequence n_1, n_2, \dots, n_K is also white. Consequently, for any given transmitted sequence $\mathbf{s}^{(m)}$, the joint pdf of r_1, r_2, \dots, r_K may be expressed as a product of K marginal pdfs, i.e.,

$$\begin{aligned} p(r_1, r_2, \dots, r_K | \mathbf{s}^{(m)}) &= \prod_{k=1}^K p(r_k | s_k^{(m)}) \\ &= \prod_{k=1}^K \frac{1}{\sqrt{2\pi}\sigma_n} \exp\left[-\frac{(r_k - s_k^{(m)})^2}{2\sigma_n^2}\right] \\ &= \left(\frac{1}{\sqrt{2\pi}\sigma_n}\right)^K \exp\left[-\sum_{k=1}^K \frac{(r_k - s_k^{(m)})^2}{2\sigma_n^2}\right] \end{aligned} \quad (5-1-59)$$

where either $s_k = \sqrt{\mathcal{E}_b}$ or $s_k = -\sqrt{\mathcal{E}_b}$. Then, given the received sequence r_1, r_2, \dots, r_K at the output of the matched filter or correlation demodulator, the detector determines the sequence $\mathbf{s}^{(m)} = \{s_1^{(m)}, s_2^{(m)}, \dots, s_K^{(m)}\}$ that maximizes the conditional pdf $p(r_1, r_2, \dots, r_K | \mathbf{s}^{(m)})$. Such a detector is called the *maximum-likelihood (ML) sequence detector*.

By taking the logarithm of (5-1-59) and neglecting the terms that are

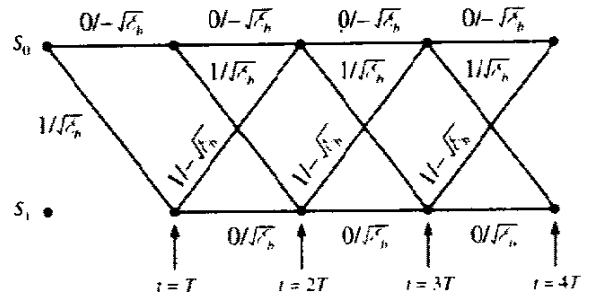


FIGURE 5-1-11 Trellis for NRZI signal.

independent of (r_1, r_2, \dots, r_K) , we find that an equivalent ML sequence detector selects the sequence $\mathbf{s}^{(m)}$ that minimizes the *euclidean distance metric*

$$D(\mathbf{r}, \mathbf{s}^{(m)}) = \sum_{k=1}^K (r_k - s_k^{(m)})^2 \tag{5-1-60}$$

In searching through the trellis for the sequence that minimizes the euclidean distance $D(\mathbf{r}, \mathbf{s}^{(m)})$, it may appear that we must compute the distance $D(\mathbf{r}, \mathbf{s}^{(m)})$ for every possible sequence. For the NRZI example, which employs binary modulation, the total number of sequences is 2^K , where K is the number of outputs obtained from the demodulator. However, this is not the case. We may reduce the number of sequences in the trellis search by using the *Viterbi algorithm* to eliminate sequences as new data is received from the demodulator.

The Viterbi algorithm is a sequential trellis search algorithm for performing ML sequence detection. It is described in Chapter 8 as a decoding algorithm for convolutional codes. We describe it below in the context of the NRZI signal. We assume that the search process begins initially at state S_0 . The corresponding trellis is shown in Fig. 5-1-11.

At time $t = T$, we receive $r_1 = s_1^{(m)} + n$ from the demodulator, and at $t = 2T$, we receive $r_2 = s_2^{(m)} + n_2$. Since the signal memory is one bit, which we denote by $L = 1$, we observe that the trellis reaches its regular (steady state) form after two transitions. Thus, upon receipt of r_2 at $t = 2T$ (and thereafter), we observe that there are two signal paths entering each of the nodes and two signal paths leaving each node. The two paths entering node S_0 at $t = 2T$ correspond to the information bits $(0, 0)$ and $(1, 1)$ or, equivalently, to the signal points $(-\sqrt{E_b}, -\sqrt{E_b})$ and $(\sqrt{E_b}, -\sqrt{E_b})$, respectively. The two paths entering node S_1 at $t = 2T$ correspond to the information bits $(0, 1)$ and $(1, 0)$ or, equivalently, to the signal points $(-\sqrt{E_b}, \sqrt{E_b})$ and $(\sqrt{E_b}, \sqrt{E_b})$, respectively.

For the two paths entering node S_0 , we compute the two Euclidean distance metrics

$$\begin{aligned} D_0(0, 0) &= (r_1 + \sqrt{E_b})^2 + (r_2 + \sqrt{E_b})^2 \\ D_0(1, 1) &= (r_1 - \sqrt{E_b})^2 + (r_2 + \sqrt{E_b})^2 \end{aligned} \tag{5-1-61}$$

by using the outputs r_1 and r_2 from the demodulator. The Viterbi algorithm compares these two metrics and discards the path having the larger (greater-distance) metric.† The other path with the lower metric is saved and is called the *survivor* at $t = 2T$. The elimination of one of the two paths may be done without compromising the optimality of the trellis search, because any extension of the path with the larger distance beyond $t = 2T$ will always have a larger metric than the survivor that is extended along the same path beyond $t = 2T$.

Similarly, for the two paths entering node S_1 at $t = 2T$, we compute the two Euclidean distance metrics

$$\begin{aligned} D_1(0, 1) &= (r_1 + \sqrt{\mathcal{E}_b})^2 + (r_2 - \sqrt{\mathcal{E}_b})^2 \\ D_1(1, 0) &= (r_1 - \sqrt{\mathcal{E}_b})^2 + (r_2 + \sqrt{\mathcal{E}_b})^2 \end{aligned} \quad (5-1-62)$$

by using the outputs r_1 and r_2 from the demodulator. The two metrics are compared and the signal path with the larger metric is eliminated. Thus, at $t = 2T$, we are left with two survivor paths, one at node S_0 and the other at node S_1 , and their corresponding metrics. The signal paths at nodes S_0 and S_1 are then extended along the two survivor paths.

Upon receipt of r_3 at $t = 3T$, we compute the metrics of the two paths entering state S_0 . Suppose the survivors at $t = 2T$ are the paths $(0, 0)$ at S_0 and $(0, 1)$ at S_1 . Then, the two metrics for the paths entering S_0 at $t = 3T$ are

$$\begin{aligned} D_0(0, 0, 0) &= D_0(0, 0) + (r_3 + \sqrt{\mathcal{E}_b})^2 \\ D_0(0, 1, 1) &= D_1(0, 1) + (r_3 + \sqrt{\mathcal{E}_b})^2 \end{aligned} \quad (5-1-63)$$

These two metrics are compared and the path with the larger (greater-distance) metric is eliminated. Similarly, the metrics for the two paths entering S_1 at $t = 3T$ are

$$\begin{aligned} D_1(0, 0, 1) &= D_0(0, 0) + (r_3 - \sqrt{\mathcal{E}_b})^2 \\ D_1(0, 1, 0) &= D_1(0, 1) + (r_3 - \sqrt{\mathcal{E}_b})^2 \end{aligned} \quad (5-1-64)$$

These two metrics are compared and the path with the larger (greater-distance) metric is eliminated.

This process is continued as each new signal sample is received from the demodulator. Thus, the Viterbi algorithm computes two metrics for the two signal paths entering a node at each stage of the trellis search and eliminates one of the two paths at each node. The two survivor paths are then extended forward to the next state. Therefore, the number of paths searched in the trellis is reduced by a factor of two at each stage.

It is relatively easy to generalize the trellis search performed by the Viterbi algorithm for M -ary modulation. For example, delay modulation employs

† Note that, for NRZI, the reception of r_2 from the demodulator neither increases nor decreases the relative difference between the two metrics, $D_0(0, 0)$ and $D_0(1, 1)$. At this point, one may ponder on the implication of this observation. In any case, we continue with the description of the ML sequence detector based on the Viterbi algorithm.

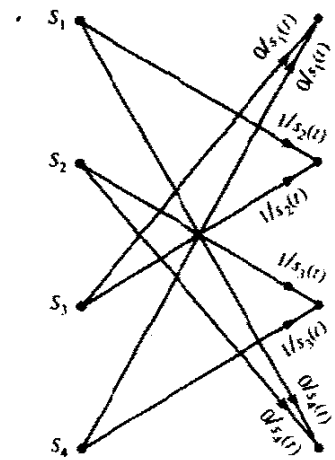


FIGURE 5-1-12 One stage of trellis diagram for delay modulation.

$M = 4$ signals and is characterized by the four-state trellis shown in Fig. 5-1-12. We observe that each state has two signal paths entering and two signal paths leaving each node. The memory of the signal is $L = 1$. Hence, the Viterbi algorithm will have four survivors at each stage and their corresponding metrics. Two metrics corresponding to the two entering paths are computed at each node, and one of the two signal paths entering the node is eliminated at each state of the trellis. Thus, the Viterbi algorithm minimizes the number of trellis paths searched in performing ML sequence detection.

From the description of the Viterbi algorithm given above, it is unclear as to how decisions are made on the individual detected information symbols given the surviving sequences. If we have advanced to some stage, say K , where $K \gg L$ in the trellis, and we compare the surviving sequences, we shall find that with probability approaching one all surviving sequences will be identical in bit (or symbol) positions $K - 5L$ and less. In a practical implementation of the Viterbi algorithm, decisions on each information bit (or symbol) are forced after a delay of $5L$ bits (or symbols), and hence, the surviving sequences are truncated to the $5L$ most recent bits (or symbols). Thus, a variable delay in bit or symbol detection is avoided. The loss in performance resulting from the suboptimum detection procedure is negligible if the delay is at least $5L$.

Example 5-1-4

Consider the decision rule for detecting the data sequence in an NRZI signal with a Viterbi algorithm having a delay of $5L$ bits. The trellis for the NRZI signal is shown in Fig. 5-1-11. In this case, $L = 1$, hence the delay in bit detection is set to five bits. Hence, at $t = 6T$, we shall have two surviving sequences, one for each of the two states and the corresponding metrics $\mu_6(b_1, b_2, b_3, b_4, b_5, b_6)$ and $\mu_6(b'_1, b'_2, b'_3, b'_4, b'_5, b'_6)$. At this stage, with probability nearly equal to one, the bit b_1 will be the same as b'_1 ; that is,

both surviving sequences will have a common first branch. If $b_1 \neq b'_1$, we may select the bit (b_1 or b'_1) corresponding to the smaller of the two metrics. Then the first bit is dropped from the two surviving sequences. At $t = 7T$, the two metrics $\mu_7(b_2, b_3, b_4, b_5, b_6, b_7)$ and $\mu_7(b'_2, b'_3, b'_4, b'_5, b'_6, b'_7)$ will be used to determine the decision on bit b_2 . This process continues at each stage of the search through the trellis for the minimum distance sequence. Thus the detection delay is fixed at five bits.†

5-1-5 A Symbol-by-Symbol Detector for Signals with Memory

In contrast to the maximum-likelihood sequence detector for detecting the transmitted information, we now describe a detector that makes symbol-by-symbol decisions based on the computation of the maximum a posteriori probability (MAP) for each detected symbol. Hence, this detector is optimum in the sense that it minimizes the probability of a symbol error. The detection algorithm that is presented below is due to Abend and Fritchman (1970), who developed it as a detection algorithm for channels with intersymbol interference, i.e., channels with memory.

We illustrate the algorithm in the context of detecting a PAM signal with M possible levels. Suppose that it is desired to detect the information symbol transmitted in the k th signal interval, and let r_1, r_2, \dots, r_{k+D} be the observed received sequence, where D is the delay parameter which is chosen to exceed the signal memory, i.e., $D \geq L$, where L is the inherent memory in the signal. On the basis of the received sequence, we compute the posterior probabilities

$$P(s^{(k)} = A_m | r_{k+D}, r_{k+D-1}, \dots, r_1) \quad (5-1-65)$$

for the M possible symbol values and choose the symbol with the largest probability. Since

$$P(s^{(k)} = A_m | r_{k+D}, \dots, r_1) = \frac{p(r_{k+D}, \dots, r_1 | s^{(k)} = A_m)P(s^{(k)} = A_m)}{p(r_{k+D}, r_{k+D-1}, \dots, r_1)} \quad (5-1-66)$$

and since the denominator is common for all M probabilities, the maximum a posteriori probability (MAP) criterion is equivalent to choosing the value of $s^{(k)}$ that maximizes the numerator of (5-1-66). Thus, the criterion for deciding on the transmitted symbol $s^{(k)}$ is

$$\hat{s}^{(k)} = \arg \left\{ \max_{s^{(k)}} p(r_{k+D}, \dots, r_1 | s^{(k)} = A_m)P(s^{(k)} = A_m) \right\} \quad (5-1-67)$$

† One may have observed by now that the ML sequence detector and the symbol-by-symbol detector that ignores the memory in the NRZI signal reach the same decisions. Hence, there is no need for a decision delay. Nevertheless, the procedure described above applies in general.

When the symbols are equally probable, the probability $P(s^{(k)} = A_m)$ may be dropped from the computation.

The algorithm for computing the probabilities in (5-1-67) recursively begins with the first symbol $s^{(1)}$. We have

$$\begin{aligned}
 \bar{s}^{(1)} &= \arg \left\{ \max_{s^{(1)}} p(r_{k+D}, \dots, r_1 | s^{(1)} = A_m) P(s^{(1)} = A_m) \right\} \\
 &= \arg \left\{ \max_{s^{(1)}} \sum_{s^{(1+D)}} \cdots \sum_{s^{(2)}} p(r_{1+D}, \dots, r_1 | s^{(1+D)}, \dots, s^{(1)}) P(s^{(1+D)}, \dots, s^{(1)}) \right\} \\
 &= \arg \left\{ \max_{s^{(1)}} \sum_{s^{(1+D)}} \cdots \sum_{s^{(2)}} p_1(s^{(1+D)}, \dots, s^{(2)}, s^{(1)}) \right\} \quad (5-1-68)
 \end{aligned}$$

where $\bar{s}^{(1)}$ denotes the decision on $s^{(1)}$ and, for mathematical convenience, we have defined

$$p_1(s^{(1+D)}, \dots, s^{(2)}, s^{(1)}) \equiv p(r_{1+D}, \dots, r_1 | s^{(1+D)}, \dots, s^{(1)}) P(s^{(1+D)}, \dots, s^{(1)}) \quad (5-1-69)$$

The joint probability $P(s^{(1+D)}, \dots, s^{(2)}, s^{(1)})$ may be omitted if the symbols are equally probable and statistically independent. As a consequence of the statistical independence of the additive noise sequence, we have

$$\begin{aligned}
 p(r_{1+D}, \dots, r_1 | s^{(1+D)}, \dots, s^{(1)}) \\
 &= p(r_{1+D} | s^{(1+D)}, \dots, s^{(1+D-L)}) p(r_D | s^{(D)}, \dots, s^{(D-L)}) \cdots \\
 &\quad p(r_2 | s^{(2)}, s^{(1)}) p(r_1 | s^{(1)}) \quad (5-1-70)
 \end{aligned}$$

where we assume that $s^{(k)} = 0$ for $k \leq 0$.

For detection of the symbol $s^{(2)}$, we have

$$\begin{aligned}
 \bar{s}^{(2)} &= \arg \left\{ \max_{s^{(2)}} p(r_{2+D}, \dots, r_1 | s^{(2)} = A_m) P(s^{(2)} = A_m) \right\} \\
 &= \arg \left\{ \max_{s^{(2)}} \sum_{s^{(2+D)}} \cdots \sum_{s^{(3)}} p(r_{2+D}, \dots, r_1 | s^{(2+D)}, \dots, s^{(2)}) P(s^{(2+D)}, \dots, s^{(2)}) \right\} \quad (5-1-71)
 \end{aligned}$$

The joint conditional probability in the multiple summation can be expressed as

$$\begin{aligned}
 p(r_{2+D}, \dots, r_1 | s^{(2+D)}, \dots, s^{(2)}) \\
 &= p(r_{2+D} | s^{(2+D)}, \dots, s^{(2+D-L)}) p(r_{1+D}, \dots, r_1 | s^{(1+D)}, \dots, s^{(2)}) \quad (5-1-72)
 \end{aligned}$$

Furthermore, the joint probability

$$p(r_{1+D}, \dots, r_1 | s^{(1+D)}, \dots, s^{(2)})P(s^{(1+D)}, \dots, s^{(2)})$$

can be obtained from the probabilities computed previously in the detection of $s^{(1)}$. That is,

$$\begin{aligned} & p(r_{1+D}, \dots, r_1 | s^{(1+D)}, \dots, s^{(2)}) \\ &= \sum_{s^{(1)}} p(r_{1+D}, \dots, r_1 | s^{(1+D)}, \dots, s^{(1)})P(s^{(1+D)}, \dots, s^{(1)}) \\ &= \sum_{s^{(1)}} p_1(s^{(1+D)}, \dots, s^{(2)}, s^{(1)}) \end{aligned} \quad (5-1-73)$$

Thus, by combining (5-1-73) and (5-1-72) and then substituting into (5-1-71), we obtain

$$\bar{s}^{(2)} = \arg \left\{ \max_{s^{(2)}} \sum_{s^{(2+D)}} \dots \sum_{s^{(3)}} p_2(s^{(2+D)}, \dots, s^{(3)}, s^{(2)}) \right\} \quad (5-1-74)$$

where, by definition,

$$\begin{aligned} & p_2(s^{(2+D)}, \dots, s^{(3)}, s^{(2)}) \\ &= p(r_{2+D} | s^{(2+D)}, \dots, s^{(2+D-L)})P(s^{(2+D)}) \sum_{s^{(1)}} p_1(s^{(1+D)}, \dots, s^{(2)}, s^{(1)}) \end{aligned} \quad (5-1-75)$$

In general, the recursive algorithm for detecting the symbol $s^{(k)}$ is as follows: upon reception of r_{k+D}, \dots, r_2, r_1 , we compute

$$\begin{aligned} \bar{s}^{(k)} &= \arg \left\{ \max_{s^{(k)}} p(r_{k+D}, \dots, r_1 | s^{(k)})P(s^{(k)}) \right\} \\ &= \arg \left\{ \max_{s^{(k)}} \sum_{s^{(k+D)}} \dots \sum_{s^{(k+1)}} p_k(s^{(k+D)}, \dots, s^{(k+1)}, s^{(k)}) \right\} \end{aligned} \quad (5-1-76)$$

where, by definition,

$$\begin{aligned} & p_k(s^{(k+D)}, \dots, s^{(k-1)}, s^{(k)}) \\ &= p(r_{k+D} | s^{(k+D)}, \dots, s^{(k+D-L)})P(s^{(k+D)}) \sum_{s^{(k-1)}} p_{k-1}(s^{(k-1+D)}, \dots, s^{(k-1)}) \end{aligned} \quad (5-1-77)$$

Thus, the recursive nature of the algorithm is established by the relations (5-1-76) and (5-1-77).

The major problem with the algorithm is its computational complexity. In particular, the averaging performed over the symbols $s^{(k+D)}, \dots, s^{(k+1)}, s^{(k)}$ in (5-1-76) involves a large amount of computation per received signal, especially if the number M of amplitude levels $\{A_m\}$ is large. On the other hand, if M is small and the memory L is relatively short, this algorithm is easily implemented.

5-2 PERFORMANCE OF THE OPTIMUM RECEIVER FOR MEMORYLESS MODULATION

In this section, we evaluate the probability of error for the memoryless modulation signals described in Section 4-3-1. First, we consider binary PAM signals and then M -ary signals of various types.

5-2-1 Probability of Error for Binary Modulation

Let us consider binary PAM signals where the two signal waveforms are $s_1(t) = g(t)$ and $s_2(t) = -g(t)$, and $g(t)$ is an arbitrary pulse that is nonzero in the interval $0 \leq t \leq T_b$ and zero elsewhere.

Since $s_1(t) = -s_2(t)$, these signals are said to be *antipodal*. The energy in the pulse $g(t)$ is \mathcal{E}_b . As indicated in Section 4-3-1, PAM signals are one-dimensional, and, hence, their geometric representation is simply the one-dimensional vector $s_1 = \sqrt{\mathcal{E}_b}$, $s_2 = -\sqrt{\mathcal{E}_b}$. Figure 5-2-1 illustrates the two signal points.

Let us assume that the two signals are equally likely and that signal $s_1(t)$ was transmitted. Then, the received signal from the (matched filter or correlation) demodulator is

$$r = s_1 + n = \sqrt{\mathcal{E}_b} + n \tag{5-2-1}$$

where n represents the additive gaussian noise component, which has zero mean and variance $\sigma_n^2 = \frac{1}{2}N_0$. In this case, the decision rule based on the correlation metric given by (5-1-44) compares r with the threshold zero. If $r > 0$, the decision is made in favor of $s_1(t)$, and if $r < 0$, the decision is made that $s_2(t)$ was transmitted. Clearly, the two conditional pdfs of r are

$$p(r | s_1) = \frac{1}{\sqrt{\pi N_0}} e^{-(r - \sqrt{\mathcal{E}_b})^2 / N_0} \tag{5-2-2}$$

$$p(r | s_2) = \frac{1}{\sqrt{\pi N_0}} e^{-(r + \sqrt{\mathcal{E}_b})^2 / N_0} \tag{5-2-3}$$

FIGURE 5-2-1 Signal points for binary antipodal signals.



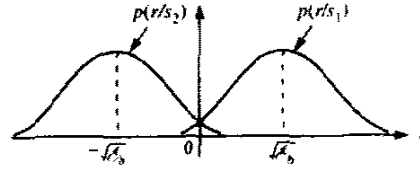


FIGURE 5-2-2 Conditional pdfs of two signals.

These two conditional pdfs are shown in Fig. 5-2-2.

Given that $s_1(t)$ was transmitted, the probability of error is simply the probability that $r < 0$, i.e.,

$$\begin{aligned}
 P(e | s_1) &= \int_{-\infty}^0 p(r | s_1) dr \\
 &= \frac{1}{\sqrt{\pi}N_0} \int_{-\infty}^0 \exp\left[-\frac{(r - \sqrt{\mathcal{E}_b})^2}{N_0}\right] dr \\
 &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-\sqrt{2\mathcal{E}_b}/N_0} e^{-x^2/2} dx \\
 &= \frac{1}{\sqrt{2\pi}} \int_{\sqrt{2\mathcal{E}_b}/N_0}^{\infty} e^{-x^2/2} dx \\
 &= Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right) \tag{5-2-4}
 \end{aligned}$$

where $Q(x)$ is the Q-function defined in (2-1-97). Similarly, if we assume that $s_2(t)$ was transmitted, $r = -\sqrt{\mathcal{E}_b} + n$ and the probability that $r > 0$ is also $P(e | s_2) = Q(\sqrt{2\mathcal{E}_b}/N_0)$. Since the signals $s_1(t)$ and $s_2(t)$ are equally likely to be transmitted, the average probability of error is

$$\begin{aligned}
 P_b &= \frac{1}{2}P(e | s_1) + \frac{1}{2}P(e | s_2) \\
 &= Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right) \tag{5-2-5}
 \end{aligned}$$

We should observe two important characteristics of this performance measure. First, we note that the probability of error depends only on the ratio \mathcal{E}_b/N_0 and not on any other detailed characteristics of the signals and the noise. Secondly, we note that $2\mathcal{E}_b/N_0$ is also the output SNR₀ from the matched-filter (and correlation) demodulator. The ratio \mathcal{E}_b/N_0 is usually called the *signal-to-noise ratio per bit*.

We also observe that the probability of error may be expressed in terms of the distance between the two signals s_1 and s_2 . From Fig. 5-2-1, we observe that the two signals are separated by the distance $d_{12} = 2\sqrt{\mathcal{E}_b}$. By substituting $\mathcal{E}_b = \frac{1}{4}d_{12}^2$ into (5-2-5), we obtain

$$P_b = Q\left(\sqrt{\frac{d_{12}^2}{2N_0}}\right) \tag{5-2-6}$$

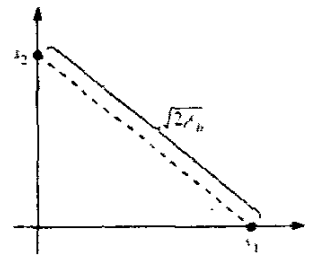


FIGURE 5-2-3 Signal points for binary orthogonal signals

This expression illustrates the dependence of the error probability on the distance between the two signal points.

Next, let us evaluate the error probability for binary orthogonal signals. Recall that the signal vectors \mathbf{s}_1 and \mathbf{s}_2 are two-dimensional, as shown in Fig. 5-2-3, and may be expressed, according to (4-3-30), as

$$\begin{aligned} \mathbf{s}_1 &= [\sqrt{\mathcal{E}_b} \quad 0] \\ \mathbf{s}_2 &= [0 \quad \sqrt{\mathcal{E}_b}] \end{aligned} \quad (5-2-7)$$

where \mathcal{E}_b denotes the energy for each of the waveforms. Note that the distance between these signal points is $d_{12} = \sqrt{2\mathcal{E}_b}$.

To evaluate the probability of error, let us assume that \mathbf{s}_1 was transmitted. Then, the received vector at the output of the demodulator is

$$\mathbf{r} = [\sqrt{\mathcal{E}_b} + n_1 \quad n_2] \quad (5-2-8)$$

We can now substitute for \mathbf{r} into the correlation metrics given by (5-1-44) to obtain $C(\mathbf{r}, \mathbf{s}_1)$ and $C(\mathbf{r}, \mathbf{s}_2)$. Then, the probability of error is the probability that $C(\mathbf{r}, \mathbf{s}_2) > C(\mathbf{r}, \mathbf{s}_1)$. Thus,

$$P(e | \mathbf{s}_1) = P[C(\mathbf{r}, \mathbf{s}_2) > C(\mathbf{r}, \mathbf{s}_1)] = P[n_2 - n_1 > \sqrt{\mathcal{E}_b}] \quad (5-2-9)$$

Since n_1 and n_2 are zero-mean statistically independent gaussian random variables each with variance $\frac{1}{2}N_0$, the random variable $x = n_2 - n_1$ is zero-mean gaussian with variance N_0 . Hence,

$$\begin{aligned} P(n_2 - n_1 > \sqrt{\mathcal{E}_b}) &= \frac{1}{\sqrt{2\pi N_0}} \int_{\sqrt{\mathcal{E}_b}}^{\infty} \frac{e^{-x^2/2N_0}}{\sqrt{2\mathcal{E}_b}} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{\sqrt{\mathcal{E}_b/N_0}}^{\infty} e^{-x^2/2} dx \\ &= Q\left(\sqrt{\frac{\mathcal{E}_b}{N_0}}\right) \end{aligned} \quad (5-2-10)$$

Due to symmetry, the same error probability is obtained when we assume that

s_2 is transmitted. Consequently, the average error probability for binary orthogonal signals is

$$P_b = Q\left(\sqrt{\frac{\mathcal{E}_b}{N_0}}\right) = Q(\sqrt{\gamma_b}) \quad (5-2-11)$$

where, by definition, γ_b is the SNR per bit.

If we compare the probability of error for binary antipodal signals with that for binary orthogonal signals, we find that orthogonal signals require a factor of two increase in energy to achieve the same error probability as antipodal signals. Since $10 \log_{10} 2 = 3$ dB, we say that orthogonal signals are 3 dB poorer than antipodal signals. The difference of 3 dB is simply due to the distance between the two signal points, which is $d_{12}^2 = 2\mathcal{E}_b$ for orthogonal signals, whereas $d_{12}^2 = 4\mathcal{E}_b$ for antipodal signals.

The error probability versus $10 \log_{10} \mathcal{E}_b/N_0$ for these two types of signals is shown in Fig. 5-2-4. As observed from this figure, at any given error probability, the \mathcal{E}_b/N_0 required for orthogonal signals is 3 dB more than that for antipodal signals.

5-2-2 Probability of Error for M -ary Orthogonal Signals

For equal energy orthogonal signals, the optimum detector selects the signal resulting in the largest cross correlation between the received vector \mathbf{r} and each of the M possible transmitted signal vectors $\{\mathbf{s}_m\}$, i.e.,

$$C(\mathbf{r}, \mathbf{s}_m) = \mathbf{r} \cdot \mathbf{s}_m = \sum_{k=1}^M r_k s_{mk}, \quad m = 1, 2, \dots, M \quad (5-2-12)$$

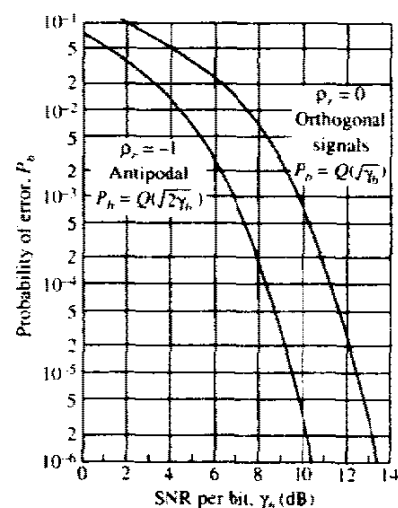


FIGURE 5-2-4 Probability of error for binary signals.

To evaluate the probability of error, let us suppose that the signal s_1 is transmitted. Then the received signal vector is

$$\mathbf{r} = [\sqrt{\mathcal{E}_s} + n_1 \quad n_2 \quad n_3 \quad \dots \quad n_M] \quad (5-2-13)$$

where n_1, n_2, \dots, n_M are zero-mean, mutually statistically independent gaussian random variables with equal variance $\sigma_n^2 = \frac{1}{2}N_0$. In this case, the outputs from the bank of M correlators are

$$\begin{aligned} C(\mathbf{r}, \mathbf{s}_1) &= \sqrt{\mathcal{E}_s}(\sqrt{\mathcal{E}_s} + n_1) \\ C(\mathbf{r}, \mathbf{s}_2) &= \sqrt{\mathcal{E}_s}n_2 \\ &\vdots \\ C(\mathbf{r}, \mathbf{s}_M) &= \sqrt{\mathcal{E}_s}n_M \end{aligned} \quad (5-2-14)$$

Note that the scale factor \mathcal{E}_s may be eliminated from the correlator outputs by dividing each output by $\sqrt{\mathcal{E}_s}$. Then, with this normalization, the pdf of the first correlator output ($r_1 = \sqrt{\mathcal{E}_s} + n_1$) is

$$p_{r_1}(x_1) = \frac{1}{\sqrt{\pi N_0}} \exp \left[-\frac{(x_1 - \sqrt{\mathcal{E}_s})^2}{N_0} \right] \quad (5-2-15)$$

and the pdfs of the other $M - 1$ correlator outputs are

$$p_{r_m}(x_m) = \frac{1}{\sqrt{\pi N_0}} e^{-x_m^2/N_0}, \quad m = 2, 3, \dots, M \quad (5-2-16)$$

It is mathematically convenient to first derive the probability that the detector makes a correct decision. This is the probability that r_1 is larger than each of the other $M - 1$ correlator outputs n_2, n_3, \dots, n_M . This probability may be expressed as

$$P_c = \int_{-\infty}^{\infty} P(n_2 < r_1, n_3 < r_1, \dots, n_M < r_1 | r_1) p(r_1) dr_1 \quad (5-2-17)$$

where $P(n_2 < r_1, n_3 < r_1, \dots, n_M < r_1 | r_1)$ denotes the joint probability that n_2, n_3, \dots, n_M are all less than r_1 , conditioned on any given r_1 . Then this joint probability is averaged over all r_1 . Since the $\{r_m\}$ are statistically independent, the joint probability factors into a product of $M - 1$ marginal probabilities of the form

$$\begin{aligned} P(n_m < r_1 | r_1) &= \int_{-\infty}^{r_1} p_{r_m}(x_m) dx_m, \quad m = 2, 3, \dots, M \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{r_1/\sqrt{2N_0}} e^{-x^2/2} dx \end{aligned} \quad (5-2-18)$$

These probabilities are identical for $m = 2, 3, \dots, M$, and, hence, the joint

probability under consideration is simply the result in (5-2-18) raised to the $(M - 1)$ th power. Thus, the probability of a correct decision is

$$P_c = \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{r_1 \sqrt{2/N_0}} e^{-x^2/2} dx \right)^{M-1} p(r_1) dr_1 \quad (5-2-19)$$

and the probability of a $(k\text{-bit})$ symbol error is

$$P_M = 1 - P_c \quad (5-2-20)$$

where

$$P_M = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[1 - \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^y e^{-x^2/2} dx \right)^{M-1} \right] \exp \left[-\frac{1}{2} \left(y - \sqrt{\frac{2\mathcal{E}_s}{N_0}} \right)^2 \right] dy \quad (5-2-21)$$

The same expression for the probability of error is obtained when any one of the other $M - 1$ signals is transmitted. Since all the M signals are equally likely, the expression for P_M given in (5-2-21) is the average probability of a symbol error. This expression can be evaluated numerically.

In comparing the performance of various digital modulation methods, it is desirable to have the probability of error expressed in terms of the SNR per bit, \mathcal{E}_b/N_0 , instead of the SNR per symbol, \mathcal{E}_s/N_0 . With $M = 2^k$, each symbol conveys k bits of information, and hence $\mathcal{E}_s = k\mathcal{E}_b$. Thus, (5-2-21) may be expressed in terms of \mathcal{E}_b/N_0 by substituting for \mathcal{E}_s .

Sometimes, it is also desirable to convert the probability of a symbol error into an equivalent probability of a binary digit error. For equiprobable orthogonal signals, all symbol errors are equiprobable and occur with probability

$$\frac{P_M}{M-1} = \frac{P_M}{2^k-1} \quad (5-2-22)$$

Furthermore, there are $\binom{k}{n}$ ways in which n bits out of k may be in error. Hence, the average number of bit errors per $k\text{-bit}$ symbol is

$$\sum_{n=1}^k \binom{k}{n} \frac{P_M}{2^k-1} = k \frac{2^{k-1}}{2^k-1} P_M \quad (5-2-23)$$

and the average bit error probability is just the result in (5-2-23) divided by k , the number of bits per symbol. Thus,

$$P_b = \frac{2^{k-1}}{2^k-1} P_M \approx \frac{P_M}{2}, \quad k \gg 1 \quad (5-2-24)$$

The graphs of the probability of a binary digit error as a function of the

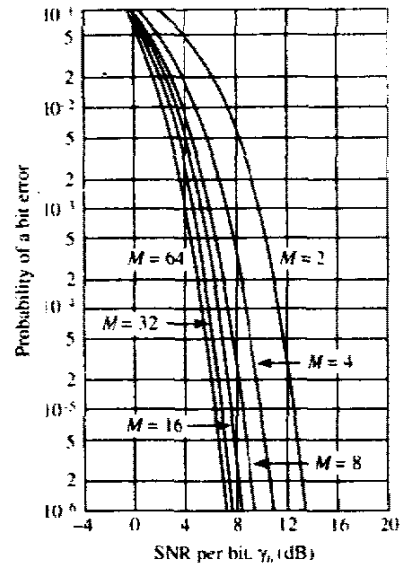


FIGURE 5-2-5 Probability of bit error for coherent detection of orthogonal signals.

SNR per bit, \mathcal{E}_b/N_0 , are shown in Fig. 5-2-5 for $M = 2, 4, 8, 16, 32$ and 64 . This figure illustrates that, by increasing the number M of waveforms, one can reduce the SNR per bit required to achieve a given probability of a bit error. For example, to achieve a $P_b = 10^{-5}$, the required SNR per bit is a little more than 12 dB for $M = 2$, but if M is increased to 64 signal waveforms ($k = 6$ bits/symbol), the required SNR per bit is approximately 6 dB. Thus, a savings of over 6 dB (a factor-of-four reduction) is realized in transmitter power (or energy) required to achieve a $P_b = 10^{-5}$ by increasing M from $M = 2$ to $M = 64$.

What is the minimum required \mathcal{E}_b/N_0 to achieve an arbitrarily small probability of error as $M \rightarrow \infty$? This question is answered below.

A Union Bound on the Probability of Error Let us investigate the effect of increasing M on the probability of error for orthogonal signals. To simplify the mathematical development, we first derive an upper bound on the probability of a symbol error that is much simpler than the exact form given in (5-2-21).

Recall that the probability of error for binary orthogonal signals is given by (5-2-11). Now, if we view the detector for M orthogonal signals as one that makes $M - 1$ binary decisions between the correlator output $C(\mathbf{r}, \mathbf{s}_1)$ that contains the signal and the other $M - 1$ correlator outputs $C(\mathbf{r}, \mathbf{s}_m)$, $m = 2, 3, \dots, M$, the probability of error is upper-bounded by the *union bound* of the $M - 1$ events. That is, if E_i represents the event that $C(\mathbf{r}, \mathbf{s}_i) > C(\mathbf{r}, \mathbf{s}_1)$ for $i \neq 1$ then we have $P_M = P(\cup_{i=1}^M E_i) \leq \sum_{i=1}^M P(E_i)$. Hence,

$$P_M \leq (M - 1)P_2 = (M - 1)Q(\sqrt{\mathcal{E}_b/N_0}) < MQ(\sqrt{\mathcal{E}_b/N_0}) \quad (5-2-25)$$

This bound can be simplified further by upper-bounding $Q(\sqrt{\mathcal{E}_b/N_0})$. We have

$$Q(\sqrt{\mathcal{E}_b/N_0}) < e^{-\mathcal{E}_b/2N_0} \quad (5-2-26)$$

Thus,

$$\begin{aligned} P_M &< M e^{-\mathcal{E}_b/2N_0} = 2^k e^{-k\mathcal{E}_b/2N_0} \\ P_M &< e^{-k(\mathcal{E}_b/N_0 - 2 \ln 2)/2} \end{aligned} \quad (5-2-27)$$

As $k \rightarrow \infty$, or equivalently, as $M \rightarrow \infty$, the probability of error approaches zero exponentially, provided that \mathcal{E}_b/N_0 is greater than $2 \ln 2$, i.e.,

$$\frac{\mathcal{E}_b}{N_0} > 2 \ln 2 = 1.39 \quad (1.42 \text{ dB}) \quad (5-2-28)$$

The simple upper bound on the probability of error given by (5-2-27) implies that, as long as $\text{SNR} > 1.42 \text{ dB}$, we can achieve an arbitrarily low P_M . However, this union bound is not a very tight upper bound at a sufficiently low SNR due to the fact that the upper bound for the Q function in (5-2-26) is loose. In fact, by more elaborate bounding techniques, it is shown in Chapter 7 that the upper bound in (5-2-27) is sufficiently tight for $\mathcal{E}_b/N_0 > 4 \ln 2$. For $\mathcal{E}_b/N_0 < 4 \ln 2$, a tighter upper bound on P_M is

$$P_M < 2e^{-k(\sqrt{\mathcal{E}_b/N_0} - \sqrt{\ln 2})^2} \quad (5-2-29)$$

Consequently, $P_M \rightarrow 0$ as $k \rightarrow \infty$, provided that

$$\frac{\mathcal{E}_b}{N_0} > \ln 2 = 0.693 \quad (-1.6 \text{ dB}) \quad (5-2-30)$$

Hence, -1.6 dB is the minimum required SNR per bit to achieve an arbitrarily small probability of error in the limit as $k \rightarrow \infty$ ($M \rightarrow \infty$). This minimum SNR per bit (-1.6 dB) is called the *Shannon limit* for an additive white Gaussian noise channel.

5-2-3 Probability of Error for M -ary Biorthogonal Signals

As indicated in Section 4-3, a set of $M = 2^k$ biorthogonal signals are constructed from $\frac{1}{2}M$ orthogonal signals by including the negatives of the orthogonal signals. Thus, we achieve a reduction in the complexity of the demodulator for the biorthogonal signals relative to that for orthogonal signals, since the former is implemented with $\frac{1}{2}M$ cross-correlators or matched filters, whereas the latter requires M matched filters or cross-correlators.

To evaluate the probability of error for the optimum detector, let us assume that the signal $s_1(t)$ corresponding to the vector $\mathbf{s}_1 = [\sqrt{\mathcal{E}_b}, 0, 0, \dots, 0]$ was transmitted. Then, the received signal vector is

$$\mathbf{r} = [\sqrt{\mathcal{E}_b} + n_1, n_2, \dots, n_{M/2}] \quad (5-2-31)$$

where the $\{n_m\}$ are zero-mean, mutually statistically independent and identically distributed gaussian random variables with variance $\sigma_n^2 = \frac{1}{2}N_0$. The

magnitude of the cross-correlators

$$C(\mathbf{r}, \mathbf{s}_m) = \mathbf{r} \cdot \mathbf{s}_m = \sum_{k=1}^{M/2} r_k s_{mk}, \quad m = 1, 2, \dots, \frac{1}{2}M \quad (5-2-32)$$

while the sign of this largest term is used to decide whether $s_m(t)$ or $-s_m(t)$ was transmitted. According to this decision rule, the probability of a correct decision is equal to the probability that $r_1 = \sqrt{E_s} + n_1 > 0$ and r_1 exceeds $|r_m| = |n_m|$ for $m = 2, 3, \dots, \frac{1}{2}M$. But

is similar to that for orthogonal signals (see Fig. 5-2-5). However, in this case, the probability of error for $M = 4$ is greater than that for $M = 2$. This is due to the fact that we have plotted the symbol error probability P_M in Fig. 5-2-6. If we plotted the equivalent bit error probability, we should find that the graphs for $M = 2$ and $M = 4$ coincide. As in the case of orthogonal signals, as $M \rightarrow \infty$ (or $k \rightarrow \infty$), the minimum required \mathcal{E}_b/N_0 to achieve arbitrarily small probability of error is -1.6 dB, the Shannon limit.

5-2-4 Probability of Error for Simplex Signals

Next we consider the probability of error for M simplex signals. Recall from Section 4-3 that simplex signals are a set of M equally correlated signals with mutual cross-correlation coefficient $\rho_{mm} = -1/(M-1)$. These signals have the same minimum separation of $\sqrt{2}\mathcal{E}$ between adjacent signal points in M -dimensional space as orthogonal signals. They achieve this mutual separation with a transmitted energy of $\mathcal{E}(M-1)/M$, which is less than that required for orthogonal signals by a factor of $(M-1)/M$. Consequently, the probability of error for simplex signals is identical to the probability of error for orthogonal signals, but this performance is achieved with a saving of

$$10 \log(1 - \rho) = 10 \log \frac{M}{M-1} \text{ dB} \quad (5-2-35)$$

in SNR. For $M = 2$, the saving is 3 db. However, as M is increased, the saving in SNR approaches 0 dB.

5-2-5 Probability of Error for M -ary Binary-Coded Signals

We have shown in Section 4-3 that binary-coded signal waveforms are represented by the signal vectors

$$\mathbf{s}_m = [s_{m1} \ s_{m2} \ \dots \ s_{mN}], \quad m = 1, 2, \dots, M$$

where $s_{mj} = \pm \sqrt{\mathcal{E}/N}$ for all m and j . N is the block length of the code, and is also the dimension of the M signal waveforms.

If $d_{\min}^{(e)}$ is the minimum euclidean distance of the M signal waveforms then the probability of a symbol error is upper-bounded as

$$\begin{aligned} P_m &< (M-1)P_b = (M-1)Q\left(\sqrt{\frac{(d_{\min}^{(e)})^2}{2N_0}}\right) \\ &< 2^k \exp\left[-\frac{(d_{\min}^{(e)})^2}{4N_0}\right] \end{aligned} \quad (5-2-36)$$

The value of the minimum euclidean distance will depend on the selection of the code words, i.e., the design of the code.

5-2-6 Probability of Error for M -ary PAM

Recall that M -ary PAM signals are represented geometrically as M one-dimensional signal points with value

$$s_m = \sqrt{\frac{1}{2}\mathcal{E}_g} A_m, \quad m = 1, 2, \dots, M \quad (5-2-37)$$

where \mathcal{E}_g is the energy of the basic signal pulse $g(t)$. The amplitude values may be expressed as

$$A_m = (2m - 1 - M)d, \quad m = 1, 2, \dots, M \quad (5-3-38)$$

where the euclidean distance between adjacent signal points is $d\sqrt{2\mathcal{E}_g}$.

$$\begin{aligned} \mathcal{E}_{av} &= \frac{1}{M} \sum_{m=1}^M \mathcal{E}_m = \frac{d^2 \mathcal{E}_g}{2M} \sum_{m=1}^M (2m - 1 - M)^2 \\ &= \frac{d^2 \mathcal{E}_g}{2M} \left[\frac{1}{3} M(M^2 - 1) \right] = \frac{1}{6} (M^2 - 1) d^2 \mathcal{E}_g \end{aligned} \quad (5-2-39)$$

Equivalently, we may characterize these signals in terms of their average power, which is

$$P_{av} = \frac{\mathcal{E}_{av}}{T} = \frac{1}{6} (M^2 - 1) \frac{d^2 \mathcal{E}_g}{T} \quad (5-2-40)$$

The average probability of error for M -ary PAM can be determined from the decision rule that maximizes the correlation metrics given by (5-1-44). Equivalently, the detector compares the demodulator output r with a set of $M - 1$ thresholds, which are placed at the midpoints of successive amplitude levels, as shown in Fig. 5-2-7. Thus, a decision is made in favor of the amplitude level that is closest to r .

The placing of the thresholds as shown in Fig. 5-2-7 helps in evaluating the probability of error. We note that if the m th amplitude level is transmitted, the demodulator output is

$$r = s_m + n = \sqrt{\frac{1}{2}\mathcal{E}_g} A_m + n \quad (5-2-41)$$

FIGURE 5-2-7 Placement of thresholds at midpoints of successive amplitude levels.



where the noise variable n has zero mean and variance $\sigma_n^2 = \frac{1}{2}N_0$. On the basis that all amplitude levels are equally likely a priori, the average probability of a symbol error is simply the probability that the noise variable n exceeds in magnitude one-half of the distance between levels. However, when either one of the two outside levels $\pm(M-1)$ is transmitted, an error can occur in one direction only. Thus, we have

$$\begin{aligned}
 P_M &= \frac{M-1}{M} P(|r - s_m| > d\sqrt{\frac{1}{2}\mathcal{E}_g}) \\
 &= \frac{M-1}{M} \frac{2}{\sqrt{\pi N_0}} \int_{d\sqrt{\mathcal{E}_g/2}}^{\infty} e^{-x^2/N_0} dx \\
 &= \frac{M-1}{M} \frac{2}{\sqrt{2\pi}} \int_{\sqrt{d^2\mathcal{E}_g/N_0}}^{\infty} e^{-x^2/2} dx \\
 &= \frac{2(M-1)}{M} Q\left(\sqrt{\frac{d^2\mathcal{E}_g}{N_0}}\right) \quad (5-2-42)
 \end{aligned}$$

The error probability in (5-2-42) can also be expressed in terms of the average transmitted power. From (5-2-40), we note that

$$d^2\mathcal{E}_g = \frac{6}{M^2-1} P_{av}T \quad (5-2-43)$$

By substituting for $d^2\mathcal{E}_g$ in (5-2-42), we obtain the average probability of a symbol error for PAM in terms of the average power as

$$P_M = \frac{2(M-1)}{M} Q\left(\sqrt{\frac{6P_{av}T}{(M^2-1)N_0}}\right) \quad (5-2-44)$$

or, equivalently,

$$P_M = \frac{2(M-1)}{M} Q\left(\sqrt{\frac{6\mathcal{E}_{av}}{(M^2-1)N_0}}\right) \quad (5-2-45)$$

where $\mathcal{E}_{av} = P_{av}T$ is the average energy.

In plotting the probability of a symbol error for M -ary signals such as M -ary PAM, it is customary to use the SNR per bit as the basic parameter. Since $T = kT_b$ and $k = \log_2 M$, (5-2-45) may be expressed as

$$P_M = \frac{2(M-1)}{M} Q\left(\sqrt{\frac{(6\log_2 M)\mathcal{E}_{bav}}{(M^2-1)N_0}}\right) \quad (5-2-46)$$

where $\mathcal{E}_{bav} = P_{av}T_b$ is the average bit energy and \mathcal{E}_{bav}/N_0 is the average SNR

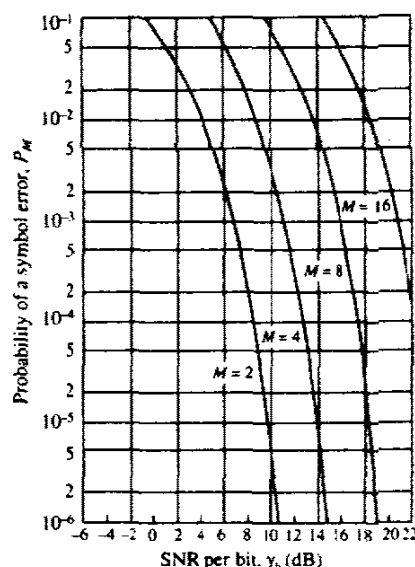


FIGURE 5-2-8 Probability of a symbol error for PAM.

per bit. Figure 5-2-8 illustrates the probability of a symbol error as a function of $10 \log_{10} \mathcal{E}_{b,av}/N_0$, with M as a parameter. Note that the case $M = 2$ corresponds to the error probability for binary antipodal signals. Also observe that the SNR per bit increases by over 4 dB for every factor-of-two increase in M . For large M , the additional SNR per bit required to increase M by a factor of two approaches 6 dB.

5-2-7 Probability of Error For M -ary PSK

Recall from Section 4-3 that digital phase-modulated signal waveforms may be expressed as

$$\mathbf{s}_m(t) = g(t) \cos \left[2\pi f_c t + \frac{2\pi}{M}(m-1) \right], \quad 1 \leq m \leq M, \quad 0 \leq t \leq T \quad (5-2-47)$$

and have the vector representation

$$\mathbf{s}_m = \left[\sqrt{\mathcal{E}_s} \cos \frac{2\pi}{M}(m-1) \quad \sqrt{\mathcal{E}_s} \sin \frac{2\pi}{M}(m-1) \right] \quad (5-2-48)$$

where $\mathcal{E}_s = \frac{1}{2}\mathcal{E}_r$ is the energy in each of the waveforms and $g(t)$ is the pulse shape of the transmitted signal. Since the signal waveforms have equal energy, the optimum detector for the AWGN channel given by (5-1-44) computes the correlation metrics

$$C(\mathbf{r}, \mathbf{s}_m) = \mathbf{r} \cdot \mathbf{s}_m, \quad m = 1, 2, \dots, M \quad (5-2-49)$$

In other words, the received signal vector $\mathbf{r} = [r_1 \quad r_2]$ is projected onto each of

the M possible signal vectors and a decision is made in favor of the signal with the largest projection.

The correlation detector described above is equivalent to a phase detector that computes the phase of the received signal from \mathbf{r} and selects the signal vector \mathbf{s}_m whose phase is closest to \mathbf{r} . Since the phase of \mathbf{r} is

$$\Theta_r = \tan^{-1} \frac{r_2}{r_1} \tag{5-2-50}$$

we will determine the pdf of Θ_r , from which we shall compute the probability of error.

Let us consider the case in which the transmitted signal phase is $\Theta_r = 0$, corresponding to the signal $s_1(t)$. Hence, the transmitted signal vector is

$$s_1 = [\sqrt{\mathcal{E}_s} \quad 0] \tag{5-2-51}$$

and the received signal vector has components

$$\begin{aligned} r_1 &= \sqrt{\mathcal{E}_s} + n_1 \\ r_2 &= n_2 \end{aligned} \tag{5-2-52}$$

Because n_1 and n_2 are jointly gaussian random variables, it follows that r_1 and r_2 are jointly gaussian random variables with $E(r_1) = \sqrt{\mathcal{E}_s}$, $E(r_2) = 0$, and $\sigma_{r_1}^2 = \sigma_{r_2}^2 = \frac{1}{2}N_0 = \sigma_r^2$. Consequently,

$$p_r(r_1, r_2) = \frac{1}{2\pi\sigma_r^2} \exp \left[-\frac{(r_1 - \sqrt{\mathcal{E}_s})^2 + r_2^2}{2\sigma_r^2} \right] \tag{5-2-53}$$

The pdf of the phase Θ_r is obtained by a change in variables from (r_1, r_2) to

$$\begin{aligned} V &= \sqrt{r_1^2 + r_2^2} \\ \Theta_r &= \tan^{-1} (r_2/r_1) \end{aligned} \tag{5-2-54}$$

This yields the joint pdf

$$p_{V,\Theta_r}(V, \Theta_r) = \frac{V}{2\pi\sigma_r^2} \exp \left(-\frac{V^2 + \mathcal{E}_s - 2\sqrt{\mathcal{E}_s} V \cos \Theta_r}{2\sigma_r^2} \right)$$

Integration of $p_{V,\Theta_r}(V, \Theta_r)$ over the range of V yields $p_{\Theta_r}(\Theta_r)$. That is,

$$\begin{aligned} p_{\Theta_r}(\Theta_r) &= \int_0^\infty p_{V,\Theta_r}(V, \Theta_r) dV \\ &= \frac{1}{2\pi} e^{-2\gamma \sin^2 \Theta_r} \int_0^\infty V e^{-(V - \sqrt{4\gamma} \cos \Theta_r)^2/2} dV \end{aligned} \tag{5-2-55}$$

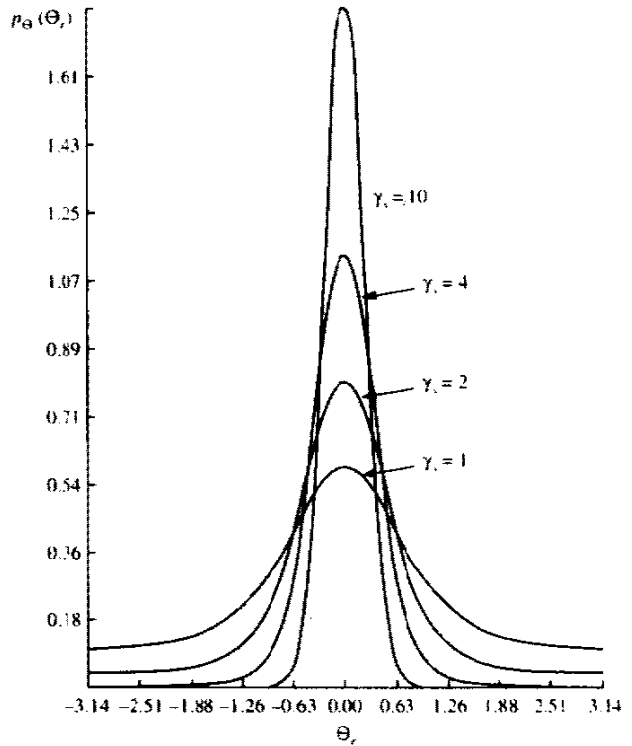


FIGURE 5-2-9 Probability density function $p_{\Theta_r}(\Theta_r)$ for $\gamma = 1, 2, 4$ and 10 .

where for convenience, we have defined the symbol SNR as $\gamma_s = \mathcal{E}_s/N_0$. Figure 5-2-9 illustrates $f_{\Theta_r}(\Theta_r)$ for several values of the SNR parameter γ , when the transmitted phase is zero. Note that $f_{\Theta_r}(\Theta_r)$ becomes narrower and more peaked about $\Theta_r = 0$ as the SNR γ , increases.

When $s_1(t)$ is transmitted, a decision error is made if the noise causes the phase to fall outside the range $-\pi/M \leq \Theta_r \leq \pi/M$. Hence, the probability of a symbol error is

$$P_M = 1 - \int_{-\pi/M}^{\pi/M} p_{\Theta_r}(\Theta_r) d\Theta_r \quad (5-2-56)$$

In general, the integral of $p_{\Theta_r}(\Theta)$ does not reduce to a simple form and must be evaluated numerically, except for $M = 2$ and $M = 4$.

For binary phase modulation, the two signals $s_1(t)$ and $s_2(t)$ are antipodal, and, hence, the error probability is

$$P_2 = Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right) \quad (5-2-57)$$

When $M = 4$, we have in effect two binary phase-modulation signals in phase

quadrature. Since there is no crosstalk or interference between the signals on the two quadrature carriers, the bit error probability is identical to that in (5-2-57). On the other hand, the symbol error probability for $M = 4$ is determined by noting that

$$P_s = (1 - P_b)^2 = \left[1 - Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right) \right]^2 \tag{5-2-58}$$

where P_b is the probability of a correct decision for the 2-bit symbol. The result (5-2-58) follows from the statistical independence of the noise on the quadrature carriers. Therefore, the symbol error probability for $M = 4$ is

$$\begin{aligned} P_s &= 1 - P_b \\ &= 2Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right) \left[1 - \frac{1}{2}Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right) \right] \end{aligned} \tag{5-2-59}$$

For $M > 4$, the symbol error probability P_M is obtained by numerically integrating (5-2-55). Figure 5-2-10 illustrates this error probability as a function of the SNR per bit for $M = 2, 4, 8, 16$, and 32. The graphs clearly illustrate the penalty in SNR per bit as M increases beyond $M = 4$. For example, at $P_M = 10^{-5}$, the difference between $M = 4$ and $M = 8$ is approximately 4 dB, and the difference between $M = 8$ and $M = 16$ is approximately 5 dB. For large values of M , doubling the number of phases requires an additional 6 dB/bit to achieve the same performance.

An approximation to the error probability for large values of M and for

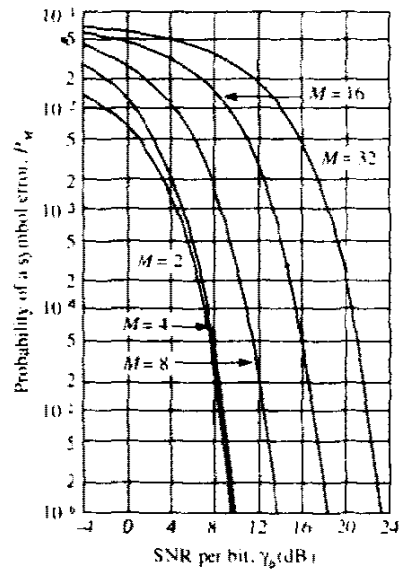


FIGURE 5-2-10 Probability of a symbol error for PSK signals.

large SNR may be obtained by first approximating $p_{\Theta_r}(\Theta_r)$. For $\mathcal{E}_s/N_0 \gg 1$ and $|\Theta_r| \leq \frac{1}{2}\pi$, $p_{\Theta_r}(\Theta_r)$ is well approximated as

$$p_{\Theta_r}(\Theta_r) \approx \sqrt{\frac{2\gamma_s}{\pi}} \cos \Theta_r e^{-2\gamma_s \sin^2 \Theta_r} \quad (5-2-60)$$

By substituting for $p_{\Theta_r}(\Theta_r)$ in (5-2-56) and performing the change in variable from Θ_r to $u = \sqrt{2\gamma_s} \sin \Theta_r$, we find that

$$\begin{aligned} P_M &\approx 1 - \int_{-\pi/M}^{\pi/M} \sqrt{\frac{2\gamma_s}{\pi}} \cos \Theta_r e^{-2\gamma_s \sin^2 \Theta_r} d\Theta_r \\ &\approx \frac{2}{\sqrt{\pi}} \int_{\sqrt{2\gamma_s} \sin(\pi/M)}^x e^{-u^2/2} du \\ &= 2Q\left(\sqrt{2\gamma_s} \sin \frac{\pi}{M}\right) = 2Q\left(\sqrt{2k\gamma_b} \sin \frac{\pi}{M}\right) \end{aligned} \quad (5-2-61)$$

where $k = \log_2 M$ and $\gamma_s = k\gamma_b$. Note that this approximation to the error probability is good for all values of M . For example, when $M = 2$ and $M = 4$, we have $P_2 = P_4 = 2Q(\sqrt{2\gamma_s})$, which compares favorably (a factor-of-two difference) with the exact probability given by (5-2-57).

The equivalent bit error probability for M -ary PSK is rather tedious to derive due to its dependence on the mapping of k -bit symbols into the corresponding signal phases. When a Gray code is used in the mapping, two k -bit symbols corresponding to adjacent signal phases differ in only a single bit. Since the most probable errors due to noise result in the erroneous selection of an adjacent phase to the true phase, most k -bit symbol errors contain only a single-bit error. Hence, the equivalent bit error probability for M -ary PSK is well approximated as

$$P_b \approx \frac{1}{k} P_M \quad (5-2-62)$$

Our treatment of the demodulation of PSK signals assumed that the demodulator had a perfect estimate of the carrier phase available. In practice, however, the carrier phase is extracted from the received signal by performing some nonlinear operation that introduces a phase ambiguity. For example, in binary PSK, the signal is often squared in order to remove the modulation, and the double-frequency component that is generated is filtered and divided by 2 in frequency in order to extract an estimate of the carrier frequency and phase ϕ . These operations result in a phase ambiguity of 180° in the carrier phase. Similarly, in four-phase PSK, the received signal is raised to the fourth power in order to remove the digital modulation, and the resulting fourth harmonic of the carrier frequency is filtered and divided by 4 in order to extract the carrier component. These operations yield a carrier frequency component containing the estimate of the carrier phase ϕ , but there are phase ambiguities of $\pm 90^\circ$ and 180° in the phase estimate. Consequently, we do not have an absolute estimate of the carrier phase for demodulation.

The phase ambiguity problem resulting from the estimation of the carrier phase ϕ can be overcome by encoding the information in phase differences between successive signal transmissions as opposed to absolute phase encoding. For example, in binary PSK, the information bit 1 may be transmitted by shifting the phase of the carrier by 180° relative to the previous carrier phase, while the information bit 0 is transmitted by a zero phase shift relative to the phase in the previous signaling interval. In four-phase PSK, the relative phase shifts between successive intervals are $0, 90^\circ, 180^\circ,$ and -90° , corresponding to the information bits 00, 01, 11, and 10, respectively. The generalization to $M > 4$ phases is straightforward. The PSK signals resulting from the encoding process are said to be *differentially encoded*. The encoding is performed by a relatively simple logic circuit preceding the modulator.

Demodulation of the differentially encoded PSK signal is performed as described above, by ignoring the phase ambiguities. Thus, the received signal is demodulated and detected to one of the M possible transmitted phases in each signaling interval. Following the detector is a relatively simple phase comparator that compares the phases of the demodulated signal over two consecutive intervals in order to extract the information.

Coherent demodulation of differentially encoded PSK results in a higher probability of error than the error probability derived for absolute phase encoding. With differentially encoded PSK, an error in the demodulated phase of the signal in any given interval will usually result in decoding errors over two consecutive signaling intervals. This is especially the case for error probabilities below 0.1. Therefore, the probability of error in differentially encoded M -ary PSK is approximately twice the probability of error for M -ary PSK with absolute phase encoding. However, this factor-of-two increase in the error probability translates into a relatively small loss in SNR.

5-2-8 Differential PSK (DPSK) and its Performance

A differentially encoded phase-modulated signal also allows another type of demodulation that does not require the estimation of the carrier phase.† Instead, the received signal in any given signaling interval is compared to the phase of the received signal from the preceding signaling interval. To elaborate, suppose that we demodulate the differentially encoded signal by multiplying $r(t)$ by $\cos 2\pi f_c t$ and $\sin 2\pi f_c t$ and integrating the two products over the interval T . At the k th signaling interval, the demodulator output is

$$r_k = [\sqrt{\mathcal{E}_s} \cos(\theta_k - \phi) + n_{k1} \quad \sqrt{\mathcal{E}_s} \sin(\theta_k - \phi) + n_{k2}]$$

or, equivalently,

$$r_k = \sqrt{\mathcal{E}_s} e^{j(\theta_k - \phi)} + n_k \quad (5-2-63)$$

† Because no phase estimation is required, DPSK is often considered to be a noncoherent communication technique. We take the view that DPSK represents a form of digital phase modulation in the extreme case where the phase estimate is derived only from the previous symbol interval.

where θ_k is the phase angle of the transmitted signal at the k th signaling interval, ϕ is the carrier phase, and $n_k = n_{k_1} + jn_{k_2}$ is the noise vector. Similarly, the received signal vector at the output of the demodulator in the preceding signaling interval is

$$r_{k-1} = \sqrt{\mathcal{E}_s} e^{j(\theta_{k-1} - \phi)} + n_{k-1} \quad (5-2-64)$$

The decision variable for the phase detector is the phase difference between these two complex numbers. Equivalently, we can project r_k onto r_{k-1} and use the phase of the resulting complex number; that is,

$$r_k r_{k-1}^* = \mathcal{E}_s e^{j(\theta_k - \theta_{k-1})} + \sqrt{\mathcal{E}_s} e^{j(\theta_k - \phi)} n_{k-1}^* + \sqrt{\mathcal{E}_s} e^{-j(\theta_{k-1} - \phi)} n_k + n_k n_{k-1}^* \quad (5-2-65)$$

which, in the absence of noise, yields the phase difference $\theta_k - \theta_{k-1}$. Thus, the mean value of $r_k r_{k-1}^*$ is independent of the carrier phase. Differentially encoded PSK signaling that is demodulated and detected as described above is called *differential PSK (DPSK)*.

The demodulation and detection of DPSK using matched filters is illustrated in Figure 5-2-11. If the pulse $g(t)$ is rectangular, the matched filters may be replaced by integrate-and-dump filters.

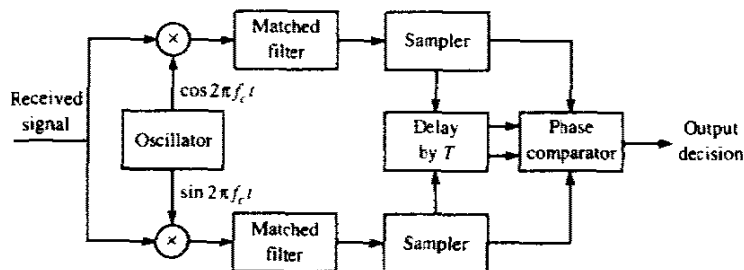
Let us now consider the evaluation of the error probability performance of a DPSK demodulator and detector. The derivation of the exact value of the probability of error for M -ary DPSK is extremely difficult, except for $M = 2$. The major difficulty is encountered in the determination of the pdf for the phase of the random variable $r_k r_{k-1}^*$, given by (5-2-65). However, an approximation to the performance of DPSK is easily obtained, as we now demonstrate.

Without loss of generality, suppose the phase difference $\theta_k - \theta_{k-1} = 0$. Furthermore, the exponential factors $e^{-j(\theta_{k-1} - \phi)}$ and $e^{j(\theta_k - \phi)}$ in (5-2-65) can be absorbed into the gaussian noise components n_{k-1} and n_k , without changing their statistical properties. Therefore, $r_k r_{k-1}^*$ in (5-2-65) can be expressed as

$$r_k r_{k-1}^* = \mathcal{E}_s + \sqrt{\mathcal{E}_s} (n_k + n_{k-1}^*) + n_k n_{k-1}^* \quad (5-2-66)$$

The complication in determining the pdf of the phase is the term $n_k n_{k-1}^*$. However, at SNRs of practical interest, the term $n_k n_{k-1}^*$ is small relative to the dominant noise term $\sqrt{\mathcal{E}_s} (n_k + n_{k-1}^*)$. If we neglect the term $n_k n_{k-1}^*$ and we

FIGURE 5-2-11 Block diagram of DPSK demodulator.



also normalize $r_k r_{k-1}^*$ by dividing through by $\sqrt{\mathcal{E}_s}$, the new set of decision metrics becomes

$$\begin{aligned} x &= \sqrt{\mathcal{E}_s} + \operatorname{Re}(n_k + n_{k-1}^*) \\ y &= \operatorname{Im}(n_k + n_{k-1}^*) \end{aligned} \quad (5-2-67)$$

The variables x and y are uncorrelated gaussian random variables with identical variances $\sigma_n^2 = N_0$. The phase is

$$\Theta_r = \tan^{-1} \frac{y}{x} \quad (5-2-68)$$

At this stage, we have a problem that is identical to the one we solved previously for phase-coherent demodulation. The only difference is that the noise variance is now twice as large as in the case of PSK. Thus we conclude that the performance of DPSK is 3 dB poorer than that for PSK. This result is relatively good for $M \geq 4$, but it is pessimistic for $M = 2$ in the sense that the loss in binary DPSK relative to binary PSK is less than 3 dB at large SNR. This is demonstrated below.

In binary DPSK, the two possible transmitted phase differences are 0 and π rad. As a consequence, only the real part of $r_k r_{k-1}^*$ is needed for recovering the information. Using (5-2-67), we express the real part as

$$\operatorname{Re}(r_k r_{k-1}^*) = \frac{1}{2}(r_k r_{k-1}^* + r_k^* r_{k-1})$$

Because the phase difference between the two successive signaling intervals is zero, an error is made if $\operatorname{Re}(r_k r_{k-1}^*) < 0$. The probability that $r_k r_{k-1}^* + r_k^* r_{k-1} < 0$ is a special case of a derivation, given in Appendix B concerned with the probability that a general quadratic form in complex-valued gaussian random variables is less than zero. The general form for this probability is given by (B-21) of Appendix B, and it depends entirely on the first and second moments of the complex-valued gaussian random variables r_k and r_{k-1} . Upon evaluating the moments and the parameters that are functions of the moments, we obtain the probability of error for binary DPSK in the form

$$P_b = \frac{1}{2} e^{-\mathcal{E}_b/N_0} \quad (5-2-69)$$

where \mathcal{E}_b/N_0 is the SNR per bit.

The graph is shown in Fig. 5-2-12. Also shown in that illustration is the probability of error for binary, coherent PSK. We observe that at error probabilities of $P_b \leq 10^{-3}$ the difference in SNR between binary PSK and binary DPSK is less than 3 dB. In fact, at $P_b \leq 10^{-5}$, the difference in SNR is less than 1 dB.

The probability of a binary digit error for four-phase DPSK with Gray coding can be expressed in terms of well-known functions, but its derivation is quite involved. We simply state the result at this point and refer the interested reader to Appendix C for the details of derivation. It is expressed in the form

$$P_b = Q_1(a, b) - \frac{1}{2} I_0(ab) \exp[-\frac{1}{2}(a^2 + b^2)] \quad (5-2-70)$$

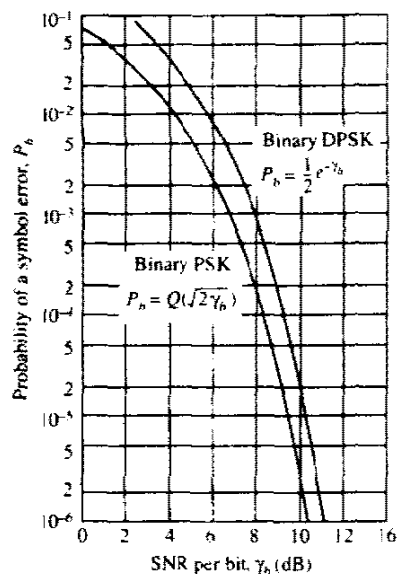


FIGURE 5-2-12 Probability of error for binary PSK and DPSK.

where $Q_1(a, b)$ is the Marcum Q function defined by (2-1-122) and (2-1-123), $I_0(x)$ is the modified Bessel function of order zero, defined by (2-1-120), and the parameters a and b are defined as

$$\begin{aligned} a &= \sqrt{2\gamma_b(1 - \sqrt{1/2})} \\ b &= \sqrt{2\gamma_b(1 + \sqrt{1/2})} \end{aligned} \tag{5-2-71}$$

Figure 5-2-13 illustrates the probability of a binary digit error for two- and

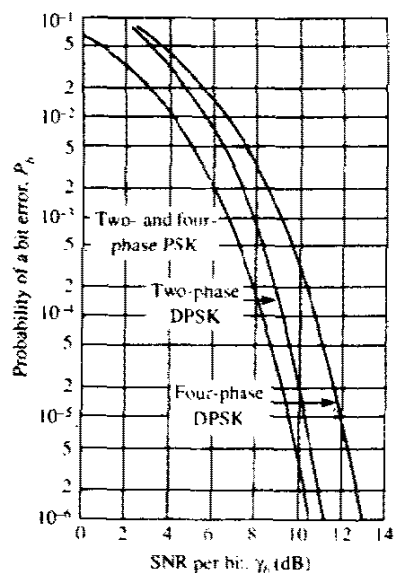


FIGURE 5-2-13 Probability of hit error for binary and four-phase PSK and DPSK.

four-phase DPSK and coherent PSK signaling obtained from evaluating the exact formulas derived in this section. Since binary DPSK is only slightly inferior to binary PSK at large SNR, and DPSK does not require an elaborate method for estimating the carrier phase, it is often used in digital communications systems. On the other hand, four-phase DPSK is approximately 2.3 dB poorer in performance than four-phase PSK at large SNR. Consequently the choice between these two four-phase systems is not as clear cut. One must weigh the 2.3 dB loss against the reduction in implementation complexity.

5-2-9 Probability of Error for QAM

Recall from Section 4-3 that QAM signal waveforms may be expressed as

$$s_m(t) = A_{mc}g(t) \cos 2\pi f_c t - A_{ms}g(t) \sin 2\pi f_c t, \quad 0 \leq t \leq T \quad (5-2-72)$$

where A_{mc} and A_{ms} are the information-bearing signal amplitudes of the quadrature carriers and $g(t)$ is the signal pulse. The vector representation of these waveforms is

$$\mathbf{s}_m = [A_{mc} \sqrt{\frac{1}{2}} \mathcal{E}_x \quad A_{ms} \sqrt{\frac{1}{2}} \mathcal{E}_x] \quad (5-2-73)$$

To determine the probability of error for QAM, we must specify the signal point constellation. We begin with QAM signal sets that have $M = 4$ points. Figure 5-2-14 illustrates two four-point signal sets. The first is a four-phase modulated signal and the second is a QAM signal with two amplitude levels, labeled A_1 and A_2 , and four phases. Because the probability of error is dominated by the minimum distance between pairs of signal points, let us impose the condition that $d_{\min}^{(e)} = 2A$ for both signal constellations and let us evaluate the average transmitter power, based on the premise that all signal points are equally probable. For the four-phase signal, we have

$$P_{av} = \frac{1}{4}(4)2A^2 = 2A^2 \quad (5-2-74)$$

For the two-amplitude, four-phase QAM, we place the points on circles of radii A and $\sqrt{3}A$. Thus, $d_{\min}^{(e)} = 2A$, and

$$P_{av} = \frac{1}{4}[2(3A^2) + 2A^2] = 2A^2 \quad (5-2-75)$$

which is the same average power as the $M = 4$ -phase signal constellation. Hence, for all practical purposes, the error rate performance of the two signal

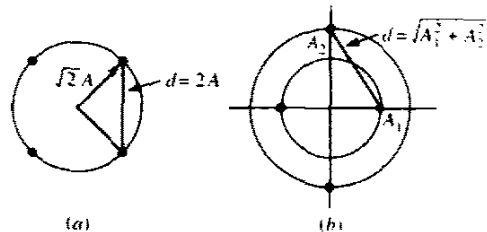


FIGURE 5-2-14 Two four-point signal constellations.

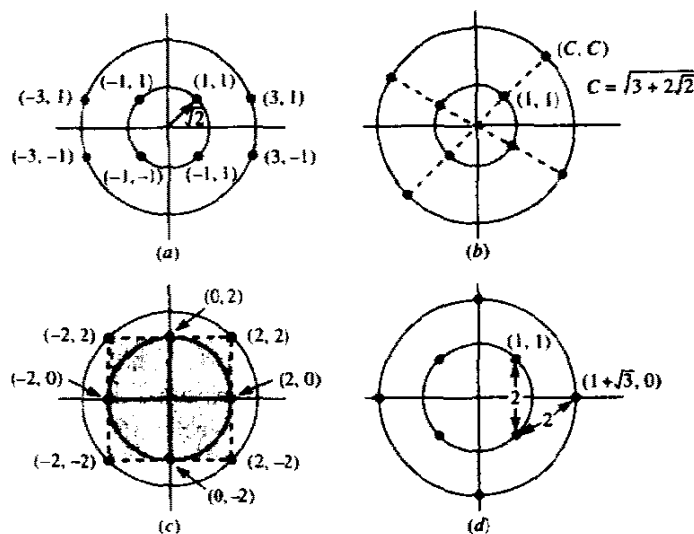


FIGURE 5-2-15 Four eight-point QAM signal constellations.

sets is the same. In other words, there is no advantage of the two-amplitude QAM signal set over $M = 4$ -phase modulation.

Next, let us consider $M = 8$ QAM. In this case, there are many possible signal constellations. We shall consider the four signal constellations shown in Fig. 5-2-15, all of which consist of two amplitudes and have a minimum distance between signal points of $2A$. The coordinates (A_{mc}, A_{ms}) for each signal point, normalized by A , are given in the figure. Assuming that the signal points are equally probable, the average transmitted signal power is

$$\begin{aligned}
 P_{av} &= \frac{1}{M} \sum_{m=1}^M (A_{mc}^2 + A_{ms}^2) \\
 &= \frac{A^2}{M} \sum_{m=1}^M (a_{mc}^2 + a_{ms}^2)
 \end{aligned} \tag{5-2-76}$$

where (a_{mc}, a_{ms}) are the coordinates of the signal points, normalized by A .

The two signal sets (a) and (c) in Fig. 5-2-15 contain signal points that fall on a rectangular grid and have $P_{av} = 6A^2$. The signal set (b) requires an average transmitted power $P_{av} = 6.83A^2$, and (d) requires $P_{av} = 4.73A^2$. Therefore, the fourth signal set requires approximately 1 dB less power than the first two and 1.6 dB less power than the third to achieve the same probability of error. This signal constellation is known to be the best eight-point QAM constellation because it requires the least power for a given minimum distance between signal points.

For $M \geq 16$, there are many more possibilities for selecting the QAM signal points in the two-dimensional space. For example, we may choose a circular multi-amplitude constellation for $M = 16$, as shown in Fig. 4-3-4. In this case,

the signal points at a given amplitude level are phase-rotated by $\frac{1}{4}\pi$ relative to the signal points at adjacent amplitude levels. This 16-QAM constellation is a generalization of the optimum 8-QAM constellation. However, the circular 16-QAM constellation is not the best 16-point QAM signal constellation for the AWGN channel.

Rectangular QAM signal constellations have the distinct advantage of being easily generated as two PAM signals impressed on phase-quadrature carriers. In addition, they are easily demodulated. Although they are not the best M -ary QAM signal constellations for $M \geq 16$, the average transmitted power required to achieve a given minimum distance is only slightly greater than the average power required for the best M -ary QAM signal constellation. For these reasons, rectangular M -ary QAM signals are most frequently used in practice.

For rectangular signal constellations in which $M = 2^k$, where k is even, the QAM signal constellation is equivalent to two PAM signals on quadrature carriers, each having $\sqrt{M} = 2^{k/2}$ signal points. Since the signals in the phase-quadrature components can be perfectly separated at the demodulator, the probability of error for QAM is easily determined from the probability of error for PAM. Specifically, the probability of a correct decision for the M -ary QAM system is

$$P_c = (1 - P_{\sqrt{M}})^2 \quad (5-2-77)$$

where $P_{\sqrt{M}}$ is the probability of error of a \sqrt{M} -ary PAM with one-half the average power in each quadrature signal of the equivalent QAM system. By appropriately modifying the probability of error for M -ary PAM, we obtain

$$P_{\sqrt{M}} = 2 \left(1 - \frac{1}{\sqrt{M}} \right) Q \left(\sqrt{\frac{3}{M-1} \frac{\mathcal{E}_{av}}{N_0}} \right) \quad (5-2-78)$$

where \mathcal{E}_{av}/N_0 is the average SNR per symbol. Therefore, the probability of a symbol error for the M -ary QAM is

$$P_M = 1 - (1 - P_{\sqrt{M}})^2 \quad (5-2-79)$$

Note that this result is exact for $M = 2^k$ when k is even. On the other hand, when k is odd, there is no equivalent \sqrt{M} -ary PAM system. This is no problem, however, because it is rather easy to determine the error rate for a rectangular signal set. If we employ the optimum detector that bases its decisions on the optimum distance metrics given by (5-1-43), it is relatively straightforward to show that the symbol error probability is tightly upper-bounded as

$$\begin{aligned} P_M &\leq 1 - \left[1 - 2Q \left(\sqrt{\frac{3\mathcal{E}_{av}}{(M-1)N_0}} \right) \right]^2 \\ &\leq 4Q \left(\sqrt{\frac{3k\mathcal{E}_{av}}{(M-1)N_0}} \right) \end{aligned} \quad (5-2-80)$$

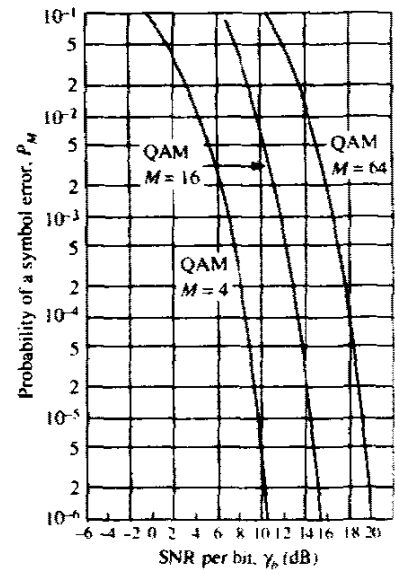


FIGURE 5-2-16 Probability of a symbol error for QAM.

for any $k \geq 1$, where $\mathcal{E}_{b,av}/N_0$ is the average SNR per bit. The probability of a symbol error is plotted in Fig. 5-2-16 as a function of the average SNR per bit.

For non-rectangular QAM signal constellations, we may upper-bound the error probability by use of a union bound. An obvious upper bound is

$$P_M < (M - 1)Q(\sqrt{[d_{\min}^{(e)}]^2/2N_0})$$

where $d_{\min}^{(e)}$ is the minimum euclidean distance between signal points. This bound may be loose when M is large. In such a case, we may approximate P_M by replacing $M - 1$ by M_n , where M_n is the largest number of neighboring points that are at distance $d_{\min}^{(e)}$ from any constellation point.

It is interesting to compare the performance of QAM with that of PSK for any given signal size M , since both types of signals are two-dimensional. Recall that for M -ary PSK, the probability of a symbol error is approximated as

$$P_M \approx 2Q\left(\sqrt{2\gamma_s} \sin \frac{\pi}{M}\right) \quad (5-2-81)$$

where γ_s is the SNR per symbol. For M -ary QAM, we may use the expression (5-2-78). Since the error probability is dominated by the argument of the Q function, we may simply compare the arguments of Q for the two signal formats. Thus, the ratio of these two arguments is

$$\mathcal{R}_M = \frac{3/(M - 1)}{2 \sin^2(\pi/M)} \quad (5-2-82)$$

For example, when $M = 4$, we have $\mathcal{R}_M = 1$. Hence, 4-PSK and 4-QAM yield comparable performance for the same SNR per symbol. On the other hand,

TABLE 5-2-1 SNR ADVANTAGE OF M -ARY QAM OVER M -ARY PSK

M	$10 \log_{10} \mathcal{R}_M$
8	1.65
16	4.20
32	7.02
64	9.95

when $M > 4$ we find that $\mathcal{R}_M > 1$, so that M -ary QAM yields better performance than M -ary PSK. Table 5-2-1 illustrates the SNR advantage of QAM over PSK for several values of M . For example, we observe that 32-QAM has a 7 dB SNR advantage over 32-PSK.

5-2-10 Comparison of Digital Modulation Methods

The digital modulation methods described in this chapter can be compared in a number of ways. For example, one can compare them on the basis of the SNR required to achieve a specified probability of error. However, such a comparison would not be very meaningful, unless it were made on the basis of some constraint, such as a fixed data rate of transmission or, equivalently, on the basis of a fixed bandwidth. With this goal in mind, let us consider the bandwidth requirements for several modulation methods.

For multiphase signals, the channel bandwidth required is simply the bandwidth of the equivalent lowpass signal pulse $g(t)$, which depends on its detailed characteristics. For our purposes, we assume that $g(t)$ is a pulse of duration T and that its bandwidth W is approximately equal to the reciprocal of T . Thus, $W = 1/T$ and, since $T = k/R = (\log_2 M)/R$, it follows that

$$W = \frac{R}{\log_2 M} \quad (5-2-83)$$

Therefore, as M is increased, the channel bandwidth required, when the bit rate R is fixed, decreases. The bandwidth efficiency is measured by the bit rate to bandwidth ratio, which is

$$\frac{R}{W} = \log_2 M \quad (5-2-84)$$

The bandwidth-efficient method for transmitting PAM is single-sideband. Then, the channel bandwidth required to transmit the signal is approximately equal to $1/2T$ and, since $T = k/R = (\log_2 M)/R$, it follows that

$$\frac{R}{W} = 2 \log_2 M \quad (5-2-85)$$

This is a factor of two better than PSK.

In the case of QAM, we have two orthogonal carriers, with each carrier having a PAM signal. Thus, we double the rate relative to PAM. However, the QAM signal must be transmitted via double sideband. Consequently, QAM and PAM have the same bandwidth efficiency when the bandwidth is referenced to the bandpass signal.

Orthogonal signals have totally different bandwidth requirements. If the $M = 2^k$ orthogonal signals are constructed by means of orthogonal carriers with minimum frequency separation of $1/2T$ for orthogonality, the bandwidth required for transmission of $k = \log_2 M$ information bits is

$$W = \frac{M}{2T} = \frac{M}{2(k/R)} = \frac{M}{2 \log_2 M} R \quad (5-2-86)$$

In this case, the bandwidth increases as M increases. Similar relationships obtain for simplex and biorthogonal signals. In the case of biorthogonal signals, the required bandwidth is one half of that for orthogonal signals.

A compact and meaningful comparison of these modulation methods is one based on the normalized data rate R/W (bits per second per hertz of bandwidth) versus the SNR per bit (\mathcal{E}_b/N_0) required to achieve a given error probability. Figure 5-2-17 illustrates the graph of R/W versus SNR per bit for PAM, QAM, PSK, and orthogonal signals, for the case in which the error probability is $P_M = 10^{-5}$. We observe that in the case of PAM, QAM, and PSK, increasing M results in a higher bit rate-to-bandwidth ratio R/W . However, the cost of achieving the higher data rate is an increase in the SNR per bit. Consequently, these modulation methods are appropriate for communication channels that are bandwidth limited, where we desire a bit rate-to-bandwidth ratio $R/W > 1$ and where there is sufficiently high SNR to support increases in M . Telephone channels and digital microwave radio channels are examples of such bandlimited channels.

In contrast, M -ary orthogonal signals yield a bit rate-to-bandwidth ratio of $R/W \leq 1$. As M increases, R/W decreases due to an increase in the required channel bandwidth. However, the SNR per bit required to achieve a given error probability (in this case, $P_M = 10^{-5}$) decreases as M increases. Consequently, M -ary orthogonal signals are appropriate for power-limited channels that have sufficiently large bandwidth to accommodate a large number of signals. In this case, as $M \rightarrow \infty$, the error probability can be made as small as desired, provided that $\mathcal{E}_b/N_0 > 0.693$ (-1.6 dB). This is the minimum SNR per bit required to achieve reliable transmission in the limit as the channel bandwidth $W \rightarrow \infty$ and the corresponding bit rate-to-bandwidth ratio $R/W \rightarrow 0$.

Also shown in Fig. 5-2-17 is the graph for the normalized capacity of the bandlimited AWGN channel, which is due to Shannon (1948). The ratio C/W , where $C (=R)$ is the capacity in bits/s, represents the highest achievable bit rate-to-bandwidth ratio on this channel. Hence, it serves as the upper bound

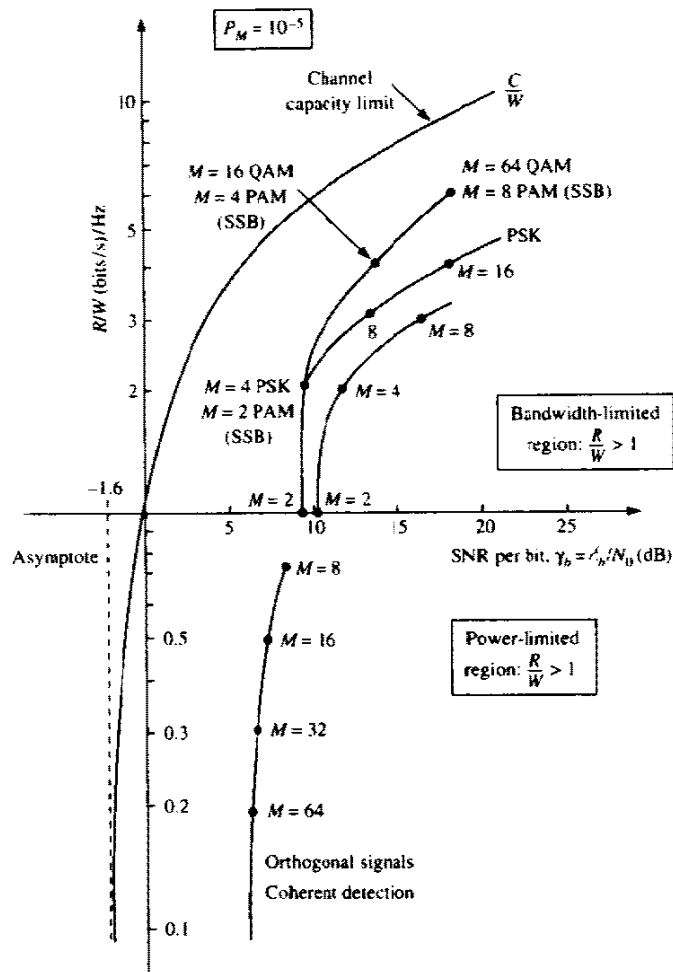


FIGURE 5-2-17 Comparison of several modulation methods at 10^{-5} symbol error probability.

on the bandwidth efficiency of any type of modulation. This bound is derived in Chapter 7 and discussed in greater detail there.

5-3 OPTIMUM RECEIVER FOR CPM SIGNALS

We recall from Section 4-3 that CPM is a modulation method with memory. The memory results from the continuity of the transmitted carrier phase from one signal interval to the next. The transmitted CPM signal may be expressed as

$$s(t) = \sqrt{\frac{2\mathcal{E}}{T}} \cos [2\pi f_c t + \phi(t; \mathbf{I})] \tag{5-3-1}$$

where $\phi(t; \mathbf{I})$ is the carrier phase. The filtered received signal for an additive gaussian noise channel is

$$r(t) = s(t) + n(t) \quad (5-3-2)$$

where

$$n(t) = n_c(t) \cos 2\pi f_c t - n_s(t) \sin 2\pi f_c t \quad (5-3-3)$$

5-3-1 Optimum Demodulation and Detection of CPM

The optimum receiver for this signal consists of a correlator followed by a maximum-likelihood sequence detector that searches the paths through the state trellis for the minimum euclidean distance path. The Viterbi algorithm is an efficient method for performing this search. Let us establish the general state trellis structure for CPM and then describe the metric computations.

Recall that the carrier phase for a CPM signal with a fixed modulation index h may be expressed as

$$\begin{aligned} \phi(t; \mathbf{I}) &= 2\pi h \sum_{k=-\infty}^n I_k q(t - kT) \\ &= \pi h \sum_{k=-\infty}^{n-L} I_k + 2\pi h \sum_{k=n-L+1}^n I_k q(t - kT) \\ &= \theta_n + \theta(t; \mathbf{I}), \quad nT \leq t \leq (n+1)T \end{aligned} \quad (5-3-4)$$

where we have assumed that $q(t) = 0$ for $t < 0$, $q(t) = \frac{1}{2}$ for $t \geq LT$, and

$$q(t) = \int_0^t g(\tau) d\tau \quad (5-3-5)$$

The signal pulse $g(t) = 0$ for $t < 0$ and $t \geq LT$. For $L = 1$, we have a full response CPM, and for $L > 1$, where L is a positive integer, we have a partial response CPM signal.

Now, when h is rational, i.e., $h = m/p$ where m and p are relatively prime positive integers, the CPM scheme can be represented by a trellis. In this case, there are p phase states

$$\Theta_s = \left\{ 0, \frac{\pi m}{p}, \frac{2\pi m}{p}, \dots, \frac{(p-1)\pi m}{p} \right\} \quad (5-3-6)$$

when m is even, and $2p$ phase states

$$\Theta_s = \left\{ 0, \frac{\pi m}{p}, \dots, \frac{(2p-1)\pi m}{p} \right\} \quad (5-3-7)$$

when m is odd. If $L = 1$, these are the only states in the trellis. On the other hand, if $L > 1$, we have an additional number of states due to the partial

response character of the signal pulse $g(t)$. These additional states can be identified by expressing $\theta(t, \mathbf{I})$ given by (5-3-4) as

$$\theta(t; \mathbf{I}) = 2\pi h \sum_{k=n-L+1}^{n-1} I_k q(t - kT) + 2\pi h I_n q(t - nT) \quad (5-3-8)$$

The first term on the right-hand side of (5-3-8) depends on the information symbols $(I_{n-1}, I_{n-2}, \dots, I_{n-L+1})$, which is called the *correlative state vector*, and represents the phase term corresponding to signal pulses that have not reached their final value. The second term in (5-3-8) represents the phase contribution due to the most recent symbol I_n . Hence, the state of the CPM signal (or the modulator) at time $t = nT$ may be expressed as the combined *phase state* and *correlative state*, denoted as

$$S_n = \{\theta_n, I_{n-1}, I_{n-2}, \dots, I_{n-L+1}\} \quad (5-3-9)$$

for a partial response signal pulse of length LT , where $L > 1$. In this case, the number of states is

$$N_s = \begin{cases} pM^{L-1} & (\text{even } m) \\ 2pM^{L-1} & (\text{odd } m) \end{cases} \quad (5-3-10)$$

when $h = m/p$.

Now, suppose the state of the modulator at $t = nT$ is S_n . The effect of the new symbol in the time interval $nT \leq t \leq (n+1)T$ is to change the state from S_n to S_{n+1} . Hence, at $t = (n+1)T$, the state becomes

$$S_{n+1} = (\theta_{n+1}, I_n, I_{n+1}, \dots, I_{n-L+2})$$

where

$$\theta_{n+1} = \theta_n + \pi h I_{n-L+1}$$

Example 5-3-1

Consider a binary CPM scheme with a modulation index $h = 3/4$ and a partial response pulse with $L = 2$. Let us determine the states S_n of the CPM scheme and sketch the phase tree and state trellis.

First, we note that there are $2p = 8$ phase states, namely,

$$\Theta_s = \{0, \pm \frac{1}{4}\pi, \pm \frac{1}{2}\pi, \pm \frac{3}{4}\pi, \pi\}$$

For each of these phase states, there are two states that result from the memory of the CPM scheme. Hence, the total number of states is $N_s = 16$, namely,

$$\begin{aligned} & (0, 1), (0, -1), (\pi, 1), (\pi, -1), (\frac{1}{4}\pi, 1), (\frac{1}{4}\pi, -1), (\frac{1}{2}\pi, 1), (\frac{1}{2}\pi, -1), \\ & (\frac{3}{4}\pi, 1), (\frac{3}{4}\pi, -1), (-\frac{1}{4}\pi, 1), (-\frac{1}{4}\pi, -1), (-\frac{1}{2}\pi, 1), (-\frac{1}{2}\pi, -1), \\ & (-\frac{3}{4}\pi, 1), (-\frac{3}{4}\pi, -1) \end{aligned}$$

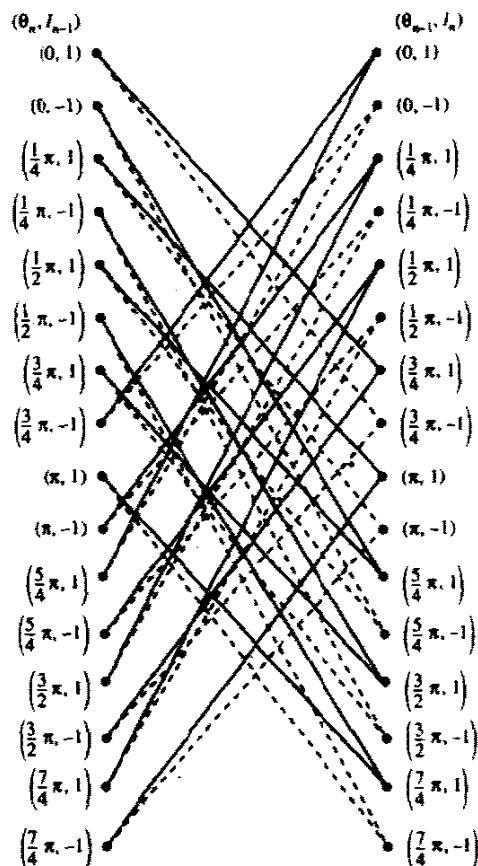


FIGURE 5-3-1 State trellis for partial response ($L = 2$) CPM with $h = \frac{3}{4}$.

If the system is in phase state $\theta_n = -\frac{1}{4}\pi$ and $I_{n-1} = -1$ then

$$\begin{aligned} \theta_{n+1} &= \theta_n + \pi h I_{n-1} \\ &= -\frac{1}{4}\pi - \frac{3}{4}\pi = -\pi \end{aligned}$$

The state trellis is illustrated in Fig. 5-3-1. A path through the state trellis corresponding to the sequence $(1, -1, -1, -1, 1, 1)$ is illustrated in Fig. 5-3-2.

In order to sketch the phase tree, we must know the signal pulse shape $g(t)$. Figure 5-3-3 illustrates the phase tree when $g(t)$ is a rectangular pulse of duration $2T$, with initial state $(0, 1)$.

Having established the state trellis representation of CPM, let us now consider the metric computations performed in the Viterbi algorithm.

Metric Computations By referring back to the mathematical development

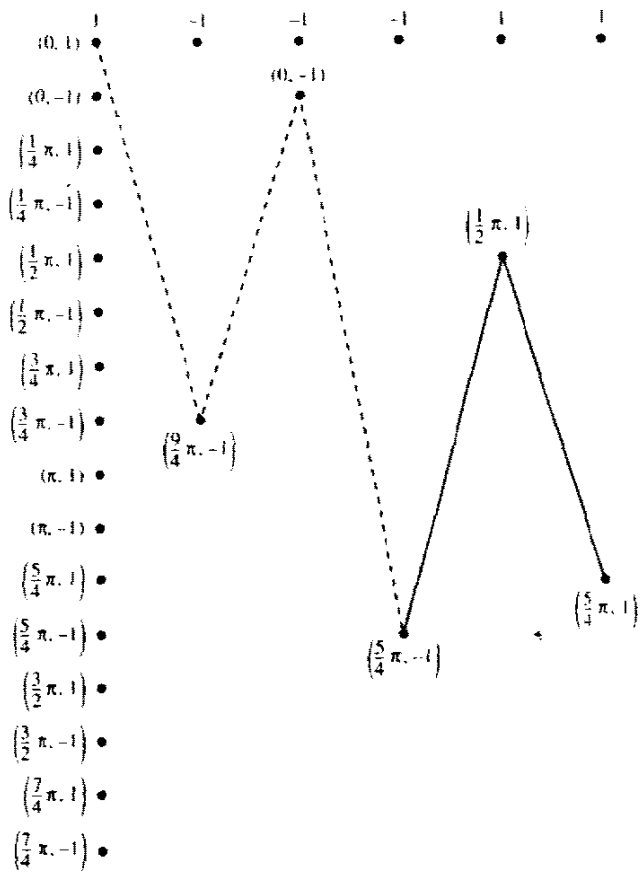


FIGURE 5-3-2 A single signal path through the trellis.

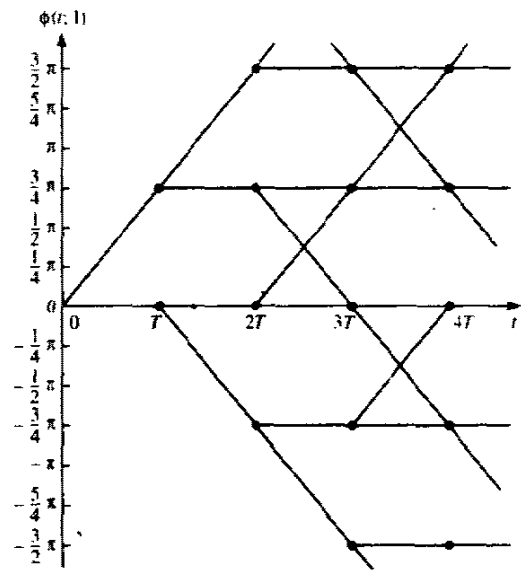


FIGURE 5-3-3 Phase tree for $L = 2$ partial response CPM with $h = \frac{1}{4}$.

for the derivation of the maximum likelihood demodulator given in Section 5-1-4, it is easy to show that the logarithm of the probability of the observed signal $r(t)$ conditioned on a particular sequence of transmitted symbols \mathbf{I} is proportional to the cross-correlation metric

$$CM_n(\mathbf{I}) = \int_{-\infty}^{(n+1)T} r(t) \cos[\omega_c t + \phi(t; \mathbf{I})] dt$$

$$= CM_{n-1}(\mathbf{I}) + \int_{nT}^{(n+1)T} r(t) \cos[\omega_c t + \theta(t; \mathbf{I}) + \theta_n] dt \quad (5-3-11)$$

The term $CM_{n-1}(\mathbf{I})$ represents the metrics for the surviving sequences up to time nT , and the term

$$v_n(\mathbf{I}; \theta_n) = \int_{nT}^{(n+1)T} r(t) \cos[\omega_c t + \theta(t; \mathbf{I}) + \theta_n] dt \quad (5-3-12)$$

represents the additional increments to the metrics contributed by the signal in the time interval $nT \leq t \leq (n+1)T$. Note that there are M^L possible sequences $\mathbf{I} = (I_n, I_{n-1}, \dots, I_{n-L+1})$ of symbols and p (or $2p$) possible phase states $\{\theta_n\}$. Therefore, there are pM^L (or $2pM^L$) different values of $v_n(\mathbf{I}, \theta_n)$, computed in each signal interval, and each value is used to increment the metrics corresponding to the pM^{L-1} surviving sequences from the previous signaling interval. A general block diagram that illustrates the computations of $v_n(\mathbf{I}; \theta_n)$ for the Viterbi decoder is shown in Fig. 5-3-4.

Note that the number of surviving sequences at each state of the Viterbi decoding process is pM^{L-1} (or $2pM^{L-1}$). For each surviving sequence, we have M new increments of $v_n(\mathbf{I}; \theta_n)$ that are added to the existing metrics to yield pM^L (or $2pM^L$) sequences with pM^L (or $2pM^L$) metrics. However, this number is then reduced back to pM^{L-1} (or $2pM^{L-1}$) survivors with corresponding metrics by selecting the most probable sequence of the M sequences merging at each node of the trellis and discarding the other $M - 1$ sequences.

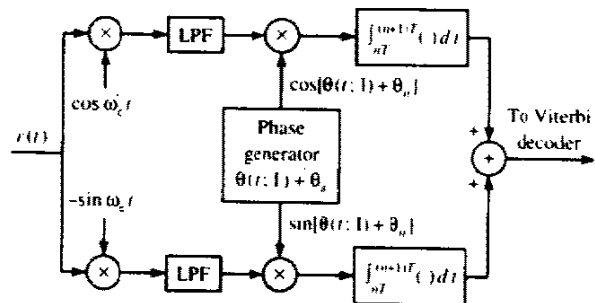


FIGURE 5-3-4 Computation of metric increments $v_n(\mathbf{I}; \theta_n)$.

5-3-2 Performance of CPM Signals

In evaluating the performance of CPM signals achieved with MLSE, we must determine the minimum euclidean distance of paths through the trellis that separate at the node at $t = 0$ and re-emerge at a later time at the same node. The distance between two paths through the trellis is related to the corresponding signals as we now demonstrate.

Suppose that we have two signals $s_i(t)$ and $s_j(t)$ corresponding to two phase trajectories $\phi(t; \mathbf{I}_i)$ and $\phi(t; \mathbf{I}_j)$. The sequences \mathbf{I}_i and \mathbf{I}_j must be different in their first symbol. Then, the euclidean distance between the two signals over an interval of length NT , where $1/T$ is the symbol rate, is defined as

$$\begin{aligned}
 d_{ij}^2 &= \int_0^{NT} [s_i(t) - s_j(t)]^2 dt \\
 &= \int_0^{NT} s_i^2(t) dt + \int_0^{NT} s_j^2(t) dt - 2 \int_0^{NT} s_i(t)s_j(t) dt \\
 &= 2N\mathcal{E} - 2 \frac{2\mathcal{E}}{T} \int_0^{NT} \cos[\omega_c t + \phi(t; \mathbf{I}_i)] \cos[\omega_c t + \phi(t; \mathbf{I}_j)] dt \\
 &= 2N\mathcal{E} - \frac{2\mathcal{E}}{T} \int_0^{NT} \cos[\phi(t; \mathbf{I}_i) - \phi(t; \mathbf{I}_j)] dt \\
 &= \frac{2\mathcal{E}}{T} \int_0^{NT} \{1 - \cos[\phi(t; \mathbf{I}_i) - \phi(t; \mathbf{I}_j)]\} dt \tag{5-3-13}
 \end{aligned}$$

Hence the euclidean distance is related to the phase difference between the paths in the state trellis according to (5-3-13).

It is desirable to express the distance d_{ij}^2 in terms of the bit energy. Since $\mathcal{E} = \mathcal{E}_b \log_2 M$, (5-3-13) may be expressed as

$$d_{ij}^2 = 2\mathcal{E}_b \delta_{ij}^2 \tag{5-3-14}$$

where δ_{ij}^2 is defined as

$$\delta_{ij}^2 = \frac{\log_2 M}{T} \int_0^{NT} \{1 - \cos[\phi(t; \mathbf{I}_i) - \phi(t; \mathbf{I}_j)]\} dt \tag{5-3-15}$$

Furthermore, we observe that $\phi(t; \mathbf{I}_i) - \phi(t; \mathbf{I}_j) = \phi(t; \mathbf{I}_i - \mathbf{I}_j)$, so that, with $\xi = \mathbf{I}_i - \mathbf{I}_j$, (5-3-15) may be written as

$$\delta_{ij}^2 = \frac{\log_2 M}{T} \int_0^{NT} [1 - \cos \phi(t; \xi)] dt \tag{5-3-16}$$

where any element of ξ can take the values $0, \pm 2, \pm 4, \pm \dots \pm 2(M-1)$, except that $\xi_0 \neq 0$.

The error rate performance for CPM is dominated by the term corresponding to the minimum euclidean distance, and it may be expressed as

$$P_M = K_{\delta_{\min}} Q\left(\sqrt{\frac{\mathcal{E}_b}{N_0}} \delta_{\min}^2\right) \quad (5-3-17)$$

where

$$\begin{aligned} \delta_{\min}^2 &= \lim_{N \rightarrow \infty} \min_{i,j} \delta_{ij}^2 \\ &= \lim_{N \rightarrow \infty} \min_{i,j} \left\{ \frac{\log_2 M}{T} \int_0^{NT} [1 - \cos \phi(t; \mathbf{I}_i - \mathbf{I}_j)] dt \right\} \end{aligned} \quad (5-3-18)$$

We note that for conventional binary PSK with no memory, $N = 1$ and $\delta_{\min}^2 = \delta_{12}^2 = 2$. Hence, (5-3-17) agrees with our previous result.

Since δ_{\min}^2 characterizes the performance of CPM with MLSE, we can investigate the effect on δ_{\min}^2 resulting from varying the alphabet size M , the modulation index h , and the length of the transmitted pulse in partial response CPM.

First, we consider full response ($L = 1$) CPM. If we take $M = 2$ as a beginning, we note that the sequences

$$\begin{aligned} \mathbf{I}_i &= +1, -1, I_2, I_3 \\ \mathbf{I}_j &= -1, +1, I_2, I_3 \end{aligned} \quad (5-3-19)$$

which differ for $k = 0, 1$ and agree for $k \geq 2$, result in two phase trajectories that merge after the second symbol. This corresponds to the difference sequence

$$\xi = \{2, -2, 0, 0, \dots\} \quad (5-3-20)$$

The euclidean distance for this sequence is easily calculated from (5-3-16), and provides an upper bound on δ_{\min}^2 . This upper bound for $M = 2$ is

$$d_B^2(h) = 2 \left(1 - \frac{\sin 2\pi h}{2\pi h}\right), \quad M = 2 \quad (5-3-21)$$

For example, where $h = \frac{1}{2}$, which corresponds to MSK, we have $d_B^2(\frac{1}{2}) = 2$, so that $\delta_{\min}^2(\frac{1}{2}) \leq 2$.

For $M > 2$ and full response CPM, it is also easily seen that phase trajectories merge at $t = 2T$. Hence, an upper bound on δ_{\min}^2 can be obtained by considering the phase difference sequence $\xi = \{\alpha, -\alpha, 0, 0, \dots\}$ where $\alpha = \pm 2, \pm 4, \dots, \pm 2(M-1)$. This sequence yields the upper bound

$$d_B^2(h) = \min_{1 \leq k \leq M-1} \left\{ (2 \log_2 M) \left(1 - \frac{\sin 2k\pi h}{2k\pi h}\right) \right\} \quad (5-3-22)$$

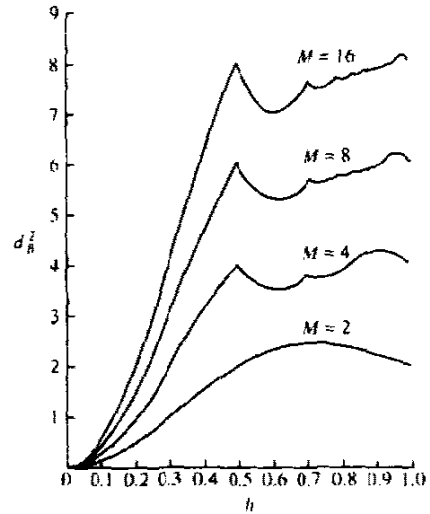


FIGURE 5-3-5 The upper bound d_B^2 as a function of the modulation index h for full response CPM with rectangular pulses. [From Aulin and Sundberg (1984). © 1984, John Wiley Ltd. Reprinted with permission of the publisher.]

The graphs of $d_B^2(h)$ versus h for $M = 2, 4, 8, 16$ are shown in Fig. 5-3-5. It is apparent from these graphs that large gains in performance can be achieved by increasing the alphabet size M . It must be remembered, however, that $\delta_{min}^2(h) \leq d_B^2(h)$. That is, the upper bound may not be achievable for all values of h .

The minimum euclidean distance $\delta_{min}^2(h)$ has been determined, by evaluating (5-3-16), for a variety of CPM signals by Aulin and Sundberg (1981). For example, Fig. 5-3-6 illustrates the dependence of the euclidean distance for binary CPFSK as a function of the modulation index h , with the number N of

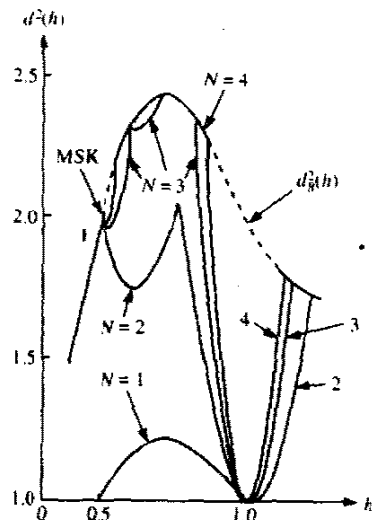


FIGURE 5-3-6 Squared minimum euclidean distance as a function of the modulation index for binary CPFSK. The upper bound is d_B^2 . [From Aulin and Sundberg (1981). © 1981 IEEE.]

bit observation (decision) intervals ($N = 1, 2, 3, 4$) as a parameter. Also shown is the upper bound $d_B^2(h)$ given by (5-3-21). In particular, we note that when $h = \frac{1}{2}$, $\delta_{\min}^2(\frac{1}{2}) = 2$, which is the same squared distance as PSK (binary or quaternary) with $N = 1$. On the other hand, the required observation interval for MSK is $N = 2$ intervals, for which we have $\delta_{\min}^2(\frac{1}{2}) = 2$. Hence, the performance of MSK with MLSE is comparable to (binary or quaternary) PSK as we have previously observed.

We also note from Fig. 5-3-6 that the optimum modulation index for binary CPFSK is $h = 0.715$ when the observation interval is $N = 3$. This yields $\delta_{\min}^2(0.715) = 2.43$, or a gain of 0.85 dB relative to MSK.

Figure 5-3-7 illustrates the euclidean distance as a function of h for $M = 4$ CPFSK, with the length of the observation interval N as a parameter. Also shown (as a dashed line where it is not reached) is the upper bound d_B^2 evaluated from (5-3-22). Note that δ_{\min}^2 achieves the upper bound for several values of h for some N . In particular, note that the maximum value of d_B^2 , which occurs at $h \approx 0.9$, is approximately reached for $N = 8$ observed symbol intervals. The true maximum is achieved at $h = 0.914$ with $N = 9$. For this case, $\delta_{\min}^2(0.914) = 4.2$, which represents a 3.2 dB gain over MSK. Also note that the euclidean distance contains minima at $h = \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, 1$, etc. These values of h are called *weak modulation indices* and should be avoided. Similar results are available for larger values of M , and may be found in the paper by Aulin and Sundberg (1981) and the text by Anderson *et al.* (1986).

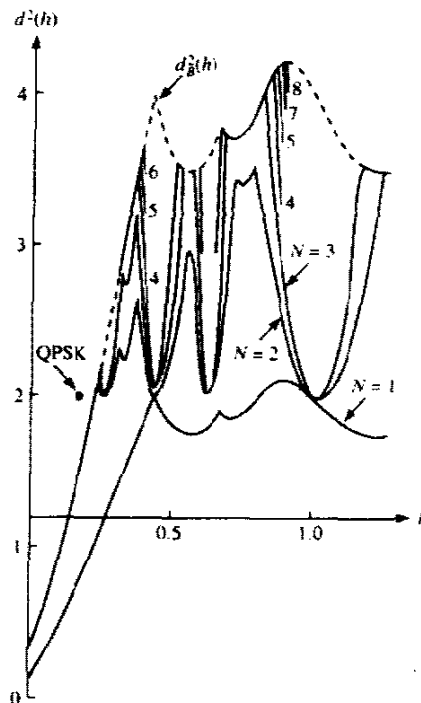


FIGURE 5-3-7 Squared minimum euclidean distance as a function of the modulation index for quaternary CPFSK. The upper bound is d_B^2 . [From Aulin and Sundberg (1981). © 1981 IEEE.]

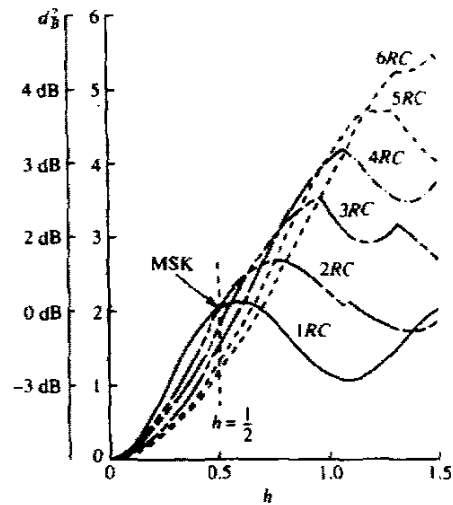


FIGURE 5-3-8 Upper bound d_B^2 on the minimum distance for partial response (raised cosine pulse) binary CPM. [From Sundberg (1986). © 1986 IEEE.]

Large performance gains can also be achieved with MLSE of CPM by using partial response signals. For example, the distance bound $d_B^2(h)$ for partial response, raised cosine pulses given by

$$g(t) = \begin{cases} \frac{1}{2LT} \left(1 - \cos \frac{2\pi t}{2LT} \right) & (0 \leq t \leq LT) \\ 0 & (\text{otherwise}) \end{cases} \quad (5-3-23)$$

is shown in Fig. 5-3-8 for $M=2$. Here, note that, as L increases, d_B^2 also achieves higher values. Clearly, the performance of CPM improves as the correlative memory L increases, but h must also be increased in order to achieve the larger values of d_B^2 . Since a larger modulation index implies a larger bandwidth (for fixed L), while a larger memory length L (for fixed h) implies a smaller bandwidth, it is better to compare the euclidean distance as a function of the normalized bandwidth $2WT_b$, where W is the 99% power bandwidth and T_b is the bit interval. Figure 5-3-9 illustrates this type of comparison with MSK used as a point of reference (0 dB). Note from this figure that there are several decibels to be gained by using partial response signals and higher signaling alphabets. The major price to be paid for this performance gain is the added exponentially increasing complexity in the implementation of the Viterbi decoder.

The performance results shown in Fig. 5-3-9 illustrate that 3–4 dB gain relative to MSK can be easily obtained with relatively no increase in bandwidth by the use of raised cosine partial response CPM and $M=4$. Although these results are for raised cosine signal pulses, similar gains can be achieved with other partial response pulse shapes. We emphasize that this gain in SNR is achieved by introducing memory into the signal modulation and exploiting the memory in the demodulation of the signal. No redundancy through coding has

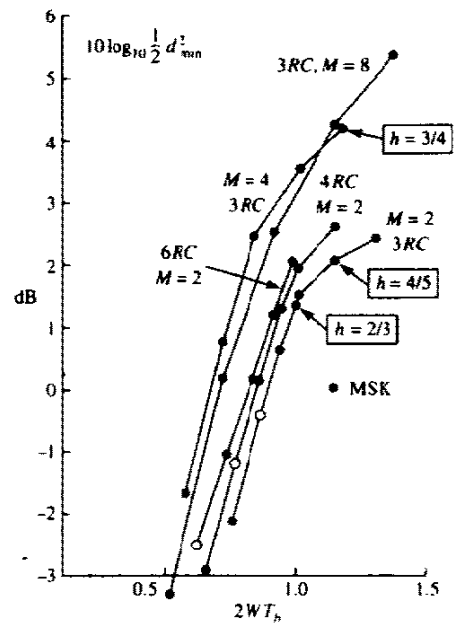


FIGURE 5-3-9 Power bandwidth tradeoff for partial response CPM signals with raised cosine pulses. W is the 99 percent in-band power bandwidth. [From Sundberg (1986). © 1986 IEEE.]

been introduced. In effect, the code has been built into the modulation and the trellis-type (Viterbi) decoding exploits the phase constraints in the CPM signal.

Additional gains in performance can be achieved by introducing additional redundancy through coding and increasing the alphabet size as a means of maintaining a fixed bandwidth. In particular, trellis-coded CPM using relatively simple convolution codes has been thoroughly investigated and many results are available in the technical literature. The Viterbi decoder for the convolutionally encoded CPM signal now exploits the memory inherent in the code and in the CPM signal. Performance gains of the order of 4–6 dB, relative to uncoded MSK with the same bandwidth, have been demonstrated by combining convolutional coding with CPM. Extensive numerical results for coded CPM are given by Lindell (1985).

Multi- h CPM By varying the modulation index from one signaling interval to another, it is possible to increase the minimum euclidean distance δ_{\min}^2 between pairs of phase trajectories and, thus, improve the performance gain over constant- h CPM. Usually, multi- h CPM employs a fixed number H of modulation indices that are varied cyclically in successive signaling intervals. Thus, the phase of the signal varies piecewise linearly.

Significant gains in SNR are achievable by using only a small number of different values of h . For example, with full response ($L = 1$) CPM and $H = 2$, it is possible to obtain a gain of 3 dB relative to binary or quaternary PSK. By increasing H to $H = 4$, a gain of 4.5 dB relative to PSK can be obtained. The performance gain can also be increased with an increase in the signal alphabet.

Table 5-3-1 lists the performance gains achieved with $M = 2, 4,$ and 8 for several values of H . The upper bounds on the minimum euclidean distance are also shown in Fig. 5-3-10 for several values of M and H . Note that the major gain in performance is obtained when H is increased from $H = 1$ to $H = 2$. For $H > 2$, the additional gain is relatively small for small values of $\{h_i\}$. On the other hand, significant performance gains are achieved by increasing the alphabet size M .

The results shown above hold for full response CPM. One can also extend the use of multi- h CPM to partial response in an attempt to further improve performance. It is anticipated that such schemes will yield some additional performance gains, but numerical results on partial response, multi- h CPM are limited. The interested reader is referred to the paper by Aulin and Sundberg (1982b).

Multiamplitude CPM Multiamplitude CPM (MACPM) is basically a combined amplitude and phase digital modulation scheme that allows us to increase the signaling alphabet relative to CPM in another dimension and, thus, to achieve higher data rates on a band-limited channel. Simultaneously, the combination of multiple amplitude in conjunction with CPM results in a bandwidth-efficient modulation technique.

We have already observed the spectral characteristics of MACPM in Section 4-3. The performance characteristics of MACPM have been investigated by Mulligan (1988) for both uncoded and trellis-coded CPM. Of particular interest is the result that trellis-coded CPM with two amplitude levels achieves a gain of 3–4 dB relative to MSK without a significant increase in the signal bandwidth.

5-3-3 Symbol-by-Symbol Detection of CPM Signals

Besides the ML sequence detector, there are other types of detectors that can be used to recover the information sequence in a CPM signal. In this section, we consider symbol-by-symbol detectors. One type of symbol-by-symbol detector is the one described in Section 5-1-5, which exploits the memory of CPM by performing matched filtering or cross-correlation over several signaling intervals. Because of its computational complexity, however, this recursive algorithm has not been directly applied to the detection of CPM. Instead, two similar, albeit suboptimal, symbol-by-symbol detection methods have been described in the papers by deBuda (1972), Osborne and Lutz (1974), and Schonhoff (1976). One of these is functionally equivalent to the algorithm given in Section 5-1-5, and the second is a suboptimum approximation of the first. We shall describe these two methods in the context of demodulation of CPFSK signals, for which these detection algorithms have been applied directly.

To describe these methods, we assume that the signal is observed over the present signaling interval and D signaling intervals into the future in deciding on the information symbol transmitted in the present signaling interval. A

TABLE 5-3-1 MAXIMUM VALUES OF THE UPPER BOUND d_B^2 FOR MULTI- h LINEAR PHASE CPM*

M	H	Max d_B^2	dB gain compared with MSK	h_1	h_2	h_3	h_4	\bar{h}
2	1	2.43	0.85	0.715				0.715
2	2	4.0	3.0	0.5	0.5			0.5
2	3	4.88	3.87	0.620	0.686	0.714		0.673
2	4	5.69	4.54	0.73	0.55	0.73	0.55	0.64
4	1	4.23	3.25	0.914				0.914
4	2	6.54	5.15	0.772	0.772			0.772
4	3	7.65	5.83	0.795	0.795	0.795		0.795
8	1	6.14	4.87	0.964				0.964
8	2	7.50	5.74	0.883	0.883			0.883
8	3	8.40	6.23	0.879	0.879	0.879		0.879

*From Aulin and Sundberg (1982b).

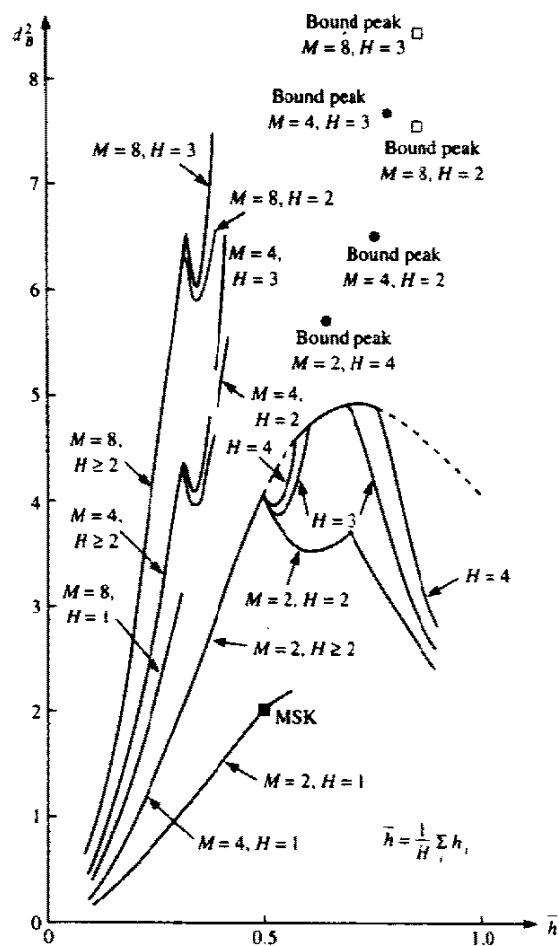


FIGURE 5-3-10 Upper bounds on minimum squared euclidean distance for various M and H values. [From Aulin and Sundberg (1982b). © 1982 IEEE.]

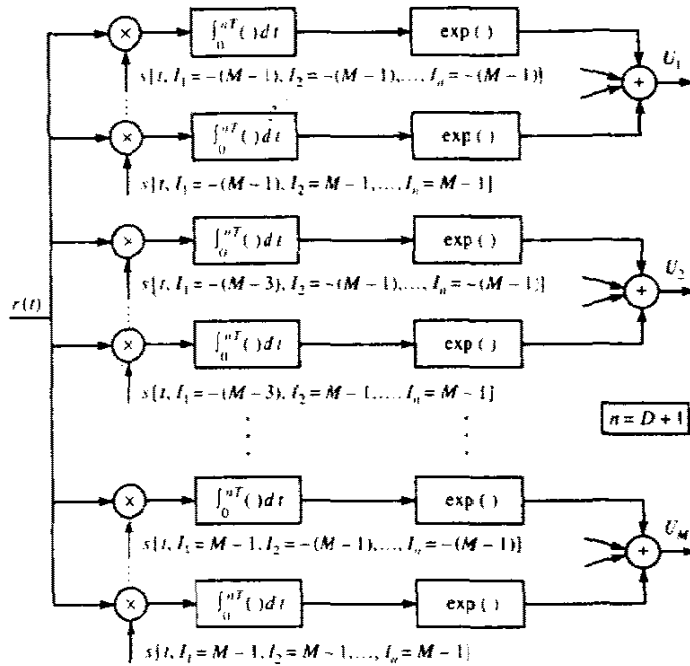


FIGURE 5-3-11 Block diagram of demodulator for detection of CPFSK.

block diagram of the demodulator, implemented as a bank of cross-correlators, is shown in Fig. 5-3-11. Recall that the transmitted CPFSK signal during the n th signaling interval is

$$s(t) = \text{Re} \{v(t)e^{j2\pi f_c t}\}$$

where

$$v(t) = \exp \left\{ j \left[\frac{\pi h [t - (n-1)T] I_n}{T} + \pi h \sum_{k=0}^{n-1} I_k + \phi_0 \right] \right\}$$

$h = 2f_d T$ is the modulation index, f_d is the peak frequency deviation, and ϕ_0 is the initial phase angle of the carrier.

In detecting the symbol I_1 , the cross-correlations shown in Fig. 5-3-11 are performed with the reference signals $s(t, I_1, I_2, \dots, I_{1+D})$ for all M^{D+1} possible values of the symbols I_1, I_2, \dots, I_{1+D} transmitted over the $D + 1$ signaling intervals. But these correlations in effect generate the variables r_1, r_2, \dots, r_{1+D} , which in turn are the arguments of the exponentials that occur in the pdf

$$p(r_1, r_2, \dots, r_{1+D} | I_1, I_2, \dots, I_{1+D})$$

Finally, the summations over the M^D possible values of the symbols I_2, I_3, \dots, I_{1+D} represent the averaging of

$$p(r_1, r_2, \dots, r_{1+D} | I_1, I_2, \dots, I_{1+D}) P(I_1, I_2, \dots, I_{1+D})$$

over the M^D possible values of these symbols. The M outputs of the demodulator constitute the decision variables from which the largest is selected to form the demodulated symbol. Consequently the metrics generated by the demodulator shown in Fig. 5-3-11 are equivalent to the decision variables given by (5-1-68) on which the decision on I_i is based.

Signals received in subsequent signaling intervals are demodulated in the same manner. That is, the demodulator cross-correlates the signal received over $D + 1$ signaling intervals with the M^{D+1} possible transmitted signals and forms the decision variables as illustrated in Fig. 5-3-11. Thus the decision made on the m th signaling interval is based on the cross-correlations performed over the signaling intervals $m, m + 1, \dots, m + D$. The initial phase in the correlation interval of duration $(D + 1)T$ is assumed to be known. On the other hand, the algorithm described by (5-1-76) and (5-1-77) involves an additional averaging operation over the previously detected symbols. In this respect, the demodulator shown in Fig. 5-3-11 differs from the recursive algorithm described above. However, the difference is insignificant.

One suboptimum demodulation method that performs almost as well as the optimum method embodied in Fig. 5-3-11 bases its decision on the largest output from the bank of M^{D+1} cross-correlators. Thus the exponential functions and the summations are eliminated. But this method is equivalent to selecting the symbol I_m for which the probability density function $p(r_m, r_{m+1}, \dots, r_{m+D} | I_m, I_{m+1}, \dots, I_{m+D})$ is a maximum.

The performance of the detector shown in Fig. 5-3-11 has been upper-bounded and evaluated numerically. Figure 5-3-12 illustrates the performance of binary CPFSK with $n = D + 1$ as a parameter. The modulation index $h = 0.715$ used in generating these results minimizes the probability of error as

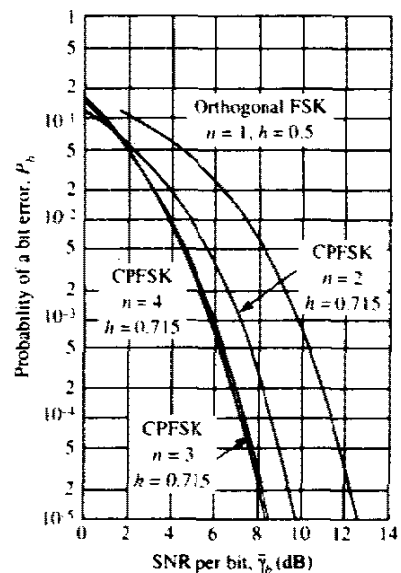


FIGURE 5-3-12 Performance of binary CPFSK with coherent detection.

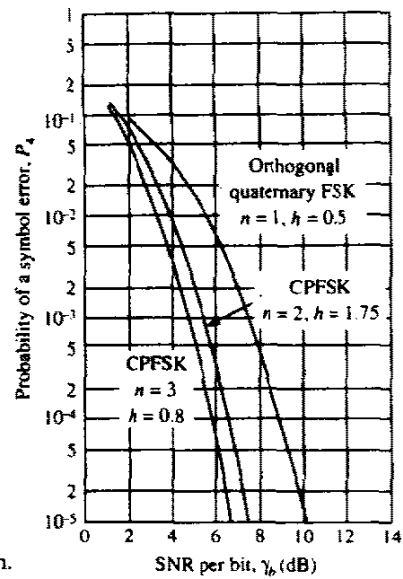


FIGURE 5-3-13 Performance of quaternary CPFSK with coherent detection.

shown by Schonhoff (1976). We note that an improvement of about 2.5 dB is obtained relative to orthogonal FSK ($n=1$) by a demodulator that cross-correlates over two symbols. An additional gain of approximately 1.5 dB is obtained by extending the correlation time to three symbols. Further extension of the correlation time results in a relatively small additional gain.

Similar results are obtained with larger alphabet sizes. For example, Figs 5-3-13 and 5-3-14 illustrate the performance improvements for quaternary and

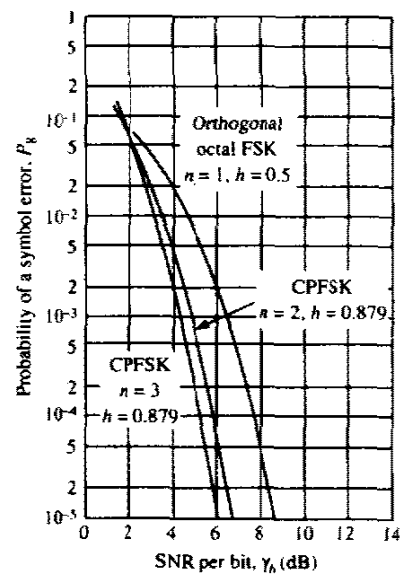


FIGURE 5-3-14 Performance of octal CPFSK with coherent detection.

octal CPFSK, respectively. The modulation indices given in these graphs are the ones that minimize the probability of a symbol error.

Instead of performing coherent detection, which requires knowledge of the carrier phase ϕ_0 , we may assume that ϕ_0 is uniformly distributed over the interval 0 to 2π , and average over it in arriving at the decision variables. Thus coherent integration (cross-correlation) is performed over the $n = D + 1$ signaling intervals, but the outputs of the correlators are envelope-detected. This is called *noncoherent detection* of CPFSK. In this detection scheme, performance is optimized by selecting n to be odd and making the decision on the middle symbol in the sequence of n symbols. The numerical results on the probability of error for noncoherent detection of CPFSK are similar to the results illustrated above for coherent detection. That is, a gain of 2–3 dB in performance is achieved by increasing the correlation interval from $n = 1$ to $n = 3$ and to $n = 5$.

5-4 OPTIMUM RECEIVER FOR SIGNALS WITH RANDOM PHASE IN AWGN CHANNEL

In this section, we consider the design of the optimum receiver for carrier modulated signals when the carrier phase is unknown at the receiver and no attempt is made to estimate its value. Uncertainty in the carrier phase of the received signal may be due to one or more of the following reasons: First, the oscillators that are used at the transmitter and the receiver to generate the carrier signals are generally not phase synchronous. Second, the time delay in the propagation of the signal from the transmitter to the receiver is not generally known precisely. To elaborate on this point, a transmitted signal of the form

$$s(t) = \text{Re} [g(t)e^{j2\pi f_c t}]$$

that propagates through a channel with delay t_0 will be received as

$$\begin{aligned} s(t - t_0) &= \text{Re} [g(t - t_0)e^{j2\pi f_c (t - t_0)}] \\ &= \text{Re} [g(t - t_0)e^{-j2\pi f_c t_0} e^{j2\pi f_c t}] \end{aligned}$$

The carrier phase shift due to the propagation delay t_0 is

$$\phi = -2\pi f_c t_0$$

Note that large changes in the carrier phase ϕ can occur due to relatively small changes in the propagation delay. For example, if the carrier frequency $f_c = 1$ MHz, an uncertainty or a change in the propagation delay of $0.5 \mu\text{s}$ will cause a phase uncertainty of π rad. In some channels (e.g., radio channels) the time delay in the propagation of the signal from the transmitter to the receiver

may change rapidly and in an apparently random manner, so that the carrier phase of the received signal varies in an apparently random fashion.

In the absence of knowledge of the carrier phase, we may treat this signal parameter as a random variable and determine the form of the optimum receiver for recovering the transmitted information from the received signal. First, we treat the case of binary signals and, then, we consider M -ary signals.

5-4-1 Optimum Receiver for Binary Signals

We consider a binary communication system that uses the two carrier modulated signals $s_1(t)$ and $s_2(t)$ to transmit the information, where

$$s_m(t) = \text{Re} [s_{im}(t)e^{j2\pi f_c t}], \quad m = 1, 2, \quad 0 \leq t \leq T \quad (5-4-1)$$

and $s_{im}(t)$, $m = 1, 2$ are the equivalent lowpass signals. The two signals are assumed to have equal energy

$$\mathcal{E} = \int_0^T s_m^2(t) dt = \frac{1}{2} \int_0^T |s_{im}(t)|^2 dt \quad (5-4-2)$$

and are characterized by the complex-valued correlation coefficient

$$\rho_{12} \equiv \rho = \frac{1}{\mathcal{E}} \int_0^T s_{1i}^*(t)s_{2i}(t) dt \quad (5-4-3)$$

The received signal is assumed to be a phase-shifted version of the transmitted signal and corrupted by the additive noise

$$\begin{aligned} n(t) &= \text{Re} \{ [n_c(t) + jn_s(t)]e^{j2\pi f_c t} \} \\ &= \text{Re} [z(t)e^{j2\pi f_c t}] \end{aligned} \quad (5-4-4)$$

Hence, the received signal may be expressed as

$$r(t) = \text{Re} \{ [s_{im}(t)e^{j\phi} + z(t)]e^{j2\pi f_c t} \} \quad (5-4-5)$$

where

$$r_i(t) = s_{im}(t)e^{j\phi} + z(t), \quad 0 \leq t \leq T \quad (5-4-6)$$

is the equivalent lowpass received signal. This received signal is now passed through a demodulator whose sampled output at $t = T$ is passed to the detector.

The Optimum Demodulator In Section 5-1-1, we demonstrated that if the received signal was correlated with a set of orthonormal functions $\{f_n(t)\}$ that

spanned the signal space, the outputs from the bank of correlators provide a set of sufficient statistics for the detector to make a decision that minimizes the probability of error. We also demonstrated that a bank of matched filters could be substituted for the bank of correlators.

A similar orthonormal decomposition can also be employed for a received signal with an unknown carrier phase. However, it is mathematically convenient to deal with the equivalent lowpass signal and to specify the signal correlators or matched filters in terms of the equivalent lowpass signal waveforms.

To be specific, the impulse response $h_l(t)$ of a filter that is matched to the complex-valued equivalent lowpass signal $s_l(t)$, $0 \leq t \leq T$, is given as (see Problem 5-6)

$$h_l(t) = s_l^*(T - t) \tag{5-4-7}$$

and the output of such a filter at $t = T$ is simply

$$\int_0^T |s_l(t)|^2 dt = 2\mathcal{E} \tag{5-4-8}$$

where \mathcal{E} is the signal energy. A similar result is obtained if the signal $s_l(t)$ is correlated with $s_l^*(t)$ and the correlator is sampled at $t = T$. Therefore, the optimum demodulator for the equivalent lowpass received signal $s_l(t)$ given in (5-4-6) may be realized by two matched filters in parallel, one matched to $s_{l1}(t)$ and the other to $s_{l2}(t)$, and shown in Fig. 5-4-1. The output of the matched filters or correlators at the sampling instant are the two complex numbers

$$r_m = r_{mc} + jr_{ms}, \quad m = 1, 2 \tag{5-4-9}$$

Suppose that the transmitted signal is $s_1(t)$. Then, it is easily shown (see Problem 5-35) that

$$\begin{aligned} r_1 &= 2\mathcal{E} \cos \phi + n_{1c} + j(2\mathcal{E} \sin \phi + n_{1s}) \\ r_2 &= 2\mathcal{E} |\rho| \cos(\phi + \alpha_0) + n_{2c} + j[2\mathcal{E} |\rho| \sin(\phi + \alpha_0) + n_{2s}] \end{aligned} \tag{5-4-10}$$

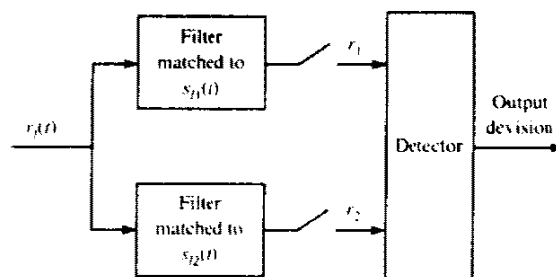


FIGURE 5-4-1 Optimum receiver for binary signals.

where ρ is the complex-valued correlation coefficient of the two signals $s_{11}(t)$ and $s_{12}(t)$, which may be expressed as $\rho = |\rho| \exp(\alpha_0)$. The random noise variables n_{1c} , n_{1s} , n_{2c} , and n_{2s} are jointly gaussian, with zero mean and equal variance.

The Optimum Detector The optimum detector observes the random variables $[r_{1c} \ r_{1s} \ r_{2c} \ r_{2s}] = \mathbf{r}$, where $r_1 = r_{1c} + jr_{1s}$ and $r_2 = r_{2c} + jr_{2s}$, and bases its decision on the posterior probabilities $P(\mathbf{s}_m | \mathbf{r})$, $m = 1, 2$. These probabilities may be expressed as

$$P(\mathbf{s}_m | \mathbf{r}) = \frac{p(\mathbf{r} | \mathbf{s}_m)P(\mathbf{s}_m)}{p(\mathbf{r})}, \quad m = 1, 2 \tag{5-4-11}$$

and, hence, the optimum decision rule may be expressed as

$$P(\mathbf{s}_1 | \mathbf{r}) \underset{s_2}{\overset{s_1}{\geq}} P(\mathbf{s}_2 | \mathbf{r})$$

or, equivalently,

$$\frac{p(\mathbf{r} | \mathbf{s}_1)}{p(\mathbf{r} | \mathbf{s}_2)} \underset{s_2}{\overset{s_1}{\geq}} \frac{P(\mathbf{s}_2)}{P(\mathbf{s}_1)} \tag{5-4-12}$$

The ratio of pdfs on the left-hand side of (5-4-12) is the *likelihood ratio*, which we denote as

$$\Lambda(\mathbf{r}) = \frac{p(\mathbf{r} | \mathbf{s}_1)}{p(\mathbf{r} | \mathbf{s}_2)} \tag{5-4-13}$$

The right-hand side of (5-4-12) is the ratio of the two prior probabilities, which takes the value of unity when the two signals are equally probable.

The probability density functions $p(\mathbf{r} | \mathbf{s}_1)$ and $p(\mathbf{r} | \mathbf{s}_2)$ can be obtained by averaging the pdfs $p(\mathbf{r} | \mathbf{s}_m, \phi)$ over the pdf of the random carrier phase, i.e.,

$$p(\mathbf{r} | \mathbf{s}_m) = \int_0^{2\pi} p(\mathbf{r} | \mathbf{s}_m, \phi) p(\phi) d\phi \tag{5-4-14}$$

We shall perform the integration indicated in (5-4-14) for the special case in which the two signals are orthogonal, i.e., $\rho = 0$. In this case, the outputs of the demodulator are

$$\begin{aligned} r_1 &= r_{1c} + jr_{1s} \\ &= 2\mathcal{E} \cos \phi + n_{1c} + j(2\mathcal{E} \sin \phi + n_{1s}) \\ r_2 &= r_{2c} + jr_{2s} \\ &= n_{2c} + jn_{2s} \end{aligned} \tag{5-4-15}$$

where $(n_{1c}, n_{1s}, n_{2c}, n_{2s})$ are mutually uncorrelated and, hence, statistically independent, zero-mean gaussian random variables (see Problem 5-25). Hence, the joint pdf of $\mathbf{r} = [r_{1c} \ r_{1s} \ r_{2c} \ r_{2s}]$ may be expressed as a product of the marginal pdfs. Consequently,

$$p(r_{1c}, r_{1s} | \mathbf{s}_1, \phi) = \frac{1}{2\pi\sigma^2} \exp \left[-\frac{(r_{1c} - 2\mathcal{E} \cos \phi)^2 + (r_{1s} - 2\mathcal{E} \sin \phi)^2}{2\sigma^2} \right] \quad (5-4-16)$$

$$p(r_{2c}, r_{2s}) = \frac{1}{2\pi\sigma^2} \exp \left(-\frac{r_{2c}^2 + r_{2s}^2}{2\sigma^2} \right)$$

where $\sigma^2 = 2\mathcal{E}N_0$.

The uniform pdf for the carrier phase ϕ represents the most ignorance that can be exhibited by the detector. This is called the *least favorable pdf* for ϕ . With $p(\phi) = 1/2\pi$, $0 \leq \phi \leq 2\pi$, substituted into the integral in (5-4-14), we obtain

$$\frac{1}{2\pi} \int_0^{2\pi} p(r_{1c}, r_{1s} | \mathbf{s}_1, \phi) d\phi$$

$$= \frac{1}{2\pi} \exp \left(-\frac{r_{1c}^2 + r_{1s}^2 + 4\mathcal{E}^2}{2\sigma^2} \right) \frac{1}{2\pi} \int \exp \left[\frac{2\mathcal{E}(r_{1c} \cos \phi + r_{1s} \sin \phi)}{\sigma^2} \right] d\phi \quad (5-4-17)$$

But

$$\frac{1}{2\pi} \int_0^{2\pi} \exp \left[\frac{2\mathcal{E}(r_{1c} \cos \phi + r_{1s} \sin \phi)}{\sigma^2} \right] d\phi = I_0 \left(\frac{2\mathcal{E} \sqrt{r_{1c}^2 + r_{1s}^2}}{\sigma^2} \right) \quad (5-4-18)$$

where $I_0(x)$ is the modified Bessel function of zeroth order, defined in (2-1-120).

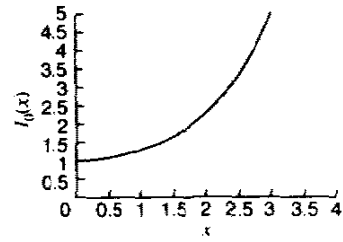
By performing a similar integration as in (5-4-17) under the assumption that the signal $s_2(t)$ was transmitted, we obtain the result

$$p(r_{2c}, r_{2s} | \mathbf{s}_2) = \frac{1}{2\pi} \exp \left(-\frac{r_{2c}^2 + r_{2s}^2 + 4\mathcal{E}^2}{2\sigma^2} \right) I_0 \left(\frac{2\mathcal{E} \sqrt{r_{2c}^2 + r_{2s}^2}}{\sigma^2} \right) \quad (5-4-19)$$

When, we substitute these results into the likelihood ratio given by (5-4-13), we obtain the result

$$\Lambda(\mathbf{r}) = \frac{I_0(2\mathcal{E} \sqrt{r_{1c}^2 + r_{1s}^2} / \sigma^2) P(\mathbf{s}_2)}{I_0(2\mathcal{E} \sqrt{r_{2c}^2 + r_{2s}^2} / \sigma^2) P(\mathbf{s}_1)} \quad (5-4-20)$$

Thus, the optimum detector computes the two envelopes $\sqrt{r_{1c}^2 + r_{1s}^2}$ and $\sqrt{r_{2c}^2 + r_{2s}^2}$ and the corresponding values of the Bessel function $I_0(2\mathcal{E} \sqrt{r_{1c}^2 + r_{1s}^2} / \sigma^2)$ and $I_0(2\mathcal{E} \sqrt{r_{2c}^2 + r_{2s}^2} / \sigma^2)$ to form the likelihood ratio. We observe that this computation requires knowledge of the noise variance σ^2 .


 FIGURE 5-4-2 Graph of $I_0(x)$.

The likelihood ratio is then compared with the threshold $P(s_2)/P(s_1)$ to determine which signal was transmitted.

A significant simplification in the implementation of the optimum detector occurs when the two signals are equally probable. In such a case the threshold becomes unity, and, due to the monotonicity of the Bessel function shown in Fig. 5-4-2, the optimum detection rule simplifies to

$$\sqrt{r_{1c}^2 + r_{1s}^2} \stackrel{s_1}{\cong} \sqrt{r_{2c}^2 + r_{2s}^2} \quad (5-4-21)$$

Thus, the optimum detector bases its decision on the two envelopes $\sqrt{r_{1c}^2 + r_{1s}^2}$ and $\sqrt{r_{2c}^2 + r_{2s}^2}$, and, hence, it is called an *envelope detector*.

We observe that the computation of the envelopes of the received signal samples at the output of the demodulator renders the carrier phase irrelevant in the decision as to which signal was transmitted. Equivalently, the decision may be based on the computation of the squared envelopes $r_{1c}^2 + r_{1s}^2$ and $r_{2c}^2 + r_{2s}^2$, in which case the detector is called a *square-law detector*.

Binary FSK signals are an example of binary orthogonal signals. Recall that in binary FSK we employ two different frequencies, say f_1 and $f_2 = f_1 + \Delta f$, to transmit a binary information sequence. The choice of minimum frequency separation $\Delta f = f_2 - f_1$ is considered below. Thus, the signal waveforms may be expressed as

$$\begin{aligned} s_1(t) &= \sqrt{2\mathcal{E}_b/T_b} \cos 2\pi f_1 t, & 0 \leq t \leq T_b \\ s_2(t) &= \sqrt{2\mathcal{E}_b/T_b} \cos 2\pi f_2 t, & 0 \leq t \leq T_b \end{aligned} \quad (5-4-22)$$

and their equivalent lowpass counterparts are

$$\begin{aligned} s_{1l}(t) &= \sqrt{2\mathcal{E}_b/T_b}, & 0 \leq t \leq T_b \\ s_{12}(t) &= \sqrt{2\mathcal{E}_b/T_b} e^{j2\pi\Delta f t}, & 0 \leq t \leq T_b \end{aligned} \quad (5-4-23)$$

The received signal may be expressed as

$$r(t) = \sqrt{\frac{2\mathcal{E}_b}{T_b}} \cos(2\pi f_m t + \phi_m) + n(t) \quad (5-4-24)$$

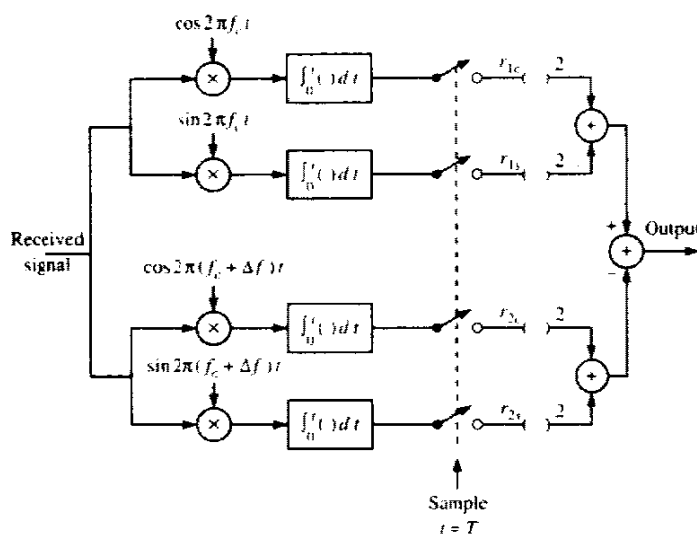


FIGURE 5-4-3 Demodulation and square-law detection of binary FSK signals.

where ϕ_m is the phase of the carrier frequency f_m . The demodulation of the real signal $r(t)$ may be accomplished, as shown in Fig. 5-4-3, by using four correlators with the basis functions

$$\begin{aligned} f_{1m}(t) &= \sqrt{\frac{2}{T_b}} \cos [(2\pi f_1 + 2\pi m \Delta f)t], \quad m = 0, 1 \\ f_{2m}(t) &= \sqrt{\frac{2}{T_b}} \sin [(2\pi f_1 + 2\pi m \Delta f)t], \quad m = 0, 1 \end{aligned} \quad (5-4-25)$$

The four outputs of the correlators are sampled at the end of each signal interval and passed to the detector. If the m th signal is transmitted, the four samples at the detector may be expressed as

$$\begin{aligned} r_{kc} = \sqrt{\mathcal{E}_b} \left[\frac{\sin [2\pi(k-m)\Delta f T]}{2\pi(k-m)\Delta f T} \cos \phi_m \right. \\ \left. - \frac{\cos [2\pi(k-m)\Delta f T] - 1}{2\pi(k-m)\Delta f T} \sin \phi_m \right] + n_{kc}, \quad k, m = 1, 2 \end{aligned} \quad (5-4-26)$$

$$\begin{aligned} r_{ks} = \sqrt{\mathcal{E}_b} \left[\frac{\cos 2\pi(k-m)\Delta f T - 1}{2\pi(k-m)\Delta f T} \cos \phi_m \right. \\ \left. + \frac{\sin [2\pi(k-m)\Delta f T]}{2\pi(k-m)\Delta f T} \sin \phi_m \right] + n_{ks}, \quad k, m = 1, 2 \end{aligned}$$

where n_{kc} and n_{ks} denote the gaussian noise components in the sampled outputs.

We observe that when $k = m$, the sampled values to the detector are

$$\begin{aligned} r_{mc} &= \sqrt{\mathcal{E}_b} \cos \phi_m + n_{mc} \\ r_{ms} &= \sqrt{\mathcal{E}_b} \sin \phi_m + n_{ms} \end{aligned} \quad (5-4-27)$$

Furthermore, we observe that when $k \neq m$, the signal components in the samples r_{kc} and r_{ks} will vanish, independently of the values of the phase shifts ϕ_k , provided that the frequency separation between successive frequencies is $\Delta f = 1/T$. In such a case, the other two correlator outputs consist of noise only, i.e.,

$$r_{kc} = n_{kc}, \quad r_{ks} = n_{ks}, \quad k \neq m \quad (5-4-28)$$

With a frequency separation of $\Delta f = 1/T$, the relations (5-4-27) and (5-4-28) are consistent with the previous result (5-4-15) for the demodulator outputs. Therefore, we conclude that for envelope or square-law detection of FSK signals, the minimum frequency separation required for orthogonality of the signals is $\Delta f = 1/T$. This separation is twice as large as that required when the detection is phase-coherent.

5-4-2 Optimum Receiver for M -ary Orthogonal Signals

The generalization of the optimum demodulator and detector to the case of M -ary orthogonal signals is straightforward. If the equal energy and equally probable signal waveforms are represented as

$$s_m(t) = \text{Re} \{ s_{lm}(t) e^{j2\pi f_c t} \}, \quad m = 1, 2, \dots, M, \quad 0 \leq t \leq T \quad (5-4-29)$$

where $s_{lm}(t)$ are the equivalent lowpass signals, the optimum correlation-type or matched-filter-type demodulator produces the M complex-valued random variables

$$r_m = r_{mc} + jr_{ms} = \int_0^T r_t(t) s_{lm}^*(t) dt, \quad m = 1, 2, \dots, M \quad (5-4-30)$$

where $r_t(t)$ is the equivalent lowpass received signal. Then, the optimum detector, based on a random, uniformly distributed carrier phase, computes the M envelopes

$$|r_m| = \sqrt{r_{mc}^2 + r_{ms}^2}, \quad m = 1, 2, \dots, M \quad (5-4-31)$$

or, equivalently, the squared envelopes $|r_m|^2$, and selects the signal with the largest envelope (or squared envelope).

In the special case of M -ary orthogonal FSK signals, the optimum receiver has the structure illustrated in Fig. 5-4-4. There are $2M$ correlators: two for each possible transmitted frequency. The minimum frequency separation between adjacent frequencies to maintain orthogonality is $\Delta f = 1/T$.

5-4-3 Probability of Error for Envelope Detection of M -ary Orthogonal Signals

Let us consider the transmission of M -ary orthogonal equal energy signals over an AWGN channel, which are envelope-detected at the receiver. We also

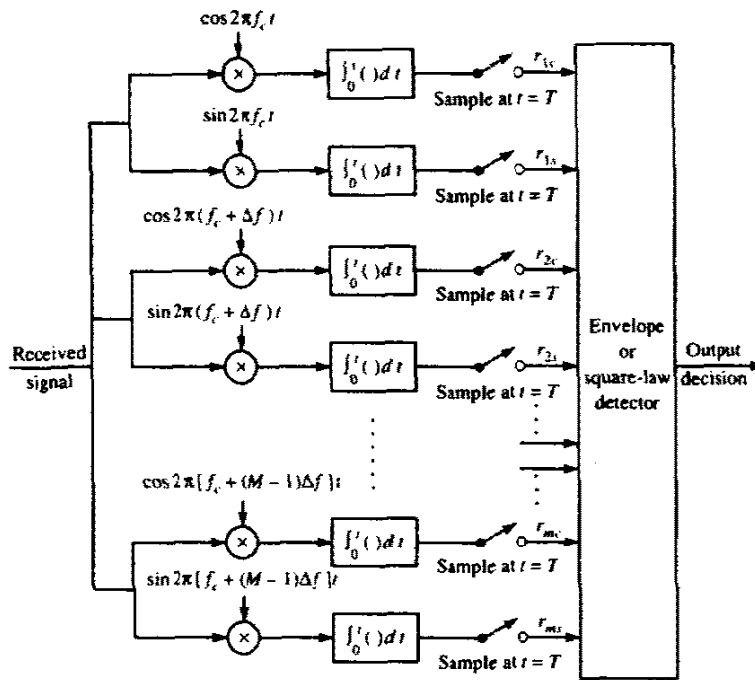


FIGURE 5-4-4 Demodulation of M -ary FSK signals for noncoherent detection.

assume that the M signals are equally probable a priori and that the signal $s_1(t)$ is transmitted in the signal interval $0 \leq t \leq T$.

The M decision metrics at the detector are the M envelopes

$$|r_m| = \sqrt{r_{mc}^2 + r_{ms}^2}, \quad m = 1, 2, \dots, M \quad (5-4-32)$$

where

$$\begin{aligned} r_{1c} &= \sqrt{\mathcal{E}_s} \cos \phi_1 + n_{1c} \\ r_{1s} &= \sqrt{\mathcal{E}_s} \sin \phi_1 + n_{1s} \end{aligned} \quad (5-4-33)$$

and

$$r_{mc} = n_{mc}, \quad r_{ms} = n_{ms}, \quad m = 2, 3, \dots, M \quad (5-4-34)$$

The additive noise components $\{n_{mc}\}$ and $\{n_{ms}\}$ are mutually statistically independent zero-mean gaussian variables with equal variance $\sigma^2 = \frac{1}{2}N_0$. Thus the pdfs of the random variables at the input to the detector are

$$P_{r_1}(r_{1c}, r_{1s}) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{r_{1c}^2 + r_{1s}^2 + \mathcal{E}_s}{2\sigma^2}\right) I_0\left(\frac{\sqrt{\mathcal{E}_s}(r_{1c}^2 + r_{1s}^2)}{\sigma^2}\right) \quad (5-4-35)$$

$$P_{r_m}(r_{mc}, r_{ms}) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{r_{mc}^2 + r_{ms}^2}{2\sigma^2}\right), \quad m = 2, 3, \dots, M \quad (5-4-36)$$

Let us make a change in variables in the joint pdfs given by (5-4-35) and (5-4-36). We define the normalized variables

$$R_m = \frac{\sqrt{r_{mc}^2 + r_{ms}^2}}{\sigma} \quad (5-4-37)$$

$$\Theta_m = \tan^{-1} \frac{r_{ms}}{r_{mc}}$$

Clearly, $r_{mc} = \sigma R_m \cos \Theta_m$ and $r_{ms} = \sigma R_m \sin \Theta_m$. The Jacobian of this transformation is

$$|J| = \begin{vmatrix} \sigma \cos \Theta_m & \sigma \sin \Theta_m \\ -\sigma R_m \sin \Theta_m & \sigma R_m \cos \Theta_m \end{vmatrix} = \sigma^2 R_m \quad (5-4-38)$$

Consequently,

$$p(R_1, \Theta_1) = \frac{R_1}{2\pi} \exp \left[-\frac{1}{2} \left(R_1^2 + \frac{2\mathcal{E}_s}{N_0} \right) \right] I_0 \left(\sqrt{\frac{2\mathcal{E}_s}{N_0}} R_1 \right) \quad (5-4-39)$$

$$p(R_m, \Theta_m) = \frac{R_m}{2\pi} \exp \left(-\frac{1}{2} R_m^2 \right), \quad m = 2, 3, \dots, M \quad (5-4-40)$$

Finally, by averaging $p(R_m, \Theta_m)$ over Θ_m , the factor of 2π is eliminated from (5-4-39) and (5-4-40). Thus, we find that R_1 has a Rice probability distribution and R_m , $m = 2, 3, \dots, M$, are each Rayleigh-distributed.

The probability of a correct decision is simply the probability that $R_1 > R_2$, and $R_1 > R_3, \dots$, and $R_1 > R_M$. Hence,

$$P_c = P(R_2 < R_1, R_3 < R_1, \dots, R_M < R_1)$$

$$= \int_0^\infty P(R_2 < R_1, R_3 < R_1, \dots, R_M < R_1 | R_1 = x) p_{R_1}(x) dx \quad (5-4-41)$$

Because the random variables R_m , $m = 2, 3, \dots, M$, are statistically iid, the joint probability in (5-4-41) conditioned on R_1 factors into a product of $M - 1$ identical terms. Thus,

$$P_c = \int_0^\infty [P(R_2 < R_1 | R_1 = x)]^{M-1} p_{R_1}(x) dx \quad (5-4-42)$$

where

$$P(R_2 < R_1 | R_1 = x) = \int_0^x p_{R_2}(r_2) dr_2$$

$$= 1 - e^{-x^2/2} \quad (5-4-43)$$

The $(M - 1)$ th power of (5-4-43) may be expressed as

$$(1 - e^{-x^2/2})^{M-1} = \sum_{n=0}^{M-1} (-1)^n \binom{M-1}{n} e^{-nx^2/2} \quad (5-4-44)$$

Substitution of this result into (5-4-42) and integration over x yields the probability of a correct decision as

$$P_C = \sum_{n=0}^{M-1} (-1)^n \binom{M-1}{n} \frac{1}{n+1} \exp \left[-\frac{n\mathcal{E}_s}{(n+1)N_0} \right] \quad (5-4-45)$$

where \mathcal{E}_s/N_0 is the SNR per symbol. Then, the probability of a symbol error, which is $P_M = 1 - P_C$, becomes

$$P_M = \sum_{n=1}^{M-1} (-1)^{n+1} \binom{M-1}{n} \frac{1}{n+1} \exp \left[-\frac{nk\mathcal{E}_b}{(n+1)N_0} \right] \quad (5-4-46)$$

where \mathcal{E}_b/N_0 is the SNR per bit.

For binary orthogonal signals ($M = 2$), (5-4-46) reduces to the simple form

$$P_2 = \frac{1}{2} e^{-\mathcal{E}_b/2N_0} \quad (5-4-47)$$

For $M > 2$, we may compute the probability of a bit error by making use of the relationship

$$P_b = \frac{2^{k-1}}{2^k - 1} P_M \quad (5-4-48)$$

which was established in Section 5-2. Figure 5-4-5 shows the bit-error probability as a function of the SNR per bit γ_b for $M = 2, 4, 8, 16$, and 32. Just as in the case of coherent detection of M -ary orthogonal signals (see Section 5-2-2), we observe that for any given bit-error probability, the SNR per bit decreases as M increases. It will be shown in Chapter 7 that, in the limit as $M \rightarrow \infty$ (or $k = \log_2 M \rightarrow \infty$), the probability of a bit error P_b can be made

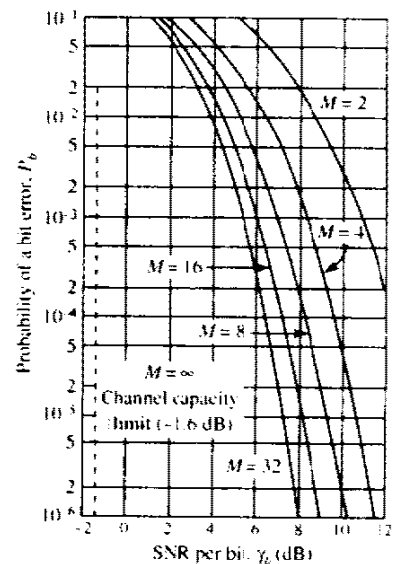


FIGURE 5-4-5 Probability of a bit error for noncoherent detection of orthogonal signals.

arbitrarily small provided that the SNR per bit is greater than the Shannon limit of -1.6 dB. The cost for increasing M is the bandwidth required to transmit the signals. For M -ary FSK, the frequency separation between adjacent frequencies is $\Delta f = 1/T$ for signal orthogonality. The bandwidth required for the M signals is $W = M \Delta f = M/T$. Also, the bit rate is $R = k/T$, where $k = \log_2 M$. Therefore, the bit-rate-to-bandwidth ratio is

$$\frac{R}{W} = \frac{\log_2 M}{M} \quad (5-4-49)$$

5-4-4 Probability of Error for Envelope Detection of Correlated Binary Signals

In this section, we consider the performance of the envelope detector for binary, equal-energy correlated signals. When the two signals are correlated, the input to the detector are the complex-valued random variables given by (5-4-10). We assume that the detector bases its decision on the envelopes $|r_1|$ and $|r_2|$, which are correlated (statistically dependent). The marginal pdfs of $R_1 = |r_1|$ and $R_2 = |r_2|$ are Ricean distributed, and may be expressed as

$$p(R_m) = \begin{cases} \frac{R_m}{2\mathcal{E}N_0} \exp\left(-\frac{R_m^2 + \beta_m^2}{4\mathcal{E}N_0}\right) I_0\left(\frac{\beta_m R_m}{2\mathcal{E}N_0}\right) & (R_m > 0) \\ 0 & (R_m < 0) \end{cases} \quad (5-4-50)$$

$m = 1, 2$, where $\beta_1 = 2\mathcal{E}$ and $\beta_2 = 2\mathcal{E}|\rho|$, based on the assumption that signal $s_1(t)$ was transmitted.

Since R_1 and R_2 are statistically dependent as a consequence of the nonorthogonality of the signals, the probability of error may be obtained by evaluating the double integral

$$P_b = P(R_2 > R_1) = \int_0^\infty \int_{x_1}^\infty p(x_1, x_2) dx_1 dx_2 \quad (5-4-51)$$

where $p(x_1, x_2)$ is the joint pdf of the envelopes R_1 and R_2 . This approach was first used by Helstrom (1955), who determined the joint pdf of R_1 and R_2 and evaluated the double integral in (5-4-51).

An alternative approach is based on the observation that the probability of error may also be expressed as

$$P_b = P(R_2 > R_1) = P(R_2^2 > R_1^2) = P(R_2^2 - R_1^2 > 0) \quad (5-4-52)$$

But $R_2^2 - R_1^2$ is a special case of a general quadratic form in complex-valued gaussian random variables, treated later in Appendix B. For the special case under consideration, the derivation yields the error probability in the form

$$P_b = Q_1(a, b) - \frac{1}{2} e^{-(a^2 + b^2)/2} I_0(ab) \quad (5-4-53)$$

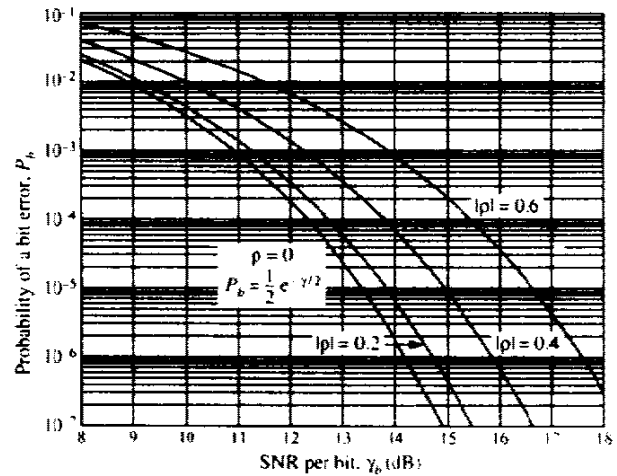


FIGURE 5-4-6 Probability of error for noncoherent detection.

where

$$a = \sqrt{\frac{\mathcal{E}_b}{2N_0} (1 - \sqrt{1 - |\rho|^2})} \quad (5-4-54)$$

$$b = \sqrt{\frac{\mathcal{E}_b}{2N_0} (1 + \sqrt{1 - |\rho|^2})}$$

$Q_1(a, b)$ is the Q function defined in (2-1-123) and $I_0(x)$ is the modified Bessel function of order zero.

The error probability P_b is illustrated in Fig. 5-4-6 for several values of $|\rho|$. P_b is minimized when $\rho = 0$; that is, when the signals are orthogonal. For this case, $a = 0$, $b = \sqrt{\mathcal{E}_b/N_0}$, and (5-4-53) reduces to

$$P_b = Q\left(0, \sqrt{\frac{\mathcal{E}_b}{N_0}}\right) = \frac{1}{2} e^{-\mathcal{E}_b/2N_0} \quad (5-4-55)$$

From the definition of $Q_1(a, b)$ in (2-1-123), it follows that

$$Q_1\left(0, \sqrt{\frac{\mathcal{E}_b}{N_0}}\right) = e^{-\mathcal{E}_b/2N_0}$$

Substitution of these relations into (5-4-55) yields the desired result given previously in (5-4-47). On the other hand, when $|\rho| = 1$, the error probability in (5-4-53) becomes $P_b = \frac{1}{2}$, as expected.

5-5 REGENERATIVE REPEATERS AND LINK BUDGET ANALYSIS

In the transmission of digital signals through an AWGN channel, we have observed that the performance of the communication system, measured in terms of the probability of error, depends solely on the received SNR, \mathcal{E}_b/N_0 .

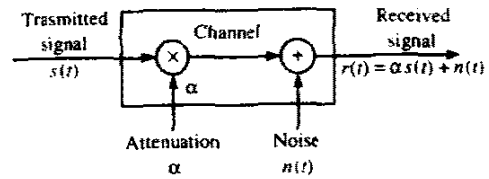


FIGURE 5-5-1 Mathematical model of channel with attenuation and additive noise.

where \mathcal{E}_b is the transmitted energy per bit and $\frac{1}{2}N_0$ is the power spectral density of the additive noise. Hence, the additive noise ultimately limits the performance of the communication system.

In addition to the additive noise, another factor that affects the performance of a communication system is channel attenuation. All physical channels, including wire lines and radio channels, are lossy. Hence, the signal is attenuated as it travels through the channel. The simple mathematical model for the attenuation shown in Fig. 5-5-1 may be used for the channel. Consequently, if the transmitted signal is $s(t)$, the received signal, with $0 < \alpha \leq 1$ is

$$r(t) = \alpha s(t) + n(t) \quad (5-5-1)$$

Then, if the energy in the transmitted signal is \mathcal{E}_b , the energy in the received signal is $\alpha^2 \mathcal{E}_b$. Consequently, the received signal has an SNR $\alpha^2 \mathcal{E}_b / N_0$. Hence, the effect of signal attenuation is to reduce the energy in the received signal and thus to render the communication system more vulnerable to additive noise.

In analog communication systems, amplifiers called repeaters are used to periodically boost the signal strength in transmission through the channel. However, each amplifier also boosts the noise in the system. In contrast, digital communication systems allow us to detect and regenerate a clean (noise-free) signal in a transmission channel. Such devices, called *regenerative repeaters*, are frequently used in wireline and fiber optic communication channels.

5-5-1 Regenerative Repeaters

The front end of each regenerative repeater consists of a demodulator/detector that demodulates and detects the transmitted digital information sequence sent by the preceding repeater. Once detected, the sequence is passed to the transmitter side of the repeater, which maps the sequence into signal waveforms that are transmitted to the next repeater. This type of repeater is called a regenerative repeater.

Since a noise-free signal is regenerated at each repeater, the additive noise does not accumulate. However, when errors occur in the detector of a repeater, the errors are propagated forward to the following repeaters in the channel. To evaluate the effect of errors on the performance of the overall system, suppose that the modulation is binary PAM, so that the probability of

a bit error for one hop (signal transmission from one repeater to the next repeater in the chain) is

$$P_b = Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right)$$

Since errors occur with low probability, we may ignore the probability that any one bit will be detected incorrectly more than once in transmission through a channel with K repeaters. Consequently, the number of errors will increase linearly with the number of regenerative repeaters in the channel, and therefore, the overall probability of error may be approximated as

$$P_b \approx KQ\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right) \quad (5-5-2)$$

In contrast, the use of K analog repeaters in the channel reduces the received SNR by K , and hence, the bit error probability is

$$P_b \approx Q\left(\sqrt{\frac{2\mathcal{E}_b}{KN_0}}\right) \quad (5-5-3)$$

Clearly, for the same probability of error performance, the use of regenerative repeaters results in a significant saving in transmitter power compared with analog repeaters. Hence, in digital communication systems, regenerative repeaters are preferable. However, in wireline telephone channels that are used to transmit both analog and digital signals, analog repeaters are generally employed.

Example 5-5-1

A binary digital communication system transmits data over a wireline channel of length 1000 km. Repeaters are used every 10 km to offset the effect of channel attenuation. Let us determine the \mathcal{E}_b/N_0 that is required to achieve a probability of a bit error of 10^{-5} if (a) analog repeaters are employed, and (b) regenerative repeaters are employed.

The number of repeaters used in the system is $K = 100$. If regenerative repeaters are used, the \mathcal{E}_b/N_0 obtained from (5-5-2) is

$$10^{-5} = 100Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right)$$

$$10^{-7} = Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right)$$

which yields approximately 11.3 dB. If analog repeaters are used, the \mathcal{E}_b/N_0 obtained from (5-5-3) is

$$10^{-5} = Q\left(\sqrt{\frac{2\mathcal{E}_b}{100N_0}}\right)$$

which yields $\mathcal{E}_b/N_0 \approx 29.6$ dB. Hence, the difference in the required SNR is

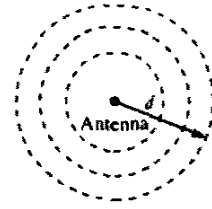


FIGURE 5-5-2 Isotropically radiating antenna.

about 18.3 dB, or approximately 70 times the transmitter power of the digital communication system.

5-5-2 Communication Link Budget Analysis

In the design of radio communications systems that transmit over line-of-sight microwave channels and satellite channels, the system designer must specify the size of the transmit and receive antennas, the transmitted power, and the SNR required to achieve a given level of performance at some desired data rate. The system design procedure is relatively straightforward and is outlined below.

Let us begin with a transmit antenna that radiates isotropically in free space at a power level of P_T watts as shown in Fig. 5-5-2. The power density at a distance d from the antenna is $P_T/4\pi d^2$ W/m². If the transmitting antenna has some directivity in a particular direction, the power density in that direction is increased by a factor called the antenna gain and denoted by G_T . In such a case, the power density at distance d is $P_T G_T/4\pi d^2$ W/m². The product $P_T G_T$ is usually called the *effective radiated power* (ERP or EIRP), which is basically the radiated power relative to an isotropic antenna, for which $G_T = 1$.

A receiving antenna pointed in the direction of the radiated power gathers a portion of the power that is proportional to its cross-sectional area. Hence, the received power extracted by the antenna may be expressed as

$$P_R = \frac{P_T G_T A_R}{4\pi d^2} \quad (5-5-4)$$

where A_R is the *effective area of the antenna*. From electromagnetic field theory, we obtain the basic relationship between the gain G_R of an antenna and its effective area as

$$A_R = \frac{G_R \lambda^2}{4\pi} \text{ m}^2 \quad (5-5-5)$$

where $\lambda = c/f$ is the wavelength of the transmitted signal, c is the speed of light (3×10^8 m/s), and f is the frequency of the transmitted signal.

If we substitute (5-5-5) for A_R into (5-5-4), we obtain an expression for the received power in the form

$$P_R = \frac{P_T G_T G_R}{(4\pi d/\lambda)^2} \quad (5-5-6)$$

The factor

$$L_s = \left(\frac{\lambda}{4\pi d}\right)^2 \tag{5-5-7}$$

is called the *free-space path loss*. If other losses, such as atmospheric losses, are encountered in the transmission of the signal, they may be accounted for by introducing an additional loss factor, say L_a . Therefore, the received power may be written in general as

$$P_R = P_T G_T G_R L_s L_a \tag{5-5-8}$$

As indicated above, the important characteristics of an antenna are its gain and its effective area. These generally depend on the wavelength of the radiated power and the physical dimensions of the antenna. For example, a parabolic (dish) antenna of diameter D has an effective area

$$A_R = \frac{1}{4}\pi D^2 \eta \tag{5-5-9}$$

where $\frac{1}{4}\pi D^2$ is the physical area and η is the *illumination efficiency factor*, which falls in the range $0.5 \leq \eta \leq 0.6$. Hence, the antenna gain for a parabolic antenna of diameter D is

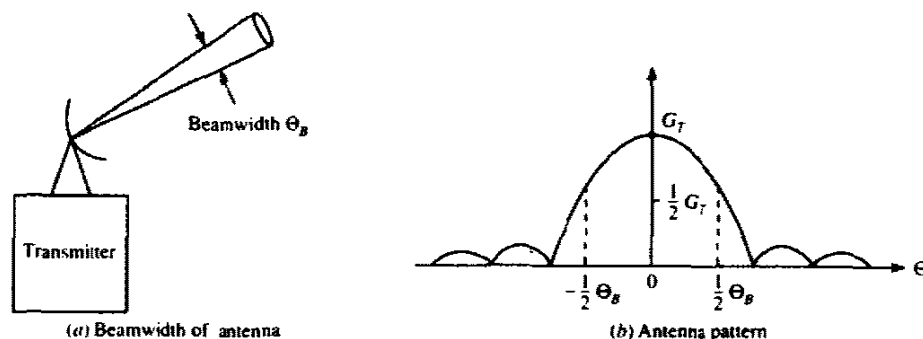
$$G_R = \eta \left(\frac{\pi D}{\lambda}\right)^2 \tag{5-5-10}$$

As a second example, a horn antenna of physical area A has an efficiency factor of 0.8, an effective area of $A_R = 0.8A$, and an antenna gain of

$$G_R = \frac{10A}{\lambda^2} \tag{5-5-11}$$

Another parameter that is related to the gain (directivity) of an antenna is its beamwidth, which we denote as Θ_B and which is illustrated graphically in Fig. 5-5-3. Usually, the beamwidth is measured as the -3 dB width of the

FIGURE 5-5-3 Antenna beamwidth and pattern.



antenna pattern. For example, the -3 dB beamwidth of a parabolic antenna is approximately

$$\Theta_B = 70(\lambda/D)^\circ \quad (5-5-12)$$

so that G_T is inversely proportional to Θ_B^2 . That is, a decrease of the beamwidth by a factor of two, which is obtained by doubling the diameter D , increases the antenna gain by a factor of four (6 dB).

Based on the general relationship for the received signal power given by (5-5-8), the system designer can compute P_R from a specification of the antenna gains and the distance between the transmitter and the receiver. Such computations are usually done on a power basis, so that

$$(P_R)_{dB} = (P_T)_{dB} + (G_T)_{dB} + (G_R)_{dB} + (L_s)_{dB} + (L_a)_{dB} \quad (5-5-13)$$

Example 5-5-2

Suppose that we have a satellite in geosynchronous orbit (36 000 km above the earth's surface) that radiates 100 W of power, i.e., 20 dB above 1 W (20 dBW). The transmit antenna has a gain of 17 dB, so that the ERP = 37 dBW. Also, suppose that the earth station employs a 3 m parabolic antenna and that the downlink is operating at a frequency of 4 GHz. The efficiency factor is $\eta = 0.5$. By substituting these numbers into (5-5-10), we obtain the value of the antenna gain as 39 dB. The free-space path loss is

$$L_s = 195.6 \text{ dB}$$

No other losses are assumed. Therefore, the received signal power is

$$\begin{aligned} (P_R)_{dB} &= 20 + 17 + 39 - 195.6 \\ &= -119.6 \text{ dBW} \end{aligned}$$

or, equivalently,

$$P_R = 1.1 \times 10^{-12} \text{ W}$$

To complete the link budget computation, we must also consider the effect of the additive noise at the receiver front end. Thermal noise that arises at the receiver front end has a relatively flat power density spectrum up to about 10^{12} Hz, and is given as

$$N_0 = k_B T_0 \text{ W/Hz} \quad (5-5-14)$$

where k_B is Boltzmann's constant (1.38×10^{-23} W s/K) and T_0 is the noise temperature in Kelvin. Therefore, the total noise power in the signal bandwidth W is $N_0 W$.

The performance of the digital communications system is specified by the \mathcal{E}_b/N_0 required to keep the error rate performance below some given value. Since

$$\frac{\mathcal{E}_b}{N_0} = \frac{T_b P_R}{N_0} = \frac{1}{R} \frac{P_R}{N_0} \quad (5-5-15)$$

it follows that

$$\frac{P_R}{N_0} = R \left(\frac{\mathcal{E}_b}{N_0} \right)_{\text{req}} \quad (5-5-16)$$

where $(\mathcal{E}_b/N_0)_{\text{req}}$ is the required SNR per bit. Hence, if we have P_R/N_0 and the required SNR per bit, we can determine the maximum data rate that is possible.

Example 5-5-3

For the link considered in Example 5-5-2, the received signal power is

$$P_R = 1.1 \times 10^{-12} \text{ W} \quad (-119.6 \text{ dBW})$$

Now, suppose the receiver front end has a noise temperature of 300 K, which is typical for receiver in the 4 GHz range. Then

$$N_0 = 4.1 \times 10^{-21} \text{ W/Hz}$$

or, equivalently, -203.9 dBW/Hz . Therefore,

$$\frac{P_R}{N_0} = -119.6 + 203.9 = 84.3 \text{ dB Hz}$$

If the required SNR per bit is 10 dB then, from (5-5-16), we have the available rate as

$$\begin{aligned} R_{\text{dB}} &= 84.3 - 10 \\ &= 74.3 \text{ dB} \quad (\text{with respect to 1 bit/s}) \end{aligned}$$

This corresponds to a rate of 26.9 megabits/s, which is equivalent to about 420 PCM channels, each operating at 64 000 bits/s.

It is a good idea to introduce some safety margin, which we shall call the *link margin* M_{dB} , in the above computations for the capacity of the communication link. Typically, this may be selected as $M_{\text{dB}} = 6 \text{ dB}$. Then, the link budget computation for the link capacity may be expressed in the simple form

$$\begin{aligned} R_{\text{dB}} &= \left(\frac{P_R}{N_0} \right)_{\text{dB Hz}} - \left(\frac{\mathcal{E}_b}{N_0} \right)_{\text{req}} - M_{\text{dB}} \\ &= (P_T)_{\text{dBW}} + (G_T)_{\text{dB}} + (G_R)_{\text{dB}} \\ &\quad + (L_a)_{\text{dB}} + (L_s)_{\text{dB}} - \left(\frac{\mathcal{E}_b}{N_0} \right)_{\text{req}} - M_{\text{dB}} \end{aligned} \quad (5-5-17)$$

BIBLIOGRAPHICAL NOTES AND REFERENCES

In the derivation of the optimum demodulator for a signal corrupted by AWGN, we applied mathematical techniques that were originally used in deriving optimum receiver structures for radar signals. For example, the

matched filter was first proposed by North (1943) for use in radar detection, and is sometimes called the North filter. An alternative method for deriving the optimum demodulator and detector is the Karhunen–Loeve expansion, which is described in the classical texts by Davenport and Root (1958), Helstrom (1968), and Van Trees (1968). Its use in radar detection theory is described in the paper by Kelly *et al.* (1960). These detection methods are based on the hypothesis testing methods developed by statisticians, e.g., Neyman and Pearson (1933) and Wald (1947).

The geometric approach to signal design and detection, which was presented in the context of digital modulation and which has its roots in Shannon's original work, is conceptually appealing and is now widely used since its introduction in the text by Wozencraft and Jacobs (1965).

Design and analysis of signal constellations for the AWGN channel have received considerable attention in the technical literature. Of particular significance is the performance analysis of two-dimensional (QAM) signal constellations that has been treated in the papers of Cahn (1960), Hancock and Lucky (1960), Campopiano and Glazer (1962), Lucky and Hancock (1962), Salz *et al.* (1971), Simon and Smith (1973), Thomas *et al.* (1974), and Foschini *et al.* (1974). Signal design based on multidimensional signal constellations has been described and analyzed in the paper by Gersho and Lawrence (1984).

The Viterbi algorithm was devised by Viterbi (1967) for the purpose of decoding convolutional codes. Its use as the optimal maximum-likelihood sequence detection algorithm for signals with memory was described by Forney (1972) and Omura (1971). Its use for carrier modulated signals was considered by Ungerboeck (1974) and MacKenzie (1973). It was subsequently applied to the demodulation of CPM by Aulin and Sundberg (1981a, b) and others.

PROBLEMS

5-1 A matched filter has the frequency response

$$H(f) = \frac{1 - e^{-j2\pi fT}}{j2\pi f}$$

- a Determine the impulse response $h(t)$ corresponding to $H(f)$.
- b Determine the signal waveform to which the filter characteristic is matched.

5-2 Consider the signal

$$s(t) = \begin{cases} (A/T)t \cos 2\pi f_c t & (0 \leq t \leq T) \\ 0 & (\text{otherwise}) \end{cases}$$

- a Determine the impulse response of the matched filter for the signal.
- b Determine the output of the matched filter at $t = T$.
- c Suppose the signal $s(t)$ is passed through a correlator that correlates the input $s(t)$ with $s(t)$. Determine the value of the correlator output at $t = T$. Compare your result with that in (b).

5-3 This problem deals with the characteristics of a DPSK signal.

a Suppose we wish to transmit the data sequence

$$1\ 1\ 0\ 1\ 0\ 0\ 0\ 1\ 0\ 1\ 1\ 0$$

by binary DPSK. Let $s(t) = A \cos(2\pi f_c t + \theta)$ represent the transmitted signal in any signaling interval of duration T . Give the phase of the transmitted signal for the data sequence. Begin with $\theta = 0$ for the phase of the first bit to be transmitted.

b If the data sequence is uncorrelated, determine and sketch the power density spectrum of the signal transmitted by DPSK.

5-4 A binary digital communication system employs the signals

$$s_0(t) = 0, \quad 0 \leq t \leq T$$

$$s_1(t) = A, \quad 0 \leq t \leq T$$

for transmitting the information. This is called *on-off signaling*. The demodulator cross-correlates the received signal $r(t)$ with $s(t)$ and samples the output of the correlator at $t = T$.

a Determine the optimum detector for an AWGN channel and the optimum threshold, assuming that the signals are equally probable.

b Determine the probability of error as a function of the SNR. How does on-off signaling compare with antipodal signaling?

5-5 The correlation metrics given by (5-1-44) are

$$C(\mathbf{r}, \mathbf{s}_m) = 2 \sum_{n=1}^N r_n s_{mn} - \sum_{n=1}^N s_{mn}^2, \quad m = 1, 2, \dots, M$$

where

$$r_n = \int_0^T r(t) f_n(t) dt$$

$$s_{mn} = \int_0^T s_m(t) f_n(t) dt$$

Show that the correlation metrics are equivalent to the metrics

$$C(\mathbf{r}, \mathbf{s}_m) = 2 \int_0^T r(t) s_m(t) dt - \int_0^T s_m^2(t) dt$$

5-6 Consider the equivalent lowpass (complex-valued) signal $s_i(t)$, $0 \leq t \leq T$, with energy

$$\mathcal{E} = \frac{1}{2} \int_0^T |s_i(t)|^2 dt$$

Suppose that this signal is corrupted by AWGN, which is represented by its equivalent lowpass form $z(t)$. Hence, the observed signal is

$$r_i(t) = s_i(t) + z(t), \quad 0 \leq t \leq T$$

The received signal is passed through a filter that has an (equivalent lowpass) impulse response $h_i(t)$. Determine $h_i(t)$ so that the filter maximizes the SNR at its output (at $t = T$).

5-7 Let $z(t) = x(t) + jy(t)$ be a complex-valued, zero-mean white gaussian noise

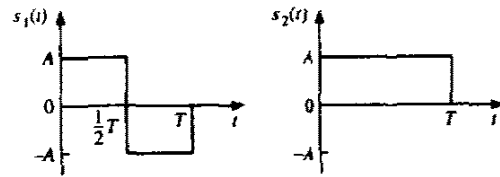


FIGURE P5-8

process with autocorrelation function $\phi_{zz}(\tau) = N_0\delta(\tau)$. Let $f_m(t)$, $m = 1, 2, \dots, M$, be a set of M orthogonal equivalent lowpass waveforms defined on the interval $0 \leq t \leq T$. Define

$$N_{mr} = \text{Re} \left[\int_0^T z(t) f_m^*(t) dt \right], \quad m = 1, 2, \dots, M$$

- a Determine the variance of N_{mr} .
 - b Show that $E(N_{mr}N_{kr}) = 0$ for $k \neq m$.
- 5-8 The two equivalent lowpass signals shown in Fig. P5-8 are used to transmit a binary sequence over an additive white gaussian noise channel. The received signal can be expressed as

$$r_i(t) = s_i(t) + z(t), \quad 0 \leq t \leq T, \quad i = 1, 2$$

where $z(t)$ is a zero-mean gaussian noise process with autocorrelation function

$$\phi_{zz}(\tau) = \frac{1}{2}E[z^*(t)z(t+\tau)] = N_0\delta(\tau)$$

- a Determine the transmitted energy in $s_1(t)$ and $s_2(t)$ and the cross-correlation coefficient ρ_{12} .
 - b Suppose the receiver is implemented by means of coherent detection using two matched filters, one matched to $s_1(t)$ and the other to $s_2(t)$. Sketch the equivalent lowpass impulse responses of the matched filters.
 - c Sketch the noise-free response of the two matched filters when the transmitted signal is $s_2(t)$.
 - d Suppose the receiver is implemented by means of two cross-correlators (multipliers followed by integrators) in parallel. Sketch the output of each integrator as a function of time for the interval $0 \leq t \leq T$ when the transmitted signal is $s_2(t)$.
 - e Compare the sketches in (c) and (d). Are they the same? Explain briefly.
 - f From your knowledge of the signal characteristics, give the probability of error for this binary communications system.
- 5-9 Suppose that we have a complex-valued gaussian random variable $z = x + jy$, where (x, y) are statistically independent variables with zero mean and variance $E(x^2) = E(y^2) = \sigma^2$. Let

$$r = z + m, \quad \text{where } m = m_r + jm_i$$

and define r as

$$r = a + jb$$

Clearly, $a = x + m_r$, and $b = y + m_i$. Determine the following probability density functions:

- a $p(a, b)$;

b $p(u, \phi)$, where $u = \sqrt{a^2 + b^2}$ and $\phi = \tan^{-1} b/a$;

c $p(u)$.

Note: In (b) it is convenient to define $\theta = \tan^{-1} (m_2/m_1)$ so that

$$m_1 = \sqrt{m_1^2 + m_2^2} \cos \theta, \quad m_2 = \sqrt{m_1^2 + m_2^2} \sin \theta.$$

Furthermore, you must use the relation

$$\frac{1}{2\pi} \int_0^{2\pi} e^{\alpha \cos(\phi - \theta)} d\phi = I_0(\alpha) = \sum_{n=0}^{\infty} \frac{\alpha^{2n}}{2^{2n}(n!)^2}$$

where $I_0(\alpha)$ is the modified Bessel function of order zero.

- 5-10** A ternary communication system transmits one of three signals, $s(t)$, 0, or $-s(t)$, every T seconds. The received signal is either $r_1(t) = s(t) + z(t)$, $r_2(t) = z(t)$, or $r_3(t) = -s(t) + z(t)$, where $z(t)$ is white gaussian noise with $E[z(t)] = 0$ and $\phi_{zz}(\tau) = \frac{1}{2}E[z(t)z^*(t+\tau)] = N_0\delta(t-\tau)$. The optimum receiver computes the correlation metric

$$U = \text{Re} \left[\int_0^T r(t)s^*(t) dt \right]$$

and compares U with a threshold A and a threshold $-A$. If $U > A$, the decision is made that $s(t)$ was sent. If $U < -A$, the decision is made in favor of $-s(t)$. If $-A < U < A$, the decision is made in favor of 0.

- a** Determine the three conditional probabilities of error; P_e given that $s(t)$ was sent, P_e given that $-s(t)$ was sent, and P_e given that 0 was sent.
b Determine the average probability of error P_e as a function of the threshold A , assuming that the three symbols are equally probable a priori.
c Determine the value of A that minimizes P_e .
- 5-11** The two equivalent lowpass signals shown in Fig. P5-11 are used to transmit a binary information sequence. The transmitted signals, which are equally probable, are corrupted by additive zero-mean white gaussian noise having an equivalent lowpass representation $z(t)$ with an autocorrelation function

$$\begin{aligned} \phi_{zz}(\tau) &= \frac{1}{2}E[z^*(t)z(t+\tau)] \\ &= N_0\delta(\tau) \end{aligned}$$

- a** What is the transmitted signal energy?
b What is the probability of a binary digit error if coherent detection is employed at the receiver?
c What is the probability of a binary digit error if noncoherent detection is employed at the receiver?
- 5-12** In Section 4-3-1 it was shown that the minimum frequency separation for orthogonality of binary FSK signals with coherent detection is $\Delta f = 1/2T$.

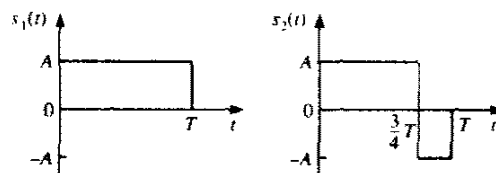


FIGURE P5-11

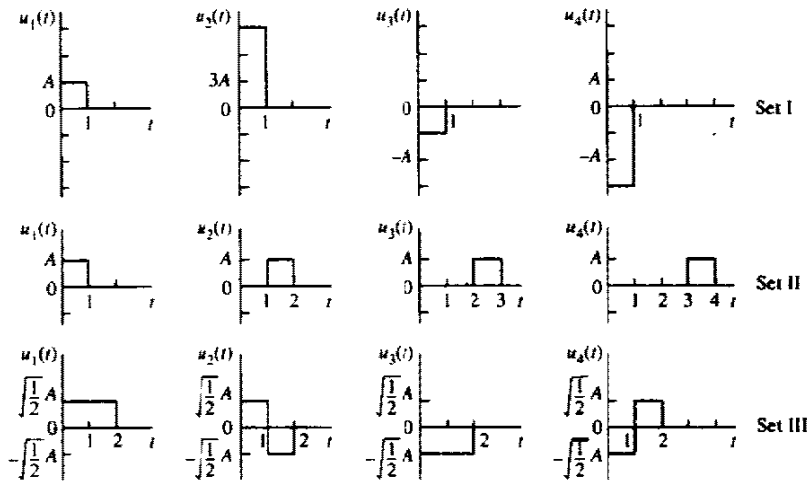


FIGURE P5-13

However, a lower error probability is possible with coherent detection of FSK if Δf is increased beyond $1/2T$. Show that the optimum value of Δf is $0.715/T$ and determine the probability of error for this value of Δf .

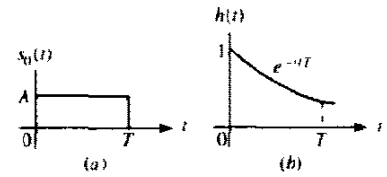
- 5-13** The equivalent lowpass waveforms for three signal sets are shown in Fig. P5-13. Each set may be used to transmit one of four equally probable messages over an additive white gaussian noise channel. The equivalent lowpass noise $z(t)$ has zero mean and autocorrelation function $\phi_{zz}(\tau) = N_0\delta(\tau)$.
- Classify the signal waveforms in sets I, II, and III. In other words, state the category or class to which each signal set belongs.
 - What is the *average* transmitted energy for each signal set?
 - For signal set I, specify the average probability of error if the signals are detected coherently.
 - For signal set II, give a union bound on the probability of a symbol error if the detection is performed (i) coherently and (ii) noncoherently.
 - Is it possible to use noncoherent detection on signal set III? Explain.
 - Which signal set or signal sets would you select if you wished to achieve a ratio of bit rate to bandwidth (R/W) of at least 2. *Briefly* explain your answer.

- 5-14** Consider a quaternary ($M=4$) communication system that transmits, every T seconds, one of four equally probable signals: $s_1(t)$, $-s_1(t)$, $s_2(t)$, $-s_2(t)$. The signals $s_1(t)$ and $s_2(t)$ are orthogonal with equal energy. The additive noise is white gaussian with zero mean and autocorrelation function $\phi_{zz}(\tau) = N_0\delta(\tau)$. The demodulator consists of two filters matched to $s_1(t)$ and $s_2(t)$, and their outputs at the sampling instant are U_1 and U_2 . The detector bases its decision on the following rule:

$$\begin{aligned}
 U_1 > |U_2| &\Rightarrow s_1(t), & U_1 < -|U_2| &\Rightarrow -s_1(t) \\
 U_2 > |U_1| &\Rightarrow s_2(t), & U_2 < -|U_1| &\Rightarrow -s_2(t)
 \end{aligned}$$

Since the signal set is biorthogonal, the error probability is given by $(1 - P_c)$ where P_c is given by (5-2-34). Express this error probability in terms of a single integral

FIGURE P5-15



and, thus, show that the symbol error probability for a biorthogonal signal set with $M = 4$ is identical to that for four-phase PSK. *Hint:* A change in variables from U_1 and U_2 to $W_1 = U_1 + U_2$ and $W_2 = U_1 - U_2$ simplifies the problem.

5-15 The input $s(t)$ to a bandpass filter is

$$s(t) = \text{Re} [s_0(t)e^{j2\pi f_c t}]$$

where $s_0(t)$ is a rectangular pulse as shown in Fig. P5-15(a).

a Determine the output $y(t)$ of the bandpass filter for all $t \geq 0$ if the impulse response of the filter is

$$g(t) = \text{Re} [2h(t)e^{j2\pi f_c t}]$$

where $h(t)$ is an exponential as shown in Fig. P5-15(b).

b Sketch the *equivalent lowpass output* of the filter.

c When would you sample the output of the filter if you wished to have the maximum output at the sampling instant? What is the value of the maximum output?

d Suppose that in addition to the input signal $s(t)$, there is additive white gaussian noise

$$n(t) = \text{Re} [z(t)e^{j2\pi f_c t}]$$

where $\phi_{z_z}(\tau) = N_0 \delta(\tau)$. At the sampling instant determined in (c), the signal sample is corrupted by an additive gaussian noise term. Determine its mean and variance.

e What is the signal-to-noise ratio γ of the sampled output?

f Determine the signal-to-noise ratio when $h(t)$ is the matched filter to $s(t)$ and compare this result with the value of γ obtained in (e).

5-16 Consider the octal signal point constellations in Fig. P5-16.

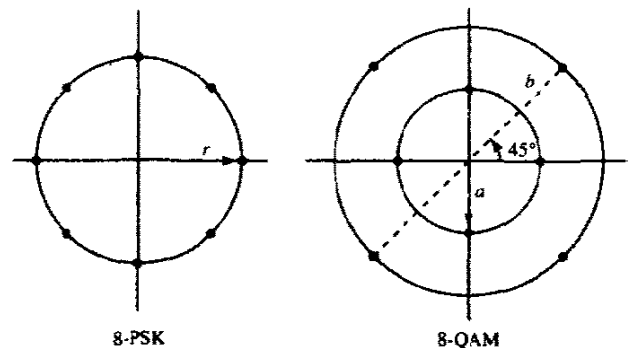


FIGURE P5-16

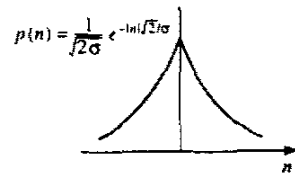


FIGURE P5-19

- a The nearest-neighbor signal points in the 8-QAM signal constellation are separated in distance by A units. Determine the radii a and b of the inner and outer circles.
- b The adjacent signal points in the 8-PSK are separated by a distance of A units. Determine the radius r of the circle.
- c Determine the average transmitter powers for the two signal constellations and compare the two powers. What is the relative power advantage of one constellation over the other? (Assume that all signal points are equally probable.)
- 5-17 Consider the 8-point QAM signal constellation shown in Fig. P5-16.
- a Is it possible to assign three data bits to each point of the signal constellation such that nearest (adjacent) points differ in only one bit position?
- b Determine the symbol rate if the desired bit rate is 90 Mbits/s.
- 5-18 Suppose that binary PSK is used for transmitting information over an AWGN with a power spectral density of $\frac{1}{2}N_0 = 10^{-10}$ W/Hz. The transmitted signal energy is $\mathcal{E}_b = \frac{1}{2}A^2T$, where T is the bit interval and A is the signal amplitude. Determine the signal amplitude required to achieve an error probability of 10^{-6} when the data rate is (a) 10 kbits/s, (b) 100 kbits/s, and (c) 1 Mbit/s.
- 5-19 Consider a signal detector with an input

$$r = \pm A + n$$

where $+A$ and $-A$ occur with equal probability and the noise variable n is characterized by the (Laplacian) pdf shown in Fig. P5-19.

- a Determine the probability of error as a function of the parameters A and σ
- b Determine the SNR required to achieve an error probability of 10^{-5} . How does the SNR compare with the result for a Gaussian pdf?
- 5-20 Consider the two 8-point QAM signal constellations shown in Fig. P5-20. The minimum distance between adjacent points is $2A$. Determine the average transmitted power for each constellation, assuming that the signal points are equally probable. Which constellation is more power-efficient?
- 5-21 For the QAM signal constellation shown in Fig. P5-21, determine the optimum decision boundaries for the detector, assuming that the SNR is sufficiently high so that errors only occur between adjacent points.

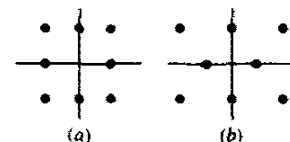


FIGURE P5-20

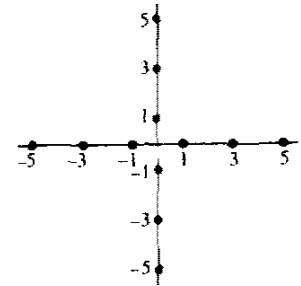


FIGURE P5-21

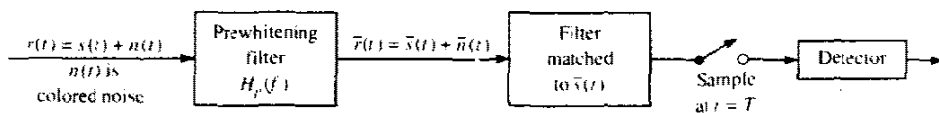
- 5-22 Specify a Gray code for the 16-QAM signal constellation shown in Fig. P5-21.
- 5-23 Two quadrature carriers $\cos 2\pi f_c t$ and $\sin 2\pi f_c t$ are used to transmit digital information through an AWGN channel at two different data rates, 10 kbits/s and 100 kbits/s. Determine the relative amplitudes of the signals for the two carriers so that the \mathcal{E}_b/N_b for the two channels is identical.
- 5-24 Three messages m_1 , m_2 , and m_3 are to be transmitted over an AWGN channel with noise power spectral density $\frac{1}{2}N_0$. The messages are

$$s_1(t) = \begin{cases} 1 & (0 \leq t \leq T) \\ 0 & (\text{otherwise}) \end{cases}$$

$$s_2(t) = -s_3(t) = \begin{cases} 1 & (0 \leq t \leq \frac{1}{2}T) \\ -1 & (\frac{1}{2}T \leq t \leq T) \\ 0 & (\text{otherwise}) \end{cases}$$

- a What is the dimensionality of the signal space?
 - b Find an appropriate basis for the signal space. [Hint: You can find the basis without using the Gram-Schmidt procedure.]
 - c Draw the signal constellation for this problem.
 - d Derive and sketch the optimal decision regions R_1 , R_2 , and R_3 .
 - e Which of the three messages is more vulnerable to errors and why? In other words, which of $P(\text{error} | m_i \text{ transmitted})$, $i = 1, 2, 3$, is larger?
- 5-25 When the additive noise at the input to the demodulator is colored, the filter matched to the signal no longer maximizes the output SNR. In such a case we may consider the use of a prefilter that “whitens” the colored noise. The prefilter is followed by a filter matched to the prefiltered signal. Towards this end, consider the configuration shown in Fig. P5-25.
- a Determine the frequency response characteristic of the prefilter that whitens the noise.

FIGURE P5-25



- b Determine the frequency response characteristic of the filter matched to $\bar{s}(t)$.
- c Consider the prefilter and the matched filter as a single "generalized matched filter." What is the frequency response characteristic of this filter?
- d Determine the SNR at the input to the detector.
- 5-26 Consider a digital communication system that transmits information via QAM over a voice-band telephone channel at a rate 2400 symbols/s. The additive noise is assumed to be white and gaussian.
- a Determine the \mathcal{E}_b/N_0 required to achieve an error probability of 10^{-5} at 4800 bits/s.
- b Repeat (a) for a rate of 9600 bits/s.
- c Repeat (a) for a rate of 19 200 bits/s.
- d What conclusions do you reach from these results?
- 5-27 Consider the four-phase and eight-phase signal constellations shown in Fig. P5-27. Determine the radii r_1 and r_2 of the circles such that the distance between two adjacent points in the two constellations is d . From this result, determine the additional transmitted energy required in the 8-PSK signal to achieve the same error probability as the four-phase signal at high SNR, where the probability of error is determined by errors in selecting adjacent points.
- 5-28 Digital information is to be transmitted by carrier modulation through an additive gaussian noise channel with a bandwidth of 100 kHz and $N_0 = 10^{-10}$ W/Hz. Determine the maximum rate that can be transmitted through the channel for four-phase PSK, binary FSK, and four-frequency orthogonal FSK, which is detected noncoherently.
- 5-29 In a MSK signal, the initial state for the phase is either 0 or π rad. Determine the terminal phase state for the following four input pairs of input data: (a) 00; (b) 01; (c) 10; (d) 11.
- 5-30 A continuous-phase FSK signal with $h = \frac{1}{2}$ is represented as

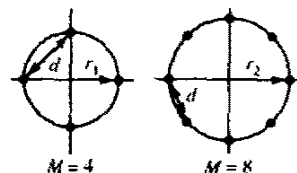
$$s(t) = \pm \sqrt{\frac{2\mathcal{E}_b}{T_b}} \cos\left(\frac{\pi t}{2T_b}\right) \cos 2\pi f_c t \pm \sqrt{\frac{2\mathcal{E}_b}{T_b}} \sin\left(\frac{\pi t}{2T_b}\right) \sin 2\pi f_c t, \quad 0 \leq t \leq 2T_b$$

where the \pm signs depend on the information bits transmitted.

- a Show that this signal has constant amplitude.
- b Sketch a block diagram of the modulator for synthesizing the signal.
- c Sketch a block diagram of the demodulator and detector for recovering the information.
- 5-31 Sketch the phase tree, the state trellis, and the state diagram for partial-response CPM with $h = \frac{1}{2}$ and

$$u(t) = \begin{cases} 1/4T & (0 \leq t \leq 2T) \\ 0 & (\text{otherwise}) \end{cases}$$

FIGURE P5-27



- 5-32 Determine the number of terminal phase states in the state trellis diagram for (a) a full response binary CPFSK with either $h = \frac{1}{2}$ or $\frac{1}{4}$ and (b) a partial-response $L = 3$ binary CPFSK with either $h = \frac{1}{2}$ or $\frac{1}{4}$.
- 5-33 Consider a biorthogonal signal set with $M = 8$ signal points. Determine a union bound for the probability of a symbol error as a function of \mathcal{E}_b/N_0 . The signal points are equally likely a priori.
- 5-34 Consider an M -ary digital communication system where $M = 2^M$, and N is the dimension of the signal space. Suppose that the M signal vectors lie on the vertices of a hypercube that is centered at the origin. Determine the average probability of a symbol error as a function of \mathcal{E}_s/N_0 where \mathcal{E}_s is the energy per symbol, $\frac{1}{2}N_0$ is the power spectral density of the AWGN, and all signal points are equally probable.
- 5-35 Consider the signal waveform

$$s(t) = \sum_{i=1}^n c_i p(t - iT_c)$$

where $p(t)$ is a rectangular pulse of unit amplitude and duration T_c . The $\{c_i\}$ may be viewed as a code vector $\mathbf{C} = [c_1 \ c_2 \ \dots \ c_n]$, where the elements $c_i = \pm 1$. Show that the filter matched to the waveform $s(t)$ may be realized as a cascade of a filter matched to $p(t)$ followed by a discrete-time filter matched to the vector \mathbf{C} . Determine the value of the output of the matched filter at the sampling instant $t = nT_c$.

- 5-36 A speech signal is sampled at a rate of 8 kHz, logarithmically compressed and encoded into a PCM format using 8 bits/sample. The PCM data is transmitted through an AWGN baseband channel via M -level PAM. Determine the bandwidth required for transmission when (a) $M = 4$, (b) $M = 8$, and (c) $M = 16$.
- 5-37 A Hadamard matrix is defined as a matrix whose elements are ± 1 and whose row vectors are pairwise orthogonal. In the case when n is a power of 2, an $n \times n$ Hadamard matrix is constructed by means of the recursion

$$\mathbf{H}_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad \mathbf{H}_{2^n} = \begin{bmatrix} \mathbf{H}_n & \mathbf{H}_n \\ \mathbf{H}_n & -\mathbf{H}_n \end{bmatrix}$$

- a Let \mathbf{C}_i denote the i th row of an $n \times n$ Hadamard matrix as defined above. Show that the waveforms constructed as

$$s_i(t) = \sum_{k=1}^n c_{ik} p(t - kT_c), \quad i = 1, 2, \dots, n$$

are orthogonal, where $p(t)$ is an arbitrary pulse confined to the time interval $0 \leq t \leq T_c$.

- b Show that the matched filters (or cross-correlators) for the n waveforms $\{s_i(t)\}$ can be realized by a single filter (or correlator) matched to the pulse $p(t)$ followed by a set of n cross-correlators using the code words $\{\mathbf{C}_i\}$.
- 5-38 The discrete sequence

$$r_k = \sqrt{\mathcal{E}_b} c_k + n_k, \quad k = 1, 2, \dots, n$$

represents the output sequence of samples from a demodulator, where $c_k = \pm 1$ are elements of one of two possible code words, $\mathbf{C}_1 = [1 \ 1 \ \dots \ 1]$ and $\mathbf{C}_2 = [1 \ 1 \ \dots \ 1 \ -1 \ \dots \ -1]$. The code word \mathbf{C}_2 has w elements that are $+1$ and $n - w$

elements that are -1 , where w is some positive integer. The noise sequence $\{n_k\}$ is white gaussian with variance σ^2 .

- a What is the optimum maximum likelihood detector for the two possible transmitted signals?
- b Determine the probability of error as a function of the parameters $(\sigma^2, \mathcal{E}_b, w)$.
- c What is the value of w that minimizes the error probability?

5-39 Derive the outputs r_1 and r_2 of the two correlators shown in Fig. 5-4-1. Assume that a signal $s_{j_1}(t)$ is transmitted and that

$$r_i(t) = s_{j_1}(t)e^{j\phi} + z(t)$$

where $z(t) = n_c(t) + jn_s(t)$ is the additive gaussian noise.

5-40 Determine the covariances and variances of the gaussian random noise variables n_{1c}, n_{2c}, n_{1s} , and n_{2s} in (5-4-15) and the joint pdf.

5-41 Derive the matched filter outputs given by (5-4-10).

5-42 In on-off keying of a carrier-modulated signal, the two possible signals are

$$s_0(t) = 0, \quad 0 \leq t \leq T_b$$

$$s_1(t) = \sqrt{\frac{2\mathcal{E}_b}{T_b}} \cos 2\pi f_c t, \quad 0 \leq t \leq T_b$$

The corresponding received signals are

$$r(t) = n(t), \quad 0 \leq t \leq T_b$$

$$r(t) = \sqrt{\frac{2\mathcal{E}_b}{T_b}} \cos (2\pi f_c t + \phi) + n(t), \quad 0 \leq t \leq T_b$$

where ϕ is the carrier phase and $n(t)$ is AWGN.

- a Sketch a block diagram of the receiver (demodulator and detector) that employs noncoherent (envelope) detection.
 - b Determine the pdfs for the two possible decision variables at the detector corresponding to the two possible received signals.
 - c Derive the probability of error for the detector.
- 5-43 In two-phase DPSK, the received signal in one signaling interval is used as a phase reference for the received signal in the following signaling interval. The decision variable is

$$D = \text{Re} (V_m V_{m-1}^*) \stackrel{>0}{\underset{<0}{\geq}} 0$$

$$V_k = 2\alpha \mathcal{E} e^{j(\theta_k - \phi)} + N_k$$

represents the complex-valued output of the filter matched to the transmitted signal $u(t)$. N_k is a complex-valued gaussian variable having zero mean and statistically independent components.

a Writing $V_k = X_k + jY_k$, show that D is equivalent to

$$d = [\frac{1}{2}(X_m + X_{m-1})]^2 + [\frac{1}{2}(Y_m + Y_{m-1})]^2 - [\frac{1}{2}(X_m - X_{m-1})]^2 - [\frac{1}{2}(Y_m - Y_{m-1})]^2$$

b For mathematical convenience; suppose that $\theta_k = \theta_{k-1}$. Show that the random variables U_1, U_2, U_3 , and U_4 are statistically independent gaussian variables, where $U_1 = \frac{1}{2}(X_m + X_{m-1})$, $U_2 = \frac{1}{2}(Y_m + Y_{m-1})$, $U_3 = \frac{1}{2}(X_m - X_{m-1})$, and $U_4 = \frac{1}{2}(Y_m - Y_{m-1})$.

c Define the random variables $W_1 = U_1^2 + U_2^2$ and $W_2 = U_3^2 + U_4^2$. Then

$$D = W_1 - W_2 \stackrel{?}{\geq} 0$$

Determine the probability density functions for W_1 and W_2 .

d Determine the probability of error P_b , where

$$P_b = P(D < 0) = P(W_1 - W_2 < 0) = \int_0^\infty P(W_2 > w_1 | w_1) p(w_1) dw_1$$

5-44 Recall that MSK can be represented as a four-phase offset PSK modulation having the lowpass equivalent form

$$v(t) = \sum_k [I_k u(t - 2kT_b) + jJ_k u(t - 2kT_b - T_b)]$$

where

$$u(t) = \begin{cases} \sin(\pi t / 2T_b) & (0 \leq t \leq 2T_b) \\ 0 & (\text{otherwise}) \end{cases}$$

and $\{I_k\}$ and $\{J_k\}$ are sequences of information symbols (± 1).

a Sketch the block diagram of an MSK demodulator for offset QPSK.

b Evaluate the performance of the four-phase demodulator for AWGN if no account is taken of the memory in the modulation.

c Compare the performance obtained in (b) with that for Viterbi decoding of the MSK signal.

d The MSK signal is also equivalent to binary FSK. Determine the performance of noncoherent detection of the MSK signal. Compare your result with (b) and (c).

5-45 Consider a transmission line channel that employs $n - 1$ regenerative repeaters plus the terminal receiver in the transmission of binary information. Assume that the probability of error at the detector of each receiver is p and that errors among repeaters are statistically independent.

a Show that the binary error probability at the terminal receiver is

$$P_n = \frac{1}{2}[1 - (1 - 2p)^n]$$

b If $p = 10^{-6}$ and $n = 100$, determine an approximate value of P_n .

5-46 A digital communication system consists of a transmission line with 100 digital (regenerative) repeaters. Binary antipodal signals are used for transmitting the information. If the overall end-to-end error probability is 10^{-6} , determine the probability of error for each repeater and the required \mathcal{E}_b/N_0 to achieve this performance in AWGN.

5-47 A radio transmitter has a power output of $P_T = 1$ W at a frequency of 1 GHz. The transmitting and receiving antennas are parabolic dishes with diameter $D = 3$ m.

a Determine the antenna gains.

b Determine the EIRP for the transmitter.

c The distance (free space) between the transmitting and receiving antennas is 20 km. Determine the signal power at the output of the receiving antenna in dBm.

- 5-48** A radio communication system transmits at a power level of 0.1 W at 1 GHz. The transmitting and receiving antennas are parabolic, each having a diameter of 1 m. The receiver is located 30 km from the transmitter.
- Determine the gains of the transmitting and receiving antennas.
 - Determine the EIRP of the transmitted signal.
 - Determine the signal power from the receiving antenna.
- 5-49** A satellite in synchronous orbit is used to communicate with an earth station at a distance of 40 000 km. The satellite has an antenna with a gain of 15 dB and a transmitter power of 3 W. The earth station uses a 10 m parabolic antenna with an efficiency of 0.6. The frequency band is at $f = 10$ GHz. Determine the received power level at the output of the receiver antenna.
- 5-50** A spacecraft located 100 000 km from the earth is sending data at a rate of R bits/s. The frequency band is centered at 2 GHz and the transmitted power is 10 W. The earth station uses a parabolic antenna, 50 m in diameter, and the spacecraft has an antenna with a gain of 10 dB. The noise temperature of the receiver front end is $T_0 = 300$ K.
- Determine the received power level.
 - If the desired $\mathcal{E}_b/N_0 = 10$ dB, determine the maximum bit rate that the spacecraft can transmit.
- 5-51** A satellite in geosynchronous orbit is used as a regenerative repeater in a digital communication system. Consider the satellite-to-earth link in which the satellite antenna has a gain of 6 dB and the earth station antenna has a gain of 50 dB. The downlink is operated at a center frequency of 4 GHz, and the signal bandwidth is 1 MHz. If the required \mathcal{E}_b/N_0 for reliable communication is 15 dB, determine the transmitted power for the satellite downlink. Assume that $N_0 = 4.1 \times 10^{-21}$ W/Hz.

CARRIER AND SYMBOL SYNCHRONIZATION

We have observed that in a digital communication system, the output of the demodulator must be sampled periodically, once per symbol interval, in order to recover the transmitted information. Since the propagation delay from the transmitter to the receiver is generally unknown at the receiver, symbol timing must be derived from the received signal in order to synchronously sample the output of the demodulator.

The propagation delay in the transmitted signal also results in a carrier offset, which must be estimated at the receiver if the detector is phase-coherent. In this chapter, we consider methods for deriving carrier and symbol synchronization at the receiver.

6-1 SIGNAL PARAMETER ESTIMATION

Let us begin by developing a mathematical model for the signal at the input to the receiver. We assume that the channel delays the signals transmitted through it and corrupts them by the addition of gaussian noise. Hence, the received signal may be expressed as

$$r(t) = s(t - \tau) + n(t)$$

where

$$s(t) = \text{Re} \{s_l(t) e^{j2\pi f_c t}\} \quad (6-1-1)$$

and where τ is the propagation delay and $s_l(t)$ is the equivalent lowpass signal.

The received signal may be expressed as

$$r(t) = \text{Re} \{[s_l(t - \tau) e^{j\phi} + z(t)] e^{j2\pi f_c t}\} \quad (6-1-2)$$

where the carrier phase ϕ , due to the propagation delay τ , is $\phi = -2\pi f_c \tau$.

Now, from this formulation, it may appear that there is only one signal parameter to be estimated, namely, the propagation delay, since one can determine ϕ from knowledge of f_c and τ . However, this is not the case. First of all, the oscillator that generates the carrier signal for demodulation at the receiver is generally not synchronous in phase with that at the transmitter. Furthermore, the two oscillators may be drifting slowly with time, perhaps in different directions. Consequently, the received carrier phase is not only dependent on the time delay τ . Furthermore, the precision to which one must synchronize in time for purpose of demodulating the received signal depends on the symbol interval T . Usually, the estimation error in estimating τ must be a relatively small fraction of T . For example, $\pm 1\%$ of T is adequate for practical applications. However, this level of precision is generally inadequate for estimating the carrier phase, even if ϕ depends only on τ . This is due to the fact that f_c is generally large, and, hence, a small estimation error in τ causes a large phase error.

In effect, we must estimate both parameters τ and ϕ in order to demodulate and coherently detect the received signal. Hence, we may express the received signal as

$$r(t) = s(t; \phi, \tau) + n(t) \quad (6-1-3)$$

where ϕ and τ represent the signal parameters to be estimated. To simplify the notation, we let ψ denote the parameter vector $\{\phi, \tau\}$, so that $s(t; \phi, \tau)$ is simply denoted by $s(t; \psi)$.

There are basically two criteria that are widely applied to signal parameter estimation: the *maximum-likelihood* (ML) criterion and the *maximum a posteriori probability* (MAP) criterion. In the MAP criterion, the signal parameter vector ψ is modeled as random, and characterized by an a priori probability density function $p(\psi)$. In the maximum-likelihood criterion, the signal parameter vector ψ is treated as deterministic but unknown.

By performing an orthonormal expansion of $r(t)$ using N orthonormal functions $\{f_n(t)\}$, we may represent $r(t)$ by the vector of coefficients $[r_1 \ r_2 \ \dots \ r_N] \equiv \mathbf{r}$. The joint pdf of the random variables $[r_1 \ r_2 \ \dots \ r_N]$ in the expansion can be expressed as $p(\mathbf{r} | \psi)$. Then, the ML estimate of ψ is the value that maximizes $p(\mathbf{r} | \psi)$. On the other hand, the MAP estimate is the value of ψ that maximizes the a posteriori probability density function

$$p(\psi | \mathbf{r}) = \frac{p(\mathbf{r} | \psi)p(\psi)}{p(\mathbf{r})} \quad (6-1-4)$$

We note that if there is no prior knowledge of the parameter vector ψ , we may assume that $p(\psi)$ is uniform (constant) over the range of values of the parameters. In such a case, the value of ψ that maximizes $p(\mathbf{r} | \psi)$ also maximizes $p(\psi | \mathbf{r})$. Therefore, the MAP and ML estimates are identical.

In our treatment of parameter estimation given below, we view the parameters ϕ and τ as unknown, but deterministic. Hence, we adopt the ML criterion for estimating them.

In the ML estimation of signal parameters, we require that the receiver extract the estimate by observing the received signal over a time interval $T_0 \geq T$, which is called the observation interval. Estimates obtained from a single observation interval are sometimes called one-shot estimates. In practice, however, the estimation is performed on a continuous basis by using tracking loops (either analog or digital) that continuously update the estimates. Nevertheless, one-shot estimates yield insight for tracking loop implementation. In addition, they prove useful in the analysis of the performance of ML estimation, and their performance can be related to that obtained with a tracking loop.

6-1-1 The Likelihood Function

Although it is possible to derive the parameter estimates based on the joint pdf of the random variables $[r_1 \ r_2 \ \dots \ r_N]$ obtained from the expansion of $r(t)$, it is convenient to deal directly with the signal waveforms when estimating their parameters. Hence, we shall develop a continuous-time equivalent of the maximization of $p(\mathbf{r} | \Psi)$.

Since the additive noise $n(t)$ is white and zero-mean gaussian, the joint pdf $p(\mathbf{r} | \Psi)$ may be expressed as

$$p(\mathbf{r} | \Psi) = \left(\frac{1}{\sqrt{2\pi\sigma}} \right)^N \exp \left\{ - \sum_{n=1}^N \frac{[r_n - s_n(\Psi)]^2}{2\sigma^2} \right\} \quad (6-1-5)$$

where

$$r_n = \int_{T_0} r(t) f_n(t) dt$$

$$s_n(\Psi) = \int_{T_0} s(t; \Psi) f_n(t) dt \quad (6-1-6)$$

where T_0 represents the integration interval in the expansion of $r(t)$ and $s(t; \Psi)$.

We note that the argument in the exponent may be expressed in terms of the signal waveforms $r(t)$ and $s(t; \Psi)$, by substituting from (6-1-6) into (6-1-5). That is,

$$\frac{1}{2\sigma^2} \sum_{n=1}^N [r_n - s_n(t; \Psi)]^2 = \frac{1}{N_0} \int_{T_0} [r(t) - s(t; \Psi)]^2 dt \quad (6-1-7)$$

where the proof is left as an exercise for the reader (see Problem 6-1). Now, the maximization of $p(\mathbf{r} | \Psi)$ with respect to the signal parameters Ψ is equivalent to the maximization of the *likelihood function*.

$$\Lambda(\Psi) = \exp \left\{ - \frac{1}{N_0} \int_{T_0} [r(t) - s(t; \Psi)]^2 dt \right\} \quad (6-1-8)$$

Below, we shall consider signal parameter estimation from the viewpoint of maximizing $\Lambda(\Psi)$.

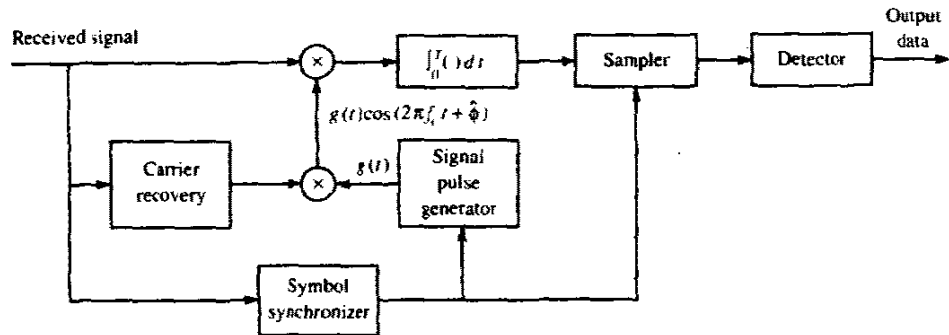


FIGURE 6-1-1 Block diagram of binary PSK receiver.

6-1-2 Carrier Recovery and Symbol Synchronization in Signal Demodulation

Symbol synchronization is required in every digital communication system which transmits information synchronously. Carrier recovery is required if the signal is detected coherently.

Figure 6-1-1 illustrates the block diagram of a binary PSK (or binary PAM) signal demodulator and detector. As shown, the carrier phase estimate $\hat{\phi}$ is used in generating the reference signal $g(t) \cos(2\pi f_c t + \hat{\phi})$ for the correlator. The symbol synchronizer controls the sampler and the output of the signal pulse generator. If the signal pulse is rectangular then the signal generator can be eliminated.

The block diagram of an M -ary PSK demodulator is shown in Fig. 6-1-2. In this case, two correlators (or matched filters) are required to correlate the received signal with the two quadrature carrier signals $g(t) \cos(2\pi f_c t + \hat{\phi})$ and $g(t) \sin(2\pi f_c t + \hat{\phi})$, where $\hat{\phi}$ is the carrier phase estimate. The detector is now a phase detector, which compares the received signal phases with the possible transmitted signal phases.

The block diagram of a PAM signal demodulator is shown in Fig. 6-1-3. In this case, a single correlator is required, and the detector is an amplitude detector, which compares the received signal amplitude with the possible transmitted signal amplitudes. Note that we have included an automatic gain control (AGC) at the front-end of the demodulator to eliminate channel gain variations, which would affect the amplitude detector. The AGC has a relatively long time constant, so that it does not respond to the signal amplitude variations that occur on a symbol-by-symbol basis. Instead, the AGC maintains a fixed average (signal plus noise) power at its output.

Finally, we illustrate the block diagram of a QAM demodulator in Fig. 6-1-4. As in the case of PAM, an AGC is required to maintain a constant average power signal at the input to the demodulator. We observe that the demodulator is similar to a PSK demodulator, in that both generate in-phase and quadrature signal samples (X, Y) for the detector. In the case of QAM,

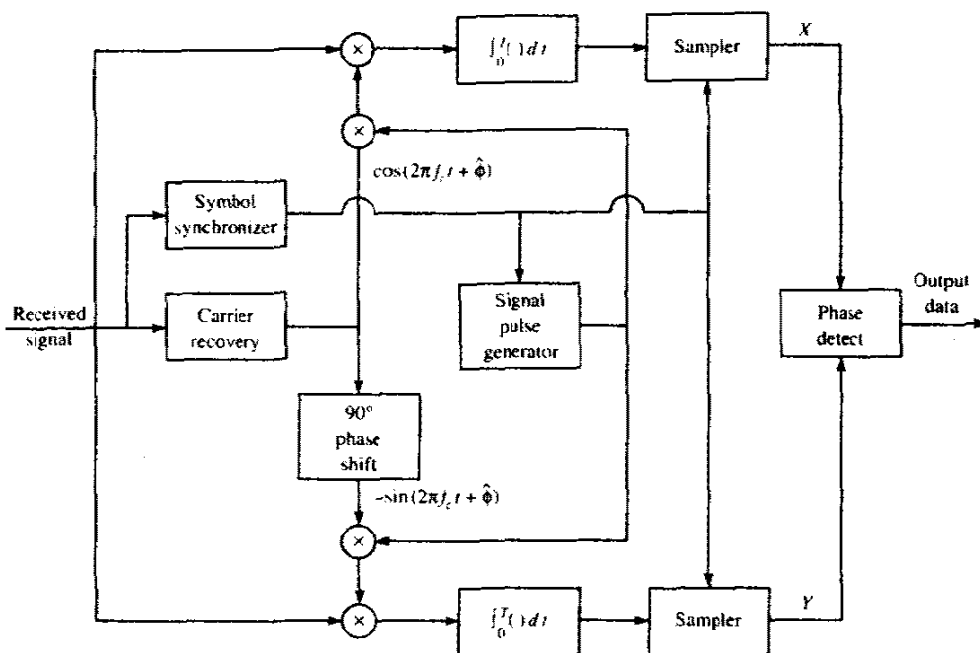


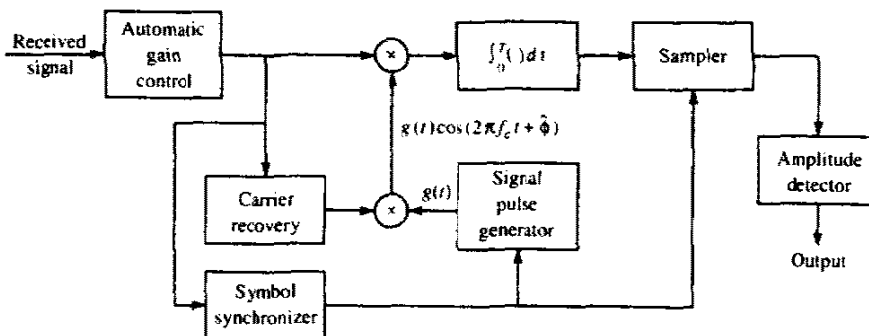
FIGURE 6-1-2 Block diagram of M -ary PSK receiver.

the detector computes the euclidean distance between the received noise-corrupted signal point and the M possible transmitted points, and selects the signal closest to the received point.

6-2 CARRIER PHASE ESTIMATION

There are two basic approaches for dealing with carrier synchronization at the receiver. One is to multiplex, usually in frequency, a special signal, called a pilot signal, that allows the receiver to extract and, thus, to synchronize its local oscillator to the carrier frequency and phase of the received signal. When

FIGURE 6-1-3 Block diagram of M -ary PAM receiver.



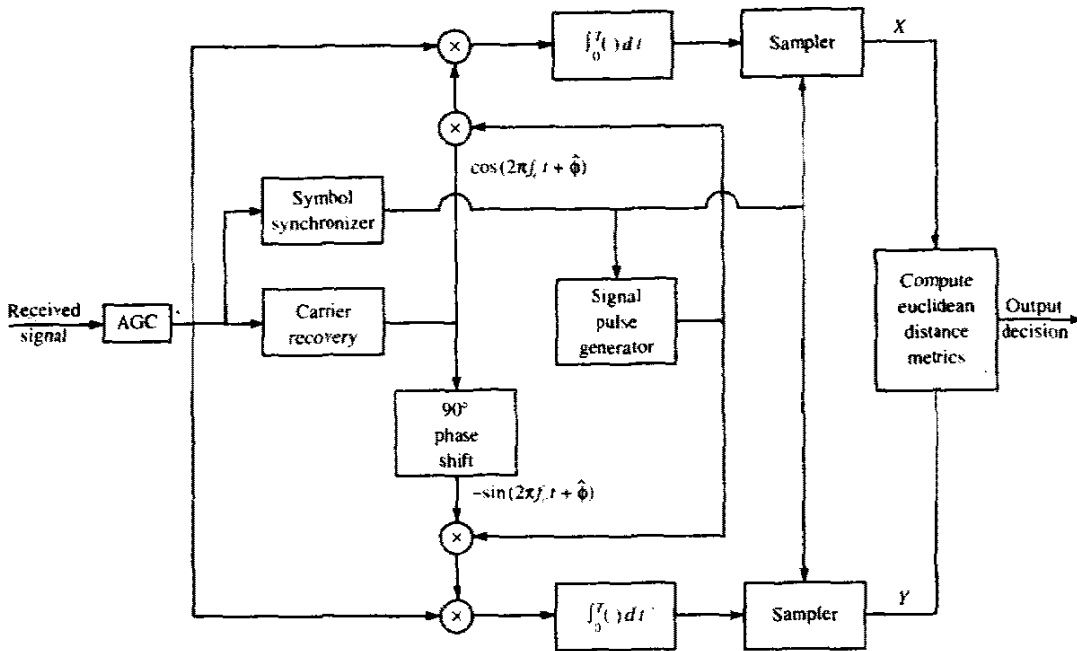


FIGURE 6-1-4 Block diagram of QAM receiver.

an unmodulated carrier component is transmitted along with the information-bearing signal, the receiver employs a phase-locked loop (PLL) to acquire and track the carrier component. The PLL is designed to have a narrow bandwidth so that it is not significantly affected by the presence of frequency components from the information-bearing signal.

The second approach, which appears to be more prevalent in practice, is to derive the carrier phase estimate directly from the modulated signal. This approach has the distinct advantage that the total transmitter power is allocated to the transmission of the information-bearing signal. In our treatment of carrier recovery, we confine our attention to the second approach; hence, we assume that the signal is transmitted via suppressed carrier.

In order to emphasize the importance of extracting an accurate phase estimate, let us consider the effect of a carrier phase error on the demodulation of a double-sideband, suppressed carrier (DSB/SC) signal. To be specific, suppose we have an amplitude-modulated signal of the form

$$s(t) = A(t) \cos(2\pi f_c t + \phi) \tag{6-2-1}$$

If we demodulate the signal by multiplying $s(t)$ with the carrier reference

$$c(t) = \cos(2\pi f_c t + \hat{\phi}) \tag{6-2-2}$$

we obtain

$$c(t)s(t) = \frac{1}{2}A(t) \cos(\phi - \hat{\phi}) + \frac{1}{2}A(t) \cos(4\pi f_c t + \phi + \hat{\phi})$$

The double-frequency component may be removed by passing the product signal $c(t)s(t)$ through a lowpass filter. This filtering yields the information-bearing signal

$$y(t) = \frac{1}{2}A(t) \cos(\phi - \hat{\phi}) \quad (6-2-3)$$

Note that the effect of the phase error $\phi - \hat{\phi}$ is to reduce the signal level in voltage by a factor $\cos(\phi - \hat{\phi})$ and in power by a factor $\cos^2(\phi - \hat{\phi})$. Hence, a phase error of 10° results in a signal power loss of 0.13 dB, and a phase error of 30° results in a signal power loss of 1.25 dB in an amplitude-modulated signal.

The effect of carrier phase errors in QAM and multiphase PSK is much more severe. The QAM and M -PSK signals may be represented as

$$s(t) = A(t) \cos(2\pi f_c t + \phi) - B(t) \sin(2\pi f_c t + \phi) \quad (6-2-4)$$

This signal is demodulated by the two quadrature carriers

$$\begin{aligned} c_c(t) &= \cos(2\pi f_c t + \hat{\phi}) \\ c_s(t) &= -\sin(2\pi f_c t + \hat{\phi}) \end{aligned} \quad (6-2-5)$$

Multiplication of $s(t)$ with $c_c(t)$ followed by lowpass filtering yields the in-phase component

$$y_i(t) = \frac{1}{2}A(t) \cos(\phi - \hat{\phi}) - \frac{1}{2}B(t) \sin(\phi - \hat{\phi}) \quad (6-2-6)$$

Similarly, multiplication of $s(t)$ by $c_s(t)$ followed by lowpass filtering yields the quadrature component

$$y_q(t) = \frac{1}{2}B(t) \cos(\phi - \hat{\phi}) + \frac{1}{2}A(t) \sin(\phi - \hat{\phi}) \quad (6-2-7)$$

The expressions (6-2-6) and (6-2-7) clearly indicate that the phase error in the demodulation of QAM and M -PSK signals has a much more severe effect than in the demodulation of a PAM signal. Not only is there a reduction in the power of the desired signal component by a factor $\cos^2(\phi - \hat{\phi})$, but there is also crosstalk interference from the in-phase and quadrature components. Since the average power levels of $A(t)$ and $B(t)$ are similar, a small phase error causes a large degradation in performance. Hence, the phase accuracy requirements for QAM and multiphase coherent PSK are much higher than DSB/SC PAM.

6-2-1 Maximum-Likelihood Carrier Phase Estimation

First, we derive the maximum-likelihood carrier phase estimate. For simplicity, we assume that the delay τ is known and, in particular, we set $\tau = 0$. The function to be maximized is the likelihood function given in (6-1-8). With ϕ substituted for ψ , this function becomes

$$\begin{aligned} \Lambda(\phi) &= \exp \left\{ -\frac{1}{N_0} \int_{\tau_0} [r(t) - s(t; \phi)]^2 dt \right\} \\ &= \exp \left\{ -\frac{1}{N_0} \int_{\tau_0} r^2(t) dt + \frac{2}{N_0} \int_{\tau_0} r(t)s(t; \phi) dt - \frac{1}{N_0} \int_{\tau_0} s^2(t; \phi) dt \right\}. \end{aligned} \quad (6-2-8)$$

Note that the first term of the exponential factor does not involve the signal parameter ϕ . The third term, which contains the integral of $s^2(t; \phi)$, is a constant equal to the signal energy over the observation interval T_0 for any value of ϕ . Only the second term, which involves the cross-correlation of the received signal $r(t)$ with the signal $s(t; \phi)$, depends on the choice of ϕ . Therefore, the likelihood function $\Lambda(\phi)$ may be expressed as

$$\Lambda(\phi) = C \exp \left[\frac{2}{N_0} \int_{T_0} r(t)s(t; \phi) dt \right] \quad (6-2-9)$$

where C is a constant independent of ϕ .

The ML estimate $\hat{\phi}_{ML}$ is the value of ϕ that maximizes $\Lambda(\phi)$ in (6-2-9). Equivalently, the value $\hat{\phi}_{ML}$ also maximizes the logarithm of $\Lambda(\phi)$, i.e., the log-likelihood function

$$\Lambda_L(\phi) = \frac{2}{N_0} \int_{T_0} r(t)s(t; \phi) dt \quad (6-2-10)$$

Note that in defining $\Lambda_L(\phi)$ we have ignored the constant term $\ln C$.

Example 6-2-1

As an example of the optimization to determine the carrier phase, let us consider the transmission of the unmodulated carrier $A \cos 2\pi f_c t$. The received signal is

$$r(t) = A \cos (2\pi f_c t + \phi) + n(t)$$

where ϕ is the unknown phase. We seek the value ϕ , say $\hat{\phi}_{ML}$, that maximizes

$$\Lambda_L(\phi) = \frac{2A}{N_0} \int_{T_0} r(t) \cos (2\pi f_c t + \phi) dt$$

A necessary condition for a maximum is that

$$\frac{d\Lambda_L(\phi)}{d\phi} = 0$$

This condition yields

$$\int_{T_0} r(t) \sin (2\pi f_c t + \hat{\phi}_{ML}) dt = 0 \quad (6-2-11)$$

or, equivalently,

$$\hat{\phi}_{ML} = -\tan^{-1} \left[\frac{\int_{T_0} r(t) \sin 2\pi f_c t dt}{\int_{T_0} r(t) \cos 2\pi f_c t dt} \right] \quad (6-2-12)$$

We observe that the optimality condition given by (6-2-11) implies the use

FIGURE 6-2-1 A PLL for obtaining the ML estimate of the phase of an unmodulated carrier.

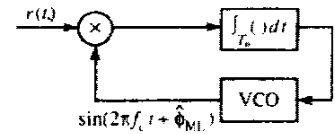
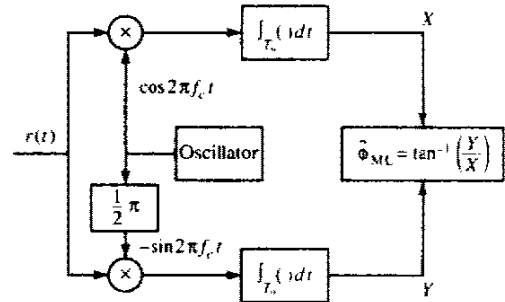


FIGURE 6-2-2 A (one-shot) ML estimate of the phase of an unmodulated carrier.



of a loop to extract the estimate as illustrated in Fig. 6-2-1. The loop filter is an integrator whose bandwidth is proportional to the reciprocal of the integration interval T_0 . On the other hand, (6-2-12) implies an implementation that uses quadrature carriers to cross-correlate with $r(t)$. Then, $\hat{\phi}_{ML}$ is the inverse tangent of the ratio of these two correlator outputs, as shown in Fig. 6-2-2. Note that this estimation scheme yields $\hat{\phi}_{ML}$ explicitly.

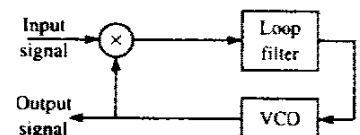
This example clearly demonstrates that the PLL provides the ML estimate of the phase of an unmodulated carrier.

6-2-2 The Phase-Locked Loop

The PLL basically consists of a multiplier, a loop filter, and a voltage-controlled oscillator (VCO), as shown in Fig. 6-2-3. If we assume that the input to the PLL is the sinusoid $\cos(2\pi f_c t + \phi)$ and the output of the VCO is $\sin(2\pi f_c t + \hat{\phi})$, where $\hat{\phi}$ represents the estimate of ϕ , the product of these two signals is

$$\begin{aligned}
 e(t) &= \cos(2\pi f_c t + \phi) \sin(2\pi f_c t + \hat{\phi}) \\
 &= \frac{1}{2} \sin(\hat{\phi} - \phi) + \frac{1}{2} \sin(4\pi f_c t + \phi + \hat{\phi})
 \end{aligned}
 \tag{6-2-13}$$

FIGURE 6-2-3 Basic elements of a phase-locked loop (PLL).



The loop filter is a lowpass filter that responds only to the low-frequency component $\frac{1}{2} \sin(\hat{\phi} - \phi)$ and removes the component at $2f_c$. This filter is usually selected to have the relatively simple transfer function

$$G(s) = \frac{1 + \tau_2 s}{1 + \tau_1 s} \quad (6-2-14)$$

where τ_1 and τ_2 are design parameters ($\tau_1 \gg \tau_2$) that control the bandwidth of the loop. A higher-order filter that contains additional poles may be used if necessary to obtain a better loop response.

The output of the loop filter provides the control voltage $v(t)$ for the VCO. The VCO is basically a sinusoidal signal generator with an instantaneous phase given by

$$2\pi f_c t + \hat{\phi}(t) = 2\pi f_c t + K \int_{-\infty}^t v(\tau) d\tau \quad (6-2-15)$$

where K is a gain constant in rad/V. Hence,

$$\hat{\phi}(t) = K \int_{-\infty}^t v(\tau) d\tau \quad (6-2-16)$$

By neglecting the double-frequency term resulting from the multiplication of the input signal with the output of the VCO, we may reduce the PLL into the equivalent closed-loop system model shown in Fig. 6-2-4. The sine function of the phase difference $\phi - \hat{\phi}$ makes this system nonlinear, and, as a consequence, the analysis of its performance in the presence of noise is somewhat involved but, nevertheless, it is mathematically tractable for some simple loop filters.

In normal operation when the loop is tracking the phase of the incoming carrier, the phase error $\phi - \hat{\phi}$ is small and, hence,

$$\sin(\hat{\phi} - \phi) \approx \hat{\phi} - \phi \quad (6-2-17)$$

With this approximation, the PLL becomes linear and is characterized by the closed-loop transfer function

$$H(s) = \frac{KG(s)/s}{1 + KG(s)/s} \quad (6-2-18)$$

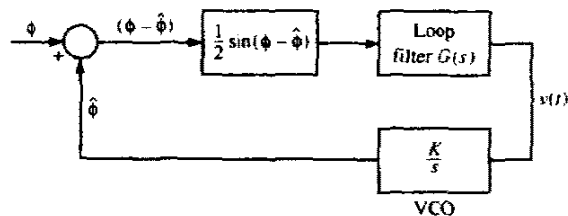


FIGURE 6-2-4 Model of phase-locked loop.

where the factor of $\frac{1}{K}$ has been absorbed into the gain parameter K . By substituting from (6-2-14) for $G(s)$ into (6-2-18), we obtain

$$H(s) = \frac{1 + \tau_2 s}{1 + (\tau_2 + 1/K)s + (\tau_1/K)s^2} \quad (6-2-19)$$

Hence, the closed-loop system for the linearized PLL is second-order when $G(s)$ is given by (6-2-14). The parameter τ_2 controls the position of the zero, while K and τ_1 are used to control the position of the closed-loop system poles. It is customary to express the denominator of $H(s)$ in the standard form

$$D(s) = s^2 + 2\zeta\omega_n s + \omega_n^2 \quad (6-2-20)$$

where ζ is called the *loop damping factor* and ω_n is the natural frequency of the loop. In terms of the loop parameters, $\omega_n = \sqrt{K/\tau_1}$, and $\zeta = (\tau_2 + 1/K)/2\omega_n$, the closed-loop transfer function becomes

$$H(s) = \frac{(2\zeta\omega_n - \omega_n^2/K)s + \omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \quad (6-2-21)$$

The (one-sided) noise-equivalent bandwidth (see Problem 2-24) of the loop is

$$\begin{aligned} B_{\text{eq}} &= \frac{\tau_2^2(1/\tau_2^2 + K/\tau_1)}{4(\tau_2 + 1/K)} \\ &= \frac{1 + (\tau_2\omega_n)^2}{8\zeta\omega_n} \end{aligned} \quad (6-2-22)$$

The magnitude response $20 \log |H(\omega)|$ as a function of the normalized frequency ω/ω_n is illustrated in Fig. 6-2-5, with the damping factor ζ as a parameter and $\tau_1 \gg 1$. Note that $\zeta = 1$ results in a critically damped loop response, $\zeta < 1$ produces an underdamped response, and $\zeta > 1$ yields an overdamped response.

In practice, the selection of the bandwidth of the PLL involves a trade-off between speed of response and noise in the phase estimate, which is the topic considered below. On the one hand, it is desirable to select the bandwidth of the loop to be sufficiently wide to track any time variations in the phase of the received carrier. On the other, a wideband PLL allows more noise to pass into the loop, which corrupts the phase estimate. Below, we assess the effects of noise in the quality of the phase estimate.

6-2-3 Effect of Additive Noise on the Phase Estimate

In order to evaluate the effects of noise on the estimate of the carrier phase, let us assume that the noise at the input to the PLL is narrowband. For this analysis, we assume that the PLL is tracking a sinusoidal signal of the form

$$s(t) = A_c \cos [2\pi f_c t + \phi(t)] \quad (6-2-23)$$

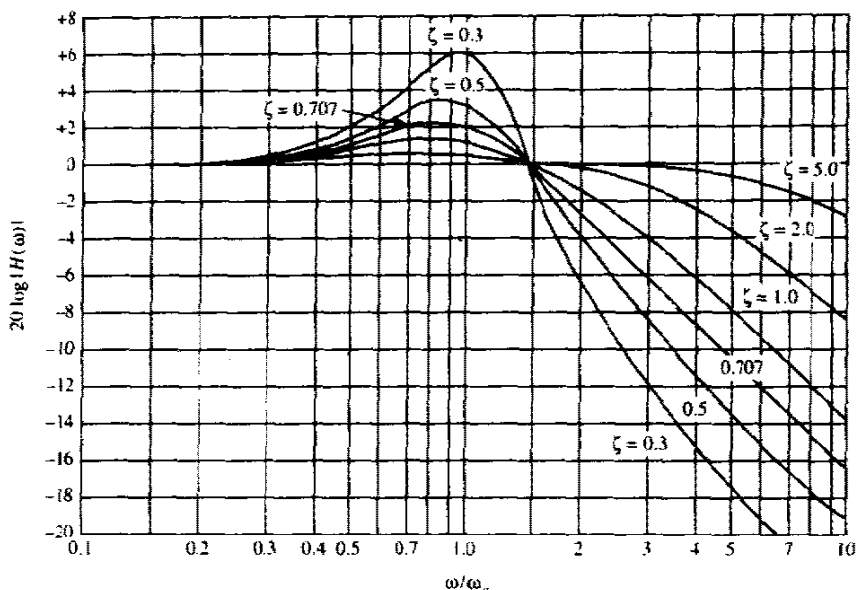


FIGURE 6-2-5 Frequency response of a second-order loop. [From Phaselock Techniques, 2nd edition, by F. M. Gardner, © 1979 by John Wiley and Sons, Inc. Reprinted with permission of the publisher.]

that is corrupted by the additive narrowband noise

$$n(t) = x(t) \cos 2\pi f_c t - y(t) \sin 2\pi f_c t \tag{6-2-24}$$

The in-phase and quadrature components of the noise are assumed to be statistically independent, stationary gaussian noise processes with (two-sided) power spectral density $\frac{1}{2}N_0$ W/Hz. By using simple trigonometric identities, the noise term in (6-2-24) can be expressed as

$$n(t) = n_c(t) \cos [2\pi f_c t + \phi(t)] - n_s(t) \sin [2\pi f_c t + \phi(t)] \tag{6-2-25}$$

where

$$\begin{aligned} n_c(t) &= x(t) \cos \phi(t) + y(t) \sin \phi(t) \\ n_s(t) &= -x(t) \sin \phi(t) + y(t) \cos \phi(t) \end{aligned} \tag{6-2-26}$$

We note that

$$n_c(t) + jn_s(t) = [x(t) + jy(t)]e^{-j\phi(t)}$$

so that the quadrature components $n_c(t)$ and $n_s(t)$ have exactly the same statistical characteristics as $x(t)$ and $y(t)$.

If $s(t) + n(t)$ is multiplied by the output of the VCO and the double-frequency terms are neglected, the input to the loop filter is the noise-corrupted signal

$$\begin{aligned} e(t) &= A_c \sin \Delta\phi + n_c(t) \sin \Delta\phi - n_s(t) \cos \Delta\phi \\ &= A_c \sin \Delta\phi + n_1(t) \end{aligned} \tag{6-2-27}$$

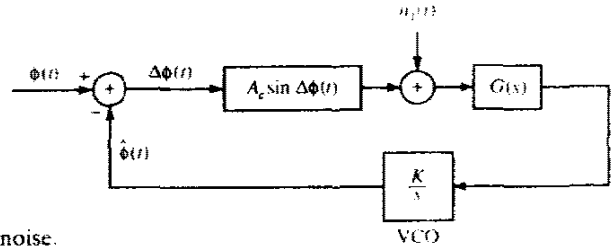


FIGURE 6-2-6 Equivalent PLL model with additive noise.

where, by definition, $\Delta\phi = \phi - \hat{\phi}$ is the phase error. Thus, we have the equivalent model for the PLL with additive noise as shown in Fig. 6-2-6.

When the power $P_c = \frac{1}{2}A_c^2$ of the incoming signal is much larger than the noise power, we may linearize the PLL and, thus, easily determine the effect of the additive noise on the quality of the estimate $\hat{\phi}$. Under these conditions, the model for the linearized PLL with additive noise is illustrated in Fig. 6-2-7. Note that the gain parameter A_c may be normalized to unity, provided that the noise terms are scaled by $1/A_c$, i.e., the noise terms become

$$n_2(t) = \frac{n_c(t)}{A_c} \sin \Delta\phi - \frac{n_s(t)}{A_c} \cos \Delta\phi \quad (6-2-28)$$

Since the noise $n_2(t)$ is additive at the input to the loop, the variance of the phase error $\Delta\phi$, which is also the variance of the VCO output phase, is

$$\sigma_{\Delta\phi}^2 = \frac{N_0 B_{\text{eq}}}{A_c^2} \quad (6-2-29)$$

where B_{eq} is the (one-sided) equivalent noise bandwidth of the loop, given in (6-2-22). Note that $\sigma_{\Delta\phi}^2$ is simply the ratio of total noise power within the bandwidth of the PLL divided by the signal power A^2 . Hence,

$$\sigma_{\Delta\phi}^2 = 1/\gamma_L \quad (6-2-30)$$

where γ_L is defined as the signal-to-noise ratio

$$\text{SNR} = \gamma_L = \frac{A_c^2}{N_0 B_{\text{eq}}} \quad (6-2-31)$$

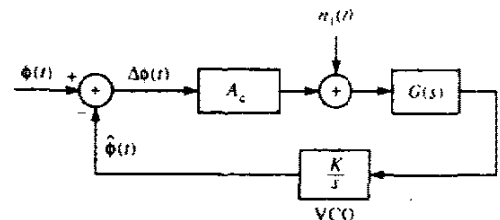


FIGURE 6-2-7 Linearized PLL model with additive noise.

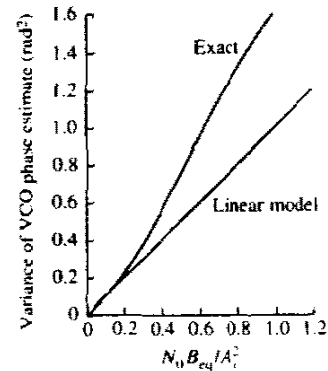


FIGURE 6-2-8 Comparison of VCO phase variance for exact and approximate (linear model) first-order PLL. [From Principles of Coherent Communication, by A. J. Viterbi; © 1966 by McGraw-Hill Book Company. Reprinted with permission of the publisher.]

The expression for the variance $\sigma_{\hat{\phi}}^2$ of the VCO phase error applies to the case where the SNR is sufficiently high that the linear model for the PLL applies. An exact analysis based on the nonlinear PLL is mathematically tractable when $G(s) = 1$, which results in a first-order loop. In this case, the probability density function for the phase error may be derived (see Viterbi, 1966) and has the form

$$p(\Delta\phi) = \frac{\exp(\gamma_L \cos \Delta\phi)}{2\pi I_0(\gamma_L)} \quad (6-2-32)$$

where γ_L is the SNR given by (6-2-31) with B_{eq} being the appropriate noise bandwidth of the first-order loop, and $I_0(\cdot)$ is the modified Bessel function of order zero.

From the expression for $p(\Delta\phi)$, we may obtain the exact value of the variance for the phase error on a first-order PLL. This is plotted in Fig. 6-2-8 as a function of $1/\gamma_L$. Also shown for comparison is the result obtained with the linearized PLL model. Note that the variance for the linear model is close to the exact variance for $\gamma_L > 3$. Hence, the linear model is adequate for practical purposes.

Approximate analyses of the statistical characteristics of the phase error for the nonlinear PLL have also been performed. Of particular importance is the transient behavior of the PLL during initial acquisition. Another important problem is the behavior of PLL at low SNR. It is known, for example, that when the SNR at the input to the PLL drops below a certain value, there is a rapid deterioration in the performance of the PLL. The loop begins to lose lock and an impulsive-type of noise, characterized as clicks, is generated which degrades the performance of the loop. Results on these topics can be found in the texts by Viterbi (1966), Lindsey (1972), Lindsey and Simon (1973), and Gardner (1979), and in the survey papers by Gupta (1975) and Lindsey and Chie (1981).

Up to this point, we have considered carrier phase estimation when the carrier signal is unmodulated. Below, we consider carrier phase recovery when the signal carries information.

6-2-4 Decision-Directed Loops

A problem arises in maximizing either (6-2-9) or (6-2-10) when the signal $s(t; \phi)$ carries the information sequence $\{I_n\}$. In this case we can adopt one of two approaches: either we assume that $\{I_n\}$ is known or we treat $\{I_n\}$ as a random sequence and average over its statistics.

In decision-directed parameter estimation, we assume that the information sequence $\{I_n\}$ over the observation interval has been estimated and, in the absence of demodulation errors, $\bar{I}_n = I_n$, where \bar{I}_n denotes the detected value of the information I_n . In this case $s(t; \phi)$ is completely known except for the carrier phase. Decision-directed phase estimation was first described by Proakis *et al.* (1964).

To be specific, let us consider the decision-directed phase estimate for the class of linear modulation techniques for which the received *equivalent lowpass signal* may be expressed as

$$\begin{aligned} r(t) &= e^{-j\phi} \sum_n I_n g(t - nT) + z(t) \\ &= s_r(t) e^{-j\phi} + z(t) \end{aligned} \quad (6-2-33)$$

where $s_r(t)$ is a known signal if the sequence $\{I_n\}$ is assumed known. The likelihood function and corresponding log-likelihood function for the equivalent lowpass signal are

$$\Lambda(\phi) = C \exp \left\{ \operatorname{Re} \left[\frac{1}{N_0} \int_{T_0} r(t) s_r^*(t) e^{j\phi} dt \right] \right\} \quad (6-2-34)$$

$$\Lambda_L(\phi) = \operatorname{Re} \left\{ \left[\frac{1}{N_0} \int_{T_0} r(t) s_r^*(t) dt \right] e^{j\phi} \right\} \quad (6-2-35)$$

If we substitute for $s_r(t)$ in (6-2-35) and assume that the observation interval $T_0 = KT$, where K is a positive integer, we obtain

$$\begin{aligned} \Lambda_L(\phi) &= \operatorname{Re} \left\{ e^{j\phi} \frac{1}{N_0} \sum_{n=0}^{K-1} I_n^* \int_{nT}^{(n+1)T} r(t) g^*(t - nT) dt \right\} \\ &= \operatorname{Re} \left\{ e^{j\phi} \frac{1}{N_0} \sum_{n=0}^{K-1} I_n^* y_n \right\} \end{aligned} \quad (6-2-36)$$

where, by definition

$$y_n = \int_{nT}^{(n+1)T} r(t) g^*(t - nT) dt \quad (6-2-37)$$

Note that y_n is the output of the matched filter in the n th signal interval. The ML estimate of ϕ is easily found from (6-2-36) by differentiating the log-likelihood

$$\Lambda_L(\phi) = \operatorname{Re} \left(\frac{1}{N_0} \sum_{n=0}^{K-1} I_n^* y_n \right) \cos \phi - \operatorname{Im} \left(\frac{1}{N_0} \sum_{n=0}^{K-1} I_n^* y_n \right) \sin \phi$$

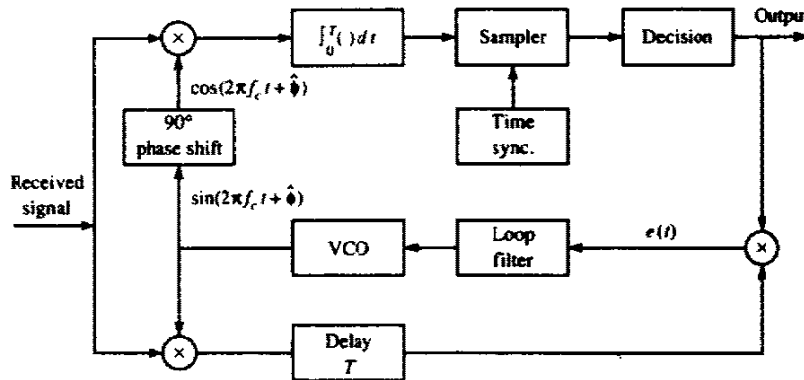


FIGURE 6-2-9 Carrier recovery with a decision-feedback PLL.

with respect to ϕ and setting the derivative equal to zero. Thus, we obtain

$$\hat{\phi}_{ML} = -\tan^{-1} \left[\text{Im} \left(\sum_{n=0}^{K-1} I_n^* y_n \right) / \text{Re} \left(\sum_{n=0}^{K-1} I_n^* y_n \right) \right] \quad (6-2-38)$$

We call $\hat{\phi}_{ML}$ in (6-2-38) the *decision-directed* (or *decision-feedback*) *carrier phase estimate*. It is easily shown (Problem 6-10) that the mean value of $\hat{\phi}_{ML}$ is ϕ , so that the estimate is unbiased. Furthermore, the pdf of $\hat{\phi}_{ML}$ can be obtained (Problem 6-11) by using the procedure described in Section 5-2-7.

A decision-feedback PLL (DFPLL) that is appropriate for a double-sideband PAM signal of the form $A(t) \cos(2\pi f_c t + \phi)$ is shown in Fig. 6-2-9. The received signal is multiplied by the quadrature carriers $c_c(t)$ and $c_s(t)$, as given by (6-2-5), which are derived from the VCO. The product signal

$$r(t) \cos(2\pi f_c t + \hat{\phi}) = \frac{1}{2} [A(t) + n_c(t)] \cos \Delta\phi - \frac{1}{2} n_s(t) \sin \Delta\phi + \text{double-frequency terms} \quad (6-2-39)$$

is used to recover the information carried by $A(t)$. The detector makes a decision on the symbol that is received every T seconds. Thus, in the absence of decision errors, it reconstructs $A(t)$ free of any noise. This reconstructed signal is used to multiply the product of the second quadrature multiplier, which has been delayed by T seconds to allow the demodulator to reach a decision. Thus, the input to the loop filter in the absence of decision errors is the error signal

$$\begin{aligned} e(t) &= \frac{1}{2} A(t) \{ [A(t) + n_c(t)] \sin \Delta\phi - n_s(t) \cos \Delta\phi \} \\ &\quad + \text{double-frequency terms} \\ &= \frac{1}{2} A^2(t) \sin \Delta\phi + \frac{1}{2} A(t) [n_c(t) \sin \Delta\phi - n_s(t) \cos \Delta\phi] \\ &\quad + \text{double-frequency terms} \end{aligned} \quad (6-2-40)$$

The loop filter is lowpass and, hence, it rejects the double-frequency term in $e(t)$. The desired component is $A^2(t) \sin \Delta\phi$, which contains the phase error for driving the loop.

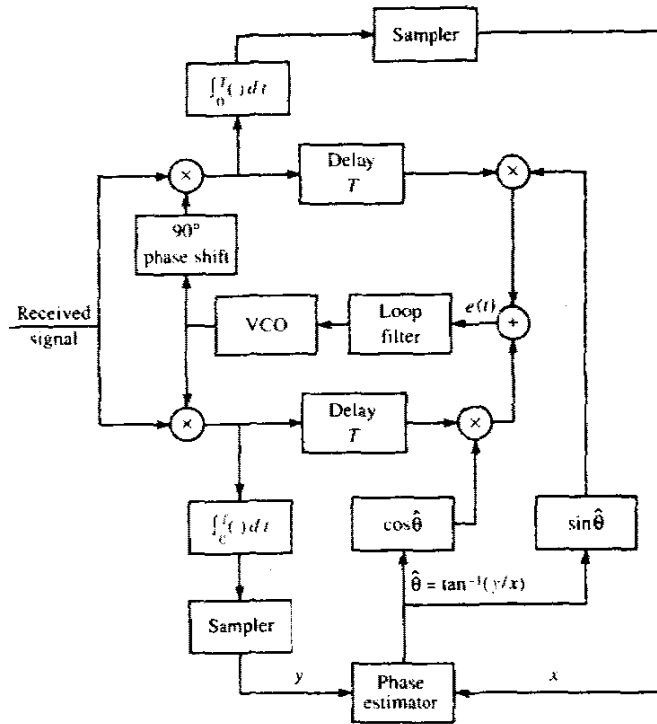


FIGURE 6-2-10 Carrier recovery for M -ary PSK using a decision-feedback PLL.

In the case of M -ary PSK, the DFPLL has the configuration shown in Fig. 6-2-10. The received signal is demodulated to yield the phase estimate

$$\hat{\theta}_m = \frac{2\pi}{M}(m - 1)$$

which, in the absence of a decision error, is the transmitted signal phase. The two outputs of the quadrature multipliers are delayed by the symbol duration T and multiplied by $\cos \theta_m$ and $\sin \theta_m$ to yield

$$\begin{aligned} & r(t) \cos(2\pi f_c t + \hat{\phi}) \sin \theta_m \\ &= \frac{1}{2}[A \cos \theta_m + n_c(t)] \sin \theta_m \cos(\phi - \hat{\phi}) \\ &\quad - \frac{1}{2}[A \sin \theta_m + n_s(t)] \sin \theta_m \sin(\phi - \hat{\phi}) \\ &\quad + \text{double-frequency terms} \\ & r(t) \sin(2\pi f_c t + \hat{\phi}) \cos \theta_m \\ &= -\frac{1}{2}[A \cos \theta_m + n_c(t)] \cos \theta_m \sin(\phi - \hat{\phi}) \\ &\quad - \frac{1}{2}[A \sin \theta_m + n_s(t)] \cos \theta_m \cos(\phi - \hat{\phi}) \\ &\quad + \text{double-frequency terms} \end{aligned} \tag{6-2-41}$$

The two signals are added to generate the error signal

$$e(t) = -\frac{1}{2}A \sin(\phi - \hat{\phi}) + \frac{1}{2}n_c(t) \sin(\phi - \hat{\phi} - \theta_m) \\ + \frac{1}{2}n_s(t) \cos(\phi - \hat{\phi} - \theta_m) + \text{double-frequency terms} \quad (6-2-42)$$

This error signal is the input to the loop filter that provides the control signal for the VCO.

We observe that the two quadrature noise components in (6-2-42) appear as additive terms. There is no term involving a product of two noise components as in an M th-power law device, described in the next section. Consequently, there is no additional power loss associated with the decision-feedback PLL.

This M -phase tracking loop has a phase ambiguity of $360^\circ/M$, necessitating the need to differentially encode the information sequence prior to transmission and differentially decode the received sequence after demodulation to recover the information.

The ML estimate in (6-2-38) is also appropriate for QAM. The ML estimate for offset QPSK is also easily obtained (Problem 6-12) by maximizing the log-likelihood function in (6-2-35), with $s_i(t)$ given as

$$s_i(t) = \sum_n I_n g(t - nT) + j \sum_n J_n g(t - nT - \frac{1}{2}T) \quad (6-2-43)$$

where $I_n = \pm 1$ and $J_n = \pm 1$.

Finally, we should also mention that carrier phase recovery for CPM signals can be accomplished in a decision-directed manner by use of a PLL. From the optimum demodulator for CPM signals, which is described in Section 5-3, we can generate an error signal that is filtered in a loop filter whose output drives a PLL.

6-2-5 Non-Decision-Directed Loops

Instead of using a decision-directed scheme to obtain the phase estimate, we may treat the data as random variables and simply average $\Lambda(\phi)$ over these random variables prior to maximization. In order to carry out this integration, we may use either the actual probability distribution function of the data, if it is known or, perhaps, we may assume some probability distribution that might be a reasonable approximation to the true distribution. The following example illustrates the first approach.

Example 6-2-2

Suppose the real signal $s(t)$ carries binary modulation. Then, in a signal interval, we have

$$s(t) = A \cos 2\pi f_c t, \quad 0 \leq t \leq T$$

where $A = \pm 1$ with equal probability. Clearly, the pdf of A is given as

$$p(A) = \frac{1}{2}\delta(A - 1) + \frac{1}{2}\delta(A + 1)$$

Now, the likelihood function $\Lambda(\phi)$ given by (6-2-9) is conditional on a given value of A and must be averaged over the two values. Thus,

$$\begin{aligned}\bar{\Lambda}(\phi) &= \int_{-\infty}^{\infty} \Lambda(\phi)p(A) dA \\ &= \frac{1}{2} \exp \left[\frac{2}{N_0} \int_0^T r(t) \cos(2\pi f_c t + \phi) dt \right] \\ &\quad + \frac{1}{2} \exp \left[-\frac{2}{N_0} \int_0^T r(t) \cos(2\pi f_c t + \phi) dt \right] \\ &= \cosh \left[\frac{2}{N_0} \int_0^T r(t) \cos(2\pi f_c t + \phi) dt \right]\end{aligned}$$

and the corresponding log-likelihood function is

$$\bar{\Lambda}_L(\phi) = \ln \cosh \left[\frac{2}{N_0} \int_0^T r(t) \cos(2\pi f_c t + \phi) dt \right] \quad (6-2-44)$$

If we differentiate $\bar{\Lambda}_L(\phi)$ and set the derivative equal to zero, we obtain the ML estimate for the non-decision-directed estimate. Unfortunately, the functional relationship in (6-2-44) is highly nonlinear and, hence, an exact solution is difficult to obtain. On the other hand, approximations are possible. In particular,

$$\ln \cosh x = \begin{cases} \frac{1}{2}x^2 & (|x| \ll 1) \\ |x| & (|x| \gg 1) \end{cases} \quad (6-2-45)$$

With these approximations, the solution for ϕ becomes tractable.

In this example, we averaged over the two possible values of the information symbol. When the information symbols are M -valued, where M is large, the averaging operation yields highly nonlinear functions of the parameter to be estimated. In such a case, we may simplify the problem by assuming that the information symbols are continuous random variables. For example, we may assume that the symbols are zero-mean gaussian. The following example illustrates this approximation and the resulting form for the average likelihood function.

Example 6-2-3

Let us consider the same signal as in Example 6-2-2, but now we assume that the amplitude A is zero-mean gaussian with unit variance. Thus,

$$p(A) = \frac{1}{\sqrt{2\pi}} e^{-A^2/2}$$

If we average $\Lambda(\phi)$ over the assumed pdf of A , we obtain the average likelihood $\bar{\Lambda}(\phi)$ in the form

$$\bar{\Lambda}(\phi) = C \exp \left\{ \left[\frac{2}{N_0} \int_0^T r(t) \cos(2\pi f_c t + \phi) dt \right]^2 \right\} \quad (6-2-46)$$

and the corresponding log-likelihood as

$$\bar{\Lambda}_L(\phi) = \left[\frac{2}{N_0} \int_0^T r(t) \cos(2\pi f_c t + \phi) dt \right]^2 \quad (6-2-47)$$

We can obtain the ML estimate of ϕ by differentiating $\bar{\Lambda}_L(\phi)$ and setting the derivative to zero.

It is interesting to note that the log-likelihood function is quadratic under the gaussian assumption and that it is approximately quadratic, as indicated in (6-2-45) for small values of the cross-correlation of $r(t)$ with $s(t; \phi)$. In other words, if the cross-correlation over a single interval is small, the gaussian assumption for the distribution of the information symbols yields a good approximation to the log-likelihood function.

In view of these results, we may use the gaussian approximation on all the symbols in the observation interval $T_0 = KT$. Specifically, we assume that the K information symbols are statistically independent and identically distributed. By averaging the likelihood function $\Lambda(\phi)$ over the gaussian pdf for each of the K symbols in the interval $T_0 = KT$, we obtain the result

$$\Lambda(\phi) = C \exp \left\{ \sum_{n=0}^{K-1} \left[\frac{2}{N_0} \int_{nT}^{(n+1)T} r(t) \cos(2\pi f_c t + \phi) dt \right]^2 \right\} \quad (6-2-48)$$

If we take the logarithm of (6-2-48), differentiate the resulting log-likelihood function, and set the derivative equal to zero, we obtain the condition for the ML estimate as

$$\sum_{n=0}^{K-1} \int_{nT}^{(n+1)T} r(t) \cos(2\pi f_c t + \hat{\phi}) dt \int_{nT}^{(n+1)T} r(t) \sin(2\pi f_c t + \hat{\phi}) dt = 0 \quad (6-2-49)$$

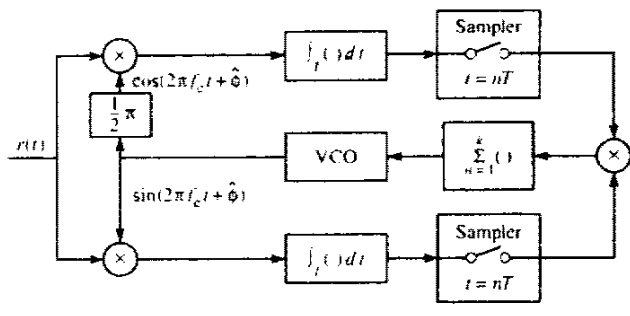


FIGURE 6-2-11 Non-decision-directed PLL for carrier phase estimations of PAM signals.

Although this equation can be manipulated further, its present form suggests the tracking loop configuration illustrated in Fig. 6-2-11. This loop resembles a Costas loop, which is described below. We note that the multiplication of the two signals from the integrators destroys the sign carried by the information symbols. The summer plays the role of the loop filter. In a tracking loop configuration, the summer may be implemented either as a sliding-window digital filter (summer) or as a lowpass digital filter with exponential weighting of the past data.

In a similar manner, one can derive non-decision directed ML phase estimates for QAM and *M*-PSK. The starting point is to average the likelihood function given by (6-2-9) over the statistical characteristics of the data. Here again, we may use the gaussian approximation (two-dimensional gaussian for complex-valued information symbols) in averaging over the information sequence.

Squaring Loop The squaring loop is a non-decision-directed loop that is widely used in practice to establish the carrier phase of double-sideband suppressed carrier signals such as PAM. To describe its operation, consider the problem of estimating the carrier phase of the digitally modulated PAM signal of the form

$$s(t) = A(t) \cos (2\pi f_c t + \phi) \tag{6-2-50}$$

where *A(t)* carries the digital information. Note that $E[s(t)] = E[A(t)] = 0$ when the signal levels are symmetric about zero. Consequently, the average value of *s(t)* does not produce any phase coherent frequency components at any frequency, including the carrier. One method for generating a carrier from the received signal is to square the signal and, thus, to generate a frequency component at $2f_c$, which can be used to drive a phase-locked loop (PLL) tuned to $2f_c$. This method is illustrated in the block diagram shown in Fig. 6-2-12.

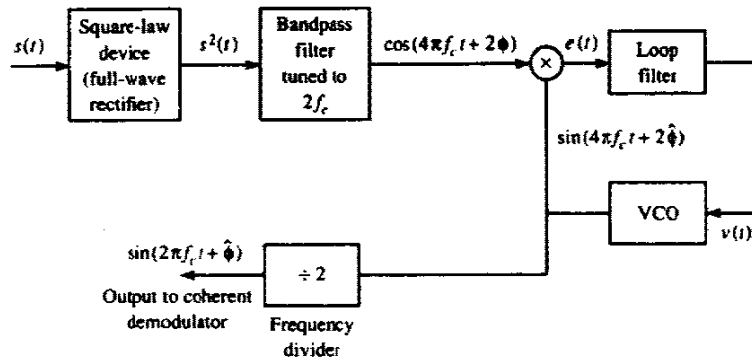


FIGURE 6-2-12 Carrier recovery using a square-law device.

The output of the square-law device is

$$\begin{aligned} s^2(t) &= A^2(t) \cos^2(2\pi f_c t + \phi) \\ &= \frac{1}{2}A^2(t) + \frac{1}{2}A^2(t) \cos(4\pi f_c t + 2\phi) \end{aligned} \quad (6-2-51)$$

Since the modulation is a cyclostationary stochastic process, the expected value of $s^2(t)$ is

$$E[s^2(t)] = \frac{1}{2}E[A^2(t)] + \frac{1}{2}E[A^2(t)] \cos(4\pi f_c t + 2\phi) \quad (6-2-52)$$

Hence, there is power at the frequency $2f_c$.

If the output of the square-law device is passed through a bandpass filter tuned to the double-frequency term in (6-2-51), the mean value of the filter is a sinusoid with frequency $2f_c$, phase 2ϕ , and amplitude $\frac{1}{2}E[A^2(t)]H(2f_c)$, where $H(2f_c)$ is the gain of the filter at $f = 2f_c$. Thus, the square-law device has produced a periodic component from the input signal $s(t)$. In effect, the squaring of $s(t)$ has removed the sign information contained in $A(t)$ and, thus, has resulted in phase-coherent frequency components at twice the carrier. The filtered frequency component at $2f_c$ is then used to drive the PLL.

The squaring operation leads to a noise enhancement that increases the noise power level at the input to the PLL and results in an increase in the variance of the phase error.

To elaborate on this point, let the input to the squarer be $s(t) + n(t)$, where $s(t)$ is given by (6-2-50) and $n(t)$ represents the bandpass additive gaussian noise process. By squaring $s(t) + n(t)$, we obtain

$$y(t) = s^2(t) + 2s(t)n(t) + n^2(t) \quad (6-2-53)$$

where $s^2(t)$ is the desired signal component and the other two components are the signal \times noise and noise \times noise terms. By computing the autocorrelation functions and power density spectra of these two noise components, one can

easily show that both components have spectral power in the frequency band centered at $2f_c$. Consequently, the bandpass filter with bandwidth B_{bp} centered at $2f_c$, which produces the desired sinusoidal signal component that drives the PLL, also passes noise due to these two terms.

Since the bandwidth of the loop is designed to be significantly smaller than the bandwidth B_{bp} of the bandpass filter, the total noise spectrum at the input to the PLL may be approximated as a constant within the loop bandwidth. This approximation allows us to obtain a simple expression for the variance of the phase error as

$$\sigma_{\hat{\theta}}^2 = 1/\gamma_L S_L \tag{6-2-54}$$

where S_L is called the squaring loss and is given by

$$S_L = \left(1 + \frac{B_{bp}/2B_{eq}}{\gamma_L}\right)^{-1} \tag{6-2-55}$$

Since $S_L < 1$, S_L^{-1} represents the increase in the variance of the phase error caused by the added noise (noise \times noise terms) that results from the squarer. Note, for example, that when $\gamma_L = B_{bp}/2B_{eq}$, the loss is 3 dB.

Finally, we observe that the output of the VCO from the squaring loop must be frequency-divided by 2 to generate the phase-locked carrier for signal demodulation. It should be noted that the output of the frequency divider has a phase ambiguity of 180° relative to the phase of the received signal. For this reason, the binary data must be differentially encoded prior to transmission and differentially decoded at the receiver.

Costas Loop Another method for generating a properly phased carrier for a double-sideband suppressed carrier signal is illustrated by the block diagram shown in Fig. 6-2-13. This scheme was developed by Costas (1956) and is

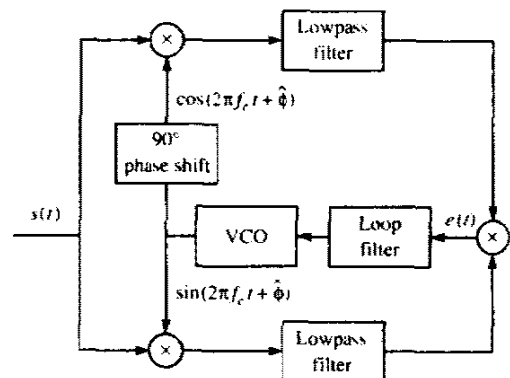


FIGURE 6-2-13 Block diagram of Costas loop.

called the *Costas loop*. The received signal is multiplied by $\cos(2\pi f_c t + \hat{\phi})$ and $\sin(2\pi f_c t + \hat{\phi})$, which are outputs from the VCO. The two products are

$$\begin{aligned}
 y_c(t) &= [s(t) + n(t)] \cos(2\pi f_c t + \hat{\phi}) \\
 &= \frac{1}{2}[A(t) + n_c(t)] \cos \Delta\phi + \frac{1}{2}n_s(t) \sin \Delta\phi \\
 &\quad + \text{double-frequency terms} \\
 y_s(t) &= [s(t) + n(t)] \sin(2\pi f_c t + \hat{\phi}) \\
 &= \frac{1}{2}[A(t) + n_c(t)] \sin \Delta\phi - \frac{1}{2}n_s(t) \cos \Delta\phi \\
 &\quad + \text{double-frequency terms}
 \end{aligned} \tag{6-2-56}$$

where the phase error $\Delta\phi = \hat{\phi} - \phi$. The double-frequency terms are eliminated by the lowpass filters following the multiplications.

An error signal is generated by multiplying the two outputs of the lowpass filters. Thus,

$$\begin{aligned}
 e(t) &= \frac{1}{4}\{[A(t) + n_c(t)]^2 - n_s^2(t)\} \sin(2\Delta\phi) \\
 &\quad - \frac{1}{4}n_s(t)[A(t) + n_c(t)] \cos(2\Delta\phi)
 \end{aligned} \tag{6-2-57}$$

This error signal is filtered by the loop filter, whose output is the control voltage that drives the VCO. The reader should note the similarity of the Costas loop to the PLL shown in Fig. 6-2-11.

We note that the error signal into the loop filter consists of the desired term $A^2(t) \sin 2(\hat{\phi} - \phi)$ plus terms that involve signal \times noise and noise \times noise. These terms are similar to the two noise terms at the input to the PLL for the squaring method. In fact, if the loop filter in the Costas loop is identical to that used in the squaring loop, the two loops are equivalent. Under this condition, the probability density function of the phase error and the performance of the two loops are identical.

It is interesting to note that the optimum lowpass filter for rejecting the double-frequency terms in the Costas loop is a filter matched to the signal pulse in the information-bearing signal. If matched filters are employed for the low pass filters, their outputs could be sampled at the bit rate, at the end of each signal interval, and the discrete-time signal samples could be used to drive the loop. The use of the matched filter results in a smaller noise into the loop.

Finally, we note that, as in the squaring PLL, the output of the VCO contains a phase ambiguity of 180° , necessitating the need for differential encoding of the data prior to transmission and differential decoding at the demodulator.

Carrier Estimation for Multiple Phase Signals When the digital information is transmitted via M -phase modulation of a carrier, the methods described above can be generalized to provide the properly phased carrier for

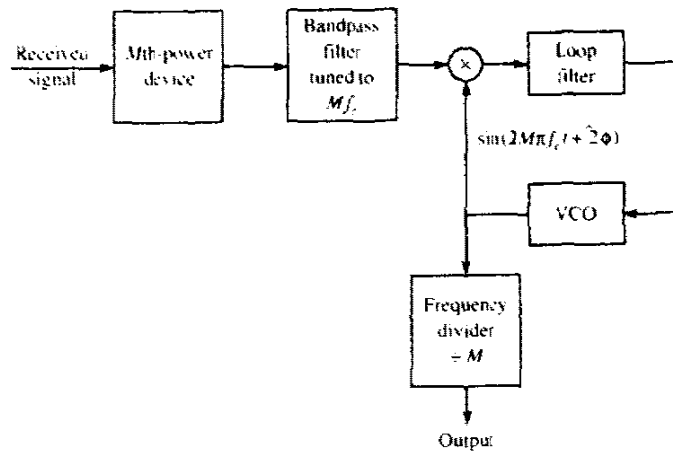


FIGURE 6-2-14 Carrier recovery with M th power law device for M -ary PSK.

demodulation. The received M -phase signal, excluding the additive noise, may be expressed as

$$s(t) = A \cos \left[2\pi f_c t + \phi + \frac{2\pi}{M}(m-1) \right], \quad m = 1, 2, \dots, M \quad (6-2-58)$$

where $2\pi(m-1)/M$ represents the information-bearing component of the signal phase. The problem in carrier recovery is to remove the information-bearing component and, thus, to obtain the unmodulated carrier $\cos(2\pi f_c t + \phi)$. One method by which this can be accomplished is illustrated in Fig. 6-2-14, which represents a generalization of the squaring loop. The signal is passed through an M th-power-law device, which generates a number of harmonics of f_c . The bandpass filter selects the harmonic $\cos(2\pi M f_c t + M\phi)$ for driving the PLL. The term

$$\frac{2\pi}{M}(m-1)M = 2\pi(m-1) \equiv 0 \pmod{2\pi}, \quad m = 1, 2, \dots, M$$

Thus, the information is removed. The VCO output is $\sin(2\pi M f_c t + M\hat{\phi})$, so this output is divided in frequency by M to yield $\sin(2\pi f_c t + \hat{\phi})$, and phase-shifted by $\frac{1}{2}\pi$ rad to yield $\cos(2\pi f_c t + \hat{\phi})$. These components are then fed to the demodulator. Although not explicitly shown, there is a phase ambiguity in these reference sinusoids of $360^\circ/M$, which can be overcome by differential encoding of the data at the transmitter and differential decoding after demodulation at the receiver.

Just as in the case of the squaring PLL, the M th-power PLL operates in the presence of noise that has been enhanced by the M th-power-law device, which results in the output

$$y(t) = [s(t) + n(t)]^M$$

The variance of the phase error in the PLL resulting from the additive noise may be expressed in the simple form

$$\sigma_{\hat{\phi}}^2 = \frac{S_{ML}^{-1}}{\gamma_L} \quad (6-2-59)$$

where γ_L is the loop SNR and S_{ML}^{-1} is the M -phase power loss. S_{ML} has been evaluated by Lindsey and Simon (1973) for $M = 4$ and 8.

Another method for carrier recovery in M -ary PSK is based on a generalization of the Costas loop. That method requires multiplying the received signal by M phase-shifted carriers of the form

$$\sin \left[2\pi f_c t + \hat{\phi} + \frac{\pi}{M}(k-1) \right], \quad k = 1, 2, \dots, M$$

lowpass-filtering each product, and then multiplying the outputs of the lowpass filters to generate the error signal. The error signal excites the loop filter, which, in turn, provides the control signal for the VCO. This method is relatively complex to implement and, consequently, has not been generally used in practice.

Comparison of Decision-Directed with Non-Decision-Directed Loops

We note that the decision-feedback phase-locked loop (DFPLL) differs from the Costas loop only in the method by which $A(t)$ is rectified for the purpose of removing the modulation. In the Costas loop, each of the two quadrature signals used to rectify $A(t)$ is corrupted by noise. In the DFPLL, only one of the signals used to rectify $A(t)$ is corrupted by noise. On the other hand, the squaring loop is similar to the Costas loop in terms of the noise effect on the estimate $\hat{\phi}$. Consequently, the DFPLL is superior in performance to both the Costas loop and the squaring loop, provided that the demodulator is operating at error rates below 10^{-2} where an occasional decision error has a negligible effect on $\hat{\phi}$. Quantitative comparisons of the variance of the phase errors in a Costas loop to those in a DFPLL have been made by Lindsey and Simon (1973), and show that the variance of the DFPLL is 4–10 times smaller for signal-to-noise ratios per bit above 0 db.

6-3 SYMBOL TIMING ESTIMATION

In a digital communication system, the output of the demodulator must be sampled periodically at the symbol rate, at the precise sampling time instants $t_m = mT + \tau$, where T is the symbol interval and τ is a nominal time delay that accounts for the propagation time of the signal from the transmitter to the receiver. To perform this periodic sampling, we require a clock signal at the

receiver. The process of extracting such a clock signal at the receiver is usually called *symbol synchronization* or *timing recovery*.

Timing recovery is one of the most critical functions that is performed at the receiver of a synchronous digital communication system. We should note that the receiver must know not only the frequency ($1/T$) at which the outputs of the matched filters or correlators are sampled, but also where to take the samples within each symbol interval. The choice of sampling instant within the symbol interval of duration T is called the *timing phase*.

Symbol synchronization can be accomplished in one of several ways. In some communication systems, the transmitter and receiver clocks are synchronized to a master clock, which provides a very precise timing signal. In this case, the receiver must estimate and compensate for the relative time delay between the transmitted and received signals. Such may be the case for radio communication systems that operate in the very low frequency (VLF) band (below 30 kHz), where precise clock signals are transmitted from a master radio station.

Another method for achieving symbol synchronization is for the transmitter to simultaneously transmit the clock frequency $1/T$ or a multiple of $1/T$ along with the information signal. The receiver may simply employ a narrowband filter tuned to the transmitted clock frequency and, thus, extract the clock signal for sampling. This approach has the advantage of being simple to implement. There are several disadvantages, however. One is that the transmitter must allocate some of its available power to the transmission of the clock signal. Another is that some small fraction of the available channel bandwidth must be allocated for the transmission of the clock signal. In spite of these disadvantages, this method is frequently used in telephone transmission systems that employ large bandwidths to transmit the signals of many users. In such a case, the transmission of a clock signal is shared in the demodulation of the signals among the many users. Through this shared use of the clock signal, the penalty in transmitter power and in bandwidth allocation is reduced proportionally by the number of users.

A clock signal can also be extracted from the received data signal. There are a number of different methods that can be used at the receiver to achieve self-synchronization. In this section, we treat both decision-directed and non-decision-directed methods.

6-3-1 Maximum-Likelihood Timing Estimation

Let us begin by obtaining the ML estimate of the time delay τ . If the signal is a baseband PAM waveform, it is represented as

$$r(t) = s(t; \tau) + n(t) \quad (6-3-1)$$

where

$$s(t; \tau) = \sum_n I_n g(t - nT - \tau) \quad (6-3-2)$$

As in the case of ML phase estimation, we distinguish between two types of timing estimators, decision-directed timing estimators and non-decision-directed estimators. In the former, the information symbols from the output of the demodulator are treated as the known transmitted sequence. In this case, the log-likelihood function has the form

$$\Lambda_L(\tau) = C_L \int_{T_0} r(t) s(t; \tau) dt \tag{6-3-3}$$

If we substitute (6-3-2) into (6-3-3), we obtain

$$\begin{aligned} \Lambda_L(\tau) &= C_L \sum_n I_n \int_{T_0} r(t) g(t - nT - \tau) dt \\ &= C_L \sum_n I_n y_n(\tau) \end{aligned} \tag{6-3-4}$$

where $y_n(t)$ is defined as

$$y_n(\tau) = \int_{T_0} r(t) g(t - nT - \tau) dt \tag{6-3-5}$$

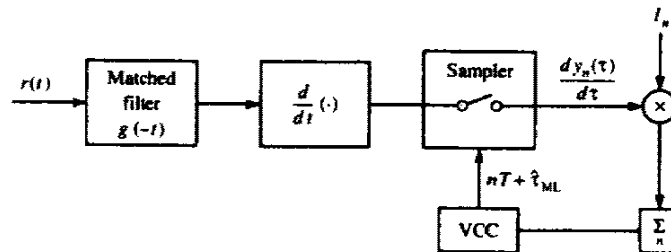
A necessary condition for $\hat{\tau}$ to be the ML estimate of τ is that

$$\begin{aligned} \frac{d\Lambda_L(\tau)}{d\tau} &= \sum_n I_n \frac{d}{d\tau} \int_{T_0} r(t) g(t - nT - \tau) dt \\ &= \sum_n I_n \frac{d}{d\tau} [y_n(\tau)] = 0 \end{aligned} \tag{6-3-6}$$

The result in (6-3-6) suggests the implementation of the tracking loop shown in Fig. 6-3-1. We should observe that the summation in the loop serves as the loop filter whose bandwidth is controlled by the length of the sliding window in the summation. The output of the loop filter drives the voltage-controlled clock (VCC), or voltage-controlled oscillator, which controls the sampling times for the input to the loop. Since the detected information sequence $\{I_n\}$ is used in the estimation of τ , the estimate is decision-directed.

The techniques described above for ML timing estimation of baseband

FIGURE 6-3-1 Decision-directed ML estimation of timing for baseband PAM.



PAM signals can be extended to carrier modulated signal formats such as QAM and PSK in a straightforward manner, by dealing with the equivalent lowpass form of the signals. Thus, the problem of ML estimation of symbol timing for carrier signals is very similar to the problem formulation for the baseband PAM signal.

6-3-2 Non-Decision-Directed Timing Estimation

A non-decision-directed timing estimate can be obtained by averaging the likelihood ratio $\Lambda(\tau)$ over the pdf of the information symbols, to obtain $\bar{\Lambda}(\tau)$, and then differentiating either $\bar{\Lambda}(\tau)$ or $\ln \bar{\Lambda}(\tau) = \bar{\Lambda}_L(\tau)$ to obtain the condition for the maximum-likelihood estimate $\hat{\tau}_{ML}$.

In the case of binary (baseband) PAM, where $I_n = \pm 1$ with equal probability, the average over the data yields

$$\bar{\Lambda}_L(\tau) = \sum_n \ln \cosh C y_n(\tau) \quad (6-3-7)$$

just as in the case of the phase estimator. Since $\ln \cosh x \approx \frac{1}{2}x^2$ for small x , the square-law approximation

$$\bar{\Lambda}_L(\tau) \approx \frac{1}{2}C^2 \sum_n y_n^2(\tau) \quad (6-3-8)$$

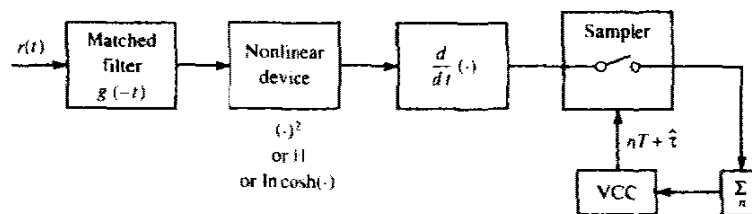
is appropriate for low signal-to-noise ratios. For multilevel PAM, we may approximate the statistical characteristics of the information symbols $\{I_n\}$ by the gaussian pdf, with zero mean and unit variance. When we average $\Lambda(\tau)$ over the gaussian pdf, the logarithm of $\bar{\Lambda}(\tau)$ is identical to $\bar{\Lambda}_L(\tau)$ given by (6-3-8). Consequently, the non-decision-directed estimate of τ may be obtained by differentiating (6-3-8). The result is an approximation to the ML estimate of the delay time. The derivative of (6-3-8) is

$$\frac{d}{d\tau} \sum_n y_n^2(\tau) = 2 \sum_n y_n(\tau) \frac{dy_n(\tau)}{d\tau} = 0 \quad (6-3-9)$$

where $y_n(\tau)$ is given by (6-3-5).

An implementation of a tracking loop based on the derivative of $\bar{\Lambda}_L(\tau)$ given by (6-3-7) is shown in Fig. 6-3-2. Alternatively, an implementation of a

FIGURE 6-3-2 Non-decision-directed estimation of timing for binary baseband PAM.



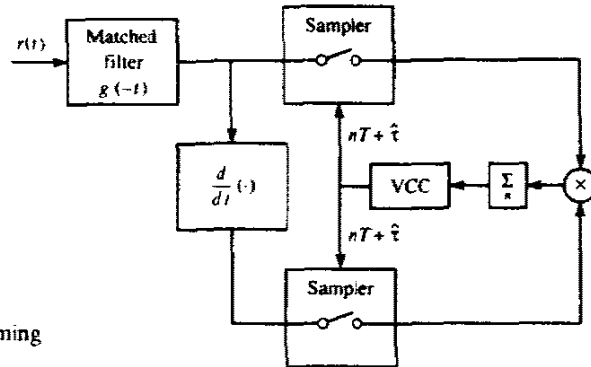


FIGURE 6-3-3 Non-decision-directed estimation of timing for baseband PAM.

tracking loop based on (6-3-9) is illustrated in Fig. 6-3-3. In both structures, we observe that the summation serves as the loop filter that drives the VCC. It is interesting to note the resemblance of the timing loop in Fig. 6-3-3 to the Costas loop for phase estimation.

Early-Late Gate Synchronizers Another non-decision-directed timing estimator exploits the symmetry properties of the signal at the output of the matched filter or correlator. To describe this method, let us consider the rectangular pulse $s(t)$, $0 \leq t \leq T$, shown in Fig. 6-3-4(a). The output of the filter matched to $s(t)$ attains its maximum value at time $t = T$, as shown in Fig. 6-3-4(b). Thus, the output of the matched filter is the time autocorrelation function of the pulse $s(t)$. Of course, this statement holds for any arbitrary pulse shape, so the approach that we describe applies in general to any signal pulse. Clearly, the proper time to sample the output of the matched filter for a maximum output is at $t = T$, i.e., at the peak of the correlation function.

In the presence of noise, the identification of the peak value of the signal is generally difficult. Instead of sampling the signal at the peak, suppose we sample early, at $t = T - \delta$ and late at $t = T + \delta$. The absolute values of the early samples $|y(m(T - \delta))|$ and the late samples $|y(m(T + \delta))|$ will be smaller (on the average in the presence of noise) than the samples of the peak value $|y(mT)|$. Since the autocorrelation function is even with respect to the optimum sampling time $t = T$, the absolute values of the correlation function at $t = T - \delta$ and $t = T + \delta$ are equal. Under this condition, the proper sampling

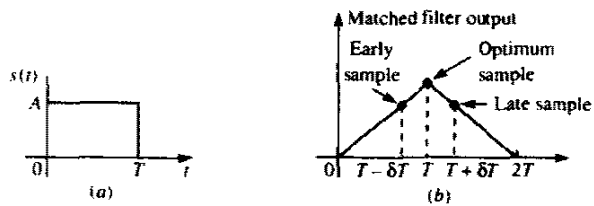


FIGURE 6-3-4 Rectangular signal pulse (a) and its matched filter output (b).

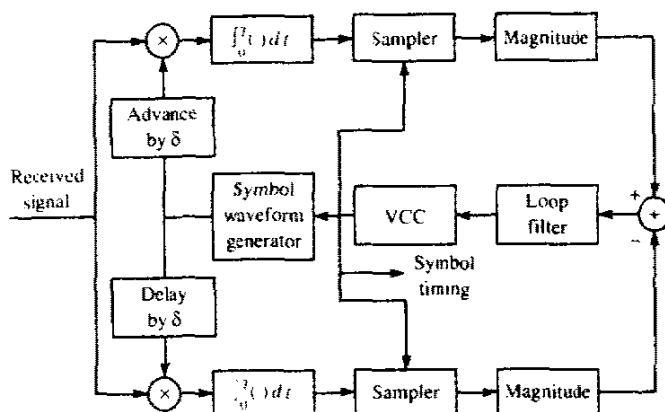


FIGURE 6-3-5 Block diagram of early-late gate synchronizer.

time is the midpoint between $t = T - \delta$ and $t = T + \delta$. This condition forms the basis for the *early-late gate symbol synchronizer*.

Figure 6-3-5 illustrates the block diagram of an early-late gate synchronizer. In this figure, correlators are used in place of the equivalent matched filters. The two correlators integrate over the symbol interval T , but one correlator starts integrating δ seconds early relative to the estimated optimum sampling time and the other integrator starts integrating δ seconds late relative to the estimated optimum sampling time. An error signal is formed by taking the difference between the absolute values of the two correlator outputs. To smooth the noise corrupting the signal samples, the error signal is passed through a lowpass filter. If the timing is off relative to the optimum sampling time, the average error signal at the output of the lowpass filter is nonzero, and the clock signal is either retarded or advanced, depending on the sign of the error. Thus, the smoothed error signal is used to drive a voltage-controlled clock (VCC), whose output is the desired clock signal that is used for sampling. The output of the VCC is also used as a clock signal for a symbol waveform generator that puts out the same basic pulse waveform as that of the transmitting filter. This pulse waveform is advanced and delayed and then fed to the two correlators, as shown in Fig. 6-3-5. Note that if the signal pulses are rectangular, there is no need for a signal pulse generator within the tracking loop.

We observe that the early-late gate synchronizer is basically a closed-loop control system whose bandwidth is relatively narrow compared to the symbol rate $1/T$. The bandwidth of the loop determines the quality of the timing estimate. A narrowband loop provides more averaging over the additive noise and, thus, improves the quality of the estimated sampling instants, provided that the channel propagation delay is constant and the clock oscillator at the transmitter is not drifting with time (or drifting very slowly with time). On the other hand, if the channel propagation delay is changing with time and/or the

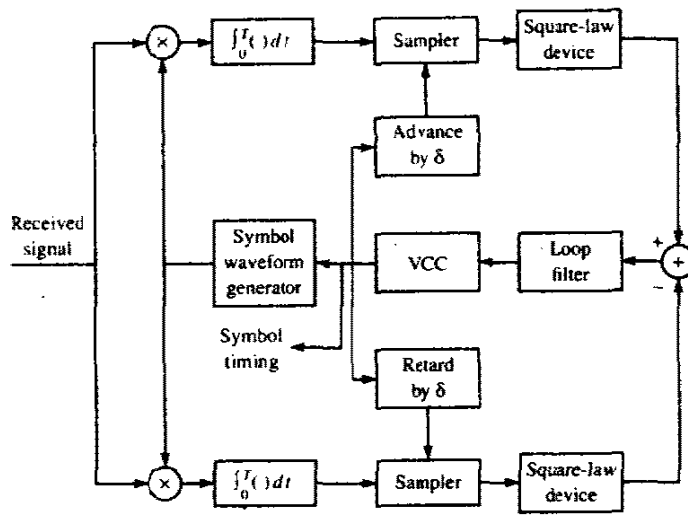


FIGURE 6-3-6 Block diagram of early-late gate synchronizer—an alternative form.

transmitter clock is also drifting with time then the bandwidth of the loop must be increased to provide for faster tracking of time variations in symbol timing.

In the tracking mode, the two correlators are affected by adjacent symbols. However, if the sequence of information symbols has zero mean, as is the case for PAM and some other signal modulations, the contribution to the output of the correlators from adjacent symbols averages out to zero in the lowpass filter.

An equivalent realization of the early-late gate synchronizer that is somewhat easier to implement is shown in Fig. 6-3-6. In this case the clock signal from the VCC is advanced and delayed by δ , and these clock signals are used to sample the outputs of the two correlators.

The early-late gate synchronizer described above is a non-decision-directed estimator of symbol timing that approximates the maximum-likelihood estimator. This assertion can be demonstrated by approximating the derivative of the log-likelihood function by the finite difference, i.e.,

$$\frac{d\bar{\Lambda}_L(\tau)}{d\tau} \approx \frac{\bar{\Lambda}_L(\tau + \delta) - \bar{\Lambda}_L(\tau - \delta)}{2\delta} \tag{6-3-10}$$

If we substitute for $\bar{\Lambda}_L(\tau)$ from (6-3-8) into (6-3-10), we obtain the approximation for the derivative as

$$\begin{aligned} \frac{d\bar{\Lambda}_L(\tau)}{d\tau} &= \frac{C^2}{4\delta} \sum_n [y_n^2(\tau + \delta) - y_n^2(\tau - \delta)] \\ &\approx \frac{C^2}{4\delta} \sum_n \left\{ \left[\int_{T_0} r(t)g(t - nT - \tau - \delta) dt \right]^2 \right. \\ &\quad \left. - \left[\int_{T_0} r(t)g(t - nT - \tau + \delta) dt \right]^2 \right\} \end{aligned} \tag{6-3-11}$$

But the mathematical expression in (6-3-11) basically describes the functions performed by the early-late gate symbol synchronizers illustrated in Figs 6-3-5 and 6-3-6.

6-4 JOINT ESTIMATION OF CARRIER PHASE AND SYMBOL TIMING

The estimation of the carrier phase and symbol timing may be accomplished separately as described above or jointly. Joint ML estimation of two or more signal parameters yields estimates that are as good and usually better than the estimates obtained from separate optimization of the likelihood function. In other words, the variances of the signal parameters obtained from joint optimization are less than or equal to the variance of parameter estimates obtained from separately optimizing the likelihood function.

Let us consider the joint estimation of the carrier phase and symbol timing. The log-likelihood function for these two parameters may be expressed in terms of the equivalent lowpass signals as

$$\Lambda_L(\phi, \tau) = \text{Re} \left[\frac{1}{N_0} \int_{\tau_0} r(t) s_l^*(t; \phi, \tau) dt \right] \quad (6-4-1)$$

where $s_l(t; \phi, \tau)$ is the equivalent lowpass signal, which has the general form

$$s_l(t; \phi, \tau) = e^{-j\phi} \left[\sum_n I_n g(t - nT - \tau) + j \sum_n J_n w(t - nT - \tau) \right] \quad (6-4-2)$$

where $\{I_n\}$ and $\{J_n\}$ are the two information sequences.

We note that, for PAM, we may set $J_n = 0$ for all n , and the sequence $\{I_n\}$ is real. For QAM and PSK, we set $J_n = 0$ for all n and the sequence $\{I_n\}$ is complex-valued. For offset QPSK, both sequences $\{I_n\}$ and $\{J_n\}$ are nonzero and $w(t) = g(t - \frac{1}{2}T)$.

For decision-directed ML estimation of ϕ and τ , the log-likelihood function becomes

$$\Lambda_L(\phi, \tau) = \text{Re} \left\{ \frac{e^{j\phi}}{N_0} \sum_n [I_n^* y_n(\tau) + j J_n^* x_n(\tau)] \right\} \quad (6-4-3)$$

where

$$y_n(\tau) = \int_{\tau_0} r(t) g^*(t - nT - \tau) dt \quad (6-4-4)$$

$$x_n(\tau) = \int_{\tau_0} r(t) w^*(t - nT - \tau) dt$$

Necessary conditions for the estimates of ϕ and τ to be the ML estimates are

$$\frac{\partial \Lambda_L(\phi, \tau)}{\partial \phi} = 0, \quad \frac{\partial \Lambda_L(\phi, \tau)}{\partial \tau} = 0 \quad (6-4-5)$$

It is convenient to define

$$A(\tau) + jB(\tau) = \frac{1}{N_0} \sum_n [I_n^* y_n(\tau) + jJ_n^* x_n(\tau)] \quad (6-4-6)$$

With this definition, (6-4-3) may be expressed in the simple form

$$\Lambda_L(\phi, \tau) = A(\tau) \cos \phi - B(\tau) \sin \phi \quad (6-4-7)$$

Now the conditions in (6-4-5) for the joint ML estimates become

$$\frac{\partial \Lambda(\phi, \tau)}{\partial \phi} = -A(\tau) \sin \phi - B(\tau) \cos \phi = 0 \quad (6-4-8)$$

$$\frac{\partial \Lambda(\phi, \tau)}{\partial \tau} = \frac{\partial A(\tau)}{\partial \tau} \cos \phi - \frac{\partial B(\tau)}{\partial \tau} \sin \phi = 0 \quad (6-4-9)$$

From (6-4-8), we obtain

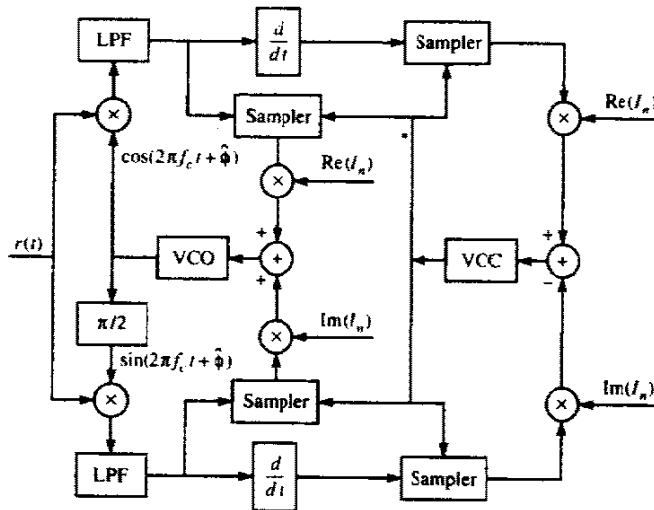
$$\hat{\phi}_{ML} = -\tan^{-1} \left[\frac{B(\hat{\tau}_{ML})}{A(\hat{\tau}_{ML})} \right] \quad (6-4-10)$$

The solution to (6-4-9) that incorporates (6-4-10) is

$$\left[A(\tau) \frac{\partial A(\tau)}{\partial \tau} + B(\tau) \frac{\partial B(\tau)}{\partial \tau} \right]_{\tau=\hat{\tau}_{ML}} = 0 \quad (6-4-11)$$

The decision-directed tracking loop for QAM (or PSK) obtained from these equations is illustrated in Fig. 6-4-1.

FIGURE 6-4-1 Decision-directed joint tracking loop for carrier phase and symbol timing in QAM and PSK.



Offset QPSK requires a slightly more complex structure for joint estimation of ϕ and τ . The structure is easily derived from (6-4-6)–(6-4-11).

In addition to the joint estimates given above, it is also possible to derive non-decision-directed estimates of the carrier phase and symbol timing, although we shall not pursue this approach.

We should also mention that one can combine the parameter estimation problem with the demodulation of the information sequence $\{I_n\}$. Thus, one can consider the joint maximum-likelihood estimation of $\{I_n\}$, the carrier phase ϕ , and the symbol timing parameter τ . Results on these joint estimation problems have appeared in the technical literature, e.g. Kobayashi (1971), Falconer (1976), and Falconer and Salz (1977).

6-5 PERFORMANCE CHARACTERISTICS OF ML ESTIMATORS

The quality of a signal parameter estimate is usually measured in terms of its bias and its variance. In order to define these terms, let us assume that we have a sequence of observations $[x_1 \ x_2 \ x_3 \ \dots \ x_n] = \mathbf{x}$, with pdf $p(\mathbf{x} | \phi)$, from which we extract an estimate of a parameter ϕ . The bias of an estimate, say $\hat{\phi}(\mathbf{x})$, is defined as

$$\text{bias} = E\{\hat{\phi}(\mathbf{x})\} - \phi \quad (6-5-1)$$

where ϕ is the true value of the parameter. When $E\{\hat{\phi}(\mathbf{x})\} = \phi$, we say that the estimate is *unbiased*. The variance of the estimate $\hat{\phi}(\mathbf{x})$ is defined as

$$\sigma_{\hat{\phi}}^2 = E\{[\hat{\phi}(\mathbf{x})]^2\} - \{E[\hat{\phi}(\mathbf{x})]\}^2 \quad (6-5-2)$$

In general $\sigma_{\hat{\phi}}^2$ may be difficult to compute. However, a well-known result in parameter estimation (see Helstrom, 1968) is the Cramér–Rao lower bound on the mean square error defined as

$$E\{[\hat{\phi}(\mathbf{x}) - \phi]^2\} \geq \left[\frac{\partial}{\partial \phi} E\{\hat{\phi}(\mathbf{x})\} \right]^2 / E\left\{ \left[\frac{\partial}{\partial \phi} \ln p(\mathbf{x} | \phi) \right]^2 \right\} \quad (6-5-3)$$

Note that when the estimate is unbiased, the numerator of (6-5-3) is unity and the bound becomes a lower bound on the variance $\sigma_{\hat{\phi}}^2$ of the estimate $\hat{\phi}(\mathbf{x})$, i.e.,

$$\sigma_{\hat{\phi}}^2 \geq 1 / E\left\{ \left[\frac{\partial}{\partial \phi} \ln p(\mathbf{x} | \phi) \right]^2 \right\} \quad (6-5-4)$$

Since $\ln p(\mathbf{x} | \phi)$ differs from the log-likelihood function by a constant factor

independent of ϕ , it follows that

$$\begin{aligned} E\left\{\left[\frac{\partial}{\partial\phi}\ln p(\mathbf{x}|\phi)\right]^2\right\} &= E\left\{\left[\frac{\partial}{\partial\phi}\ln\Lambda(\phi)\right]^2\right\} \\ &= -E\left\{\frac{\partial^2}{\partial\phi^2}\ln\Lambda(\phi)\right\} \end{aligned} \quad (6-5-5)$$

Therefore, the lower bound on the variance is

$$\sigma_{\hat{\phi}}^2 \geq 1/E\left\{\left[\frac{\partial}{\partial\phi}\ln\Lambda(\phi)\right]^2\right\} = -1/E\left\{\frac{\partial^2}{\partial\phi^2}\ln\Lambda(\phi)\right\} \quad (6-5-6)$$

This lower bound is a very useful result. It provides a benchmark for comparing the variance of any practical estimate to the lower bound. Any estimate that is unbiased and whose variance attains the lower bound is called an *efficient estimate*.

In general, efficient estimates are rare. When they exist, they are maximum-likelihood estimates. A well-known result from parameter estimation theory is that any ML parameter estimate is asymptotically (arbitrarily large number of observations) unbiased and efficient. To a large extent, these desirable properties constitute the importance of ML parameter estimates. It also known that an ML estimate is asymptotically gaussian-distributed [with mean ϕ and variance equal to the lower bound given by (6-5-6).]

In the case of the ML estimates described in this chapter for the two signal parameters, their variance is generally inversely proportional to the signal-to-noise ratio, or, equivalently, inversely proportional to the signal power multiplied by the observation interval T_0 . Furthermore, the variance of the decision-directed estimates, at low error probabilities, are generally lower than the variance of non-decision-directed estimates. In fact, the performance of the ML decision-directed estimates for ϕ and τ attain the lower bound.

The following example is concerned with the evaluation of the Cramér-Rao lower bound for the ML estimate of the carrier phase.

Example 6-5-1

The ML estimate of the phase of an unmodulated carrier was shown in (6-2-11) to satisfy the condition

$$\int_{T_0} r(t) \sin(2\pi f_c t + \hat{\phi}_{\text{ML}}) dt = 0 \quad (6-5-7)$$

where

$$\begin{aligned} r(t) &= s(t; \phi) + n(t) \\ &= A \cos(2\pi f_c t + \phi) + n(t) \end{aligned} \quad (6-5-8)$$

The condition in (6-5-7) was derived by maximizing the log likelihood function

$$\Lambda_L(\phi) = \frac{2}{N_0} \int_{T_0} r(t) s(t; \phi) dt \quad (6-5-9)$$

The variance of $\hat{\phi}_{ML}$ is lower-bounded as

$$\begin{aligned} \sigma_{\hat{\phi}_{ML}}^2 &\geq \left\{ \frac{2A}{N_0} \int_{T_0} E[r(t)] \cos(2\pi f_c t + \phi) dt \right\}^{-1} \\ &\geq \left\{ \frac{A^2}{N_0} \int_{T_0} dt \right\}^{-1} = \frac{N_0}{A^2 T_0} \\ &\geq \frac{N_0/2T_0}{\frac{1}{2}A^2} = \frac{N_0 B_{eq}}{\frac{1}{2}A^2} \end{aligned} \tag{6-5-10}$$

The factor $1/2T_0$ is simply the (one-sided) equivalent noise bandwidth of the ideal integrator.

From this example, we observe that the variance of the ML phase estimate is lower-bounded as

$$\sigma_{\hat{\phi}_{ML}}^2 \geq 1/\gamma_L \tag{6-5-11}$$

where $\gamma_L = A^2/2N_0B_{eq}$ is the loop SNR. This is also the variance obtained for the phase estimate from a PLL with decision-directed estimation. As we have already observed, non-decision-directed estimates do not perform as well due to losses in the nonlinearities required to remove the modulation, e.g., the squaring loss and the M th-power loss.

Similar results can be obtained on the quality of the symbol timing estimates derived above. In addition to their dependence on the SNR, the quality of symbol timing estimates is a function of the signal pulse shape. For example, a pulse shape that is commonly used in practice is one that has a raised cosine spectrum (see Section 9-2). For such a pulse, the rms timing error (σ_t) as a function of SNR is illustrated in Fig. 6-5-1, for both decision-directed and

FIGURE 6-5-1 Performance of baseband symbol timing estimate for fixed signal and loop bandwidths. [From Synchronization Subsystems: Analysis and Design, by L. Franks, 1983. Reprinted with permission of the author.]

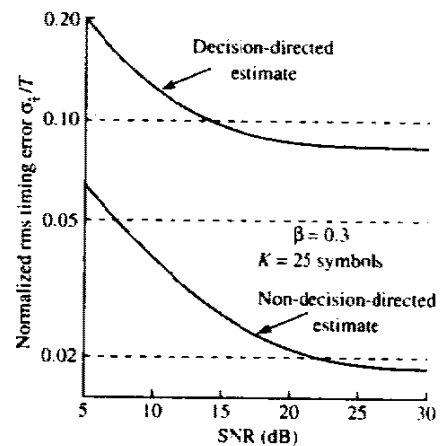
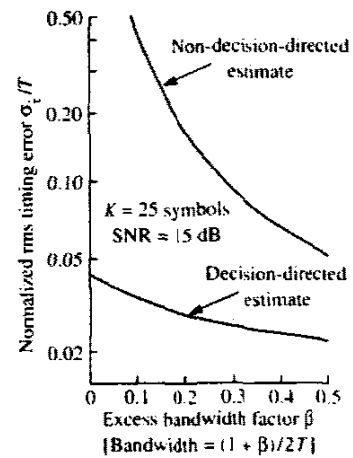


FIGURE 6-5-2 Performance of baseband symbol timing estimate for fixed SNR and fixed loop bandwidth. [From *Synchronization Subsystems: Analysis and Design*, by L. Franks, 1983. Reprinted with permission of the author.]



non-decision-directed estimates. Note the significant improvement in performance of the decision-directed estimate compared with the non-decision-directed estimate. Now, if the bandwidth of the pulse is varied, the pulse shape is changed and, hence, the rms value of the timing error also changes. For example, when the bandwidth of the pulse that has a raised cosine spectrum is varied, the rms timing error varies as shown in Fig. 6-5-2. Note that the error decreases as the bandwidth of the pulse increases.

In conclusion, we have presented the ML method for signal parameter estimation and have applied it to the estimation of the carrier phase and symbol timing. We have also described their performance characteristics.

6-6 BIBLIOGRAPHICAL NOTES AND REFERENCES

Carrier recovery and timing synchronization are two topics that have been thoroughly investigated over the past three decades. The Costas loop was invented in 1956 and the decision-directed phase estimation methods were described by Proakis *et al.* (1964) and by Natali and Walbesser (1969). The work on decision-directed estimation was motivated by earlier work of Price (1962a, b). Comprehensive treatments of phase-locked loops first appeared in the books by Viterbi (1966) and Gardner (1979). Books that cover carrier phase recovery and time synchronization techniques have been written by Stiffler (1971), Lindsey (1972), Lindsey and Simon (1973), and Meyr and Ascheid (1990).

A number of tutorial papers have appeared in IEEE journals on the PLL and on time synchronization. We cite, for example, the paper by Gupta (1975), which treats both analog and digital implementation of PLLs, and the paper by Lindsey and Chie (1981), which is devoted to the analysis of digital PLLs. In addition, the tutorial paper by Franks (1980) describes both carrier phase and symbol synchronization methods, including methods based on the maximum-likelihood estimation criterion. The paper by Franks is contained in a special

issue of the *IEEE Transactions on Communications* (August 1980) devoted to synchronization. The paper by Mueller and Muller (1976) describes digital signal processing algorithms for extracting symbol timing.

Application of the maximum-likelihood criterion to parameter estimation was first described in the context of radar parameter estimation (range and range rate). Subsequently, this optimal criterion was applied to carrier phase and symbol timing estimation as well as to joint parameter estimation with data symbols. Papers on these topics have been published by several researchers, including Falconer (1976), Mengali (1977), Falconer and Salz (1977), and Meyers and Franks (1980).

The Cramér–Rao lower bound on the variance of a parameter estimate is derived and evaluated in a number of standard texts on detection and estimation theory, such as Helstrom (1968) and Van Trees (1968). It is also described in several books on mathematical statistics, such as the book by Cramér (1946).

PROBLEMS

- 6-1 Prove the relation (6-1-7).
 6-2 Sketch the equivalent realization of the binary PSK receiver in Fig. 6-1-1 that employs a matched filter instead of a correlator.
 6-3 Suppose that the loop filter [see (6-2-14)] for a PLL has the transfer function

$$G(s) = \frac{1}{s + \sqrt{2}}$$

- a Determine the closed-loop transfer function $H(s)$ and indicate if the loop is stable.
 b Determine the damping factor and the natural frequency of the loop.
 6-4 Consider the PLL for estimating the carrier phase of a signal in which the loop filter is specified as

$$G(s) = \frac{K}{1 + \tau_1 s}$$

- a Determine the closed-loop transfer function $H(s)$ and its gain at $f = 0$.
 b For what range of values of τ_1 and K is the loop stable?
 6-5 The loop filter $G(s)$ in a PLL is implemented by the circuit shown in Fig. P6-5. Determine the system function $G(s)$ and express the time constants τ_1 and τ_2 in terms of the circuit parameters.

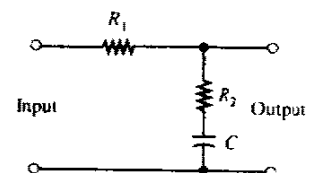


FIGURE P6-5

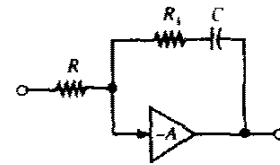


FIGURE P6-6

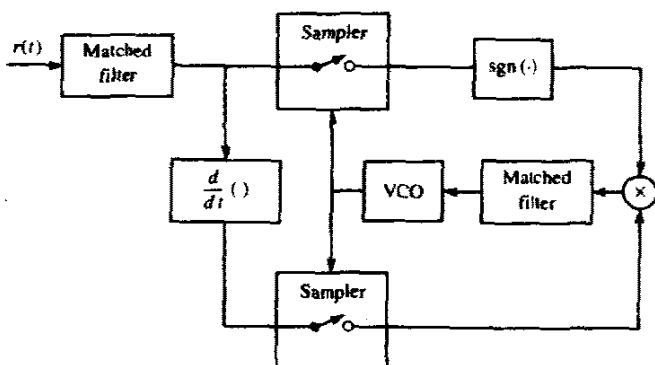
- 6-6 The loop filter $G(s)$ in a PLL is implemented with the active filter shown in Fig. P6-6. Determine the system function $G(s)$ and express the time constants τ_1 and τ_2 in terms of the circuit parameters.
- 6-7 Show that the early-late gate synchronizer illustrated in Fig. 6-3-5 is a close approximation to the timing recovery system illustrated in Fig. P6-7.
- 6-8 Based on a ML criterion, determine a carrier phase estimation method for binary on-off keying modulation.
- 6-9 In the transmission and reception of signals to and from moving vehicles, the transmitted signal frequency is shifted in direct proportion to the speed of the vehicle. The so-called *Doppler frequency shift* imparted to a signal that is received in a vehicle traveling at a velocity v relative to a (fixed) transmitter is given by the formula

$$f_D = \pm \frac{v}{\lambda}$$

where λ is the wavelength, and the sign depends on the direction (moving toward or moving away) that the vehicle is traveling relative to the transmitter. Suppose that a vehicle is traveling at a speed of 100 km/h relative to a base station in a mobile cellular communication system. The signal is a narrowband signal transmitted at a carrier frequency of 1 GHz.

- a Determine the Doppler frequency shift.
- b What should be the bandwidth of a Doppler frequency tracking loop if the loop is designed to track Doppler frequency shifts for vehicles traveling at speeds up to 100 km/h?
- c Suppose the transmitted signal bandwidth is 2 MHz centered at 1 GHz.

FIGURE P6-7



Determine the Doppler frequency spread between the upper and lower frequencies in the signal.

- 6-10** Show that the mean value of the ML estimate in (6-2-38) is ϕ , i.e., that the estimate is unbiased.
- 6-11** Determine the pdf of the ML phase estimate in (6-2-38).
- 6-12** Determine the ML phase estimate for offset QPSK.
- 6-13** A single-sideband PAM signal may be represented as

$$u_m(t) = A_m [g_r(t) \cos 2\pi f_c t - \hat{g}_r(t) \sin 2\pi f_c t]$$

where $\hat{g}_r(t)$ is the Hilbert transform of $g_r(t)$ and A_m is the amplitude level that conveys the information. Demonstrate mathematically that a Costas loop can be used to demodulate the SSB PAM signal.

- 6-14** A carrier component is transmitted on the quadrature carrier in a communication system that transmits information via binary PSK. Hence, the received signal has the form

$$r(t) = \pm \sqrt{2P_s} \cos(2\pi f_c t + \phi) + \sqrt{2P_p} \sin(2\pi f_c t + \phi) + n(t)$$

where ϕ is the carrier phase and $n(t)$ is AWGN. The unmodulated carrier component is used as a pilot signal at the receiver to estimate the carrier phase.

- a** Sketch a block diagram of the receiver, including the carrier phase estimator.
- b** Illustrate mathematically the operations involved in the estimation of the carrier phase ϕ .
- c** Express the probability of error for the detection of the binary PSK signal as a function of the total transmitted power $P_T = P_s + P_p$. What is the loss in performance due to the allocation of a portion of the transmitted power to the pilot signal? Evaluate the loss for $P_p/P_T = 0.1$.
- 6-15** Determine the signal and noise components at the input to a fourth-order ($M = 4$) PLL that is used to generate the carrier phase for demodulation of QPSK. By ignoring all noise components except those that are linear in the noise $n(t)$, determine the variance of the phase estimate at the output of the PLL.
- 6-16** The probability of error for binary PSK demodulation and detection when there is a carrier phase error ϕ_e is

$$P_2(\phi_e) = Q\left(\sqrt{\frac{2E_b}{N_0}} \cos^2 \phi_e\right)$$

Suppose that the phase error from the PLL is modeled as a zero-mean gaussian random variable with variance $\sigma_\phi^2 \ll \pi$. Determine the expression for the average probability of error (in integral form).

7

CHANNEL CAPACITY AND CODING

In Chapter 5, we considered the problem of digital modulation by means of $M = 2^k$ signal waveforms, where each waveform conveys k bits of information. We observed that some modulation methods provide better performance than others. In particular, we demonstrated that orthogonal signaling waveforms allow us to make the probability of error arbitrarily small by letting the number of waveforms $M \rightarrow \infty$, provided that the SNR per bit $\gamma_b \geq -1.6$ dB. Thus, we can operate at the capacity of the additive, white gaussian noise channel in the limit as the bandwidth expansion factor $B_e = W/R \rightarrow \infty$. This is a heavy price to pay, because B_e grows exponentially with the block length k . Such inefficient use of channel bandwidth is highly undesirable.

In this and the following chapter, we consider signal waveforms generated from either binary or nonbinary sequences. The resulting waveforms are generally characterized by a bandwidth expansion factor that grows only linearly with k . Consequently, coded waveforms offer the potential for greater bandwidth efficiency than orthogonal M -ary waveforms. We shall observe that, in general, coded waveforms offer performance advantages not only in power-limited applications where $R/W < 1$, but also in bandwidth-limited systems where $R/W > 1$.

We begin by establishing several channel models that will be used to evaluate the benefits of channel coding, and we shall introduce the concept of channel capacity for the various channel models. Then, we treat the subject of code design for efficient communications.

7-1 CHANNEL MODELS AND CHANNEL CAPACITY

In the model of a digital communication system described in Section 1-1, we recall that the transmitter building blocks consist of the discrete-input, discrete-output channel encoder followed by the modulator. The function of the discrete channel encoder is to introduce, in a controlled manner, some redundancy in the binary information sequence, which can be used at the receiver to overcome the effects of noise and interference encountered in the transmission of the signal through the channel. The encoding process generally involves taking k information bits at a time and mapping each k -bit sequence into a unique n -bit sequence, called a *code word*. The amount of redundancy introduced by the encoding of the data in this manner is measured by the ratio n/k . The reciprocal of this ratio, namely k/n , is called the *code rate*.

The binary sequence at the output of the channel encoder is fed to the modulator, which serves as the interface to the communication channel. As we have discussed, the modulator may simply map each binary digit into one of two possible waveforms, i.e., a 0 is mapped into $s_1(t)$ and a 1 is mapped into $s_2(t)$. Alternatively, the modulator may transmit q -bit blocks at a time by using $M = 2^q$ possible waveforms.

At the receiving end of the digital communication system, the demodulator processes the channel-corrupted waveform and reduces each waveform to a scalar or a vector that represents an estimate of the transmitted data symbol (binary or M -ary). The detector, which follows the demodulator, may decide on whether the transmitted bit is a 0 or a 1. In such a case, the detector has made a *hard decision*. If we view the decision process at the detector as a form of quantization, we observe that a hard decision corresponds to binary quantization of the demodulator output. More generally, we may consider a detector that quantizes to $Q > 2$ levels, i.e., a Q -ary detector. If M -ary signals are used then $Q \geq M$. In the extreme case when no quantization is performed, $Q = \infty$. In the case where $Q > M$, we say that the detector has made a *soft decision*.

The quantized output from the detector is then fed to the channel decoder, which exploits the available redundancy to correct for channel disturbances.

In the following sections, we describe three channel models that will be used to establish the maximum achievable bit rate for the channel.

7-1-1 Channel Models

In this section we describe channel models that will be useful in the design of codes. The simplest is the *binary symmetric channel* (BSC), which corresponds to the case with $M = 2$ and hard decisions at the detector.

Binary Symmetric Channel Let us consider an additive noise channel and let the modulator and the demodulator/detector be included as parts of the

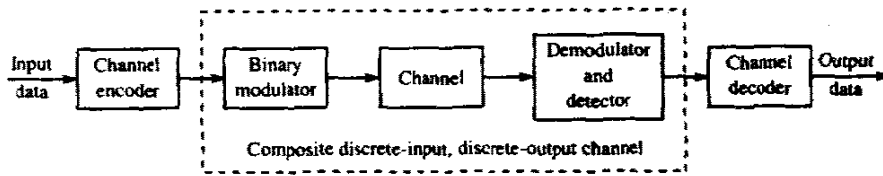


FIGURE 7-1-1 A composite discrete-input, discrete-output channel formed by including the modulator and the demodulator/detector as part of the channel.

channel. If the modulator employs binary waveforms and the detector makes hard decisions, then the composite channel, shown in Fig. 7-1-1, has a discrete-time binary input sequence and a discrete-time binary output sequence. Such a composite channel is characterized by the set $X = \{0, 1\}$ of possible inputs, the set of $Y = \{0, 1\}$ of possible outputs, and a set of conditional probabilities that relate the possible outputs to the possible inputs. If the channel noise and other disturbances cause statistically independent errors in the transmitted binary sequence with average probability p then

$$P(Y = 0 | X = 1) = P(Y = 1 | X = 0) = p \tag{7-1-1}$$

$$P(Y = 1 | X = 1) = P(Y = 0 | X = 0) = 1 - p$$

Thus, we have reduced the cascade of the binary modulator, the waveform channel, and the binary demodulator and detector into an equivalent discrete-time channel which is represented by the diagram shown in Fig. 7-1-2. This binary-input, binary-output, symmetric channel is simply called a *binary symmetric channel (BSC)*. Since each output bit from the channel depends only on the corresponding input bit, we say that the channel is memoryless.

Discrete Memoryless Channels The BSC is a special case of a more general discrete-input, discrete-output channel. Suppose that the output from the channel encoder are q -ary symbols, i.e., $X = \{x_0, x_1, \dots, x_{q-1}\}$ and the output of the detector consists of Q -ary symbols, where $Q \geq M = 2^q$. If the

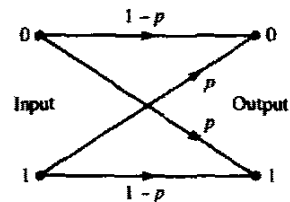


FIGURE 7-1-2 Binary symmetric channel.

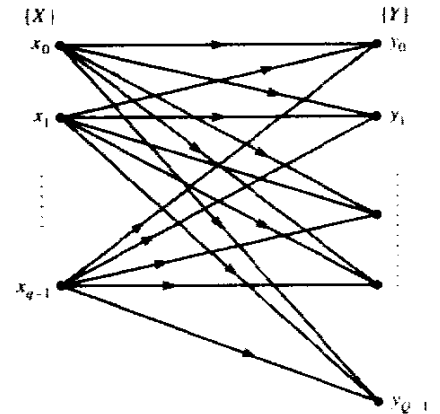


FIGURE 7-1-3 Discrete q -ary input, Q -ary output channel.

channel and the modulation are memoryless, then the input-output characteristics of the composite channel, shown in Fig. 7-1-1, are described by a set of qQ conditional probabilities

$$P(Y = y_i | X = x_j) \equiv P(y_i | x_j) \quad (7-1-2)$$

where $i = 0, 1, \dots, Q - 1$ and $j = 0, 1, \dots, q - 1$. Such a channel is called a *discrete memoryless channel* (DMC), and its graphical representation is shown in Fig. 7-1-3. Hence, if the input to a DMC is a sequence of n symbols u_1, u_2, \dots, u_n selected from the alphabet X and the corresponding output is the sequence v_1, v_2, \dots, v_n of symbols from the alphabet Y , the joint conditional probability is

$$\begin{aligned} P(Y_1 = v_1, Y_2 = v_2, \dots, Y_n = v_n | X = u_1, \dots, X = u_n) \\ = \prod_{k=1}^n P(Y = v_k | X = u_k) \end{aligned} \quad (7-1-3)$$

This expression is simply a mathematical statement of the memoryless condition.

In general, the conditional probabilities $\{P(y_i | x_j)\}$ that characterize a DMC can be arranged in the matrix form $\mathbf{P} = [p_{ji}]$, where, by definition, $p_{ji} \equiv P(y_i | x_j)$. \mathbf{P} is called the *probability transition matrix* for the channel.

Discrete-Input, Continuous-Output Channel Now, suppose that the input to the modulator comprises symbols selected from a finite and discrete input alphabet $X = \{x_0, x_1, \dots, x_{q-1}\}$ and the output of the detector is unquantized ($Q = \infty$). Then, the input to the channel decoder can assume any value on the real line, i.e., $Y = \{-\infty, \infty\}$. This leads us to define a composite discrete-time

memoryless channel that is characterized by the discrete input X , the continuous output Y , and the set of conditional probability density functions

$$p(y | X = x_k), \quad k = 0, 1, \dots, q - 1$$

The most important channel of this type is the additive white gaussian noise channel (AWGN), for which

$$Y = X + G \quad (7-1-4)$$

where G is a zero-mean gaussian random variable with variance σ^2 and $X = x_k$, $k = 0, 1, \dots, q - 1$. For a given X , it follows that Y is gaussian with mean x_k and variance σ^2 . That is,

$$p(y | X = x_k) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(y-x_k)^2/2\sigma^2} \quad (7-1-5)$$

For any given input sequence, X_i , $i = 1, 2, \dots, n$, there is a corresponding output sequence

$$Y_i = X_i + G_i, \quad i = 1, 2, \dots, n \quad (7-1-6)$$

The condition that the channel is memoryless may be expressed as

$$p(y_1, y_2, \dots, y_n | X_1 = u_1, X_2 = u_2, \dots, X_n = u_n) = \prod_{i=1}^n p(y_i | X_i = u_i) \quad (7-1-7)$$

Waveform Channels We may separate the modulator and demodulator from the physical channel, and consider a channel model in which the inputs are waveforms and the outputs are waveforms. Let us assume that such a channel has a given bandwidth W , with ideal frequency response $C(f) = 1$ within the bandwidth W , and the signal at its output is corrupted by additive white gaussian noise. Suppose, that $x(t)$ is a band-limited input to such a channel and $y(t)$ is the corresponding output. Then,

$$y(t) = x(t) + n(t) \quad (7-1-8)$$

where $n(t)$ represents a sample function of the additive noise process. A suitable method for defining a set of probabilities that characterize the channel is to expand $x(t)$, $y(t)$, and $n(t)$ into a complete set of orthonormal functions. That is, we express $x(t)$, $y(t)$, and $n(t)$ in the form

$$\begin{aligned} y(t) &= \sum_i y_i f_i(t) \\ x(t) &= \sum_i x_i f_i(t) \\ n(t) &= \sum_i n_i f_i(t) \end{aligned} \quad (7-1-9)$$

where $\{y_i\}$, $\{x_i\}$, and $\{n_i\}$ are the sets of coefficients in the corresponding expansions, e.g.,

$$\begin{aligned} y_i &= \int_0^T y(t) f_i^*(t) dt \\ &= \int_0^T [x(t) + n(t)] f_i^*(t) dt \\ &= x_i + n_i \end{aligned} \quad (7-1-10)$$

The functions $\{f_i(t)\}$ form a complete orthonormal set over the interval $(0, T)$, i.e.,

$$\int_0^T f_i(t) f_j^*(t) dt = \delta_{ij} = \begin{cases} 1 & (i = j) \\ 0 & (i \neq j) \end{cases} \quad (7-1-11)$$

where δ_{ij} is the Kronecker delta function. Since the gaussian noise is white, any complete set of orthonormal functions may be used in the expansions (7-1-9).

We may now use the coefficients in the expansion for characterizing the channel. Since

$$y_i = x_i + n_i$$

where n_i is gaussian, it follows that

$$p(y_i | x_i) = \frac{1}{\sqrt{2\pi} \sigma_i} e^{-(y_i - x_i)^2 / 2\sigma_i^2}, \quad i = 1, 2, \dots \quad (7-1-12)$$

Since the functions $\{f_i(t)\}$ in the expansion are orthonormal, it follows that the $\{n_i\}$ are uncorrelated. Since they are gaussian, they are also statistically independent. Hence,

$$p(y_1, y_2, \dots, y_N | x_1, x_2, \dots, x_N) = \prod_{i=1}^N p(y_i | x_i) \quad (7-1-13)$$

for any N . In this manner, the waveform channel is reduced to an equivalent discrete-time channel characterized by the conditional pdf given in (7-1-12).

When the additive noise is white and gaussian with spectral density $\frac{1}{2}N_0$, the variances $\sigma_i^2 = \frac{1}{2}N_0$ for all i in (7-1-12). In this case, samples of $x(t)$ and $y(t)$ may be taken at the Nyquist rate of $2W$ samples/s, so that $x_i = x(i/2W)$ and $y_i = y(i/2W)$. Since the noise is white, the noise samples are statistically independent. Thus, (7-1-12) and (7-1-13) describe the statistics of the sampled signal. We note that in a time interval of length T , there are $N = 2WT$ samples. This parameter is used below in obtaining the capacity of the band-limited AWGN waveform channel.

The choice of which channel model to use at any one time depends on our objectives. If we are interested in the design and analysis of the performance

of the discrete channel encoder and decoder, it is appropriate to consider channel models in which the modulator and demodulator are a part of the composite channel. On the other hand, if our intent is to design and analyze the performance of the digital modulator and digital demodulator, we use a channel model for the waveform channel.

7-1-2 Channel Capacity

Now let us consider a DMC having an input alphabet $X = \{x_0, x_1, \dots, x_{q-1}\}$, an output alphabet $Y = \{y_0, y_1, \dots, y_{Q-1}\}$, and the set of transition probabilities $P(y_i | x_j)$ as defined in (7-1-2). Suppose that the symbol x_j is transmitted and the symbol y_i is received. The mutual information provided about the event $X = x_j$ by the occurrence of the event $Y = y_i$ is $\log [P(y_i | x_j) / P(y_i)]$, where

$$P(y_i) \equiv P(Y = y_i) = \sum_{k=0}^{q-1} P(x_k) P(y_i | x_k) \quad (7-1-14)$$

Hence, the average mutual information provided by the output Y about the input X is

$$I(X; Y) = \sum_{j=0}^{q-1} \sum_{i=0}^{Q-1} P(x_j) P(y_i | x_j) \log \frac{P(y_i | x_j)}{P(y_i)} \quad (7-1-15)$$

The channel characteristics determine the transition probabilities $P(y_i | x_j)$, but the probabilities of the input symbols are under the control of the discrete channel encoder. The value of $I(X; Y)$ maximized over the set of input symbol probabilities $P(x_j)$ is a quantity that depends only on the characteristics of the DMC through the conditional probabilities $P(y_i | x_j)$. This quantity is called the *capacity* of the channel and is denoted by C . That is, the capacity of a DMC is defined as

$$\begin{aligned} C &= \max_{P(x_j)} I(X; Y) \\ &= \max_{P(x_j)} \sum_{j=0}^{q-1} \sum_{i=0}^{Q-1} P(x_j) P(y_i | x_j) \log \frac{P(y_i | x_j)}{P(y_i)} \end{aligned} \quad (7-1-16)$$

The maximization of $I(X; Y)$ is performed under the constraints that

$$\begin{aligned} P(x_j) &\geq 0 \\ \sum_{j=0}^{q-1} P(x_j) &= 1 \end{aligned}$$

The units of C are bits per input symbol into the channel (bits/channel use)

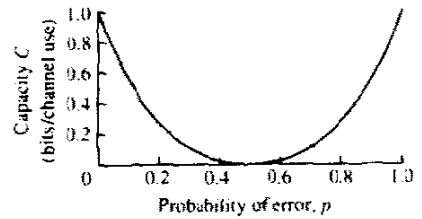


FIGURE 7-1-4 The capacity of a BSC as a function of the error probability p .

when the logarithm is base 2, and nats/input symbol when the natural logarithm (base e) is used. If a symbol enters the channel every τ_s seconds, the channel capacity in bits/s or nats/s is C/τ_s .

Example 7-1-1

For the BSC with transition probabilities

$$P(0 | 1) = P(1 | 0) = p$$

the average mutual information is maximized when the input probabilities $P(0) = P(1) = \frac{1}{2}$. Thus, the capacity of the BSC is

$$C = p \log 2p + (1 - p) \log 2(1 - p) = 1 - H(p) \quad (7-1-17)$$

where $H(p)$ is the binary entropy function. A plot of C versus p is illustrated in Fig. 7-1-4. Note that for $p = 0$, the capacity is 1 bit/channel use. On the other hand, for $p = \frac{1}{2}$, the mutual information between input and output is zero. Hence, the channel capacity is zero. For $\frac{1}{2} < p \leq 1$, we may reverse the position of 0 and 1 at the output of the BSC, so that C becomes symmetric with respect to the point $p = \frac{1}{2}$. In our treatment of binary modulation and demodulation given in Chapter 5, we showed that p is a monotonic function of the signal-to-noise ratio (SNR) as illustrated in Fig. 7-1-5(a). Consequently when C is plotted as a function of the SNR, it increases monotonically as the SNR increases. This characteristic behavior of C versus SNR is illustrated in Fig. 7-1-5(b).

Next let us consider the discrete-time AWGN memoryless channel described by the transition probability density functions defined by (7-1-5). The

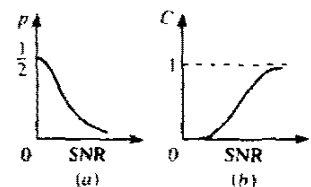


FIGURE 7-1-5 General behavior of error probability and channel capacity as a function of SNR.

average mutual information between the discrete input $X = \{x_0, x_1, \dots, x_{q-1}\}$ and the output $Y = \{-\infty, \infty\}$ is given by the capacity of this channel in bits/channel use is

$$C = \max_{P(x_i)} \sum_{i=0}^{q-1} \int_{-\infty}^{\infty} p(y | x_i) P(x_i) \log_2 \frac{p(y | x_i)}{p(y)} dy \quad (7-1-18)$$

where

$$p(y) = \sum_{k=0}^{q-1} p(y | x_k) P(x_k) \quad (7-1-19)$$

Example 7-1-2

Let us consider a binary-input AWGN memoryless channel with possible inputs $X = A$ and $X = -A$. The average mutual information $I(X; Y)$ is maximized when the input probabilities are $P(X = A) = P(X = -A) = \frac{1}{2}$. Hence, the capacity of this channel in bits/channel use is

$$C = \frac{1}{2} \int_{-\infty}^{\infty} p(y | A) \log_2 \frac{p(y | A)}{p(y)} dy + \frac{1}{2} \int_{-\infty}^{\infty} p(y | -A) \log_2 \frac{p(y | -A)}{p(y)} dy \quad (7-1-20)$$

Figure 7-1-6 illustrates C as a function of the ratio $A^2/2\sigma^2$. Note that C increases monotonically from 0 to 1 bit/symbol as this ratio increases.

It is interesting to note that in the two channel models described above, the choice of equally probable input symbols maximizes the average mutual information. Thus, the capacity of the channel is obtained when the input symbols are equally probable. This is not always the solution for the capacity formulas given in (7-1-16) and (7-1-18), however. Nothing can be said in general about the input probability assignment that maximizes the average mutual information. However, in the two channel models considered above,

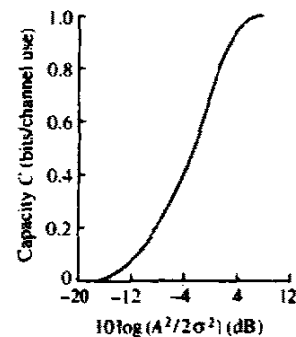


FIGURE 7-1-6 Channel capacity as a function of $A^2/2\sigma^2$ for binary-input AWGN memoryless channel.

the channel transition probabilities exhibit a form of symmetry that results in the maximum of $I(X; Y)$ being obtained when the input symbols are equally probable. The symmetry condition can be expressed in terms of the elements of the probability transition matrix \mathbf{P} of the channel. When each row of this matrix is a permutation of any other row and each column is a permutation of any other column, the probability transition matrix is symmetric and input symbols with equal probability maximize $I(X; Y)$.

In general, necessary and sufficient conditions for the set of input probabilities $\{P(x_j)\}$ to maximize $I(X; Y)$ and, thus, to achieve capacity on a DMC are that (Problem 7-1)

$$\begin{aligned} I(x_j; Y) &= C \quad \text{for all } j \text{ with } P(x_j) > 0 \\ I(x_j; Y) &\leq C \quad \text{for all } j \text{ with } P(x_j) = 0 \end{aligned} \quad (7-1-21)$$

where C is the capacity of the channel and

$$I(x_j; Y) = \sum_{i=0}^{Q-1} P(y_i | x_j) \log \frac{P(y_i | x_j)}{P(y_i)} \quad (7-1-22)$$

Usually, it is relatively easy to check if the equally probable set of input symbols satisfy the conditions (7-1-21). If they do not, then one must determine the set of unequal probabilities $\{P(x_j)\}$ that satisfy (7-1-21).

Now let us consider a band-limited waveform channel with additive white gaussian noise. Formally, the capacity of the channel per unit time has been defined by Shannon (1948b) as

$$C = \lim_{T \rightarrow \infty} \max_{p(x)} \frac{1}{T} I(X; Y) \quad (7-1-23)$$

where the average mutual information $I(X; Y)$ is given in (3-2-17). Alternatively, we may use the samples or the coefficients $\{y_i\}$, $\{x_i\}$, and $\{n_i\}$ in the series expansions of $y(t)$, $x(t)$, and $n(t)$, respectively, to determine the average mutual information between $\mathbf{x}_N = [x_1 \ x_2 \ \dots \ x_N]$ and $\mathbf{y}_N = [y_1 \ y_2 \ \dots \ y_N]$, where $N = 2WT$, $y_i = x_i + n_i$, and $p(y_i | x_i)$ is given by (7-1-12). The average mutual information between \mathbf{x}_N and \mathbf{y}_N for the AWGN channel is

$$\begin{aligned} I(\mathbf{X}_N; \mathbf{Y}_N) &= \int_{\mathbf{x}_N} \dots \int_{\mathbf{y}_N} \dots \int p(\mathbf{y}_N | \mathbf{x}_N) p(\mathbf{x}_N) \log \frac{p(\mathbf{y}_N | \mathbf{x}_N)}{p(\mathbf{y}_N)} d\mathbf{x}_N d\mathbf{y}_N \\ &= \sum_{i=1}^N \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(y_i | x_i) p(x_i) \log \frac{p(y_i | x_i)}{p(y_i)} dy_i dx_i \end{aligned} \quad (7-1-24)$$

where

$$p(y_i | x_i) = \frac{1}{\sqrt{\pi N_0}} e^{-(y_i - x_i)^2 / N_0} \quad (7-1-25)$$

The maximum of $I(X; Y)$ over the input pdfs $p(x_i)$ is obtained when the $\{x_i\}$ are statistically independent zero-mean gaussian random variables, i.e.,

$$p(x_i) = \frac{1}{\sqrt{2\pi}\sigma_x} e^{-x_i^2/2\sigma_x^2} \quad (7-1-26)$$

where σ_x^2 is the variance of each x_i . Then, it follows from (7-1-24) that

$$\begin{aligned} \max_{p(x)} I(\mathbf{X}_N; \mathbf{Y}_N) &= \sum_{i=1}^N \frac{1}{2} \log \left(1 + \frac{2\sigma_x^2}{N_0} \right) \\ &= \frac{1}{2} N \log \left(1 + \frac{2\sigma_x^2}{N_0} \right) \\ &= WT \log \left(1 + \frac{2\sigma_x^2}{N_0} \right) \end{aligned} \quad (7-1-27)$$

Suppose that we put a constraint on the average power in $x(t)$. That is,

$$\begin{aligned} P_{av} &= \frac{1}{T} \int_0^T E[x^2(t)] dt \\ &= \frac{1}{T} \sum_{i=1}^N E(x_i^2) \\ &= \frac{N\sigma_x^2}{T} \end{aligned} \quad (7-1-28)$$

Hence,

$$\begin{aligned} \sigma_x^2 &= \frac{TP_{av}}{N} \\ &= \frac{P_{av}}{2W} \end{aligned} \quad (7-1-29)$$

Substitution of this result into (7-1-27) for σ_x^2 yields

$$\max_{p(x)} I(\mathbf{X}_N; \mathbf{Y}_N) = WT \log \left(1 + \frac{P_{av}}{WN_0} \right) \quad (7-1-30)$$

Finally, the channel capacity per unit time is obtained by dividing the result in (7-1-30) by T . Thus

$$C = W \log \left(1 + \frac{P_{av}}{WN_0} \right) \quad (7-1-31)$$

This is the basic formula for the capacity of the band-limited AWGN

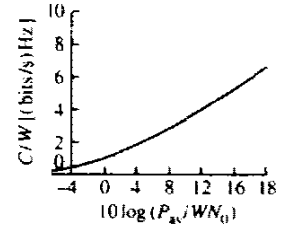


FIGURE 7-1-7 Normalized channel capacity as a function of SNR for band-limited AWGN channel.

waveform channel with a band-limited and average power-limited input. It was originally derived by Shannon (1948b).

A plot of the capacity in bits/s normalized by the bandwidth W is plotted in Fig. 7-1-7 as a function of the ratio of signal power P_{av} to noise power WN_0 . Note that the capacity increases monotonically with increasing SNR. Thus, for a fixed bandwidth, the capacity of the waveform channel increases with an increase in the transmitted signal power. On the other hand, if P_{av} is fixed, the capacity can be increased by increasing the bandwidth W . Figure 7-1-8 illustrates a graph of C versus W . Note that as W approaches infinity, the capacity of the channel approaches the asymptotic value

$$C_{\infty} = \frac{P_{av}}{N_0} \log_2 e = \frac{P_{av}}{N_0 \ln 2} \text{ bits/s} \tag{7-1-32}$$

It is instructive to express the normalized channel capacity C/W as a function of the SNR per bit. Since P_{av} represents the average transmitted power and C is the ratio in bits/s, it follows that

$$P_{av} = C \mathcal{E}_b \tag{7-1-33}$$

where \mathcal{E}_b is the energy per bit. Hence, (7-1-31) may be expressed as

$$\frac{C}{W} = \log_2 \left(1 + \frac{C \mathcal{E}_b}{W N_0} \right) \tag{7-1-34}$$

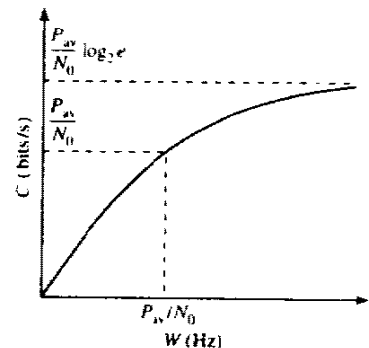


FIGURE 7-1-8 Channel capacity as a function of bandwidth with a fixed transmitted average power.

Consequently,

$$\frac{\mathcal{E}_b}{N_0} = \frac{2^{C/W} - 1}{C/W} \quad (7-1-35)$$

When $C/W = 1$, $\mathcal{E}_b/N_0 = 1$ (0 dB). As $C/W \rightarrow \infty$,

$$\begin{aligned} \frac{\mathcal{E}_b}{N_0} &\approx \frac{2^{C/W}}{C/W} \\ &\approx \exp\left(\frac{C}{W} \ln 2 - \ln \frac{C}{W}\right) \end{aligned} \quad (7-1-36)$$

Thus, \mathcal{E}_b/N_0 increases exponentially as $C/W \rightarrow \infty$. On the other hand, as $C/W \rightarrow 0$,

$$\frac{\mathcal{E}_b}{N_0} = \lim_{C/W \rightarrow 0} \frac{2^{C/W} - 1}{C/W} = \ln 2 \quad (7-1-37)$$

which is -1.6 dB. A plot of C/W versus \mathcal{E}_b/N_0 is shown in Fig. 5-2-17.

Thus, we have derived the channel capacities of three important channel models that are considered in this book. The first is the discrete-input, discrete-output channel, of which the BSC is a special case. The second is a discrete-input, continuous-output memoryless additive white gaussian noise channel. From these two channel models, we can obtain benchmarks for the coded performance with hard- and soft-decision decoding in digital communications systems.

The third channel model focuses on the capacity in bits/s of a waveform channel. In this case, we assumed that we have a bandwidth limitation on the channel, an additive gaussian noise that corrupts the signal, and an average power constraint at the transmitter. Under these conditions, we derived the result given in (7-1-31).

The major significance of the channel capacity formulas given above is that they serve as upper limits on the transmission rate for reliable communication over a noisy channel. The fundamental rate that the channel capacity plays is given by the *noisy channel coding theorem* due to Shannon (1948a).

Noisy Channel Coding Theorem

There exist channel codes (and decoders) that make it possible to achieve reliable communication, with as small an error probability as desired, if the transmission rate $R < C$, where C is the channel capacity. If $R > C$, it is not possible to make the probability of error tend toward zero with any code.

In the following section, we explore the benefits of coding for the additive

noise channel models described above, and use the channel capacity as the benchmark for accessing code performance.

7-1-3 Achieving Channel Capacity with Orthogonal Signals

In Section 5-2, we used a simple union bound to show that, for orthogonal signals, the probability of error can be made as small as desired by increasing the number M of waveforms, provided that $\mathcal{E}_b/N_0 > 2 \ln 2$. We indicated that the simple union bound does not produce the smallest lower bound on the SNR per bit. The problem is that the upper bound used on $Q(x)$ is very loose for small x .

An alternative approach is to use two different upper bounds for $Q(x)$, depending on the value of x . Beginning with (5-2-21), we observe that

$$1 - [1 - Q(y)]^M \leq (M-1)Q(y) < Me^{-y^2/2} \quad (7-1-38)$$

This is just the union bound, which is tight when y is large, i.e., for $y > y_0$, where y_0 depends on M . When y is small, the union bound exceeds unity for large M . Since

$$1 - [1 - Q(y)]^M \leq 1 \quad (7-1-39)$$

for all y , we may use this bound for $y < y_0$ because it is tighter than the union bound. Thus (5-2-21) may be upper-bounded as

$$P_M < \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y_0} e^{-(y - \sqrt{2\gamma})^2/2} dy + \frac{M}{\sqrt{2\pi}} \int_{y_0}^{\infty} e^{-y^2/2} e^{-(y - \sqrt{2\gamma})^2/2} dy \quad (7-1-40)$$

The value of y_0 that minimizes this upper bound is found by differentiating the right-hand side of (7-1-40) and setting the derivative equal to zero. It is easily verified that the solution is

$$e^{y_0^2/2} = M \quad (7-1-41)$$

or, equivalently,

$$\begin{aligned} y_0 &= \sqrt{2 \ln M} = \sqrt{2 \ln 2 \log_2 M} \\ &= \sqrt{2k \ln 2} \end{aligned} \quad (7-1-42)$$

Having determined y_0 , let us now compute simple exponential upper bounds for the integrals in (7-1-40). For the first integral, we have

$$\begin{aligned} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y_0} e^{-(y - \sqrt{2\gamma})^2/2} dy &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{-(\sqrt{2\gamma} - y_0)/\sqrt{2}} e^{-x^2} dx \\ &= Q(\sqrt{2\gamma} - y_0), \quad y_0 \leq \sqrt{2\gamma} \\ &< e^{-(\sqrt{2\gamma} - y_0)^2/2}, \quad y_0 \leq \sqrt{2\gamma} \end{aligned} \quad (7-1-43)$$

The second integral is upper-bounded as follows:

$$\begin{aligned} \frac{M}{\sqrt{2\pi}} \int_{y_0}^{\infty} e^{-y^2/2} e^{-(y-\sqrt{2\gamma})^2/2} dy &= \frac{M}{\sqrt{2\pi}} e^{-\gamma/2} \int_{x_0-\sqrt{\gamma/2}}^{\infty} e^{-x^2} dx \\ &< \begin{cases} Me^{-\gamma/2} & (y_0 \leq \sqrt{\frac{1}{2}\gamma}) \\ Me^{-\gamma/2} e^{-(y_0-\sqrt{\gamma/2})^2} & (y_0 \geq \sqrt{\frac{1}{2}\gamma}) \end{cases} \end{aligned} \quad (7-1-44)$$

Combining the bounds for the two integrals and substituting $e^{y_0^2/2}$ for M , we obtain

$$P_M < \begin{cases} e^{-(\sqrt{2\gamma}-y_0)^2/2} + e^{(y_0^2-\gamma)/2} & (0 \leq y_0 \leq \sqrt{\frac{1}{2}\gamma}) \\ e^{-(\sqrt{2\gamma}-y_0)^2/2} + e^{(y_0^2-\gamma)/2} e^{-(y_0-\sqrt{\gamma/2})^2} & (\sqrt{\frac{1}{2}\gamma} \leq y_0 \leq \sqrt{2\gamma}) \end{cases} \quad (7-1-45)$$

In the range $0 \leq y_0 \leq \sqrt{\frac{1}{2}\gamma}$, the bound may be expressed as

$$\begin{aligned} P_M &< e^{(y_0^2-\gamma)/2} (1 + e^{-(y_0-\sqrt{\gamma/2})^2}) \\ &< 2e^{(y_0^2-\gamma)/2}, \quad 0 \leq y_0 \leq \sqrt{\frac{1}{2}\gamma} \end{aligned} \quad (7-1-46)$$

In the range $\sqrt{\frac{1}{2}\gamma} \leq y_0 \leq \sqrt{2\gamma}$, the two terms in (7-1-45) are identical. Hence,

$$P_M < 2e^{-(\sqrt{2\gamma}-y_0)^2/2}, \quad \sqrt{\frac{1}{2}\gamma} \leq y_0 \leq \sqrt{2\gamma} \quad (7-1-47)$$

Now we substitute for y_0 and γ . Since $y_0 = \sqrt{2 \ln M} = \sqrt{2k \ln 2}$ and $\gamma = k\gamma_b$, the bounds in (7-1-46) and (7-1-47) may be expressed as

$$P_M < \begin{cases} 2e^{-k(\gamma_b - 2 \ln 2)/2} & (\ln M \leq \frac{1}{4}\gamma_b) \\ 2e^{-k(\sqrt{\gamma_b} - \sqrt{\ln 2})^2} & (\frac{1}{4}\gamma_b \leq \ln M \leq \gamma_b) \end{cases} \quad (7-1-48)$$

The first upper bound coincides with the union bound presented earlier, but it is loose for large values of M . The second upper bound is better for large values of M . We note that $P_M \rightarrow 0$ as $k \rightarrow \infty$ ($M \rightarrow \infty$) provided that $\gamma_b > \ln 2$. But, $\ln 2$ is the limiting value of the SNR per bit required for reliable transmission when signaling at a rate equal to the capacity of the infinite-bandwidth AWGN channel as shown in Section 7-1-2. In fact, when the substitutions

$$\begin{aligned} y_0 &= \sqrt{2k \ln 2} = \sqrt{2RT \ln 2} \\ \gamma &= \frac{TP_{av}}{N_0} = TC_x \ln 2 \end{aligned} \quad (7-1-49)$$

are made into the two upper bounds given in (7-1-46) and (7-1-47), where $C_x = P_{av}/(N_0 \ln 2)$ is the capacity of the infinite-bandwidth AWGN channel, the result is

$$P_M < \begin{cases} 2 \cdot 2^{-T(\frac{1}{4}C_x - R)} & (0 \leq R \leq \frac{1}{4}C_x) \\ 2 \cdot 2^{-T(\sqrt{C_x} - \sqrt{R})^2} & (\frac{1}{4}C_x \leq R \leq C_x) \end{cases} \quad (7-1-50)$$

Thus we have expressed the bounds in terms of C_∞ and the bit rate in the channel. The first upper bound is appropriate for rates below $\frac{1}{4}C_\infty$, while the second is tighter than the first for rates between $\frac{1}{4}C_\infty$ and C_∞ . Clearly, the probability of error can be made arbitrarily small by making $T \rightarrow \infty$ ($M \rightarrow \infty$ for fixed R), provided that $R < C_\infty = P_{av}/(N_0 \ln 2)$. Furthermore, we observe that the set of orthogonal waveforms achieves the channel capacity bound as $M \rightarrow \infty$, when the rate $R < C_\infty$.

7-1-4 Channel Reliability Functions

The exponential bounds on the error probability for M -ary orthogonal signals on an infinite-bandwidth AWGN channel given by (7-1-50) may be expressed as

$$P_M < 2 \cdot 2^{-TE(R)} \quad (7-1-51)$$

The exponential factor

$$E(R) = \begin{cases} \frac{1}{2}C_\infty - R & (0 \leq R \leq \frac{1}{4}C_\infty) \\ (\sqrt{C_\infty} - \sqrt{R})^2 & (\frac{1}{4}C_\infty \leq R \leq C_\infty) \end{cases} \quad (7-1-52)$$

in (7-1-51) is called the *channel reliability function* for the infinite-bandwidth AWGN channel. A plot of $E(R)/C_\infty$ is shown in Fig. 7-1-9. Also shown is the exponential factor for the union bound on P_M , given by (5-2-27), which may be expressed as

$$P_M \leq \frac{1}{2} \cdot 2^{-T(\frac{1}{2}C_\infty - R)}, \quad 0 \leq R \leq \frac{1}{2}C_\infty \quad (7-1-53)$$

Clearly, the exponential factor in (7-1-53) is not as tight as $E(R)$, due to the looseness of the union bound.

The bound given by (7-1-51) and (7-1-52) has been shown by Gallager (1965) to be *exponentially tight*. This means that there does not exist another reliability function, say $E_1(R)$, satisfying the condition $E_1(R) > E(R)$ for any R . Consequently, the error probability is bounded from above and below as

$$K_l 2^{-TE(R)} \leq P_e \leq K_u 2^{-TE(R)} \quad (7-1-54)$$

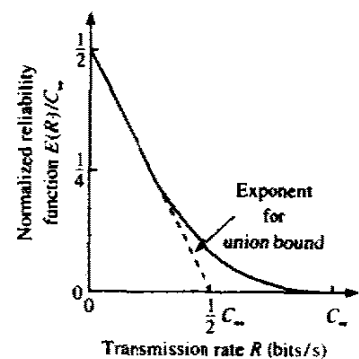


FIGURE 7-1-9 Channel reliability function for the infinite-bandwidth AWGN channel.

where the constants have only a weak dependence on T , i.e., they vary slowly with T .

Since orthogonal signals provide essentially the same performance as the optimum simplex signals for large M , the lower bound in (7-1-54) applies for any signal set. Hence, the reliability function $E(R)$ given by (7-1-52) determines the exponential characteristics of the error probability for digital signaling over the infinite-bandwidth AWGN channel.

Although the error probability can be made arbitrarily small by increasing the number of either orthogonal, biorthogonal, or simplex signals, with $R < C_{\infty}$, for a relatively modest number of signals, there is a large gap between the actual performance and the best achievable performance given by the channel capacity formula. For example, from Fig. 5-2-17, we observe that a set of $M = 16$ orthogonal signals detected coherently requires a SNR per bit of approximately 7.5 dB to achieve a bit error rate of $P_e = 10^{-5}$. In contrast, the channel capacity formula indicates that for a $C/W = 0.5$, reliable transmission is possible with a SNR of -0.8 dB. This represents a rather large difference of 8.3 dB/bit and serves as a motivation for searching for more efficient signaling waveforms. In this chapter and in Chapter 8, we demonstrate that coded waveforms can reduce this gap considerably.

Similar gaps in performance also exist in the bandwidth-limited region of Fig. 5-2-17, where $R/W > 1$. In this region, however, we must be more clever in how we use coding to improve performance, because we cannot expand the bandwidth as in the power-limited region. The use of coding techniques for bandwidth-efficient communication is also treated in Chapter 8.

7-2 RANDOM SELECTION OF CODES

The design of coded modulation for efficient transmission of information may be divided into two basic approaches. One is the algebraic approach, which is primarily concerned with the design of coding and decoding techniques for specific classes of codes, such as cyclic block codes and convolutional codes. The second is the probabilistic approach, which is concerned with the analysis of the performance of a general class of coded signals. This approach yields bounds on the probability of error that can be attained for communication over a channel having some specified characteristic.

In this section, we adopt the probabilistic approach to coded modulation. The algebraic approach, based on block codes and on convolutional codes, is treated in Chapter 8.

7-2-1 Random Coding Based on M -ary Binary Coded Signals

Let us consider a set of M coded signal waveforms constructed from a set of n -dimensional binary code words of the form

$$\mathbf{C}_i = [c_{i1} c_{i2} \dots c_{in}], \quad i = 1, 2, \dots, M \quad (7-2-1)$$

where $c_{ij} = 0$ or 1. Each bit in the code word is mapped into a binary PSK waveform, so that the signal waveform corresponding to the code word \mathbf{C}_i may be expressed as

$$s_i(t) = \sum_{j=1}^n s_{ij} f_j(t), \quad i = 1, 2, \dots, M \quad (7-2-2)$$

where

$$s_{ij} = \begin{cases} \sqrt{\mathcal{E}_c} & \text{when } c_{ij} = 1 \\ -\sqrt{\mathcal{E}_c} & \text{when } c_{ij} = 0 \end{cases} \quad (7-2-3)$$

and \mathcal{E}_c is the energy per code bit. Thus, the waveforms $s_i(t)$ are equivalent to the n -dimensional vectors

$$\mathbf{s}_i = [s_{i1} \quad s_{i2} \quad \dots \quad s_{in}], \quad i = 1, 2, \dots, M \quad (7-2-4)$$

which correspond to the vertices of a hypercube in n -dimensional space.

Now, suppose that the information rate into the encoder is R bits/s and we encode blocks of k bits at a time into one of the M waveforms. Hence, $k = RT$ and $M = 2^k = 2^{RT}$ signals are required. It is convenient to define a parameter D as

$$D = \frac{n}{T} \text{ dimensions/s} \quad (7-2-5)$$

Thus, $n = DT$ is the dimensionality of the signal space.

The hypercube has $2^n = 2^{DT}$ vertices, of which $M = 2^{RT}$ may be used to transmit the information. If we impose the condition that $D > R$, the fraction of the vertices that we use as signal points is

$$F = \frac{2^k}{2^n} = \frac{2^{RT}}{2^{DT}} = 2^{-(D-R)T} \quad (7-2-6)$$

Clearly, if $D > R$, we have $F \rightarrow 0$ as $T \rightarrow \infty$.

The question that we wish to pose is the following. Can we choose a subset $M = 2^{RT}$ vertices out of the $2^n = 2^{DT}$ available vertices such that the probability of error $P \rightarrow 0$ as $T \rightarrow \infty$ or, equivalently, as $n \rightarrow \infty$? Since the fraction F of vertices used approaches zero as $T \rightarrow \infty$, it should be possible to select M signal waveforms having a minimum distance that increases as $T \rightarrow \infty$ and, thus, $P_e \rightarrow 0$.

Instead of attempting to find a single set of M coded waveforms for which we compute the error probability, let us consider the ensemble of $(2^n)^M$ distinct ways in which we can select M vertices from the 2^n available vertices of the hypercube. Associated with each of the 2^{nM} selections, there is a communication system, consisting of a modulator, a channel, and a demodulator, that is optimum for the selected set of M waveforms. Thus, there are 2^{nM}

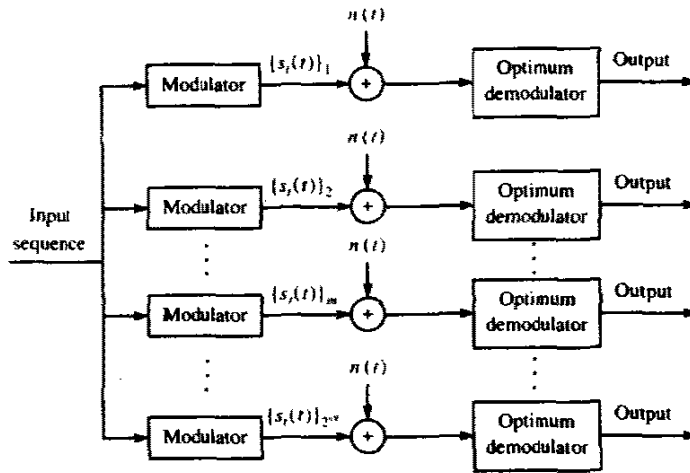


FIGURE 7-2-1 An ensemble at 2^{nM} communications system. Each system employs a different set of M signals from the set of 2^{nM} possible choices.

communication systems, one for each choice of the M coded waveforms, as illustrated in Fig. 7-2-1. Each communication system is characterized by its probability of error.

Suppose that our choice of M coded waveforms is based on random selection from the set of 2^{nM} possible sets of codes. Thus, the random selection of the m th code, denoted by $\{s_i\}_m$, occurs with probability

$$P(\{s_i\}_m) = 2^{-nM} \tag{7-2-7}$$

and the corresponding conditional probability of error for this choice of coded signals is $P_e(\{s_i\}_m)$. Then, the average probability of error over the ensemble of codes is

$$\begin{aligned} \bar{P}_e &= \sum_{m=1}^{2^{nM}} P_e(\{s_i\}_m) P(\{s_i\}_m) \\ &= 2^{-nM} \sum_{m=1}^{2^{nM}} P_e(\{s_i\}_m) \end{aligned} \tag{7-2-8}$$

where the overbar on P_e denotes an average over the ensemble of codes.

It is clear that some choices of codes will result in large probability of error. For example, the code that assigns all M k -bit sequences to the same vertex of the hypercube will result in a large probability of error. In such a case, $P_e(\{s_i\}_m) > \bar{P}_e$. However, there will also be choices of codes for which $P_e(\{s_i\}_m) < \bar{P}_e$. Consequently, if we obtain an upper bound on \bar{P}_e , this bound will also hold for those codes for which $P_e(\{s_i\}_m) < \bar{P}_e$. Furthermore, if $\bar{P}_e \rightarrow 0$ as $T \rightarrow \infty$ then we conclude that, for these codes, $P_e(\{s_i\}_m) \rightarrow 0$ as $T \rightarrow \infty$.

In order to determine an upper bound on \bar{P}_e , we consider the transmission

of a k -bit message $\mathbf{X}_k \equiv [x_1, x_2, x_3, \dots, x_k]$, where $x_j = 0$ or 1 for $j = 1, 2, \dots, k$. The conditional probability of error averaged over the ensemble of codes is

$$\overline{P_e(\mathbf{X}_k)} = \sum_{\text{all codes}} P_e(\mathbf{X}_k, \{\mathbf{s}_i\}_m) P(\{\mathbf{s}_i\}_m) \quad (7-2-9)$$

where $P_e(\mathbf{X}_k, \{\mathbf{s}_i\}_m)$ is the conditional probability of error for a given k -bit message \mathbf{X}_k , which is transmitted by use of the code $\{\mathbf{s}_i\}_m$. For the m th code, the probability of error $P_e(\mathbf{X}_k, \{\mathbf{s}_i\}_m)$ is upper-bounded as

$$P_e(\mathbf{X}_k, \{\mathbf{s}_i\}_m) \leq \sum_{\substack{l=1 \\ l \neq k}}^M P_{2m}(\mathbf{s}_l, \mathbf{s}_k) \quad (7-2-10)$$

where $P_{2m}(\mathbf{s}_l, \mathbf{s}_k)$ is the probability of error for a binary communication system that employs the signal vectors \mathbf{s}_l and \mathbf{s}_k to communicate one of two equally likely k -bit messages. Hence,

$$\overline{P_e(\mathbf{X}_k)} \leq \sum_{\text{all codes}} P_e(\{\mathbf{s}_i\}_m) \sum_{\substack{l=1 \\ l \neq k}}^M P_{2m}(\mathbf{s}_l, \mathbf{s}_k) \quad (7-2-11)$$

If we interchange the order of the summations in (7-2-11) we obtain

$$\begin{aligned} \overline{P_e(\mathbf{X}_k)} &\leq \sum_{\substack{l=1 \\ l \neq k}}^M \left[\sum_{\text{all codes}} P_e(\{\mathbf{s}_i\}_m) P_{2m}(\mathbf{s}_l, \mathbf{s}_k) \right] \\ &\leq \sum_{\substack{l=1 \\ l \neq k}}^M \overline{P_{2m}(\mathbf{s}_l, \mathbf{s}_k)} \end{aligned} \quad (7-2-12)$$

where $\overline{P_{2m}(\mathbf{s}_l, \mathbf{s}_k)}$ represents the ensemble average of $P_{2m}(\mathbf{s}_l, \mathbf{s}_k)$ over the 2^{nM} codes or the 2^{nM} communication systems.

For the additive white gaussian noise channel, the binary error probability $P_{2m}(\mathbf{s}_l, \mathbf{s}_k)$ is

$$P_{2m}(\mathbf{s}_l, \mathbf{s}_k) = Q\left(\sqrt{\frac{d_{lk}^2}{2N_0}}\right) \quad (7-2-13)$$

where $d_{lk}^2 = |\mathbf{s}_l - \mathbf{s}_k|^2$. If \mathbf{s}_l and \mathbf{s}_k differ in d coordinates,

$$d_{lk}^2 = |\mathbf{s}_l - \mathbf{s}_k|^2 = \sum_{j=1}^n (s_{lj} - s_{kj})^2 = d(2\sqrt{\mathcal{E}_c})^2 = 4d\mathcal{E}_c \quad (7-2-14)$$

Hence,

$$P_{2m}(\mathbf{s}_l, \mathbf{s}_k) = Q\left(\sqrt{\frac{2d\mathcal{E}_c}{N_0}}\right) \quad (7-2-15)$$

Now, we can average $P_{2m}(\mathbf{s}_l, \mathbf{s}_k)$ over the ensemble of codes. Since all the codes are equally probable, the signal vector \mathbf{s}_l is equally likely to be any of the 2^n possible vertices of the hypercube and it is statistically independent

of the signal vector \mathbf{s}_k . Therefore, $P(s_{li} = s_{ki}) = \frac{1}{2}$ and $P(s_{li} \neq s_{ki}) = \frac{1}{2}$, independently for all $i = 1, 2, \dots, n$. Consequently, the probability that \mathbf{s}_l and \mathbf{s}_k differ in d positions is simply

$$P(d) = \left(\frac{1}{2}\right)^n \binom{n}{d} \quad (7-2-16)$$

Hence, the expected value of $P_{2m}(\mathbf{s}_l, \mathbf{s}_k)$ over the ensemble of codes may be expressed as

$$\begin{aligned} \overline{P_2(\mathbf{s}_l, \mathbf{s}_k)} &= \sum_{d=0}^n P(d) Q\left(\sqrt{\frac{2d\mathcal{E}_c}{N_0}}\right) \\ &= \frac{1}{2^n} \sum_{d=0}^n \binom{n}{d} Q\left(\sqrt{\frac{2d\mathcal{E}_c}{N_0}}\right) \end{aligned} \quad (7-2-17)$$

The result (7-2-17) can be simplified if we upper-bound the Q -function as

$$Q\left(\sqrt{\frac{2d\mathcal{E}_c}{N_0}}\right) < e^{-d\mathcal{E}_c/N_0}$$

Thus,

$$\begin{aligned} \overline{P_2(\mathbf{s}_l, \mathbf{s}_k)} &\leq 2^{-n} \sum_{d=0}^n \binom{n}{d} e^{-d\mathcal{E}_c/N_0} \\ &\leq 2^{-n} (1 + e^{-\mathcal{E}_c/N_0})^n \\ &\leq \left[\frac{1}{2}(1 + e^{-\mathcal{E}_c/N_0})\right]^n \end{aligned} \quad (7-2-18)$$

We observe that the right-hand side of (7-2-18) is independent of the indices l and k . Hence, when we substitute the bound (7-2-18) into (7-2-12), we obtain

$$\begin{aligned} \overline{P_e(\mathbf{X}_k)} &\leq \sum_{\substack{l=1 \\ l \neq k}}^M \overline{P_2(\mathbf{s}_l, \mathbf{s}_k)} = (M-1) \left[\frac{1}{2}(1 + e^{-\mathcal{E}_c/N_0})\right]^n \\ &< M \left[\frac{1}{2}(1 + e^{-\mathcal{E}_c/N_0})\right]^n \end{aligned}$$

Finally, the unconditional average error probability \bar{P}_e is obtained by averaging $\overline{P_e(\mathbf{X}_k)}$ over all possible k -bit information sequences. Thus,

$$\begin{aligned} \bar{P}_e &= \sum_k \overline{P_e(\mathbf{X}_k)} P(\mathbf{X}_k) < M \left[\frac{1}{2}(1 + e^{-\mathcal{E}_c/N_0})\right]^n \sum_k P(\mathbf{X}_k) \\ &< M \left[\frac{1}{2}(1 + e^{-\mathcal{E}_c/N_0})\right]^n \end{aligned} \quad (7-2-19)$$

This result can be expressed in a more convenient form by first defining a parameter R_0 , which is called the *cutoff rate* and has units of bits/dimension, as

$$\begin{aligned} R_0 &= \log_2 \frac{2}{1 + e^{-\mathcal{E}_c/N_0}} \\ &= 1 - \log_2 (1 + e^{-\mathcal{E}_c/N_0}), \quad \text{antipodal signaling} \end{aligned} \quad (7-2-20)$$

Then, (7-2-19) becomes

$$\bar{P}_e < M 2^{-nR_0} = 2^{RT} 2^{-nR_0} \quad (7-2-21)$$

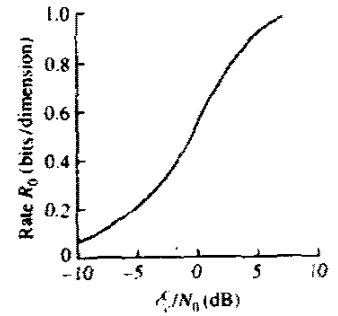


FIGURE 7-2-2 The cutoff rate R_0 as a function of the SNR per dimension in decibels.

Since $n = DT$, (7-2-21) may be expressed as

$$\bar{P}_e < 2^{-T(DR_0 - R)} \quad (7-2-22)$$

The parameter R_0 is plotted as a function of \mathcal{E}_c/N_0 in Fig. 7-2-2. We observe that $0 \leq R_0 \leq 1$. Consequently, $\bar{P}_e \rightarrow 0$ as $T \rightarrow \infty$, provided that the information rate $R < DR_0$.

Alternatively, (7-2-21) may be expressed as

$$\bar{P}_e < 2^{-n(R_0 - R/D)} \quad (7-2-23)$$

The ratio R/D also has units of bits/dimension and may be defined as

$$R_c = \frac{R}{D} = \frac{R}{n/T} = \frac{RT}{n} = \frac{k}{n} \quad (7-2-24)$$

Hence, R_c is the code rate and

$$\bar{P}_e < 2^{-n(R_0 - R_c)} \quad (7-2-25)$$

We conclude that when $R_c < R_0$, the average probability of error $\bar{P}_e \rightarrow 0$ as the code block length $n \rightarrow \infty$. Since the average value of the probability error can be made arbitrarily small as $n \rightarrow \infty$, it follows that there exist codes in the ensemble of 2^{nM} codes that have a probability of error no larger than \bar{P}_e .

From the derivation of the average error probability given above, we conclude that good codes exist. Although we do not normally select codes at random, it is interesting to consider the question of whether or not a randomly selected code is likely to be a good code. In fact, we can easily show that there are many good codes in the ensemble. First, we note that \bar{P}_e is an ensemble average of error probabilities over all codes and that all these probabilities are obviously positive quantities. If a code is selected at random, the probability that its error probability $P_e > \alpha \bar{P}_e$ is less than $1/\alpha$. Consequently, no more than 10% of the codes have an error probability that exceeds $10\bar{P}_e$ and no more than 1% of the codes have an error probability that exceeds $100\bar{P}_e$.

We should emphasize that codes with error probabilities exceeding \bar{P}_e are not necessarily poor codes. For example, suppose that an average error rate of $\bar{P}_e < 10^{-10}$ can be attained by using codes with dimensionality n_0 when $R_0 > R_c$. Then, if we select a code with error probability $1000\bar{P}_e = 10^{-7}$, we may compensate for this reduction in error probability by increasing n from n_0 to $n = 10n_0/7$. Thus, by a modest increase in dimensionality, we have a code with $\bar{P}_e < 10^{-10}$. In summary, good codes are abundant and, hence, they are easily found even by random selection.

It is also interesting to express the average error probability in (7-2-25) in terms of the SNR per bit, γ_b . To accomplish this, we express the energy per signal waveform as

$$\mathcal{E} = n\mathcal{E}_c = k\mathcal{E}_b \tag{7-2-26}$$

Hence, $n = k\mathcal{E}_b/\mathcal{E}_c$. We also note that $R_c\mathcal{E}_b/\mathcal{E}_c = 1$. Therefore, (7-2-25) may be expressed as

$$\bar{P}_e < 2^{-k(\gamma_b/\gamma_0-1)} \tag{7-2-27}$$

where γ_0 is a normalized SNR parameter, defined as

$$\begin{aligned} \gamma_0 &= \frac{R_c}{R_0} \gamma_b \\ &= \frac{R_c \gamma_b}{1 - \log_2(1 + e^{-R_c \gamma_b})} \end{aligned} \tag{7-2-28}$$

Now, we note that $\bar{P}_e \rightarrow 0$ as $k \rightarrow \infty$, provided that the SNR per bit, $\gamma_b > \gamma_0$.

The parameter γ_0 is plotted in Fig. 7-2-3 as a function of $R_c \gamma_b$. Note that as $R_c \gamma_b \rightarrow 0$, $\gamma_0 \rightarrow 2 \ln 2$. Consequently, the error probability for M -ary binary coded signals is equivalent to the error probability obtained from the union bound for M -ary orthogonal signals, provided that the signal dimensionality is sufficiently large so that $\gamma_0 \approx 2 \ln 2$.

The dimensionality parameter D that we introduced in (7-2-5) is proportional to the channel bandwidth required to transmit the signals. Recall from the sampling theorem that a signal of bandwidth W may be represented by samples taken at a rate of $2W$ samples/s. Thus, in the time interval of length T

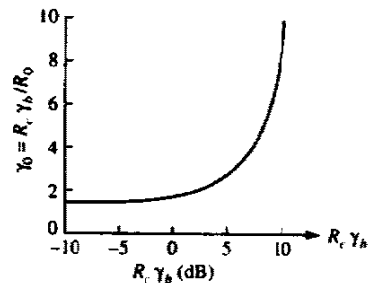


FIGURE 7-2-3 Lower bound on SNR per bit, γ_b , for binary antipodal signals.

there are $n = 2WT$ samples or, equivalently, n degrees of freedom (dimensions). Consequently, D may be equated with $2W$.

Finally, we note that the binary coded signals considered in this section are appropriate when the SNR per dimension is small, e.g., $\mathcal{E}_c/N_0 < 10$. However, when $\mathcal{E}_c/N_0 > 10$, R_0 saturates at 1 bit/dimension. Since the code rate is restricted to be less than R_0 , binary coded signals become inefficient at $\mathcal{E}_c/N_0 > 10$. In such a case, we may use nonbinary-coded signals to achieve an increase in the number of bits per dimension. For example, multiple-amplitude coded signal sets can be constructed from nonbinary codes by mapping each code element into one of q possible amplitude levels (as in PAM). Such codes are considered below.

7-2-2 Random Coding Based on M -ary Multi-amplitude Signals

Instead of constructing binary-coded signals, suppose we employ nonbinary codes with code words of the form given by (7-2-1), where the code elements c_{ij} are selected from the set $\{0, 1, \dots, q-1\}$. Each code element is mapped into one of q possible amplitude levels. Thus, we construct signals corresponding to n -dimensional vectors $\{\mathbf{s}_{ij}\}$ as in (7-2-4), where the components $\{s_{ij}\}$ are selected from a multi-amplitude set of q possible values. Now, we have q^n possible signals, from which we select $M = 2^{kT}$ signals to transmit k -bit blocks of information. The q amplitudes corresponding to the code elements $\{0, 1, \dots, q-1\}$ may be denoted by $\{a_1, a_2, \dots, a_q\}$, and they are assumed to be selected according to some specified probabilities $\{p_i\}$. The amplitude levels are assumed to be equally spaced over the interval $[-\sqrt{\mathcal{E}_c}, \sqrt{\mathcal{E}_c}]$. For example, Fig. 7-2-4 illustrates the amplitude values for $q = 4$. In general, adjacent amplitude levels are separated by $2\sqrt{\mathcal{E}_c}/(q-1)$. This assignment guarantees not only that each component s_{ij} is peak-energy-limited to $\sqrt{\mathcal{E}_c}$, but, also, each code word is constrained in average energy to satisfy the condition

$$|\mathbf{s}_{ij}|^2 < n\mathcal{E}_c \quad (7-2-29)$$

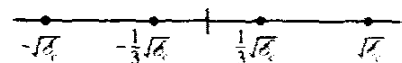
By repeating the derivation given above for random selection of codes in an AWGN channel, we find that the average probability of error is upper-bounded as

$$\bar{P}_e < M 2^{-nR_0} = 2^{kT} 2^{-nT\mathcal{R}} = 2^{-n(R_0 - R/D)} \quad (7-2-30)$$

where R_0 is defined as

$$R_0 = -\log_2 \left(\sum_{l=1}^q \sum_{m=1}^q p_l p_m e^{-d_{lm}^2/4N_0} \right) \quad (7-2-31)$$

FIGURE 7-2-4 Signal alphabet consisting of four amplitude levels.



and

$$d_{lm} = |a_l - a_m|, \quad l, m = 1, 2, \dots, q \quad (7-2-32)$$

In the special case where all the amplitude levels are equally likely, $p_l = p_m = 1/q$ and (7-2-31) reduces to

$$R_0 = -\log_2 \left(\frac{1}{q^2} \sum_{l=1}^q \sum_{m=1}^q e^{-d_{lm}^2/4N_0} \right) \quad (7-2-33)$$

For example, where $q = 2$ and $a_1 = -\sqrt{\mathcal{E}_c}$, $a_2 = \sqrt{\mathcal{E}_c}$, we have $d_{11} = d_{22} = 0$, $d_{12} = d_{21} = 2\sqrt{\mathcal{E}_c}$, and, hence,

$$R_0 = \log_2 \frac{2}{1 + e^{-\mathcal{E}_c/N_0}}, \quad q = 2$$

which agrees with our previous result. When $q = 4$, $a_1 = -\sqrt{\mathcal{E}_c}$, $a_2 = -\sqrt{\mathcal{E}_c}/3$, $a_3 = \sqrt{\mathcal{E}_c}/3$, and $a_4 = \sqrt{\mathcal{E}_c}$, we have $d_{mm} = 0$ for $m = 1, 2, 3, 4$, $d_{12} = d_{23} = d_{34} = d_{21} = d_{32} = d_{43} = 2\sqrt{\mathcal{E}_c}/3$, $d_{13} = d_{31} = d_{24} = d_{42} = 4\sqrt{\mathcal{E}_c}/3$, and $d_{14} = d_{41} = 2\sqrt{\mathcal{E}_c}$. Hence,

$$R_0 = \log_2 \frac{8}{2 + 3e^{-\mathcal{E}_c/9N_0} + 2e^{-4\mathcal{E}_c/9N_0} + e^{-\mathcal{E}_c/N_0}}, \quad q = 4 \quad (7-2-34)$$

Clearly, R_0 now saturates at 2 bits/dimension as \mathcal{E}_c/N_0 increases.

The graphs of R_0 as a function of \mathcal{E}_c/N_0 for equally spaced and equally likely amplitude levels are shown in Fig. 7-2-5 for $q = 2, 3, 4, 8, 16, 32$, and 64. Note that the saturation level now occurs at $\log_2 q$ bits/dimension. Consequently, for high SNR, $\bar{P}_e \rightarrow 0$ as $n \rightarrow \infty$, provided that $R < DR_0 = 2WR_0$ bits/s.

If we remove the peak energy constraint on each of the elements, but retain

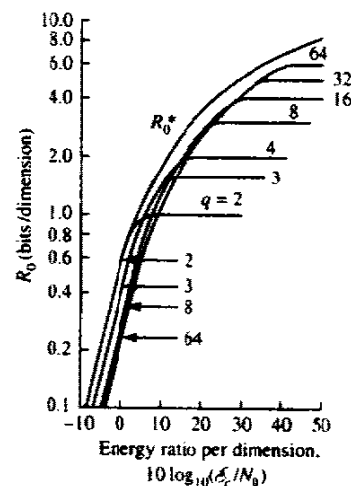


FIGURE 7-2-5 Cutoff rate R_0 for equally spaced q -level amplitude modulation with equal probabilities $p_c = 1/q$. [From Principles of Communication Engineering, by J. M. Wozencraft and I. M. Jacobs, © 1965 by John Wiley and Sons, Inc. Reprinted with permission of the publisher.]

the average energy constraint per code word as given by (7-2-29) it is possible to obtain a larger upper bound on the number of bits per dimension. For this case, the result obtained by Shannon (1959b) is

$$R_0^* = \frac{1}{2} \left[1 + \frac{\mathcal{E}_c}{N_0} - \sqrt{1 + \left(\frac{\mathcal{E}_c}{N_0} \right)^2} \right] \log_2 e + \frac{1}{2} \log_2 \left[\frac{1}{2} \left(1 + \sqrt{1 + \left(\frac{\mathcal{E}_c}{N_0} \right)^2} \right) \right] \quad (7-2-35)$$

The graph of R_0^* as a function of the SNR per dimension, \mathcal{E}_c/N_0 , is also shown in Fig. 7-2-5. It is clear that our selection of the equally spaced, equally likely amplitude levels that result in R_0 is suboptimum. However, these coded signals are easily generated and implemented in practice. This is an important advantage that justifies their use.

7-2-3 Comparison of R_0^* with the Capacity of the AWGN Channel

The channel capacity of the band-limited additive white gaussian noise channel with an average power constraint on the input signal was derived in Section 7-1-2, and is given by

$$C = W \log_2 \left(1 + \frac{P_{av}}{WN_0} \right) \text{ bits/s} \quad (7-2-36)$$

where P_{av} is the average power of the input signal and W is the channel bandwidth. It is interesting to express the capacity of this channel in terms of bits/dimension and the average power in terms of energy/dimension. With $D = 2W$ and

$$\mathcal{E}_c = \frac{\mathcal{E}}{n} = \frac{P_{av} T}{n}$$

we have

$$P_{av} = \frac{n}{T} \mathcal{E}_c = D \mathcal{E}_c \quad (7-2-37)$$

By defining $C_n = C/2W = C/D$ and substituting for W and P_{av} , (7-2-36) may be expressed as

$$\begin{aligned} C_n &= \frac{1}{2} \log_2 \left(1 + 2 \frac{\mathcal{E}_c}{N_0} \right) \\ &= \frac{1}{2} \log_2 (1 + 2R_c \gamma_h) \text{ bits/dimension} \end{aligned} \quad (7-2-38)$$

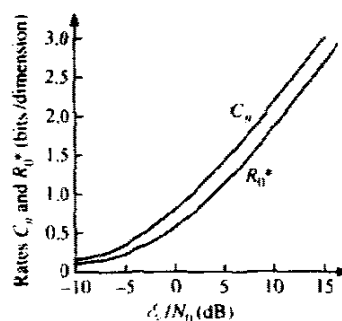


FIGURE 7-2-6 Comparison of cutoff rate R_0^* with the channel capacity for an AWGN channel.

This expression for the normalized capacity may be compared with R_0^* , as shown in Fig. 7-2-6. Since C_n is the ultimate upper limit on the transmission rate R/D , $R_0^* < C_n$ as expected. We also observe that for small values of \mathcal{E}_c/N_0 , the difference between R_0^* and C_n is approximately 3 dB. Therefore, the use of randomly selected, optimum average power-limited, multi-amplitude signals yields a rate function R_0^* that is within 3 dB of the channel capacity. More elaborate bounding techniques are required to show that the probability of error can be made arbitrarily small when $R < DC_n = 2WC_n = C$.

7-3 COMMUNICATION SYSTEM DESIGN BASED ON THE CUTOFF RATE

In the foregoing discussion, we characterized coding and modulation performance in terms of the error probability, which is certainly a meaningful criterion for system design. However, in many cases, the computation of the error probability is extremely difficult, especially if nonlinear operations such as signal quantization are performed in processing the signal at the receiver, or if the additive noise is nongaussian.

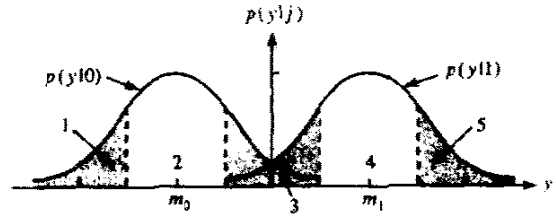
Instead of attempting to compute the exact probability of error for specific codes, we may use the ensemble average probability of error for randomly selected code words. The channel is assumed to have q input symbols $\{0, 1, \dots, q-1\}$ and Q output symbols $\{0, 1, \dots, Q-1\}$, and to be characterized by the transition probabilities $P(i|j)$, where $j = 0, 1, \dots, q-1$ and $i = 0, 1, \dots, Q-1$, with $Q \geq q$. The input symbols occur with probabilities $\{p_j\}$ and are assumed to be statistically independent. In addition, the noise on the channel is assumed to be statistically independent in time, so that there is no dependence among successive received symbols. Under these conditions, the ensemble average probability of error for random selected code words may be derived by applying the Chernoff bound (see Viterbi and Omura, 1979).

The general result that is obtained for the discrete memoryless channel is

$$\bar{P}_e < 2^{-n(R_Q - R/D)} \quad (7-3-1)$$

where n is the block length of the code, R is the information rate in bits/s, D is

FIGURE 7-3-1 Example of quantization of the demodulator output into five levels.



the number of dimensions per second, and R_Q is the cutoff rate for a quantizer with Q levels, defined as

$$R_Q = \max_{\{p_i\}} \left\{ -\log_2 \sum_{i=0}^{Q-1} \left[\sum_{j=0}^{Q-1} p_j \sqrt{P(i|j)} \right]^2 \right\} \quad (7-3-2)$$

From the viewpoint of code design, the combination of modulator, waveform channel, and demodulator constitutes a discrete-time channel with q inputs and Q outputs. The transition probabilities $\{P(i|j)\}$ depend on the channel noise characteristics, the number of quantization levels, and the type of quantizer, e.g., uniform or nonuniform. For example, in the binary-input AWGN channel, the output of the correlator at the sampling instant may be expressed as

$$p(y|j) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(y-m_j)^2/2\sigma^2}, \quad j = 0, 1 \quad (7-3-3)$$

where $m_0 = -\sqrt{\mathcal{E}_c}$, $m_1 = \sqrt{\mathcal{E}_c}$, and $\sigma^2 = \frac{1}{2}N_0$. These two pdfs are shown in Fig. 7-3-1. Also illustrated in the figure is a quantization scheme that subdivides the real line into five regions. From such a subdivision, we may compute the transition probabilities and optimally select the thresholds that subdivide the regions in a way that maximizes R_Q for any given Q . Thus,

$$P(i|j) = \int_{r_i} p(y|j) dy \quad (7-3-4)$$

where the integral of $p(y|j)$ is evaluated over the region r_i that corresponds to the transition probability $P(i|j)$.

The value of the rate R_Q in the limit as $Q \rightarrow \infty$ yields the cutoff rate for the unquantized decoder. It is relatively straightforward to show that as $Q \rightarrow \infty$, the first summation (sum from $i=0$ to $Q-1$) in (7-3-2) becomes an integral and the transition probabilities are replaced by the corresponding pdfs. Thus, when the channel consists of q discrete inputs and one continuous output y , which represents the unquantized output from a matched filter or a cross-correlator in a system that employs either PSK or a multi-amplitude (PAM) modulation, the cutoff rate is given by

$$R_0 = \max_{\{p_j\}} \left\{ -\log_2 \int_{-\infty}^{\infty} dy \left[\sum_{j=0}^{q-1} p_j \sqrt{p(y|j)} \right]^2 \right\} \quad (7-3-5)$$

where p_j , $0 \leq j \leq q-1$, is the probability of transmitting the j th symbol and

$p(y | j)$ is the conditional probability density function of the output y from the matched filter or cross-correlator when the j th signal is transmitted. This is the desired expression for unquantized (soft-decision) decoding.

We observe that when the input signal is binary PSK with $p_0 = p_1 = \frac{1}{2}$ and the noise is additive, white, and gaussian, (7-3-5) reduces to the familiar result given previously in (7-2-20).

The general expressions in (7-3-5) and (7-3-2) allow us to compare the performance of various receiver implementations based on a different number of quantization levels.

Example 7-3-1

Let us compare the performance of a binary PSK input signal in an AWGN channel when the receiver quantizes the output to $Q = 2, 4,$ and 8 levels. To simplify the optimization problem for the quantization of the signal at the output of the demodulator, the quantization levels are placed at $0, \pm\tau_h, \pm 2\tau_h, \dots, \pm(2^{b-1} - 1)\tau_h$, where τ_h is the *quantizer step-size parameter*, which is to be selected, and b is the number of bits of the quantizer. A good strategy for the selection of τ_h is to choose it to minimize the SNR per bit γ_b that is required for operation at a code rate R_0 . This implies that the step-size parameter must be optimized for every SNR, which in a practical implementation of the receiver means that the SNR must be measured. Fortunately, τ_h does not exhibit high sensitivity to small changes in SNR, so that it is possible to optimize τ_h for one SNR and obtain good performance for a wide range of SNRs about this nominal value by using a fixed τ_h .

Based on this approach, the expression for R_Q given by (7-3-2) was evaluated for $b = 1$ (hard-decision decoding), 2, and 3 bits, corresponding to $Q = 2, 4,$ and 8 levels of quantization. The results are plotted in Fig. 7-3-2. The value of R_0 for unquantized soft-decision decoding, obtained by evaluating (7-3-5) is also shown in Fig. 7-3-2. We observe that two-bit quantization with $\tau_h = 1.0$ gains about 1.4 dB over hard-decision decoding, and three-bit quantization with $\tau_h = 0.5$ yields an additional 0.4 dB improvement. Thus, with a three-bit quantizer, we are within 0.2 dB of the

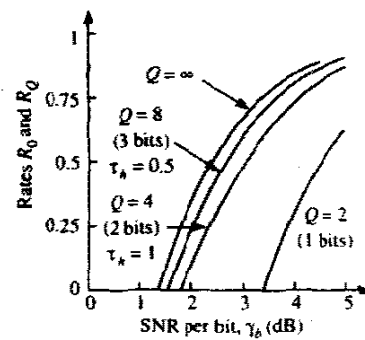


FIGURE 7-3-2 Effect of quantization on the performance of a coded communications system operating at a rate $R = R_0$ or $R = R_Q$, with binary PSK modulation on an AWGN channel.

unquantized soft-decision decoding limit. Clearly, there is little to be gained by increasing the precision any further.

When a nonbinary code is used in conjunction with M -ary ($M = q$) signaling, the received signal at the output of the M matched filters may be represented by the vector $\mathbf{y} = [y_1 \ y_2 \ \cdots \ y_M]$. The cutoff rate for this M -input, M -output (unquantized) channel is

$$R_0 = \max_{\{p_i\}} \left\{ -\log_2 \sum_{j=0}^{M-1} \sum_{i=0}^{M-1} p_i p_j \int_{-\infty}^{\infty} \sqrt{p(\mathbf{y}|j)p(\mathbf{y}|i)} dy \right\} \quad (7-3-6)$$

where $p(\mathbf{y}|j)$ is the conditional probability density function of the output vector \mathbf{y} from the demodulator given that the j th signal was transmitted. Note that (7-3-6) is similar in form to (7-3-5) except that we now have an M -fold integral to perform because there are M outputs from the demodulator.

Let us assume that the M signals are orthogonal so that the M outputs conditioned on a particular input signal are statistically independent. As a consequence,

$$p(\mathbf{y}|j) = p_{s+n}(y_j) \prod_{\substack{i=0 \\ i \neq j}}^{M-1} p_n(y_i) \quad (7-3-7)$$

where $p_{s+n}(y_j)$ is the pdf of the matched filter output corresponding to the transmitted signal and $\{p_n(y_i)\}$ corresponds to the noise-only outputs from the other $M - 1$ matched filters. When (7-3-7) is incorporated into (7-3-6) we obtain

$$R_0 = \max_{\{p_i\}} \left\{ -\log_2 \left[\sum_{j=0}^{M-1} p_j^2 + \sum_{j=0}^{M-1} \sum_{\substack{i=0 \\ i \neq j}}^{M-1} p_i p_j \left(\int_{-\infty}^{\infty} dy \sqrt{p_{s+n}(y)p_n(y)} \right)^2 \right] \right\} \quad (7-3-8)$$

The maximization of R_0 over the set of input probabilities yields $p_i = 1/M$ for $1 \leq j \leq M$. Consequently, (7-3-8) reduces to

$$\begin{aligned} R_0 &= \log_2 \left\{ \frac{M}{1 + (M-1) \left[\int_{-\infty}^{\infty} \sqrt{p_{s+n}(y)p_n(y)} dy \right]^2} \right\} \\ &= \log_2 M - \log_2 \left\{ 1 + (M-1) \left[\int_{-\infty}^{\infty} \sqrt{p_{s+n}(y)p_n(y)} dy \right]^2 \right\} \end{aligned} \quad (7-3-9)$$

This is the desired result for the cutoff rate of an M -ary input, M -ary vector output unquantized channel.

For phase coherent detection of the M -ary orthogonal signals the appropriate pdfs are

$$\begin{aligned} P_{s+n}(y) &= \frac{1}{\sqrt{2\pi}\sigma} e^{-(y-m)^2/2\sigma^2} \\ P_n(y) &= \frac{1}{\sqrt{2\pi}\sigma} e^{-y^2/2\sigma^2} \end{aligned} \quad (7-3-10)$$

where $m = \sqrt{\mathcal{E}}$ and $\sigma^2 = \frac{1}{2}N_0$. Substituting these relations into (7-3-9) and evaluating the integral yields

$$R_0 = \log_2 \left[\frac{M}{1 + (M-1)e^{-\mathcal{E}/2N_0}} \right] \\ = \log_2 \left[\frac{M}{1 + (M-1)e^{-R_w \gamma_b/2}} \right] \quad (7-3-11)$$

where \mathcal{E} is the received energy per waveform, R_w is the information rate in bits/waveform, and $\gamma_b = \mathcal{E}_b/N_0$ is the SNR per bit.

We should emphasize that the rate parameter R_w has imbedded in it the code rate R_c . For example, if $M = 2$ and the code is binary then $R_w = R_c$. More generally, if the code is binary and $M = 2^v$ then each M -ary waveform conveys $R_w = vR_c$ bits of information. It is also interesting to note that if the code is binary and $M = 2$ then (7-3-11) reduces to

$$R_0 = \log_2 \left(\frac{2}{1 + e^{-R_w \gamma_b/2}} \right), \quad M = 2 \text{ orthogonal signals} \quad (7-3-12)$$

which is 3 dB worse than the cutoff rate for antipodal signals. If we set $R_w = R_0$ in (7-3-11) and solve for γ_b , we obtain

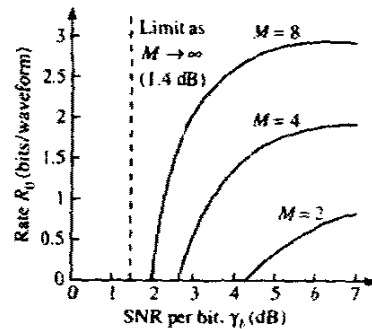
$$\gamma_b = \frac{2}{R_0} \ln \left(\frac{M-1}{2^{-R_0} M - 1} \right) \quad (7-3-13)$$

Graphs of R_0 versus γ_b for several values of M are illustrated in Fig. 7-3-3. Note that the curve for any value of M saturates at $R_0 = \log_2 M$.

It is also interesting to consider the limiting form of (7-3-11) in the limit as $M \rightarrow \infty$. This yields

$$\lim_{M \rightarrow \infty} R_0 = \frac{\mathcal{E}}{2 N_0 \ln 2} \text{ bits/waveform} \quad (7-3-14)$$

FIGURE 7-3-3 SNR per bit required to operate at a rate R_0 with M -ary orthogonal signals detected coherently in an AWGN channel.



Since $\mathcal{E} = P_{av}T$, where T is the time interval per waveform, it follows that

$$\lim_{M \rightarrow \infty} \frac{R_0}{T} = \frac{P_{av}}{2 N_0 \ln 2} = \frac{1}{2} C_z \tag{7-3-15}$$

Hence, in the limit as $M \rightarrow \infty$, the cutoff rate is one-half of the capacity for the infinite bandwidth AWGN channel. Alternatively, the substitution of $\mathcal{E} = R_0 \mathcal{E}_b$ into (7-3-14) yields $\gamma_b = 2 \ln 2$ (1.4 dB), which is the minimum SNR required to operate at R_0 (as $M \rightarrow \infty$). Hence, signaling at a rate R_0 requires 3 dB more power than the Shannon limit.

The value of R_0 given in (7-3-11) is based on the use of M -ary orthogonal signals, which are clearly suboptimal when M is small. If we attempt to maximize R_0 by selecting the best set of M waveforms, we should not be surprised to find that the simplex set of waveforms is optimum. In fact, R_0 for these optimum waveforms is simply given as

$$R_0 = \log_2 \left[\frac{M}{1 + (M - 1)e^{-M/2(M-1)N_0}} \right] \tag{7-3-16}$$

If we compare this expression with (7-3-11) we observe that R_0 in (7-3-16) simply reflects the fact that the simplex set is more energy-efficient by a factor $M/(M - 1)$.

In the case of noncoherent detection, the probability density functions corresponding to signal-plus-noise and noise alone may be expressed as

$$\begin{aligned} P_{s+n}(y) &= ye^{-(y^2+a^2)/2} I_0(ay), \quad y \geq 0 \\ p_n(y) &= ye^{-y^2/2}, \quad y \geq 0 \end{aligned} \tag{7-3-17}$$

where, by definition, $a = \sqrt{2\mathcal{E}/N_0}$. The computation of R_0 given by (7-3-9) does not yield a closed-form solution. Instead, the integral in (7-3-9) must be evaluated numerically. Results for this case have been given by Jordan (1966) and Bucher (1980). For example, the (normalized) cutoff rate R_0 for M -ary orthogonal signals with noncoherent detection is shown in Fig. 7-3-4 for

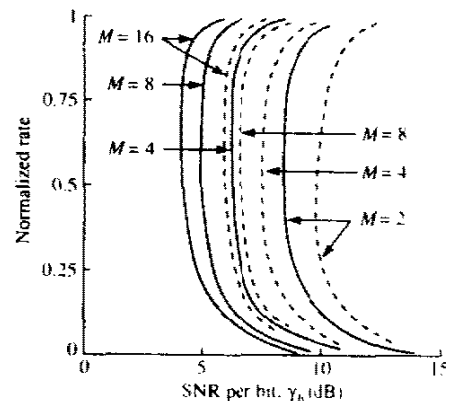


FIGURE 7-3-4 SNR per bit required to operate at a rate R_0 , with M -ary orthogonal signals detected noncoherently in an AWGN channel.

$M = 2, 4, 8,$ and 16 . For purposes of comparison we also plot the cutoff rate for hard-decision decoding ($Q = M$) of the M -ary symbols. In this case, we have

$$R_Q = \log_2 \left\{ \frac{M}{[\sqrt{(1 - P_M)} + \sqrt{(M - 1)P_M}]^2} \right\}, \quad Q = M \quad (7-3-18)$$

where P_M is the probability of a symbol error. For a relatively broad range of rates, the difference between soft- and hard-decision decoding is approximately 2 dB.

The most striking characteristic of the performance curves in Fig. 7-3-4 is that there is an optimum code rate for any given M . Unlike the case of coherent detection, where the SNR per bit decreases monotonically with a decrease in code rate, the SNR per bit for noncoherent detection reaches a minimum in the vicinity of a normalized rate of 0.5, and increases for both high and low rates. The minimum is rather broad, so there is really a range of rates from 0.2 to 0.9 where the SNR per bit is within 1 dB of the minimum. This characteristic behavior in the performance with noncoherent detection is attributed to the nonlinear characteristic of the detector.

7-4 BIBLIOGRAPHICAL NOTES AND REFERENCES

The pioneering work on channel characterization in terms of channel capacity and random coding was done by Shannon (1948a, b, 1949). Additional contributions were subsequently made by Gilbert (1952), Elias (1955), Gallager (1965), Wyner (1965), Shannon *et al.* (1967), Forney (1968) and Viterbi (1969). All of these early publications are contained in the IEEE Press book entitled *Key Papers in the Development of Information Theory*, edited by Slepian (1974).

The use of the cutoff rate parameter as a design criterion was proposed and developed by Wozencraft and Kennedy (1966) and by Wozencraft and Jacobs (1965). It was used by Jordan (1966) in the design of coded waveforms for M -ary orthogonal signals with coherent and noncoherent detection. Following these pioneering works, the cutoff rate has been widely used as a design criterion for coded signals in a variety of different channel conditions.

PROBLEMS

7-1 Show that the following two relations are necessary and sufficient conditions for the set of input probabilities $\{P(x_j)\}$ to maximize $I(X; Y)$ and, thus, to achieve capacity for a DMC:

$$I(x_j; Y) = C \quad \text{for all } j \text{ with } P(x_j) > 0$$

$$I(x_j; Y) \leq C \quad \text{for all } j \text{ with } P(x_j) = 0$$

where C is the capacity of the channel and

$$I(x_j; Y) = \sum_{i=0}^{Q-1} P(y_i | x_j) \log \frac{P(y_i | x_j)}{P(y_i)}$$

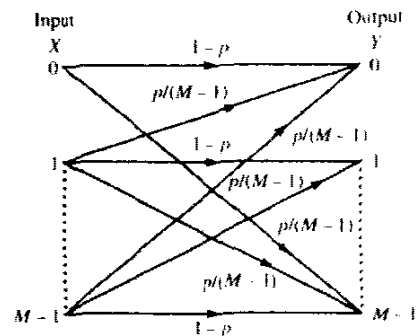


FIGURE P7-2

7-2 Figure P7-2 illustrates an M -ary symmetric DMC with transition probabilities $P(y | x) = 1 - p$ when $x = y = k$ for $k = 0, 1, \dots, M - 1$, and $P(y | x) = p / (M - 1)$ when $x \neq y$.

- a Show that this channel satisfies the condition given in Problem 7-1 when $P(x_k) = 1/M$.
 - b Determine and plot the channel capacity as a function of p .
- 7-3 Determine the capacities of the channels shown in Fig. P7-3.

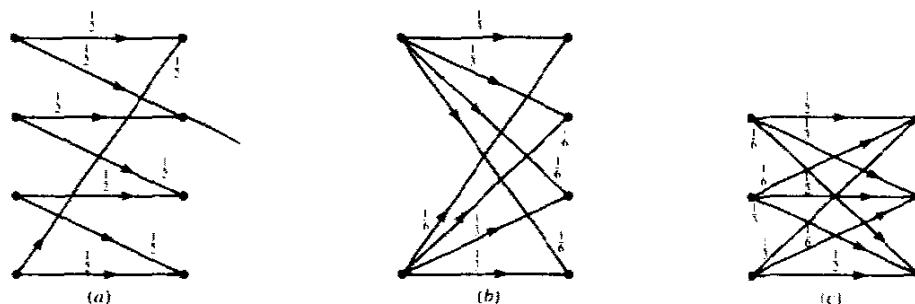


FIGURE P7-3

7-4 Consider the two channels with the transition probabilities as shown in Fig. P7-4. Determine if equally probable input symbols maximize the information rate through the channel.

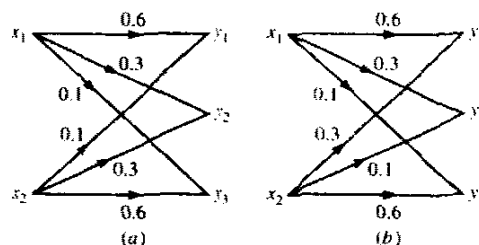


FIGURE P7-4

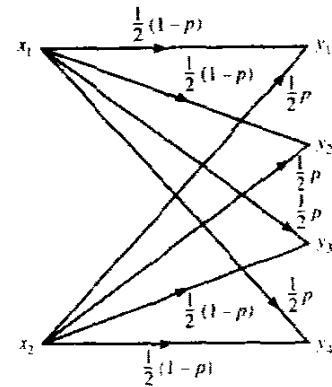


FIGURE P7-6

- 7-5 A telephone channel has a bandwidth $W = 3000$ Hz and a signal-to-noise power ratio of 400 (26 dB). Suppose we characterize the channel as a band-limited AWGN waveform channel with $P_{sn}/WN_0 = 400$.
- Determine the capacity of the channel in bits/s.
 - Is the capacity of the channel sufficient to support the transmission of a speech signal that has been sampled and encoded by means of logarithmic PCM?
 - Usually, channel impairments other than additive noise limit the transmission rate over the telephone channel to less than the channel capacity of the equivalent band-limited AWGN channel considered in (a). Suppose that a transmission rate of $0.7C$ is achievable in practice without channel encoding. Which of the speech source encoding methods described in Section 3-5 provide sufficient compression to fit the bandwidth restrictions of the telephone channel?
- 7-6 Consider the binary-input, quaternary-output DMC shown in Fig. P7-6.
- Determine the capacity of the channel.
 - Show that this channel is equivalent to a BSC.
- 7-7 Determine the channel capacity for the channel shown in Fig. P7-7.
- 7-8 Consider a BSC with crossover probability of error p . Suppose that R is the number of bits in a source code word that represents one of 2^R possible levels at the output of a quantizer. Determine
- the probability that a code word transmitted over the BSC is received correctly;
 - the probability of having at least one bit error in a code word transmitted over the BSC;
 - the probability of having n_e or less bit errors in a code word;

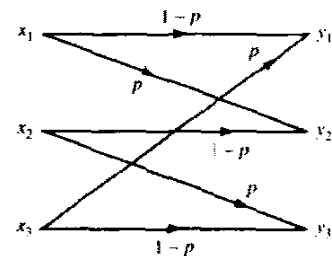


FIGURE P7-7

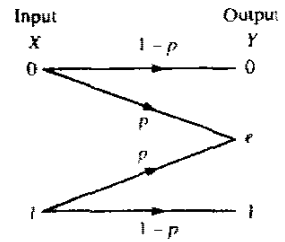


FIGURE P7-10

- d** Evaluate the probability in (a), (b), and (c) for $R = 5$, $p = 0.01$, and $n_c = 5$.
- 7-9** Show that, for a DMC, the average mutual information between a sequence $X_1 X_2 \cdots X_n$ of channel inputs and the corresponding channel outputs satisfy the condition

$$I(X_1 X_2 \cdots X_n; Y_1, Y_2, \dots, Y_n) \leq \sum_{i=1}^n I(X_i; Y_i)$$

with equality if and only if the set of input symbols is statistically independent.

- 7-10** Figure P7-10 illustrates a binary erasure channel with transition probabilities $P(0 | 0) = P(1 | 1) = 1 - p$ and $P(e | 0) = P(e | 1) = p$. The probabilities for the input symbols are $P(X = 0) = \alpha$ and $P(X = 1) = 1 - \alpha$.
- Determine the average mutual information $I(X; Y)$ in bits.
 - Determine the value of α that maximizes $I(X; Y)$, i.e., the channel capacity C in bits/channel use, and plot C as a function of p for the optimum value of α .
 - For the value of α found in (b), determine the mutual information $I(x; y) = I(0; 0)$, $I(1; 1)$, $I(0; e)$, and $I(1; e)$.
- 7-11** Consider the binary-input, ternary-output channel with transition probabilities shown in Fig. P7-11, where e denotes an erasure. For the AWGN channel, α and p are defined as

$$\alpha = \frac{1}{\sqrt{\pi N_0}} \int_{-\beta}^{\beta} e^{-(x + \sqrt{\mathcal{E}_c}/N_0)^2 / N_0} dx$$

$$p = \frac{1}{\sqrt{\pi N_0}} \int_{\beta}^{\infty} e^{-(x + \sqrt{\mathcal{E}_c}/N_0)^2 / N_0} dx$$

- Determine R_Q for $Q = 3$ as a function of the probabilities α and p .
- The rate parameter R_Q depends on the choice of the threshold β through the probabilities α and p . For any \mathcal{E}_c/N_0 , the value of β that maximizes R_Q can be determined by trial and error. For example, it can be shown that for \mathcal{E}_c/N_0 below 0 dB, $\beta_{opt} = 0.65\sqrt{\mathcal{E}_c/N_0}$, for $1 \leq \mathcal{E}_c/N_0 \leq 10$, β_{opt} varies approximately

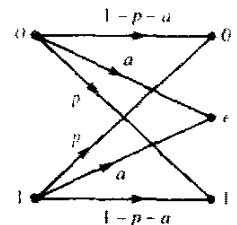


FIGURE P7-11

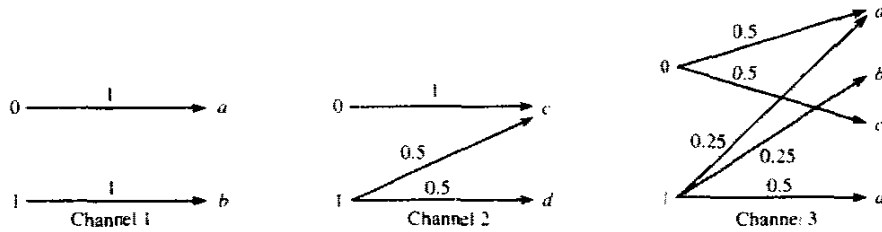


FIGURE P7-13

linearly between $0.65\sqrt{\frac{1}{2}N_0}$ and $1.0\sqrt{\frac{1}{2}N_0}$. By using $\beta = 0.65\sqrt{\frac{1}{2}N_0}$ for the entire range of \mathcal{E}_c/N_0 , plot R_Q versus \mathcal{E}_c/N_0 and compare this result with R_Q ($Q = \infty$).

- 7-12 Find the capacity of the cascade connection of n binary-symmetric channels with the same crossover probability ϵ . What is the capacity when the number of channels goes to infinity?
- 7-13 Channels 1, 2, and 3 are shown in Fig. P7-13.
 - a Find the capacity of channel 1. What input distribution achieves capacity?
 - b Find the capacity of channel 2. What input distribution achieves capacity?
 - c Let C denote the capacity of the third channel and C_1 and C_2 represent the capacities of the first and second channel. Which of the following relations holds true and why?

$$C < \frac{1}{2}(C_1 + C_2) \tag{i}$$

$$C = \frac{1}{2}(C_1 + C_2) \tag{ii}$$

$$C > \frac{1}{2}(C_1 + C_2) \tag{iii}$$

- 7-14 Let C denote the capacity of a discrete memoryless channel with input alphabet $\mathcal{X} = \{x_1, x_2, \dots, x_N\}$ and output alphabet $\mathcal{Y} = \{y_1, y_2, \dots, y_M\}$. Show that $C \leq \min \{\log M, \log N\}$.
- 7-15 The channel C (known as the Z channel) is shown in Fig. P7-15.
 - a Find the input probability distribution that achieves capacity.
 - b What is the input distribution and capacity for the special cases $\epsilon = 0$, $\epsilon = 1$, and $\epsilon = 0.5$?
 - c Show that if n such channels are cascaded, the resulting channel will be equivalent to a Z channel with $\epsilon_1 = \epsilon^n$.
 - d What is the capacity of the equivalent Z channel when $n \rightarrow \infty$.
- 7-16 Find the capacity of an additive white Gaussian noise channel with a bandwidth 1 MHz, power 10 W, and noise power spectral density $\frac{1}{2}N_0 = 10^{-9}$ W/Hz.
- 7-17 Channel C_1 is an additive white gaussian noise channel with a bandwidth W , average transmitter power P , and noise power spectral density $\frac{1}{2}N_0$. Channel C_2 is

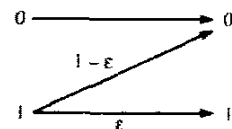


FIGURE P7-15

an additive gaussian noise channel with the same bandwidth and power as channel C_1 but with noise power spectral density $\Phi_n(f)$. It is further assumed that the total noise power for both channels is the same; that is,

$$\int_{-W}^W \Phi_n(f) df = \int_{-W}^W \frac{1}{2} N_0 df = N_0 W$$

Which channel do you think has a larger capacity? Give an intuitive reasoning.

- 7-18** A discrete-time memoryless gaussian source with mean 0 and variance σ^2 is to be transmitted over a binary-symmetric channel with crossover probability p .
- What is the minimum value of the distortion attainable at the destination (distortion is measured in mean-squared error)?
 - If the channel is a discrete-time memoryless additive gaussian noise channel with input power P and noise power P_n , what is the minimum attainable distortion?
 - Now assume that the source has the same basic properties but is not memoryless. Do you expect the distortion in transmission over the binary-symmetric channel to be decreased or increased? Why?
- 7-19** X is a binary memoryless source with $p(X=0) = 0.3$. This source is transmitted over a binary-symmetric channel with crossover probability $p = 0.1$.
- Assume that the source is directly connected to the channel, i.e., no coding is employed. What is the error probability at the destination?
 - If coding is allowed, what is the minimum possible error probability in the reconstruction of the source.
 - For what values of p is reliable transmission possible (with coding, of course)?
- 7-20** Plot the capacity of an AWGN channel that employs binary antipodal signaling, with optimal bit-by-bit detection at the receiver, as a function of \mathcal{E}_b/N_0 . On the same axis, plot the capacity of the same channel when binary orthogonal signaling is employed.
- 7-21** In a coded communication system, M messages $1, 2, \dots, M = 2^k$ are transmitted by M baseband signals $x_1(t), x_2(t), \dots, x_M(t)$, each of duration nT . The general form of $x_i(t)$ is given by

$$x_i(t) = \sum_{j=0}^{n-1} f_{ij}(t - jT)$$

where $f_{ij}(t)$ can be either of the two signals $f_1(t)$ or $f_2(t)$, where $f_1(t) = f_2(t) = 0$ for all $t \notin [0, T]$. We further assume that $f_1(t)$ and $f_2(t)$ have equal energy \mathcal{E} and the channel is ideal (no attenuation) with additive white gaussian noise of power spectral density $\frac{1}{2}N_0$. This means that the received signal is $r(t) = x(t) + n(t)$, where $x(t)$ is one of the $x_i(t)$ and $n(t)$ represents the noise.

- With $f_1(t) = -f_2(t)$, show that N , the dimensionality of the signal space, satisfies $N \leq n$.
- Show that, in general, $N \leq 2n$.
- With $M = 2$, show that, for general $f_1(t)$ and $f_2(t)$,

$$p(\text{error} | x_1(t) \text{ sent}) \leq \int \cdots \int_{R^N} \sqrt{p(\mathbf{r} | \mathbf{x}_1) p(\mathbf{r} | \mathbf{x}_2)} d\mathbf{r}$$

where \mathbf{r} , \mathbf{x}_1 , and \mathbf{x}_2 are the vector representations of $r(t)$, $x_1(t)$, and $x_2(t)$ in the N -dimensional space.

d Using the result of (c), show that, for general M ,

$$p(\text{error} | x_m(t) \text{ sent}) \leq \sum_{\substack{1 \leq m' \leq M \\ m' \neq m}} \int_{R^N} \cdots \int \sqrt{p(\mathbf{r} | \mathbf{x}_m)p(\mathbf{r} | \mathbf{x}_{m'})} d\mathbf{r}$$

e Show that

$$\int_{R^N} \cdots \int \sqrt{p(\mathbf{r} | \mathbf{x}_m)p(\mathbf{r} | \mathbf{x}_{m'})} d\mathbf{r} = \exp\left(-\frac{|\mathbf{x}_m - \mathbf{x}_{m'}|^2}{4N_0}\right)$$

and, therefore,

$$p(\text{error} | x_m(t) \text{ sent}) \leq \sum_{\substack{1 \leq m' \leq M \\ m' \neq m}} \exp\left(-\frac{|\mathbf{x}_m - \mathbf{x}_{m'}|^2}{4N_0}\right)$$

BLOCK AND CONVOLUTIONAL CHANNEL CODES

In Chapter 7, we treated channel coding and decoding from a general viewpoint, and showed that even randomly selected codes on the average yield performances close to the capacity of a channel. In the case of orthogonal signals, we demonstrated that the channel capacity limit can be achieved as the number of signals approaches infinity.

In this chapter, we describe specific codes and evaluate their performance for the additive white gaussian noise channel. In particular, we treat two classes of codes, namely, linear block codes and convolutional codes. The code performance is evaluated for both hard-decision decoding and soft-decision decoding.

8-1 LINEAR BLOCK CODES

A block code consists of a set of fixed-length vectors called *code words*. The length of a code word is the number of elements in the vector and is denoted by n . The elements of a code word are selected from an alphabet of q elements. When the alphabet consists of two elements, 0 and 1, the code is a binary code and the elements of any code word are called bits. When the elements of a code word are selected from an alphabet having q elements ($q > 2$), the code is nonbinary. It is interesting to note that when q is a power of 2, i.e., $q = 2^b$ where b is a positive integer, each q -ary element has an equivalent binary representation consisting of b bits, and, thus, a nonbinary code of block length N can be mapped into a binary code of block length $n = bN$.

There are 2^n possible code words in a binary block code of length n . From

these 2^n code words, we may select $M = 2^k$ code words ($k < n$) to form a code. Thus, a block of k information bits is mapped into a code word of length n selected from the set of $M = 2^k$ code words. We refer to the resulting block code as an (n, k) code, and the ratio $k/n \equiv R_c$ is defined to be the *rate* of the code. More generally, in a code having q elements, there are q^n possible code words. A subset of $M = 2^k$ code words may be selected to transmit k -bit blocks of information.

Besides the code rate parameter R_c , an important parameter of a code word is its *weight*, which is simply the number of nonzero elements that it contains. In general, each code word has its own weight. The set of all weights in a code constitutes the *weight distribution* of the code. When all the M code words have equal weight, the code is called a *fixed-weight code* or a *constant-weight code*.

The encoding and decoding functions involve the arithmetic operations of addition and multiplication performed on code words. These arithmetic operations are performed according to the conventions of the algebraic field that has as its elements the symbols contained in the alphabet. For example, the symbols in a binary alphabet are 0 and 1; hence, the field has two elements. In general, a field F consists of a set of elements that has two arithmetic operations defined on its elements, namely, addition and multiplication, that satisfy the following properties (axioms).

Addition

- 1 The set F is closed under addition, i.e., if $a, b \in F$ then $a + b \in F$.
- 2 Addition is associative, i.e., if $a, b,$ and c are elements of F then $a + (b + c) = (a + b) + c$.
- 3 Addition is commutative, i.e., $a + b = b + a$.
- 4 The set contains an element called *zero* that satisfies the condition $a + 0 = a$.
- 5 Every element in the set has its own negative element. Hence, if b is an element, its negative is denoted by $-b$. The subtraction of two elements, such as $a - b$, is defined as $a + (-b)$.

Multiplication

- 1 The set F is closed under multiplication, i.e., if $a, b \in F$ then $ab \in F$.
- 2 Multiplication is associative, i.e., $a(bc) = (ab)c$.
- 3 Multiplication is commutative, i.e., $ab = ba$.
- 4 Multiplication is distributive over addition, i.e., $(a + b)c = ac + bc$.
- 5 The set F contains an element, called the *identity*, that satisfies the condition $a(1) = a$, for any element $a \in F$.
- 6 Every element of F , except zero, has an inverse. Hence, if $b \in F$ ($b \neq 0$)

then its inverse is defined as b^{-1} , and $bb^{-1} = 1$. The division of two elements, such as $a \div b$, is defined as ab^{-1} .

We are very familiar with the field of real numbers and the field of complex numbers. These fields have an infinite number of elements. However, as indicated above, codes are constructed from fields with a finite number of elements. A finite field with q elements is generally called a *Galois field* and denoted by $GF(q)$.

Every field must have a *zero* element and a *one* element. Hence, the simplest field is $GF(2)$. In general, when q is a prime, we can construct the finite field $GF(q)$ consisting of the elements $\{0, 1, \dots, q - 1\}$. The addition and multiplication operations on the elements of $GF(q)$ are defined modulo q and denoted as $(\text{mod } q)$. For example, the addition and multiplication tables for $GF(2)$ are

+	0	1
0	0	1
1	1	0

·	0	1
0	0	0
1	0	1

which are operations $(\text{mod } 2)$. Similarly, the field $GF(5)$ is a set consisting of the elements $\{0, 1, 2, 3, 4\}$. The addition and multiplication tables for $GF(5)$ are

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

·	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

In general, the finite field $GF(q)$ can be constructed only if q is a prime or a power of a prime. When q is a prime, multiplication and addition are based on modulo- q arithmetic as illustrated above. If $q = p^m$ where p is a prime and m is any positive integer, it is possible to extend the field $GF(p)$ to the field $GF(p^m)$. This is called the *extension field* of $GF(p)$. Multiplication and addition of the elements in the extension field are based on modulo- p arithmetic.

With this brief introduction to the arithmetic operations that may be performed on the elements of code words, let us now consider some basic characteristics of block codes.

Suppose C_i and C_j are any two code words in an (n, k) block code. A measure of the difference between the code words is the number of corresponding elements or positions in which they differ. This measure is called the *Hamming distance* between the two code words and is denoted as d_{ij} .

Clearly, d_{ij} for $i \neq j$ satisfies the condition $0 < d_{ij} \leq n$. The smallest value of the set $\{d_{ij}\}$ for the M code words is called the *minimum distance* of the code and is denoted as d_{\min} . Since the Hamming distance is a measure of the separation between pairs of code words, it is intimately related to the cross-correlation coefficient between corresponding pairs of waveforms generated from the code words. The relationship is discussed in Section 8-1-4.

Besides characterizing a code as being binary or nonbinary, one can also describe it as either linear or nonlinear. Suppose C_i and C_j are two code words in an (n, k) block code and let α_1 and α_2 be any two elements selected from the alphabet. Then the code is said to be linear if and only if $\alpha_1 C_i + \alpha_2 C_j$ is also a code word. This definition implies that a linear code must contain the all-zero code word. Consequently a constant-weight code is nonlinear.

Suppose we have a binary linear block code, and let $C_i, i = 1, 2, \dots, M$, denote the M code words. For convenience, let C_1 denote the all-zero code word, i.e., $C_1 = [00 \dots 0]$, and let w_r denote the weight of the r th code word. It follows that w_r is the Hamming distance between the code words C_r and C_1 . Thus, the distance $d_{1r} = w_r$. In general, the distance d_{ij} between any pair of code words C_i and C_j is simply equal to the weight of the code word formed by taking the difference between C_i and C_j . Since the code is linear, the difference (equivalent to taking the modulo-2 sum for a binary code) between C_i and C_j is also a code word having a weight included in the set $\{w_r\}$. Hence, the weight distribution of a linear code completely characterizes the distance properties of the code. The minimum distance of the code is, therefore,

$$d_{\min} = \min_{r, r \neq 1} \{w_r\} \quad (8-1-1)$$

A number of elementary concepts from linear algebra are particularly useful in dealing with linear block codes. Specifically, the set of all n -tuples (vectors with n elements) form a vector space S . If we select a set of $k < n$ linearly independent vectors from S and from these construct the set of all linear combinations of these vectors, the resulting set forms a subspace of S , say S_c , of dimension k . Any set of k linearly independent vectors in the subspace S_c constitutes a basis. Now consider the set of vectors in S that are orthogonal to every vector in a basis for S_c (and, hence, orthogonal to all vectors in S_c). This set of vectors is also a subspace of S and is called the *null space* of S_c . If the dimension of S_c is k , the dimension of the null space is $n - k$.

Expressed in terms appropriate for binary block codes, the vector space S consists of the 2^n binary valued n -tuples. The linear (n, k) code is a set of 2^k n -tuples called *code words*, which forms a subspace S_c over the field of two elements. Since there are 2^k code words in S_c , a basis for S_c has k code words. That is, k linearly independent code words are required to construct 2^k linear combinations, thus generating the entire code. The null space of S_c is another linear code, which consists of 2^{n-k} code words of block length n and $n - k$ information bits. Its dimension is $n - k$. In Section 8-1-1, we consider these relationships in greater detail.

8-1-1 The Generator Matrix and the Parity Check Matrix

Let $x_{m1}, x_{m2}, \dots, x_{mk}$ denote the k information bits encoded into the code word \mathbf{C}_m . Throughout this chapter, we follow the established convention in coding of representing code words as row vectors. Thus, the vector of k information bits into the encoder is denoted by

$$\mathbf{X}_m = [x_{m1} \ x_{m2} \ \dots \ x_{mk}]$$

and the output of the encoder is the vector

$$\mathbf{C}_m = [c_{m1} \ c_{m2} \ \dots \ c_{mn}]$$

The encoding operation performed in a linear binary block encoder can be represented by a set of n equations of the form

$$c_{mj} = x_{m1}g_{1j} + x_{m2}g_{2j} + \dots + x_{mk}g_{kj}, \quad j = 1, 2, \dots, n \quad (8-1-2)$$

where $g_{ij} = 0$ or 1 and $x_{mi}g_{ij}$ represents the product of x_{mi} and g_{ij} . The linear equations (8-1-2) may also be represented in a matrix form as

$$\mathbf{C}_m = \mathbf{X}_m \mathbf{G} \quad (8-1-3)$$

where \mathbf{G} , called the *generator matrix* of the code, is

$$\mathbf{G} = \begin{bmatrix} \leftarrow \mathbf{g}_1 \rightarrow \\ \leftarrow \mathbf{g}_2 \rightarrow \\ \vdots \\ \leftarrow \mathbf{g}_k \rightarrow \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} & \dots & g_{1n} \\ g_{21} & g_{22} & \dots & g_{2n} \\ \vdots & \vdots & & \vdots \\ g_{k1} & g_{k2} & \dots & g_{kn} \end{bmatrix} \quad (8-1-4)$$

Note that any code word is simply a linear combination of the vectors $\{\mathbf{g}_i\}$ of \mathbf{G} , i.e.,

$$\mathbf{C}_m = x_{m1}\mathbf{g}_1 + x_{m2}\mathbf{g}_2 + \dots + x_{mk}\mathbf{g}_k \quad (8-1-5)$$

Since the linear (n, k) code with 2^k code words is a subspace of dimension k , the row vectors $\{\mathbf{g}_i\}$ of the generator matrix \mathbf{G} must be linearly independent, i.e., they must span a subspace of k dimensions. In other words, the $\{\mathbf{g}_i\}$ must be a basis for the (n, k) code. We note that the set of basis vectors is not unique, and, hence, \mathbf{G} is not unique. We also note that, since the subspace has dimension k , the rank of \mathbf{G} is k .

Any generator matrix of an (n, k) code can be reduced by row operations (and column permutations) to the "systematic form."

$$\mathbf{G} = [\mathbf{I}_k \mid \mathbf{P}] = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & p_{11} & p_{12} & \dots & p_{1n-k} \\ 0 & 1 & 0 & \dots & 0 & p_{21} & p_{22} & \dots & p_{2n-k} \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 & p_{k1} & p_{k2} & \dots & p_{kn-k} \end{bmatrix} \quad (8-1-6)$$

where \mathbf{I}_k is the $k \times k$ identity matrix and \mathbf{P} is a $k \times (n - k)$ matrix that

determines the $n - k$ redundant bits or parity check bits. Note that a generator matrix of the systematic form generates a linear block code in which the first k bits of each code word are identical to the information bits to be transmitted, and the remaining $n - k$ bits of each code word are linear combinations of the k information bits. These $n - k$ redundant bits are called *parity check bits*. The resulting (n, k) code is called a *systematic code*.

An (n, k) code generated by a generator matrix that is not in the systematic form (8-1-6) is called *nonsystematic*. However, such a generator matrix is equivalent to a generator matrix of the systematic form in the sense that one can be obtained from the other by elementary row operations and column permutations. The two (n, k) linear codes generated by the two equivalent generator matrices are said to be *equivalent*, and one can be obtained from the other by a permutation of the places of every element. Thus, every linear (n, k) code is equivalent to a linear systematic (n, k) code.

Example 8-1-1

Consider a $(7, 4)$ code with generator matrix

$$\mathbf{G} = \left[\begin{array}{cccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{array} \right] = [\mathbf{I}_4 \mid \mathbf{P}] \quad (8-1-7)$$

A typical code word may be expressed as

$$\mathbf{C}_m = [x_{m1} \ x_{m2} \ x_{m3} \ x_{m4} \ c_{m5} \ c_{m6} \ c_{m7}]$$

where the $\{x_{mj}\}$ represents the four information bits and the $\{c_{mj}\}$ represent the three parity check bits given by

$$\begin{aligned} c_{m5} &= x_{m1} + x_{m2} + x_{m3} \\ c_{m6} &= x_{m2} + x_{m3} + x_{m4} \\ c_{m7} &= x_{m1} + x_{m2} + x_{m4} \end{aligned} \quad (8-1-8)$$

A linear systematic (n, k) binary block encoder may be implemented by using a k -bit shift register and $n - k$ modulo-2 adders tied to the appropriate stages of the shift register. The $n - k$ adders generate the parity check bits, which are subsequently stored temporarily in a second shift register of length $n - k$. The k -bit block of information bits shifted into the k -bit shift register and the $n - k$ parity check bits are computed. Then the k information bits followed by the $n - k$ parity check bits are shifted out of the two shift registers

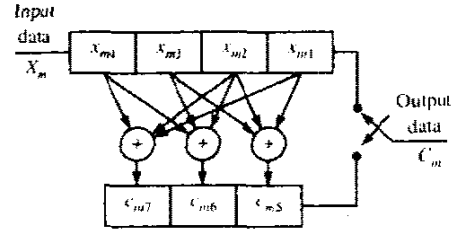


FIGURE 8-1-1 A linear shift register for generating a (7, 4) binary code.

and fed to the modulator. This encoding is illustrated in Fig. 8-1-1 for the (7, 4) code of Example 8-1-1.

Associated with any linear (n, k) code is the dual code of dimension $n - k$. The dual code is a linear $(n, n - k)$ code with 2^{n-k} code vectors, which is the null space of the (n, k) code. The generator matrix for the dual code, denoted by \mathbf{H} , consists of $n - k$ linearly independent code vectors selected from the null space. Any code word \mathbf{C}_m of the (n, k) code is orthogonal to any code word in the dual code. Hence, any code word of the (n, k) code is orthogonal to every row of the matrix \mathbf{H} , i.e.,

$$\mathbf{C}_m \mathbf{H}' = \mathbf{0} \tag{8-1-9}$$

where $\mathbf{0}$ denotes an all-zero row vector with $n - k$ elements, and \mathbf{C}_m is a code word of the (n, k) code. Since (8-1-9) holds for every code word of the (n, k) code, it follows that

$$\mathbf{G} \mathbf{H}' = \mathbf{0} \tag{8-1-10}$$

where $\mathbf{0}$ is now a $k \times (n - k)$ matrix with all-zero elements.

Now suppose that the linear (n, k) code is systematic and its generator matrix \mathbf{G} is given by the systematic form (8-1-6). Then, since $\mathbf{G} \mathbf{H}' = \mathbf{0}$, it follows that

$$\mathbf{H} = [-\mathbf{P}' \mid \mathbf{I}_{n-k}] \tag{8-1-11}$$

The negative sign in (8-1-11) may be dropped when dealing with binary codes, since modulo-2 subtraction is identical to modulo-2 addition.

Example 8-1-2

For the systematic (7, 4) code generated by matrix \mathbf{G} given by (8-1-7), we have, according to (8-1-11), the matrix \mathbf{H} in the form

$$\mathbf{H} = \left[\begin{array}{cccc|ccc} 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{array} \right] \tag{8-1-12}$$

Now, the product $C_m H'$ yields the three equations

$$\begin{aligned} x_{m1} + x_{m2} + x_{m3} + c_{m5} &= 0 \\ x_{m2} + x_{m3} + x_{m4} + c_{m6} &= 0 \\ x_{m1} + x_{m2} + x_{m4} + c_{m7} &= 0 \end{aligned} \tag{8-1-13}$$

Thus, we observe that the product $C_m H'$ is equivalent to adding the parity check bits to the corresponding linear combinations of the information bits used to compute c_{mj} , $j = 5, 6, 7$. That is, (8-1-13) are equivalent to (8-1-8). The matrix H may be used by the decoder to check that a received code word Y satisfies the condition (8-1-13), i.e., $YH' = 0$. In so doing, the decoder checks the received parity check bits with the corresponding linear combination of the bits y_1, y_2, y_3 , and y_4 that formed the parity check bits at the transmitter. It is, therefore, appropriate to call H the *parity check matrix* associated with the (n, k) code.

We make the following observation regarding the relation of the minimum distance of a code to its parity check matrix H . The product $C_m H'$ with $C_m \neq 0$ represents a linear combination of the n columns of H' . Since $C_m H' = 0$, the column vectors of H are linearly dependent. Suppose C_j denotes the minimum weight code word of a linear (n, k) code. It must satisfy the condition $C_j H' = 0$. Since the minimum weight is equal to the minimum distance, it follows that d_{min} of the columns of H are linearly dependent. Alternatively, we may say that no more than $d_{min} - 1$ columns of H are linearly independent. Since the rank of H is at most $n - k$, we have $n - k \geq d_{min} - 1$. Therefore, d_{min} is upper-bounded as

$$d_{min} \leq n - k + 1 \tag{8-1-14}$$

Given a linear binary (n, k) code with minimum distance d_{min} , we can construct a linear binary $(n + 1, k)$ code by appending one additional parity check bit to each code word. The check bit is usually selected to be a check bit on all the bits in the code word. Thus the added check bit is a 0 if the original code word has an even number of 1s and it is a 1 if the code word has an odd number of 1s. Consequently, if the minimum weight and, hence, the minimum distance of the code is odd, the added parity check bit increases the minimum distance by 1. We call the $(n + 1, k)$ code an *extended code*. Its parity check matrix is

$$H_e = \left[\begin{array}{cccc|c} & & & & 0 \\ & & & & 0 \\ & & & & \vdots \\ & & & & 0 \\ \hline 1 & 1 & 1 & \dots & 1 & 1 \end{array} \right] \tag{8-1-15}$$

where H is the parity check matrix of the original code.

A systematic (n, k) code can also be shortened by setting a number of the information bits to zero. That is, a linear (n, k) code consisting of k information bits and $n - k$ check bits can be shortened into a $(n - l, k - l)$ linear code by setting the first l bits to zero. These l bits are not transmitted. The $n - k$ check bits are computed in the usual manner, as in the original code. Since

$$\mathbf{C}_m = \mathbf{X}_m \mathbf{G}$$

the effect of setting the first l bits of \mathbf{X}_m to 0 is equivalent to reducing the number of rows of \mathbf{G} by removing the first l rows. Equivalently, since

$$\mathbf{C}_m \mathbf{H}' = \mathbf{0}$$

we may remove the first l columns of \mathbf{H} . The shortened $(n - l, k - l)$ code consists of 2^{k-l} code words. The minimum distance of these 2^{k-l} code words is at least as large as the minimum distance of the original (n, k) code.

8-1-2 Some Specific Linear Block Codes

In this subsection, we shall briefly describe three types of linear block codes that are frequently encountered in practice and list their important parameters.

Hamming Codes There are both binary and nonbinary Hamming codes. We limit our discussion to the properties of binary Hamming codes. These comprise a class of codes with the property that

$$(n, k) = (2^m - 1, 2^m - 1 - m) \quad (8-1-16)$$

where m is any positive integer. For example, if $m = 3$, we have a $(7, 4)$ code.

The parity check matrix \mathbf{H} of a Hamming code has a special property that allows us to describe the code rather easily. Recall that the parity check matrix of an (n, k) code has $n - k$ rows and n columns. For the binary (n, k) Hamming code, the $n = 2^m - 1$ columns consist of all possible binary vectors with $n - k = m$ elements, except the all-zero vector. For example, the $(7, 4)$ code considered in Examples 8-1-1 and 8-1-2 is a Hamming code. Its parity check matrix consists of the seven column vectors (001) , (010) , (011) , (100) , (101) , (110) , (111) .

If we desire to generate a systematic Hamming code, the parity check matrix \mathbf{H} can be easily arranged in the systematic form (8-1-11). Then the corresponding generator matrix \mathbf{G} can be obtained from (8-1-11).

We make the observation that no two columns of \mathbf{H} are linearly dependent, for otherwise the two columns would be identical. However, for $m > 1$, it is possible to find three columns of \mathbf{H} that add to zero. Consequently, $d_{\min} = 3$ for an (n, k) Hamming code.

By adding an overall parity bit, a Hamming (n, k) code can be modified to yield an $(n + 1, k)$ code with $d_{\min} = 4$. On the other hand, an (n, k) Hamming code may be shortened to $(n - l, k - l)$ by removing l rows of its generator matrix \mathbf{G} or, equivalently, by removing l columns of its parity check matrix \mathbf{H} .

The weight distribution for the class of Hamming (n, k) codes is known and is expressed in compact form by the weight enumerating polynomial

$$\begin{aligned} A(z) &= \sum_{i=0}^n A_i z^i \\ &= \frac{1}{n+1} [(1+z)^n + n(1+z)^{(n-1)/2}(1-z)^{(n+1)/2}] \end{aligned} \quad (8-1-17)$$

where A_i is the number of code words of weight i .

Hadamard Codes A Hadamard code is obtained by selecting as code words the rows of a Hadamard matrix. A Hadamard matrix \mathbf{M}_n is an $n \times n$ matrix (n an even integer) of 1s and 0s with the property that any row differs from any other row in exactly $\frac{1}{2}n$ positions.† One row of the matrix contains all zeros. The other rows contain $\frac{1}{2}n$ zeros and $\frac{1}{2}n$ ones.

For $n = 2$, the Hadamard matrix is

$$\mathbf{M}_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad (8-1-18)$$

Furthermore, from \mathbf{M}_n , we can generate the Hadamard matrix \mathbf{M}_{2n} according to the relation

$$\mathbf{M}_{2n} = \begin{bmatrix} \mathbf{M}_n & \mathbf{M}_n \\ \mathbf{M}_n & \bar{\mathbf{M}}_n \end{bmatrix} \quad (8-1-19)$$

where $\bar{\mathbf{M}}_n$ denotes the complement (0s replaced by 1s and vice versa) of \mathbf{M}_n . Thus, by substituting (8-1-18) into (8-1-19), we obtain

$$\mathbf{M}_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \quad (8-1-20)$$

The complement of \mathbf{M}_4 is

$$\bar{\mathbf{M}}_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \quad (8-1-21)$$

Now the rows of \mathbf{M}_4 and $\bar{\mathbf{M}}_4$ form a linear binary code of block length $n = 4$ having $2n = 8$ code words. The minimum distance of the code is $d_{\min} = \frac{1}{2}n = 2$.

By repeated application of (8-1-19), we can generate Hadamard codes with block length $n = 2^m$, $k = \log_2 2n = \log_2 2^{m+1} = m + 1$, and $d_{\min} = \frac{1}{2}n = 2^{m-1}$, where m is a positive integer. In addition to the important special case where $n = 2^m$, Hadamard codes of other block lengths are possible, but the codes are not linear.

† Sometimes the elements of the Hadamard matrix are denoted by -1 and $+1$. Then the rows of the Hadamard matrix are mutually orthogonal. We also note that the $M = 2^k$ signal waveforms, constructed from Hadamard code words by mapping each bit in a code word into a binary PSK signal, are orthogonal.

TABLE 8-1-1 WEIGHT DISTRIBUTION OF GOLAY (23, 12) AND EXTENDED GOLAY (24, 12) CODES

Weight	Number of code words	
	(23, 12) code	(24, 12) code
0	1	1
7	253	0
8	506	759
11	1288	0
12	1288	2576
15	506	0
16	253	759
23	1	0
24	0	1

Source: Peterson and Weldon (1972).

Golay Code The Golay code is a binary linear (23, 12) code with $d_{\min} = 7$. The extended Golay code obtained by adding an overall parity to the (23, 12) is a binary linear (24, 12) code with $d_{\min} = 8$. Table 8-1-1 lists the weight distribution of the code words in the Golay (23, 12) and the extended Golay (24, 12) codes. We discuss the generation of the Golay code in Section 8-1-3.

8-1-3 Cyclic Codes

Cyclic codes are a subset of the class of linear codes that satisfy the following cyclic shift property: if $\mathbf{C} = [c_{n-1}c_{n-2} \dots c_1c_0]$ is a code word of a cyclic code then $[c_{n-2}c_{n-3} \dots c_0c_{n-1}]$, obtained by a cyclic shift of the elements of \mathbf{C} , is also a code word. That is, all cyclic shifts of \mathbf{C} are code words. As a consequence of the cyclic property, the codes possess a considerable amount of structure which can be exploited in the encoding and decoding operations. A number of efficient encoding and hard-decision decoding algorithms have been devised for cyclic codes that make it possible to implement long block codes with a large number of code words in practical communications systems. A description of specific algorithms is beyond the scope of this book. Our primary objective is to briefly describe a number of characteristics of cyclic codes.

In dealing with cyclic codes, it is convenient to associate with a code word $\mathbf{C} = [c_{n-1}c_{n-2} \dots c_1c_0]$ a polynomial $C(p)$ of degree $\leq n - 1$, defined as

$$C(p) = c_{n-1}p^{n-1} + c_{n-2}p^{n-2} + \dots + c_1p + c_0 \quad (8-1-22)$$

For a binary code, each of the coefficients of the polynomial is either zero or one.

Now suppose we form the polynomial

$$pC(p) = c_{n-1}p^n + c_{n-2}p^{n-1} + \dots + c_1p^2 + c_0p$$

This polynomial cannot represent a code word, since its degree may be equal to n (when $c_{n-1} = 1$). However, if we divide $pC(p)$ by $p^n + 1$, we obtain

$$\frac{pC(p)}{p^n + 1} = c_{n-1} + \frac{C_1(p)}{p^n + 1} \quad (8-1-23)$$

where

$$C_1(p) = c_{n-2}p^{n-1} + c_{n-3}p^{n-2} + \dots + c_0p + c_{n-1}$$

Note that the polynomial $C_1(p)$ represents the code word $\mathbf{C}_1 = [c_{n-2} \dots c_0 c_{n-1}]$, which is just the code word \mathbf{C} shifted cyclicly by one position. Since $C_1(p)$ is the remainder obtained by dividing $pC(p)$ by $p^n + 1$, we say that

$$C_1(p) = pC(p) \pmod{p^n + 1} \quad (8-1-24)$$

In a similar manner, if $C(p)$ represents a code word in a cyclic code then $p^i C(p) \pmod{p^n + 1}$ is also a code word of the cyclic code. Thus we may write

$$p^i C(p) = Q(p)(p^n + 1) + C_i(p) \quad (8-1-25)$$

where the remainder polynomial $C_i(p)$ represents a code word of the cyclic code and $Q(p)$ is the quotient.

We can generate a cyclic code by using a *generator polynomial* $g(p)$ of degree $n - k$. The generator polynomial of an (n, k) cyclic code is a factor of $p^n + 1$ and has the general form

$$g(p) = p^{n-k} + g_{n-k-1}p^{n-k-1} + \dots + g_1p + 1 \quad (8-1-26)$$

We also define a *message polynomial* $X(p)$ as

$$X(p) = x_{k-1}p^{k-1} + x_{k-2}p^{k-2} + \dots + x_1p + x_0 \quad (8-1-27)$$

where $[x_{k-1}x_{k-2} \dots x_1x_0]$ represent the k information bits. Clearly, the product $X(p)g(p)$ is a polynomial of degree less than or equal to $n - 1$, which may represent a code word. We note that there are 2^k polynomials $\{X_i(p)\}$, and, hence, there are 2^k possible code words that can be formed from a given $g(p)$.

Suppose we denote these code words as

$$C_m(p) = X_m(p)g(p), \quad m = 1, 2, \dots, 2^k \quad (8-1-28)$$

To show that the code words in (8-1-28) satisfy the cyclic property, consider any code word $C(p)$ in (8-1-28). A cyclic shift of $C(p)$ produces

$$C_1(p) = pC(p) + c_{n-1}(p^n + 1) \quad (8-1-29)$$

and, since $g(p)$ divides both $p^n + 1$ and $C(p)$, it also divides $C_1(p)$, i.e., $C_1(p)$ can be represented as

$$C_1(p) = X_1(p)g(p)$$

Therefore, a cyclic shift of any code word $C(p)$ generated by (8-1-28) yields another code word.

From the above, we see that code words possessing the cyclic property can

be generated by multiplying the 2^k message polynomials with a unique polynomial $g(p)$, called the generator polynomial of the (n, k) cyclic code, which divides $p^n + 1$ and has degree $n - k$. The cyclic code generated in this manner is a subspace S_c of the vector space S . The dimension of S_c is k .

Example 8-1-3

Consider a code with block length $n = 7$. The polynomial $p^7 + 1$ has the following factors:

$$p^7 + 1 = (p + 1)(p^3 + p^2 + 1)(p^3 + p + 1) \tag{8-1-30}$$

To generate a $(7, 4)$ cyclic code, we may take as a generator polynomial one of the following two polynomials:

$$\begin{aligned} g_1(p) &= p^3 + p^2 + 1 \\ g_2(p) &= p^3 + p + 1 \end{aligned} \tag{8-1-31}$$

The codes generated by $g_1(p)$ and $g_2(p)$ are equivalent. The code words in the $(7, 4)$ code generated by $g_1(p) = p^3 + p^2 + 1$ are given in Table 8-1-2.

In general, the polynomial $p^n + 1$ may be factored as

$$p^n + 1 = g(p)h(p)$$

where $g(p)$ denotes the generator polynomial for the (n, k) cyclic code and

TABLE 8-1-2 (7, 4) CYCLIC CODE
Generator Polynomial: $g_1(p) = p^3 + p^2 + 1$

Information bits				Code words						
p^3	p^2	p^1	p^0	p^6	p^5	p^4	p^3	p^2	p^1	p^0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	1	1	0	1
0	0	1	0	0	0	1	1	0	1	0
0	0	1	1	0	0	1	0	1	1	1
0	1	0	0	0	1	1	0	1	0	0
0	1	0	1	0	1	1	1	0	0	1
0	1	1	0	0	1	0	1	1	1	0
0	1	1	1	0	1	0	0	0	1	1
1	0	0	0	1	1	0	1	0	0	0
1	0	0	1	1	1	0	0	1	0	1
1	0	1	0	1	1	1	0	0	1	0
1	0	1	1	1	1	1	1	1	1	1
1	1	0	0	1	0	1	1	1	0	0
1	1	0	1	1	0	1	0	0	0	1
1	1	1	0	1	0	0	0	1	1	0
1	1	1	1	1	0	0	1	0	1	1

$h(p)$ denotes the *parity polynomial* that has degree k . The latter may be used to generate the dual code.

For this purpose, we define the *reciprocal polynomial* of $h(p)$ as

$$\begin{aligned} p^k h(p^{-1}) &= p^k (p^{-k} + h_{k-1} p^{-k+1} + h_{k-2} p^{-k+2} + \dots + h_1 p^{-1} + 1) \\ &= 1 + h_{k-1} p + h_{k-2} p^2 + \dots + h_1 p^{k-1} + p^k \end{aligned} \tag{8-1-32}$$

Clearly, the reciprocal polynomial is also a factor of $p^n + 1$. Hence, $p^k h(p^{-1})$ is the generator polynomial of an $(n, n - k)$ cyclic code. This cyclic code is the dual code to the (n, k) code generated from $g(p)$. Thus, the $(n, n - k)$ dual code constitutes the null space of the (n, k) cyclic code.

Example 8-1-4

Let us consider the dual code to the $(7, 4)$ cyclic code generated in Example 8-1-3. This dual code is a $(7, 3)$ cyclic code associated with the parity polynomial

$$\begin{aligned} h_1(p) &= (p+1)(p^3 + p + 1) \\ &= p^4 + p^3 + p^2 + 1 \end{aligned} \tag{8-1-33}$$

The reciprocal polynomial is

$$p^4 h_1(p^{-1}) = 1 + p + p^2 + p^4$$

This polynomial generates the $(7, 3)$ dual code given in Table 8-1-3. The reader can verify that the code words in the $(7, 3)$ dual code are orthogonal to the code words in the $(7, 4)$ cyclic code of Example 8-1-3. Note that neither the $(7, 4)$ nor the $(7, 3)$ codes are systematic.

It is desirable to show how a generator matrix can be obtained from the generator polynomial of a cyclic (n, k) code. As previously indicated, the generator matrix for an (n, k) code can be constructed from any set of k

TABLE 8-1-3 $(7, 3)$ DUAL CODE
Generator Polynomial $p^4 h_1(p^{-1}) = p^4 + p^2 + p + 1$

Information bits			Code words						
p^2	p^1	p^0	p^6	p^5	p^4	p^3	p^2	p^1	p^0
0	0	0	0	0	0	0	0	0	0
0	0	1	0	0	1	0	1	1	1
0	1	0	0	1	0	1	1	1	0
0	1	1	0	1	1	1	0	0	1
1	0	0	1	0	1	1	1	0	0
1	0	1	1	0	0	1	0	1	1
1	1	0	1	1	1	0	0	1	0
1	1	1	1	1	0	0	1	0	1

linearly independent code words. Hence, given the generator polynomial $g(p)$, an easily generated set of k linearly independent code words is the code words corresponding to the set of k linearly independent polynomials

$$p^{k-1}g(p), p^{k-2}g(p), \dots, pg(p), g(p)$$

Since any polynomial of degree less than or equal to $n-1$ and divisible by $g(p)$ can be expressed as a linear combination of this set of polynomials, the set forms a basis of dimension k . Consequently, the code words associated with these polynomials form a basis of dimension k for the (n, k) cyclic code.

Example 8-1-5

The four rows of the generator matrix for the $(7, 4)$ cyclic code with generator polynomial $g_1(p) = p^3 + p^2 + 1$ are obtained from the polynomials

$$p^i g_1(p) = p^{3+i} + p^{2+i} + p^i, \quad i = 3, 2, 1, 0$$

It is easy to see that the generator matrix is

$$\mathbf{G}_1 = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix} \quad (8-1-34)$$

Similarly, the generator matrix for the $(7, 4)$ cyclic code generated by the polynomial $g_2(p) = p^3 + p + 1$ is

$$\mathbf{G}_2 = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} \quad (8-1-35)$$

The parity check matrices corresponding to \mathbf{G}_1 and \mathbf{G}_2 can be constructed in the same manner by using the respective reciprocal polynomials (Problem 8-8).

Note that the generator matrix obtained by this construction is not in systematic form. We can construct the generator matrix of a cyclic code in the systematic form $\mathbf{G} = [\mathbf{I}_k \mid \mathbf{P}]$ from the generator polynomial as follows. First, we observe that the l th row of \mathbf{G} corresponds to a polynomial of the form $p^{n-l} + R_l(p)$, $l = 1, 2, \dots, k$, where $R_l(p)$ is a polynomial of degree less than $n-k$. This form can be obtained by dividing p^{n-l} by $g(p)$. Thus, we have

$$\frac{p^{n-l}}{g(p)} = Q_l(p) + \frac{R_l(p)}{g(p)}, \quad l = 1, 2, \dots, k$$

or, equivalently,

$$p^{n-l} = Q_l(p)g(p) + R_l(p), \quad l = 1, 2, \dots, k \quad (8-1-36)$$

where $Q_l(p)$ is the quotient. But $p^{n-l} + R_l(p)$ is a code word of the cyclic code since $p^{n-l} + R_l(p) = Q_l(p)g(p)$. Therefore the desired polynomial corresponding to the l th row of \mathbf{G} is $p^{n-l} + R_l(p)$.

Example 8-1-6

For the (7,4) cyclic code with generator polynomial $g_2(p) = p^3 + p + 1$, previously discussed in Example 8-1-5, we have

$$\begin{aligned} p^6 &= (p^3 + p + 1)g_2(p) + p^2 + 1 \\ p^5 &= (p^2 + 1)g_2(p) + p^2 + p + 1 \\ p^4 &= pg_2(p) + p^2 + p \\ p^3 &= g_2(p) + p + 1 \end{aligned}$$

Hence, the generator matrix of the code in systematic form is

$$\mathbf{G}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} \quad (8-1-37)$$

and the corresponding parity check matrix is

$$\mathbf{H}_2 = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \quad (8-1-38)$$

It is left as an exercise for the reader to demonstrate that the generator matrix \mathbf{G}_2 given by (8-1-35) and the systematic form given by (8-1-37) generate the same set of code words (Problem 8-2).

The method for constructing the generator matrix \mathbf{G} in systematic form according to (8-1-36) also implies that a systematic code can be generated directly from the generator polynomial $g(p)$. Suppose that we multiply the message polynomial $X(p)$ by p^{n-k} . Thus, we obtain

$$p^{n-k}X(p) = x_{k-1}p^{n-1} + x_{k-2}p^{n-2} + \dots + x_1p^{n-k+1} + x_0p^{n-k}$$

In a systematic code, this polynomial represents the first k bits in the code word $C(p)$. To this polynomial we must add a polynomial of degree less than $n - k$ representing the parity check bits. Now, if $p^{n-k}X(p)$ is divided by $g(p)$, the result is

$$\frac{p^{n-k}X(p)}{g(p)} = Q(p) + \frac{r(p)}{g(p)}$$

or, equivalently,

$$p^{n-k}X(p) = Q(p)g(p) + r(p) \quad (8-1-39)$$

where $r(p)$ has degree less than $n - k$. Clearly, $Q(p)g(p)$ is a code word of the cyclic code. Hence, by adding (modulo-2) $r(p)$ to both sides of (8-1-39), we obtain the desired systematic code.

To summarize, the systematic code may be generated by

- 1 multiplying the message polynomial $X(p)$ by p^{n-k} ;
- 2 dividing $p^{n-k}X(p)$ by $g(p)$ to obtain the remainder $r(p)$; and
- 3 adding $r(p)$ to $p^{n-k}X(p)$.

Below we demonstrate how these computations can be performed by using shift registers with feedback.

Since $p^n + 1 = g(p)h(p)$ or, equivalently, $g(p)h(p) = 0 \pmod{p^n + 1}$, we say that the polynomials $g(p)$ and $h(p)$ are *orthogonal*. Furthermore, the polynomials $p^i g(p)$ and $p^j h(p)$ are also orthogonal for all i and j . However, the vectors corresponding to the polynomials $g(p)$ and $h(p)$ are orthogonal only if the ordered elements of one of these vectors are reversed. The same statement applies to the vectors corresponding to $p^i g(p)$ and $p^j h(p)$. In fact, if the parity polynomial $h(p)$ is used as a generator for the $(n, n - k)$ dual code, the set of code words obtained just comprises the same code words generated by the reciprocal polynomial except that the code vectors are reversed. This implies that the generator matrix for the dual code obtained from the reciprocal polynomial $p^k h(p^{-1})$ can also be obtained indirectly from $h(p)$. Since the parity check matrix \mathbf{H} for the (n, k) cyclic code is the generator matrix for the dual code, it follows that \mathbf{H} can also be obtained from $h(p)$. The following example illustrates these relationships.

Example 8-1-7

The dual code to the $(7, 4)$ cyclic code generated by $g_1(p) = p^3 + p^2 + 1$ is the $(7, 3)$ dual code that is generated by the reciprocal polynomial $p^4 h_1(p^{-1}) = p^4 + p^2 + p + 1$. However, we may also use $h_1(p)$ to obtain the generator matrix for the dual code. Then, the matrix corresponding to the polynomials $p^i h_1(p)$, $i = 2, 1, 0$, is

$$\mathbf{G}_{h_1} = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{bmatrix}$$

The generator matrix for the $(7, 3)$ dual code, which is the parity check matrix for the $(7, 4)$ cyclic code, consists of the rows of \mathbf{G}_{h_1} taken in reverse order. Thus,

$$\mathbf{H}_1 = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}$$

The reader may verify that $\mathbf{G}_1\mathbf{H}_1^t = 0$.

Note that the column vectors of \mathbf{H}_1 consist of all seven binary vectors of length 3, except the all-zero vector. But this is just the description of the parity check matrix for a (7, 4) Hamming code. Therefore, the (7, 4) cyclic code is equivalent to the (7, 4) Hamming code discussed previously in Examples 8-1-1 and 8-1-2.

Encoders for Cyclic Codes The encoding operations for generating a cyclic code may be performed by a linear feedback shift register based on the use of either the generator polynomial or the parity polynomial. First, let us consider the use of $g(p)$.

As indicated above, the generation of a systematic cyclic code involves three steps, namely multiplying the message polynomial $X(p)$ by p^{n-k} , dividing the product by $g(p)$, and, finally, adding the remainder to $p^{n-k}X(p)$. Of these three steps, only the division is nontrivial.

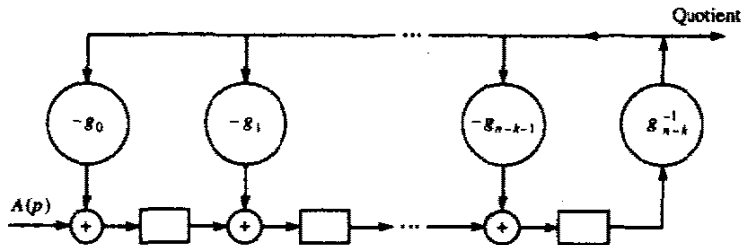
The division of the polynomial $A(p) = p^{n-k}X(p)$ of degree $n - 1$ by the polynomial

$$g(p) = g_{n-k}p^{n-k} + g_{n-k-1}p^{n-k-1} + \dots + g_1p + g_0$$

may be accomplished by the $(n - k)$ stage feedback shift register illustrated in Fig. 8-1-2. Initially, the shift register contains all zeros. The coefficients of $A(p)$ are clocked into the shift register one (bit) coefficient at a time, beginning with the higher-order coefficients, i.e., with a_{n-1} , followed by a_{n-2} , and so on. After the k th shift, the first nonzero output of the quotient is $q_1 = g_{n-k}a_n$. Subsequent outputs are generated as illustrated in Fig. 8-1-2. For each output coefficient in the quotient, we must subtract the polynomial $g(p)$ multiplied by that coefficient, as in ordinary long division. This subtraction is performed by means of the feedback part of the shift register. Thus, the feedback shift register in Fig. 8-1-2 performs division of two polynomials.

In our case, $g_{n-k} = g_0 = 1$, and, for binary codes, the arithmetic operations are performed in modulo-2 arithmetic. Consequently, the subtraction operations reduce to modulo-2 addition. Furthermore, we are only interested in

FIGURE 8-1-2 A feedback shift register for dividing the polynomial $A(p)$ by $g(p)$.



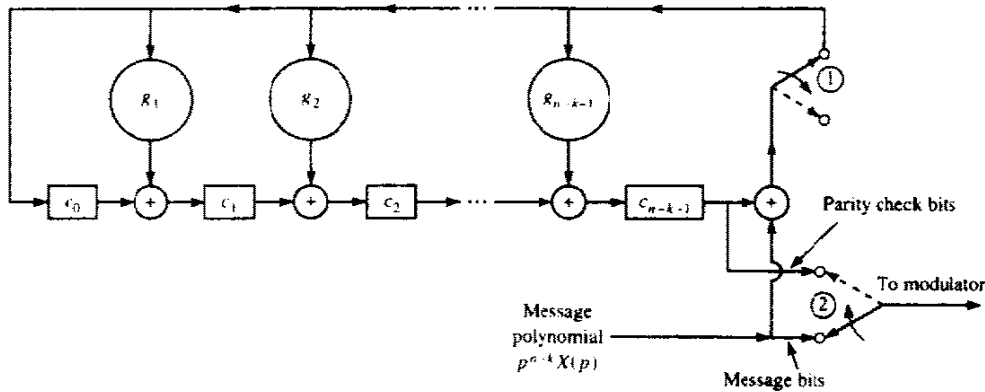


FIGURE 8-1-3 Encoding of a cyclic code by use of the generator polynomial $g(p)$.

generating the parity check bits for each code word, since the code is systematic. Consequently, the encoder for the cyclic code takes the form illustrated in Fig. 8-1-3. The first k bits at the output of the encoder are simply the k information bits. These k bits are also clocked simultaneously into the shift register, since the switch 1 is in the closed position. Note that the polynomial multiplication of p^{n-k} with $X(p)$ is not performed explicitly. After the k information bits are all clocked into the encoder, the positions of the two switches are reversed. At this time, the contents of the shift register are simply the $n - k$ parity check bits, which correspond to the coefficients of the remainder polynomial. These $n - k$ bits are clocked out one at a time and sent to the modulator.

Example 8-1-8

The shift register for encoding the (7, 4) cyclic code with generator polynomial $g(p) = p^3 + p + 1$ is illustrated in Fig. 8-1-4. Suppose the input

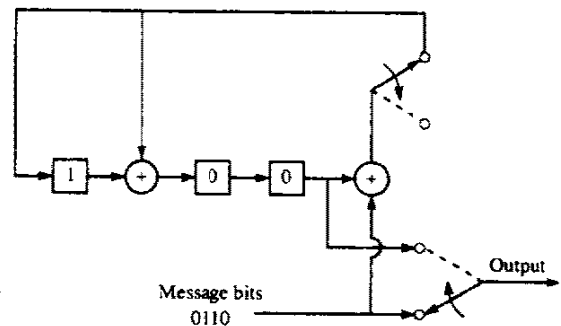


FIGURE 8-1-4 The encoder for the (7, 4) cyclic code with generator polynomial $g(p) = p^3 + p + 1$.

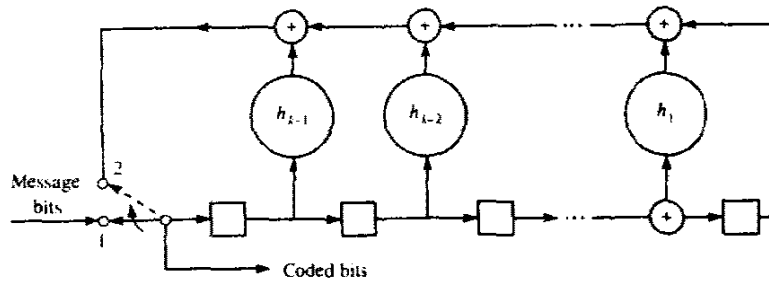


FIGURE 8-1-5 Encoder for an (n, k) cyclic code based on parity polynomial $h(p)$.

message bits are 0110. The contents of the shift register are as follows:

Input	Shift	Shift register contents
	0	0 0 0
0	1	0 0 0
1	2	1 1 0
1	3	1 0 1
0	4	1 0 0

Hence, the three parity check bits are 100, which correspond to the code bits $c_5 = 0$, $c_6 = 0$, and $c_7 = 1$.

Instead of using the generator polynomial, we may implement the encoder for the cyclic code by making use of the parity polynomial

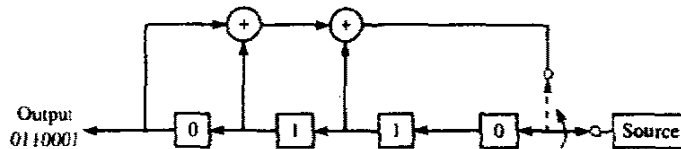
$$h(p) = p^k + h_{k-1}p^{k-1} + \dots + h_1p + 1$$

The encoder is shown in Fig. 8-1-5. Initially, the k information bits are shifted into the shift register and simultaneously fed to the modulator. After all k information bits are in the shift register, the switch is thrown into position 2 and the shift register is clocked $n - k$ times to generate the $n - k$ parity check bits as illustrated in Fig. 8-1-5.

Example 8-1-9

The parity polynomial for the $(7, 4)$ cyclic code generated by $g(p) = p^3 + p + 1$ is $h(p) = p^4 + p^2 + p + 1$. The encoder for this code based on the parity polynomial is illustrated in Fig. 8-1-6. If the input to the encoder is

FIGURE 8-1-6 The encoder for the $(7, 4)$ cyclic code based on the parity polynomial $h(p) = p^4 + p^2 + 1$.



the message bits 0110, the parity check bits are $c_5 = 0$, $c_6 = 0$, and $c_7 = 1$, as is easily verified.

It should be noted that the encoder based on the generator polynomial is simpler when $n - k < k$ ($k > \frac{1}{2}n$), i.e., for high rate codes ($R_c > \frac{1}{2}$), while the encoder based on the parity polynomial is simpler when $k < n - k$ ($k < \frac{1}{2}n$), which corresponds to low rate codes ($R_c < \frac{1}{2}$).

Cyclic Hamming Codes The class of cyclic codes include the Hamming codes, which have a block length $n = 2^m - 1$ and $n - k = m$ parity check bits, where m is any positive integer. The cyclic Hamming codes are equivalent to the Hamming codes described in Section 8-1-2.

Cyclic (23, 12) Golay Code The linear (23, 12) Golay code described in Section 8-1-2 can be generated as a cyclic code by means of the generator polynomial

$$g(p) = p^{11} + p^9 + p^7 + p^6 + p^5 + p + 1 \tag{8-1-40}$$

The code words have a minimum distance $d_{\min} = 7$.

Maximum-Length Shift-Register Codes Maximum-length shift-register codes are a class of cyclic codes with

$$(n, k) = (2^m - 1, m) \tag{8-1-41}$$

where m is a positive integer. The code words are usually generated by means of an m -stage digital shift register with feedback, based on the parity polynomial. For each code word to be transmitted, the m information bits are loaded into the shift register, and the switch is thrown from position 1 to position 2. The contents of the shift register are shifted to the left one bit at a time for a total of $2^m - 1$ shifts. This operation generates a systematic code with the desired output length $n = 2^m - 1$. For example, the code words generated by the $m = 3$ stage shift register in Fig. 8-1-7 are listed in Table 8-1-4.

Note that, with the exception of the all-zero code word, all the code words generated by the shift register are different cyclic shifts of a single code word. The reason for this structure is easily seen from the state diagram of the shift register, which is illustrated in Fig. 8-1-8 for $m = 3$. When the shift register is loaded initially and shifted $2^m - 1$ times, it will cycle through all possible $2^m - 1$ states. Hence, the shift register is back to its original state in $2^m - 1$ shifts.

FIGURE 8-1-7 Three-stage ($m = 3$) shift register with feedback.

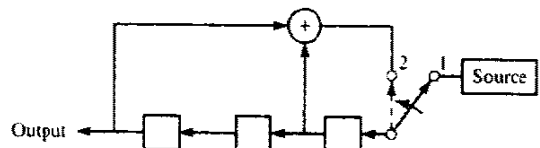


TABLE 8-1-4 MAXIMUM-LENGTH SHIFT-REGISTER CODE FOR $m = 3$

Information bits			Code words						
0	0	0	0	0	0	0	0	0	0
0	0	1	0	0	1	1	1	0	1
0	1	0	0	1	0	0	1	1	1
0	1	1	0	1	1	1	0	1	0
1	0	0	1	0	0	1	1	1	0
1	0	1	1	0	1	0	0	1	1
1	1	0	1	1	0	1	0	0	1
1	1	1	1	1	1	0	1	0	0

Consequently, the output sequence is periodic with length $n = 2^m - 1$. Since there are $2^m - 1$ possible states, this length corresponds to the largest possible period. This explains why the $2^m - 1$ code words are different cyclic shifts of a single code word.

Maximum-length shift-register codes exist for any positive value of m .

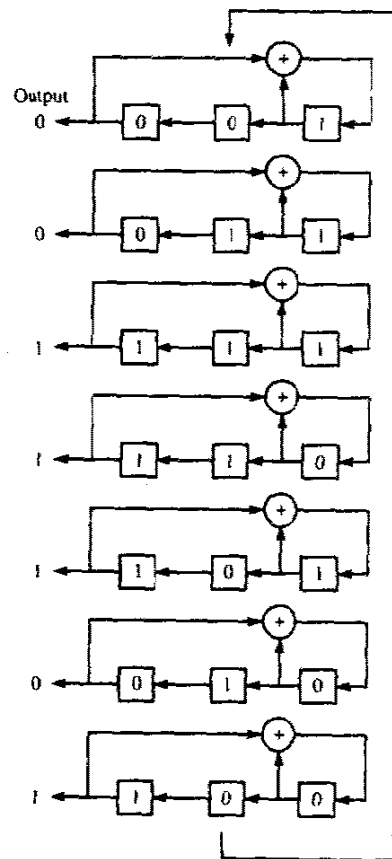


FIGURE 8-1-8 The seven states for the $m = 3$ maximum length shift register.

TABLE 8-1-5 SHIFT-REGISTER CONNECTIONS FOR GENERATING MAXIMUM-LENGTH SEQUENCES

m	Stages connected to modulo-2 adder	m	Stages connected to modulo-2 adder	m	Stages connected to modulo-2 adder
2	1, 2	13	1, 10, 11, 13	24	1, 18, 23, 24
3	1, 3	14	1, 5, 9, 14	25	1, 23
4	1, 4	15	1, 15	26	1, 21, 25, 26
5	1, 4	16	1, 5, 14, 16	27	1, 23, 26, 27
6	1, 6	17	1, 15	28	1, 26
7	1, 7	18	1, 12	29	1, 28
8	1, 5, 6, 7	19	1, 15, 18, 19	30	1, 8, 29, 30
9	1, 6	20	1, 18	31	1, 29
10	1, 8	21	1, 20	32	1, 11, 31, 32
11	1, 10	22	1, 22	33	1, 21
12	1, 7, 9, 12	23	1, 19	34	1, 8, 33, 34

Source: Forney (1970).

Table 8-1-5 lists the stages connected to the modulo-2 adder that result in a maximum-length shift register for $2 \leq m \leq 34$.

Another characteristic of the code words in a maximum-length shift-register code is that each code word, with the exception of the all-zero code word, contains 2^{m-1} ones and 2^{m-1} zeros. Hence all these code words have identical weights, namely, $w = 2^{m-1}$. Since the code is linear, this weight is also the minimum distance of the code, i.e.,

$$d_{\min} = 2^{m-1}$$

Finally, note that the (7, 3) maximum-length shift-register code shown in Table 8-1-4 is identical to the (7, 3) code given in Table 8-1-3, which is the dual of the (7, 4) Hamming code given in Table 8-1-2. This is not a coincidence. The maximum-length shift-register codes are the dual codes of the cyclic Hamming $(2^m - 1, 2^m - 1 - m)$ codes.

The shift register for generating the maximum-length code may also be used to generate a periodic binary sequence with period $n = 2^m - 1$. The binary periodic sequence exhibits a periodic autocorrelation $\phi(m)$ with values $\phi(m) = n$ for $m = 0, \pm n, \pm 2n, \dots$, and $\phi(m) = -1$ for all other shifts as described in Section 13-2-4. This impulse-like autocorrelation implies that the power spectrum is nearly white and, hence, the sequence resembles white noise. As a consequence, maximum-length sequences are called pseudo-noise (PN) sequences and find use in the scrambling of data and in the generation of spread spectrum signals.

Bose-Chaudhuri-Hocquenghem (BCH) Codes BCH codes comprise a large class of cyclic codes that include both binary and nonbinary alphabets.

Binary BCH codes may be constructed with parameters

$$\begin{aligned} n &= 2^m - 1 \\ n - k &\leq mt \\ d_{\min} &= 2t + 1 \end{aligned} \quad (8-1-42)$$

where m ($m \geq 3$) and t are arbitrary positive integers. Hence, this class of binary codes provides the communications system designer with a large selection of block lengths and code rates. Nonbinary BCH codes include the powerful Reed-Solomon codes that are described later.

The generator polynomials for BCH codes can be constructed from factors of $p^{2^m-1} + 1$. Table 8-1-6 lists the coefficients of generator polynomials for BCH codes of block lengths $7 \leq n \leq 255$, corresponding to $3 \leq m \leq 8$. The coefficients are given in octal form, with the left-most digit corresponding to the highest-degree term of the generator polynomial. Thus, the coefficients of the generator polynomial for the (15, 5) code are 2467, which in binary form is 10 100 110 111. Consequently, the generator polynomial is $g(p) = p^{10} + p^8 + p^5 + p^4 + p^2 + p + 1$.

A more extensive list of generator polynomials for BCH codes is given by Peterson and Weldon (1972), who tabulate the polynomial factors of $p^{2^m-1} + 1$ for $m \leq 34$.

8-1-4 Optimum Soft-Decision Decoding of Linear Block Codes

In this subsection, we derive the performance of linear binary block codes on an AWGN channel when optimum (unquantized) soft-decision decoding is employed at the receiver. The bits of a code word may be transmitted by any one of the binary signaling methods described in Chapter 5. For our purposes, we consider binary (or quaternary) coherent PSK, which is the most efficient method, and binary orthogonal FSK either with coherent detection or noncoherent detection.

Let \mathcal{E} denote the transmitted signal energy per code word and let \mathcal{E}_c denote the signal energy required to transmit a single element (bit) in the code word. Since there are n bits per code word, $\mathcal{E} = n\mathcal{E}_c$, and since each code word conveys k bits of information, the energy per information bit is

$$\mathcal{E}_b = \frac{\mathcal{E}}{k} = \frac{n}{k} \mathcal{E}_c = \frac{\mathcal{E}_c}{R_c} \quad (8-1-43)$$

The code words are assumed to be equally likely a priori with prior probability $1/M$.

Suppose the bits of a code word are transmitted by binary PSK. Thus each code word results in one of M signaling waveforms. From Chapter 5, we know that the optimum receiver, in the sense of minimizing the average probability

TABLE 8-1-6 COEFFICIENTS OF GENERATOR POLYNOMIALS (IN OCTAL FORM) FOR BCH CODES OF LENGTHS $7 \leq n \leq 255$

n	k	t	$g(p)$
7	4	1	13
15	11	1	23
	7	2	721
	5	3	2467
31	26	1	45
	21	2	3551
	16	3	107657
	11	5	5423325
	6	7	313365047
63	57	1	103
	51	2	12471
	45	3	1701317
	39	4	166623567
	36	5	1033500423
	30	6	157464165547
	24	7	17323260404441
	18	10	1363026512351725
	16	11	6331141367235453
	10	13	472622305527250155
	7	15	5231045543503271737
127	120	1	211
	113	2	41567
	106	3	11554743
	99	4	3447023271
	92	5	624730022327
	85	6	130704476322273
	78	7	26230002166130115
	71	9	6255010713253127753
	64	10	1206534025570773100045
	57	11	335265252505705053517721
	50	13	54446512523314012421501421
	43	14	17721772213651227521220574343
	36	15	3146074666522075044764574721735
	29	21	403114461367670603667530141176155
	22	23	123376070404722522435445626637647043
	15	27	22057042445604554770523013762217604353
	8	31	7047264052751030651476224271567733130217
255	247	1	435
	239	2	267543
	231	3	156720665
	223	4	75626641375
	215	5	23157564726421
	207	6	16176560567636227
	199	7	7633031270420722341
	191	8	2663470176115333714567
	187	9	52755313540001322236351
	179	10	22624710717340432416300455
	171	11	1541621421234235607706163067

TABLE 8-1-6 (Continued)

n	k	t	$g(p)$
163	12		7500415510075602551574724514601
155	13		3757513005407665015722506464677633
147	14		1642130173537165525304165305441011711
139	15		461401732060175561570722730247453567445
131	18		2157133314715101512612502774421420241 65471
123	19		1206140522420660037172103265161412262 72506267
115	21		6052666557210024726363640460027635255 6313472737
107	22		2220577232206625631241730023534742017 6574750154441
99	23		1065666725347317422274141620157433225 2411076432303431
91	25		6750265030327444172723631724732511075 550762720724344561
87	26		1101367634147432364352316343071720462 06722545273311721317
79	27		6670003563765750002027034420736617462 1015326711766541342355
71	29		2402471052064432151555417211233116320 5444250362557643221706035
63	30		1075447505516354432531521735770700366 6111726455267613656702543301
55	31		7315425203501100133015275306032054325 414326755010557044426035473617
47	42		2533542017062646563033041377406233175 123334145446045005066024552543173
45	43		1520205605523416113110134637642370156 3670024470762373033202157025051541
37	45		5136330255067007414177447245437530420 735706174323432347644354737403044003
29	47		3025715536673071465527064012361377115 34224232420117411406025475741040356 5037
21	55		1256215257060332656001773153607612103 22734140565307454252115312161446651 3473725
13	59		4641732005052564544426573714250066004 33067744547656140317467721357026134 460500547
9	63		1572602521747246320103104325535513461 41623672120440745451127661155477055 61677516057

Source: Stenbit (1964), © 1964 IEEE.

of a code word error, for the AWGN channel, can be realized as a parallel bank of M filters matched to the M possible transmitted waveforms. The outputs of the M matched filters at the end of each signaling interval, which encompasses the transmission of n bits in the code word, are compared and the code word corresponding to the largest matched filter output is selected. Alternatively, M cross-correlators can be employed. In either case, the receiver implementation can be simplified. That is, an equivalent optimum receiver can be realized by use of a single filter (or cross-correlator) matched to the binary PSK waveform used to transmit each bit in the code word, followed by a decoder that forms the M decision variables corresponding to the M code words.

To be specific, let r_j , $j = 1, 2, \dots, n$, represent the n sampled outputs of the matched filter for any particular code word. Since the signaling is binary coherent PSK, the output r_j may be expressed either as

$$r_j = \sqrt{\mathcal{E}_c} + n_j \quad (8-1-44)$$

when the j th bit of a code word is a 1, or as

$$r_j = -\sqrt{\mathcal{E}_c} + n_j \quad (8-1-45)$$

when the j th bit is a 0. The variables $\{n_j\}$ represent additive white gaussian noise at the sampling instants. Each n_j has zero mean and variance $\frac{1}{2}N_0$. From knowledge of the M possible transmitted code words and upon reception of $\{r_j\}$, the optimum decoder forms the M correlation metrics

$$CM_i = C(\mathbf{r}, \mathbf{C}_i) = \sum_{j=1}^n (2c_{ij} - 1)r_j, \quad i = 1, 2, \dots, M \quad (8-1-46)$$

where c_{ij} denotes the bit in the j th position of the i th code word. Thus, if $c_{ij} = 1$, the weighting factor $2c_{ij} - 1 = 1$, and if $c_{ij} = 0$, the weighting factor $2c_{ij} - 1 = -1$. In this manner, the weighting $2c_{ij} - 1$ aligns the signal components in $\{r_j\}$ such that the correlation metric corresponding to the actual transmitted code word will have a mean value $\sqrt{\mathcal{E}_c}n$, while the other $M - 1$ metrics will have smaller mean values.

Although the computations involved in forming the correlation metrics for soft-decision decoding according to (8-1-46) are relatively simple, it may still be impractical to compute (8-1-46) for all the possible code words when the number of code words is large, e.g., $M > 2^{10}$. In such a case it is still possible to implement soft-decision decoding using algorithms which employ techniques for discarding improbable code words without computing their entire correlation metrics as given by (8-1-46). Several different types of soft-decision decoding algorithms have been described in the technical literature. The interested reader is referred to the papers by Forney (1966b), Weldon (1971), Chase (1972), Wainberg and Wolf (1973), Wolf (1978), and Matis and Modestino (1982).

In determining the probability of error for a linear block code, note that

when such a code is employed on a binary-input, symmetric channel such as the AWGN channel with optimum soft-decision decoding, the error probability for the transmission of the m th code word is the same for all m . Hence, we assume for simplicity that the all-zero code word C_1 is transmitted. For correct decoding of C_1 , the correlation metric CM_1 must exceed all the other $M - 1$ correlation metrics CM_m , $m = 2, \dots, M$. All the CM are gaussian distributed. The mean value of CM_1 is $\sqrt{\mathcal{E}_c n}$, while the mean values of CM_m , $m = 2, \dots, M$ is $\sqrt{\mathcal{E}_c n}(1 - 2w_m/n)$. The variance of each decision variable is $\frac{1}{2}N_0$. The derivation of the exact expression for the probability of correct decoding or, equivalently, the probability of a code word error is complicated by the correlations among the M correlation metrics. The cross-correlation coefficients between C_1 and the other $M - 1$ code words are

$$\rho_m = 1 - 2w_m/n, \quad m = 2, \dots, M \quad (8-1-47)$$

where w_m denotes the weight of the m th code word.

Instead of attempting to derive the exact error probability, we resort to a union bound. The probability that $CM_m > CM_1$ is

$$P_2(m) = Q\left(\sqrt{\frac{\mathcal{E}}{N_0}}(1 - \rho_m)\right) \quad (8-1-48)$$

where $\mathcal{E} = k\mathcal{E}_b$ is the transmitted energy per waveform. Substitution for ρ_m from (8-1-47) and for \mathcal{E} yields

$$\begin{aligned} P_2(m) &= Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0} R_c w_m}\right) \\ &= Q(\sqrt{2\gamma_b R_c w_m}) \end{aligned} \quad (8-1-49)$$

where γ_b is the SNR per bit and R_c is the code rate. Then the average probability of a code word error is bounded from above by the sum of the binary error events given by (8-1-49). Thus,

$$\begin{aligned} P_M &\leq \sum_{m=2}^M P_2(m) \\ &\leq \sum_{m=2}^M Q(\sqrt{2\gamma_b R_c w_m}) \end{aligned} \quad (8-1-50)$$

The computation of the probability of error for soft-decision decoding according to (8-1-50) requires knowledge of the weight distribution of the code. Weight distributions of many codes are given in a number of texts on coding theory, e.g., Berlekamp (1968) and MacWilliams and Sloane (1977).

A somewhat looser bound is obtained by noting that

$$Q(\sqrt{2\gamma_b R_c w_m}) \leq Q(\sqrt{2\gamma_b R_c d_{\min}}) < \exp(-\gamma_b R_c d_{\min}) \quad (8-1-51)$$

Consequently,

$$P_M \leq (M - 1)Q(\sqrt{2\gamma_b R_c d_{\min}}) < \exp(-\gamma_b R_c d_{\min} + k \ln 2) \quad (8-1-52)$$

This bound is particularly useful since it does not require knowledge of the weight distribution of the code. When the upper bound in (8-1-52) is compared with the performance of an uncoded binary PSK system, which is upper-bounded as $\frac{1}{2} \exp(-\gamma_b)$, we find that coding yields a gain of approximately $10 \log(R, d_{\min} - k \ln 2/\gamma_b)$ dB. We may call this the *coding gain*. We note that its value depends on the code parameters and also on the SNR per bit γ_b .

The expression for the probability of error for equicorrelated waveforms that can be obtained for the simplex signals described in Section 5-2 gives us yet a third approximation to the error probabilities for coded waveforms. We know that the maximum cross-correlation coefficient between a pair of coded waveforms is

$$\rho_{\max} = 1 - \frac{2}{n} d_{\min} \quad (8-1-53)$$

If we assume as a worst case that all the M code words have a cross-correlation coefficient equal to ρ_{\max} then the code word error probability can easily be manipulated. Since some code words are separated by more than the minimum distance, the error probability evaluated for $\rho_r = \rho_{\max}$ is actually an upper bound. Thus,

$$P_M \leq 1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-v^2/2} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{v + \sqrt{4\gamma_b R, d_{\min}}} e^{-x^2/2} dx \right)^{M-1} dv \quad (8-1-54)$$

The bounds on the performance of linear block codes given above are in terms of the block error or code word error probability. The evaluation of the equivalent bit error probability P_b is much more complicated. In general, when a block error is made, some of the k information bits in the block will be correct and some will be in error. For orthogonal waveforms, the conversion factor that multiplies P_M to yield P_b is $2^{k-1}/(2^k - 1)$. This factor is unity for $k = 1$ and approaches $\frac{1}{2}$ as k increases, which is equivalent to assuming that, on the average, half of the k bits will be in error when a block error occurs. The conversion factor for coded waveforms depends in a complicated way on the distance properties of the code, but is certainly no worse than assuming that, on the average, half of the k bits will be in error when a block error occurs. Consequently, $P_b \leq \frac{1}{2} P_M$.

The bounds on performance given by (8-1-50), (8-1-52), and (8-1-54) also apply to the case in which a pair of bits of a code word are transmitted by quaternary PSK, since quaternary PSK may be viewed as being equivalent to two independent binary PSK waveforms transmitted in phase quadrature. Furthermore, the bounds in (8-1-52) and (8-1-54), which depend only on the minimum distance of the code, apply also to nonlinear binary block codes.

If binary orthogonal FSK is used to transmit each bit of a code word on the AWGN channel, the optimum receiver can be realized by means of two matched filters, one matched to the frequency corresponding to a transmission of a 0, and the other to the frequency corresponding to a transmission of a 1, followed by a decoder that forms the M correlation metrics corresponding to

the M possible code words. The detection at the receiver may be coherent or noncoherent. In either case, let r_{0j} and r_{1j} denote the input samples to the combiner. The correlation metrics formed by the decoder may be expressed as

$$CM_i = \sum_{j=1}^n [c_{ij}r_{1j} + (1 - c_{ij})r_{0j}], \quad i = 1, 2, \dots, M \quad (8-1-55)$$

where c_{ij} represents the j th bit in the i th code word. The code word corresponding to the largest of the $\{CM_i\}$ is selected as the transmitted code word.

If the detection of the binary FSK waveforms is coherent, the random variables $\{r_{0j}\}$ and $\{r_{1j}\}$ are gaussian and, hence, the correlation metrics $\{CM_i\}$ are also gaussian. In this case, bounds on the performance of the code are easily obtained. To be specific, suppose that the all-zero code word C_1 is transmitted. Then,

$$\left. \begin{aligned} r_{0j} &= \sqrt{\mathcal{E}_c} + n_{0j} \\ r_{1j} &= n_{1j} \end{aligned} \right\} \quad j = 1, 2, \dots, n \quad (8-1-56)$$

where the $\{n_{ij}\}$, $i = 0, 1$, $j = 1, 2, \dots, n$, are mutually statistically independent gaussian random variables with zero mean and variance $\frac{1}{2}N_0$. Consequently CM_1 is gaussian with mean $\sqrt{\mathcal{E}_c}n$ and variance $\frac{1}{2}nN_0$. On the other hand, the correlation metric CM_m , corresponding to the code word having weight w_m , is gaussian with mean $\sqrt{\mathcal{E}_c}n(1 - w_m/n)$ and variance $\frac{1}{2}nN_0$. Since the $\{CM_m\}$ are correlated, we again resort to a union bound. The correlation coefficients are given by

$$\rho_m = 1 - w_m/n \quad (8-1-57)$$

Hence, the probability that $CM_m > CM_1$ is

$$P_2(m) = Q(\sqrt{\gamma_b R_c w_m}) \quad (8-1-58)$$

Comparison of this result with that given in (8-1-49) for coherent PSK reveals that coherent PSK requires 3 dB less SNR to achieve the same performance. This is not surprising in view of the fact that uncoded PSK is 3 dB better than binary orthogonal FSK with coherent detection. Hence, the advantage of PSK over FSK is maintained in the coded waveforms. We conclude, then, that the bounds given in (8-1-50), (8-1-52), and (8-1-54) apply to coded waveforms transmitted by binary orthogonal coherent FSK with γ_b replaced by $\frac{1}{2}\gamma_b$.

If square-law detection of the binary orthogonal FSK signal is employed at the receiver, the performance is further degraded by the noncoherent combining loss, as shown in Chapter 12. Suppose again that the all-zero code word is transmitted. Then the correlation metrics are given by (8-1-55), where the input variables to the decoder are now

$$\left. \begin{aligned} r_{0j} &= |\sqrt{\mathcal{E}_c} + N_{0j}|^2 \\ r_{1j} &= |N_{1j}|^2 \end{aligned} \right\} \quad j = 1, 2, \dots, n \quad (8-1-59)$$

where $\{N_{0j}\}$ and $\{N_{1j}\}$ represent complex-valued mutually statistically independent gaussian random variables with zero mean and variance N_0 . The correlation metric CM_1 is given as

$$CM_1 = \sum_{j=1}^n r_{0j} \quad (8-1-60)$$

while the correlation metric corresponding to the code word having weight w_m is statistically equivalent to the correlation metric of a code word in which $c_{mj} = 1$ for $1 \leq j \leq w_m$ and $c_{mj} = 0$ for $w_m + 1 \leq j \leq n$. Hence, CM_m may be expressed as

$$CM_m = \sum_{j=1}^{w_m} r_{1j} + \sum_{j=w_m+1}^n r_{0j} \quad (8-1-61)$$

The difference between CM_1 and CM_m is

$$CM_1 - CM_m = \sum_{j=1}^{w_m} (r_{0j} - r_{1j}) \quad (8-1-62)$$

and the probability of error is simply the probability that $CM_1 - CM_m < 0$. But this difference is a special case of the general quadratic form in complex-valued gaussian random variables considered in Chapter 12 and Appendix B. The expression for the probability of error in deciding between CM_1 and CM_m is (see Section 12-1-1)

$$P_2(m) = \frac{1}{2^{2w_m-1}} \exp(-\frac{1}{2}\gamma_b R_c w_m) \sum_{i=0}^{w_m-1} K_i(\frac{1}{2}\gamma_b R_c w_m)^i \quad (8-1-63)$$

where, by definition,

$$K_i = \frac{1}{i!} \sum_{r=0}^{w_m-1-i} \binom{2w_m-1}{r} \quad (8-1-64)$$

The union bound obtained by summing $P_2(m)$ over $2 \leq m \leq M$ provides us with an upper bound on the probability of a code word error.

As an alternative, we may use the minimum distance instead of the weight distribution to obtain the looser upper bound

$$P_M \leq \frac{M-1}{2^{2d_{\min}-1}} \exp(-\frac{1}{2}\gamma_b R_c d_{\min}) \sum_{i=0}^{d_{\min}-1} K_i(\frac{1}{2}\gamma_b R_c d_{\min})^i \quad (8-1-65)$$

A measure of the noncoherent combining loss inherent in the square-law detection and combining of the n elementary binary FSK waveforms in a code word can be obtained from Fig. 12-1-1, where d_{\min} is used in place of L . The loss obtained is relative to the case in which the n elementary binary FSK waveforms are first detected coherently and combined as in (8-1-55) and then the sums are square-law-detected or envelope-detected to yield the M decision variables. The binary error probability for the latter case is

$$P_2(m) = \frac{1}{2} \exp(-\frac{1}{2}\gamma_b R_c w_m) \quad (8-1-66)$$

and, hence,

$$P_M \leq \sum_{m=2}^M P_2(m)$$

If d_{\min} is used instead of the weight distribution, the union bound for the code word error probability in the latter case is

$$P_M \leq \frac{1}{2}(M-1) \exp(-\frac{1}{2}\gamma_b R_c d_{\min}) \quad (8-1-67)$$

The channel bandwidth required to transmit the coded waveforms can be determined as follows. If binary PSK is used to transmit each bit in a code word, the required bandwidth is approximately equal to the reciprocal of the time interval devoted to the transmission of each bit. For an information rate of R bits/s, the time available to transmit k information bits and $n-k$ redundant (parity) bits (n total bits) is $T = k/R$. Hence,

$$W = \frac{1}{T/n} = \frac{n}{k/R} = \frac{R}{R_c} \quad (8-1-68)$$

Therefore, the bandwidth expansion factor B_e for the coded waveform is

$$\begin{aligned} B_e &= \frac{W}{R} \\ &= \frac{n}{k} = \frac{1}{R_c} \end{aligned} \quad (8-1-69)$$

On the other hand, if binary FSK with noncoherent detection is employed for transmitting the bits in a code word, $W \approx 2n/T$, and, hence, the bandwidth expansion factor increases by approximately a factor of 2 relative to binary PSK. In any case, B_e increases inversely with the code rate, or, equivalently, it increases linearly with the block size n .

We are now in a position to compare the performance characteristics and bandwidth requirements of coded signaling waveforms with orthogonal signaling waveforms. A comparison of the expression for P_M given in (5-2-21) for orthogonal waveforms and in (8-1-54) for coded waveforms with coherent PSK indicates that the coded waveforms result in a loss of at most $10 \log(n/2d_{\min})$ dB relative to orthogonal waveforms having the same number of waveforms. On the other hand, if we compensate for the loss in SNR due to coding by increasing the number of code words so that coded transmission requires $M_c = 2^k$ waveforms and orthogonal signaling requires $M_o = 2^k$ waveforms then [from the union bounds in (5-2-27) and (8-1-52)], the performance obtained with the two sets of signaling waveforms at high SNR is about equal if

$$k_o = 2R_c d_{\min} \quad (8-1-70)$$

Under this condition, the bandwidth expansion factor for orthogonal signaling can be expressed as

$$B_{eo} = \frac{M_o}{2 \log_2 M_o} = \frac{2^{k_o}}{2k_o} = \frac{2^{2R_c d_{\min}}}{4R_c d_{\min}} \quad (8-1-71)$$

while, for coded signaling waveforms, we have $B_{ec} = 1/R_c$. The ratio of B_{eo} given in (8-1-71) to B_{ec} , which is

$$\frac{B_{eo}}{B_{ec}} = \frac{2^{2R_c d_{\min}}}{4d_{\min}} \quad (8-1-72)$$

provides a measure of the relative bandwidth between orthogonal signaling and signaling with coded coherent PSK waveforms.

For example, suppose we use a (63, 33) binary cyclic code that has a minimum distance $d_{\min} = 12$. The bandwidth ratio for orthogonal signaling relative to this code, given by (8-1-72), is 127. This is indicative of the bandwidth efficiency obtained through coding relative to orthogonal signaling.

8-1-5 Hard-Decision Decoding

The bounds given in Section 8-1-4 on the performance of coded signaling waveforms on the AWGN channel are based on the premise that the samples from the matched filter or cross correlator are not quantized. Although this processing yields the best performance, the basic limitation is the computational burden of forming M correlation metrics and comparing these to obtain the largest. The amount of computation becomes excessive when the number M of code words is large.

To reduce the computational burden, the analog samples can be quantized and the decoding operations are then performed digitally. In this subsection, we consider the extreme situation in which each sample corresponding to a single bit of a code word is quantized to two levels: zero and one. That is, a (hard) decision is made as to whether each transmitted bit in a code word is a 0 or a 1. The resulting discrete-time channel (consisting of the modulator, the AWGN channel, and the demodulator) constitutes a BSC with crossover probability p . If coherent PSK is employed in transmitting and receiving the bits in each code word then

$$\begin{aligned} p &= Q\left(\sqrt{\frac{2\mathcal{E}_c}{N_0}}\right) \\ &= Q(\sqrt{2\gamma_b R_c}) \end{aligned} \quad (8-1-73)$$

On the other hand, if FSK is used to transmit the bits in each code word then

$$p = Q(\sqrt{\gamma_b R_c}) \quad (8-1-74)$$

for coherent detection and

$$p = \frac{1}{2} \exp\left(-\frac{1}{2}\gamma_b R_c\right) \quad (8-1-75)$$

for noncoherent detection.

Minimum-Distance (Maximum-Likelihood) Decoding The n bits from the demodulator corresponding to a received code word are passed to the decoder, which compares the received code word with the M possible transmitted code words and decides in favor of the code word that is closest in Hamming distance (number of bit positions in which two code words differ) to the received code word. This minimum distance decoding rule is optimum in the sense that it results in a minimum probability of a code word error for the binary symmetric channel.

A conceptually simple, albeit computationally inefficient, method for decoding is to first add (modulo 2) the received code word vector to all the M possible transmitted code words C_i to obtain the error vectors e_i . Hence, e_i represents the error event that must have occurred on the channel in order to transform the code word C_i into the particular received code word. The number of errors in transforming C_i into the received code word is just equal to the number of 1s in e_i . Thus, if we simply compute the weight of each of the M error vectors $\{e_i\}$ and decide in favor of the code word that results in the smallest weight error vector, we have, in effect, a realization of the minimum distance decoding rule.

A more efficient method for hard-decision decoding makes use of the parity check matrix H . To elaborate, suppose that C_m is the transmitted code word and Y is the received code word at the output of the demodulator. In general, Y may be expressed as

$$Y = C_m + e$$

where e denotes an arbitrary binary error vector. The product YH' yields

$$\begin{aligned} YH' &= (C_m + e)H' \\ &= C_m H' + eH' \\ &= eH' = S \end{aligned} \quad (8-1-76)$$

where the $(n - k)$ -dimensional vector S is called the *syndrome of the error pattern*. In other words, the vector S has components that are zero for all parity check equations that are satisfied and nonzero for all parity check equations that are not satisfied. Thus, S contains the pattern of failures in the parity checks.

We emphasize that the syndrome S is a characteristic of the error pattern and not of the transmitted code word. Furthermore, we observe that there are 2^n possible error patterns and only 2^{n-k} syndromes. Consequently, different error patterns result in the same syndrome.

Suppose we construct a decoding table in which we list all the 2^k possible code words in the first row, beginning with the all-zero code word in the first (left-most) column. This all-zero code word also represents the all-zero error pattern. We fill in the first column by listing first all $n - 1$ error patterns $\{e_i\}$ of weight 1. If $n < 2^{n-k}$, we may then list all double error patterns, then all triple

error patterns, etc., until we have a total of 2^{n-k} entries in the first column. Thus, the number of rows that we can have is 2^{n-k} , which is equal to the number of syndromes. Next, we add each error pattern in the first column to the corresponding code words. Thus, we fill in the remainder of the $n \times (n - k)$ table as follows:

C_1	C_2	C_3	...	C_{2^k}
e_1	$C_2 + e_1$	$C_3 + e_1$...	$C_{2^k} + e_1$
e_2	$C_2 + e_2$	$C_3 + e_2$...	$C_{2^k} + e_2$
e_3	$C_2 + e_3$	$C_3 + e_3$...	$C_{2^k} + e_3$
\vdots	\vdots	\vdots		\vdots
$e_{2^{n-k}}$	$C_2 + e_{2^{n-k}}$	$C_3 + e_{2^{n-k}}$...	$C_{2^k} + e_{2^{n-k}}$

This table is called a *standard array*. Each row, including the first, consists of k possible received code words that would result from the corresponding error pattern in the first column. Each row is called a *coset* and the first (left-most) code word (or error pattern) is called a *coset leader*. Therefore, a coset consists of all the possible received code words resulting from a particular error pattern (coset leader).

Example 8-1-10

Let us construct the standard array for the (5, 2), systematic code with generator matrix given by

$$G = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

This code has a minimum distance $d_{min} = 3$. The standard array is given in Table 8-1-7. Note that in this code, the coset leaders consist of the all-zero error pattern, five error patterns of weight 1, and two error patterns of

TABLE 8-1-7 STANDARD ARRAY FOR THE (5, 2) CODE

Code words			
0 0 0 0 0	0 1 0 1 1	1 0 1 0 1	1 1 1 1 0
0 0 0 0 1	0 1 0 1 0	1 0 1 0 0	1 1 1 1 1
0 0 0 1 0	0 1 0 0 1	1 0 1 1 1	1 1 1 0 0
0 0 1 0 0	0 1 1 1 1	1 0 0 0 1	1 1 0 1 0
0 1 0 0 0	0 0 0 1 1	1 1 1 0 1	1 0 1 1 0
1 0 0 0 0	1 1 0 1 1	0 0 1 0 1	0 1 1 1 0
1 1 0 0 0	1 0 0 1 1	0 1 1 0 1	0 0 1 1 0
1 0 0 1 0	1 1 0 0 1	0 0 1 1 1	0 1 1 0 0

weight 2. Although many more double error patterns exist, there is only room for two to complete the table. These were selected such that their corresponding syndromes are distinct from those of the single error patterns.

Now, suppose that \mathbf{e}_i is a coset leader and that \mathbf{C}_m was the transmitted code word. Then, the error pattern \mathbf{e}_i would result in the received code word

$$\mathbf{Y} = \mathbf{C}_m + \mathbf{e}_i$$

The syndrome is

$$\mathbf{S} = (\mathbf{C}_m + \mathbf{e}_i)\mathbf{H}' = \mathbf{C}_m\mathbf{H}' + \mathbf{e}_i\mathbf{H}' = \mathbf{e}_i\mathbf{H}'$$

Clearly, all received code words in the same coset have the same syndrome, since the latter depends only on the error pattern. Furthermore, each coset has a different syndrome. Having established this characteristic of the standard array, we may simply construct a syndrome decoding table in which we list the 2^{n-k} syndromes and the corresponding 2^{n-k} coset leaders that represent the minimum weight error patterns. Then, given a received code vector \mathbf{Y} , we compute the syndrome

$$\mathbf{S} = \mathbf{Y}\mathbf{H}'$$

For the computed \mathbf{S} , we find the corresponding (most likely) error vector, say $\hat{\mathbf{e}}_m$. This error vector is added to \mathbf{Y} to yield the decoded word

$$\hat{\mathbf{C}}_m = \mathbf{Y} \oplus \hat{\mathbf{e}}_m$$

Example 8-1-11

Consider the (5, 2) code with the standard array given in Table 8-1-7. The syndromes versus the most likely error patterns are given in Table 8-1-8. Now suppose the actual error vector on the channel is

$$\mathbf{e} = [1 \ 0 \ 1 \ 0 \ 0]$$

TABLE 8-1-8 SYNDROME TABLE FOR THE (5, 2) CODE

Syndrome	Error pattern
0 0 0	0 0 0 0 0
0 0 1	0 0 0 0 1
0 1 0	0 0 0 1 0
1 0 0	0 0 1 0 0
0 1 1	0 1 0 0 0
1 0 1	1 0 0 0 0
1 1 0	1 1 0 0 0
1 1 1	1 0 0 1 0

The syndrome computed for the error is $\mathbf{S} = [0 \ 0 \ 1]$. Hence, the error determined from the table is $\hat{\mathbf{e}} = [0 \ 0 \ 0 \ 0 \ 1]$. When $\hat{\mathbf{e}}$ is added to \mathbf{Y} , the result is a decoding error. In other words the (5, 2) code corrects all single errors and only two double errors, namely $[1 \ 1 \ 0 \ 0 \ 0]$ and $[1 \ 0 \ 0 \ 1 \ 0]$.

Syndrome Decoding of Cyclic Codes As described above, hard-decision decoding of a linear block code may be accomplished by first computing the syndrome $\mathbf{S} = \mathbf{YH}'$, then using a table lookup to find the most probable error pattern \mathbf{e} corresponding to the computed syndrome \mathbf{S} , and, finally, adding the error pattern \mathbf{e} to the received vector \mathbf{Y} to obtain the most probable code word $\hat{\mathbf{C}}_m$. When the code is cyclic, the syndrome computation may be performed by a shift register similar in form to that used for encoding.

To elaborate, let us consider a systematic cyclic code and let us represent the received code vector \mathbf{Y} by the polynomial $Y(p)$. In general, $\mathbf{Y} = \mathbf{C} + \mathbf{e}$, where \mathbf{C} is the transmitted code word and \mathbf{e} is the error vector. Hence, we have

$$\begin{aligned} Y(p) &= C(p) + e(p) \\ &= X(p)g(p) + e(p) \end{aligned} \quad (8-1-77)$$

Now, suppose we divide $Y(p)$ by the generator polynomial $g(p)$. This division will yield

$$\frac{Y(p)}{g(p)} = Q(p) + \frac{R(p)}{g(p)}$$

or, equivalently,

$$Y(p) = Q(p)g(p) + R(p) \quad (8-1-78)$$

The remainder $R(p)$ is a polynomial of degree less than or equal to $n - k - 1$. If we combine (8-1-77) with (8-1-78), we obtain

$$e(p) = [X(p) + Q(p)]g(p) + R(p) \quad (8-1-79)$$

This relationship illustrates that the remainder $R(p)$ obtained from dividing $Y(p)$ by $g(p)$ depends only on the error polynomial $e(p)$, and, hence, $R(p)$ is simply the syndrome associated with the error pattern \mathbf{e} . Therefore,

$$Y(p) = Q(p)g(p) + S(p) \quad (8-1-80)$$

where $S(p)$ is the syndrome polynomial of degree less than or equal to $n - k - 1$. If $g(p)$ divides $Y(p)$ exactly then $S(p) = 0$ and the received decoded word is $\hat{\mathbf{C}}_m = \mathbf{Y}$.

The division of $Y(p)$ by the generator polynomial $g(p)$ may be carried out by means of a shift register which performs division as described previously. First the received vector \mathbf{Y} is shifted into an $(n - k)$ -stage shift register as

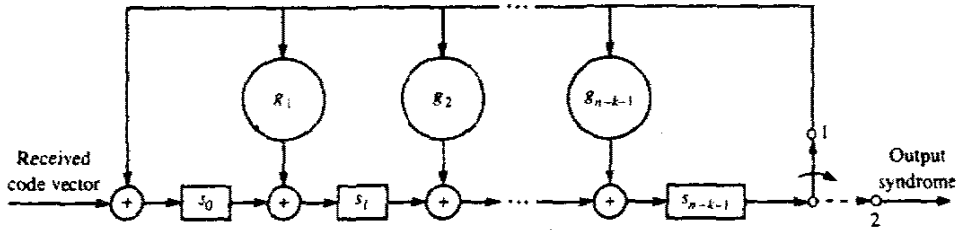


FIGURE 8-1-9 An $(n - k)$ -stage shift register for computing the syndrome.

illustrated in Fig. 8-1-9. Initially, all the shift-register contents are zero and the switch is closed in position 1. After the entire n -bit received vector has been shifted into the register, the contents of the $n - k$ stages constitute the syndrome with the order of the bits numbered as shown in Fig. 8-1-9. These bits may be clocked out by throwing the switch into position 2. Given the syndrome from the $(n - k)$ -stage shift register, a table lookup may be performed to identify the most probable error vector.

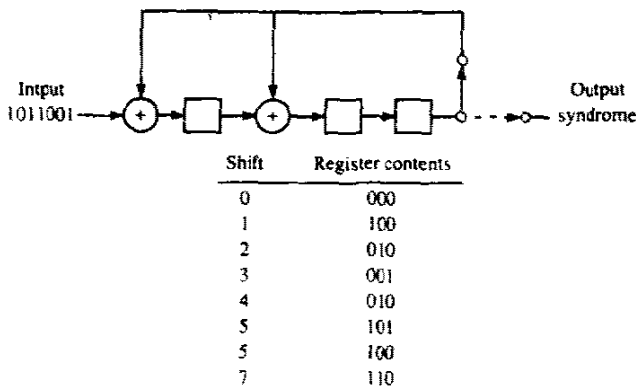
Example 8-1-12

Let us consider the syndrome computation for the $(7, 4)$ cyclic Hamming code generated by the polynomial $g(p) = p^3 + p + 1$. Suppose that the received vector is $\mathbf{Y} = [1\ 0\ 0\ 1\ 1\ 0\ 1]$. This is fed into the three-stage register shown in Fig. 8-1-10. After seven shifts the contents of the shift register are 110, which corresponds to the syndrome $\mathbf{S} = [0\ 1\ 1]$. The most probable error vector corresponding to this syndrome is $\mathbf{e} = [0\ 0\ 0\ 1\ 0\ 0\ 0]$ and, hence,

$$\hat{\mathbf{C}}_m = \mathbf{Y} + \mathbf{e} = [1\ 0\ 0\ 0\ 1\ 0\ 1]$$

The information bits are 1 0 0 0.

FIGURE 8-1-10 Syndrome computation for the $(7, 4)$ cyclic code with generator polynomial $g(p) = p^3 + p + 1$ and received vector $\mathbf{Y} = [1\ 0\ 0\ 1\ 1\ 0\ 1]$.



The table lookup decoding method using the syndrome is practical only when $n - k$ is small, e.g., $n - k < 10$. This method is impractical for many interesting and powerful codes. For example, if $n - k = 20$, the table has 2^{20} (approximately 1 million) entries. Such a large amount of storage and the time required to locate an entry in such a large table renders the table lookup decoding method impractical for long codes having large numbers of check bits.

More efficient and practical hard-decision decoding algorithms have been devised for the class of cyclic codes and, more specifically, the BCH codes. A description of these algorithms requires further development of computational methods with finite fields, which is beyond the scope of our treatment of coding theory. It suffices to indicate that efficient decoding algorithms exist which make it possible to implement long BCH codes with high redundancy in practical digital communications systems. The interested reader is referred to the texts of Peterson and Weldon (1972), Lin and Costello (1983), Blahut (1983), and Berlekamp (1968), and to the paper by Forney (1965).

Error Detection and Error Correction Capability It is clear from the discussion above that when the syndrome consists of all zeros, the received code word is one of the 2^k possible transmitted code words. Since the minimum separation between a pair of code words is d_{\min} , it is possible for an error pattern of weight d_{\min} to transform one of these 2^k code words in the code into another code word. When this happens we have an *undetected error*. On the other hand, if the actual number of errors is less than d_{\min} , the syndrome will have a nonzero weight. When this occurs, we have detected the presence of one or more errors on the channel. Clearly, the (n, k) block code is capable of detecting $d_{\min} - 1$ errors. Error detection may be used in conjunction with an automatic repeat-request (ARQ) scheme for retransmission of the code word.

The *error correction capability* of a code also depends on the minimum distance. However, the number of correctable error patterns is limited by the number of possible syndromes or coset leaders in the standard array. To determine the error correction capability of an (n, k) code, it is convenient to view the 2^k code words as points in an n -dimensional space. If each code word is viewed as the center of a sphere of radius (Hamming distance) t , the largest value that t may have without intersection (or tangency) of any pair of the 2^k spheres is $t = \lfloor \frac{1}{2}(d_{\min} - 1) \rfloor$, where $\lfloor x \rfloor$ denotes the largest integer contained in x . Within each sphere lie all the possible received code words of distance less than or equal to t from the valid code word. Consequently, any received code vector that falls within a sphere is decoded into the valid code word at the center of the sphere. This implies that an (n, k) code with minimum distance d_{\min} is capable of correcting $t = \lfloor \frac{1}{2}(d_{\min} - 1) \rfloor$ errors. Figure 8-1-11 is a two-dimensional representation of the code words and the spheres.

As described above, a code may be used to detect $d_{\min} - 1$ errors or to correct $t = \lfloor \frac{1}{2}(d_{\min} - 1) \rfloor$ errors. Clearly, to correct t error implies that we have

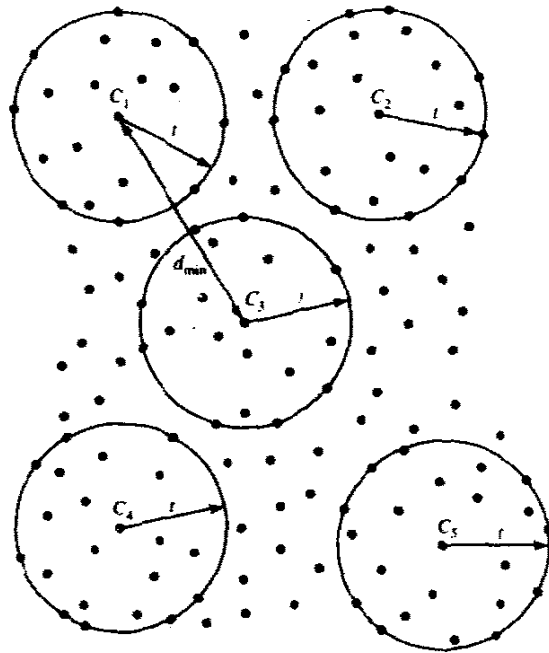


FIGURE 8-1-11 A representation of code words as centers of spheres of radius $t = \lfloor \frac{1}{2}(d_{\min} - 1) \rfloor$.

detected t errors. However, it is also possible to detect more than t errors if we compromise in the error correction capability of the code. For example, a code with $d_{\min} = 7$ can correct $t = 3$ errors. If we wish to detect four errors, we can do so by reducing the radius of the sphere around each code word from 3 to 2. Thus, patterns with four errors are detectable but only patterns of two errors are correctable. In other words, when only two errors occur, these are corrected, and when three or four errors occur, the receiver may ask for a retransmission. If more than four errors occur, they will go undetected if the code word falls within a sphere of radius 2. Similarly, for $d_{\min} = 7$, five errors can be detected and one error corrected. In general, a code with minimum distance d_{\min} can detect e_d errors and correct e_c errors, where

$$e_d + e_c \leq d_{\min} - 1$$

and

$$e_c \leq e_d$$

Probability of Error Based on Error Correction We conclude this section with the derivation of the probability of error for hard-decision decoding of linear binary block codes based on error correction only.

From the above discussion, it is clear that the optimum decoder for a binary

symmetric channel will decode correctly if (but not necessarily only if) the number of errors in a code word is less than half the minimum distance d_{\min} of the code. That is, any number of errors up to

$$t = \lfloor \frac{1}{2}(d_{\min} - 1) \rfloor$$

are always correctable. Since the binary symmetric channel is memoryless, the bit errors occur independently. Hence, the probability of m errors in a block of n bits is

$$P(m, n) = \binom{n}{m} p^m (1-p)^{n-m} \quad (8-1-81)$$

and, therefore, the probability of a code word error is upper-bounded by the expression

$$P_M \leq \sum_{m=t+1}^n P(m, n) \quad (8-1-82)$$

Equality holds in (8-1-82) if the linear block code is a perfect code. In order to describe the basic characteristics of a perfect code, suppose we place a sphere of radius t around each of the possible transmitted code words. Each sphere around a code word contains the set of all code words of Hamming distance less than or equal to t from the code word. Now, the number of code words in a sphere of radius $t = \lfloor \frac{1}{2}(d_{\min} - 1) \rfloor$ is

$$1 + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{t} = \sum_{i=0}^t \binom{n}{i}$$

Since there are $M = 2^k$ possible transmitted code words, there are 2^k nonoverlapping spheres each having a radius t . The total number of code words enclosed in the 2^k spheres cannot exceed the 2^n possible received code words. Thus, a t -error correcting code must satisfy the inequality

$$2^k \sum_{i=0}^t \binom{n}{i} \leq 2^n$$

or, equivalently,

$$2^{n-k} \geq \sum_{i=0}^t \binom{n}{i} \quad (8-1-83)$$

A perfect code has the property that all spheres of Hamming distance $t = \lfloor \frac{1}{2}(d_{\min} - 1) \rfloor$ around the $M = 2^k$ possible transmitted code words are disjoint and every received code word falls in one of the spheres. Thus, every received code word is at most, at distance t from one of the possible transmitted code words and (8-1-83) holds with equality. For such a code, all error

patterns of weight less than or equal to t are corrected by the optimum (minimum distance) decoder. On the other hand, any error pattern of weight $t + 1$ or greater cannot be corrected. Consequently, the expression for the error probability given in (8-1-82) holds with equality. The Golay (23, 12) code, having $d_{\min} = 7$ and $t = 3$, is a perfect code. The Hamming codes, which have the parameters $n = 2^{n-k} - 1$, $d_{\min} = 3$, and $t = 1$, are also perfect codes. These two nontrivial codes and the trivial code consisting of two code words of odd length n and $d_{\min} = n$ are the only perfect binary block codes. These codes are optimum on the BSC in the sense that they result in a minimum error probability among all codes having the same block length and the same number of information bits.

The optimality property defined above also holds for quasiperfect codes. A quasiperfect code is characterized by the property that all spheres of Hamming radius t around the M possible transmitted code words are disjoint and every received code word is at most at distance $t + 1$ from one of the possible transmitted code words. For such a code, all error patterns of weight less than or equal to t and some error patterns of weight $t + 1$ are correctable, but any error pattern of weight $t + 2$ or greater leads to incorrect decoding of the code word. Clearly, (8-1-82) is an upper bound on the error probability and

$$P_M \geq \sum_{m=t+2}^n P(m, n) \quad (8-1-84)$$

is a lower bound.

A more precise measure of the performance for quasiperfect codes can be obtained by making use of the inequality in (8-1-83). That is, the total number of code words outside the 2^k spheres of radius t is

$$N_{t+1} = 2^n - 2^k \sum_{i=0}^t \binom{n}{i}$$

If these code words are equally subdivided into 2^k sets and each set is associated with one of the 2^k spheres then each sphere is enlarged by the addition of

$$\beta_{t+1} = 2^{n-k} - \sum_{i=0}^t \binom{n}{i} \quad (8-1-85)$$

code words having distance $t + 1$ from the transmitted code word. Consequently, of the $\binom{n}{t+1}$ error patterns of distance $t + 1$ from each code word, we can correct β_{t+1} error patterns. Thus, the error probability for decoding the quasiperfect code may be expressed as

$$P_M = \sum_{m=t+2}^n P(m, n) + \left[\binom{n}{t+1} - \beta_{t+1} \right] p^{t+1} (1-p)^{n-t-1} \quad (8-1-86)$$

There are many known quasiperfect codes, although they do not exist for

all choices of n and k . Since such codes are optimum for the binary symmetric channel, any (n, k) linear block code must have an error probability that is at least as large as (8-1-86). Consequently, (8-1-86) is a lower bound on the probability of error for any (n, k) linear block code, where t is the largest integer such that $\beta_{t+1} \geq 0$.

Another pair of upper and lower bounds is obtained by considering two code words that differ by the minimum distance. First, we note that P_M cannot be less than the probability of erroneously decoding the transmitted code word as its nearest neighbor, which is at distance d_{\min} from the transmitted code word. That is,

$$P_M \geq \sum_{m=\lfloor d_{\min}/2 \rfloor + 1}^{d_{\min}} \binom{d_{\min}}{m} p^m (1-p)^{d_{\min}-m} \quad (8-1-87)$$

On the other hand, P_M cannot be greater than $M - 1$ times the probability of erroneously decoding the transmitted code word as its nearest neighbor, which is at distance d_{\min} from the transmitted code word. That is a union bound, which is expressed as

$$P_M \leq (M - 1) \sum_{m=\lfloor d_{\min}/2 \rfloor + 1}^{d_{\min}} \binom{d_{\min}}{m} p^m (1-p)^{d_{\min}-m} \quad (8-1-88)$$

When M is large, the lower bound in (8-1-87) and the upper bound in (8-1-88) are very loose.

A tight upper bound on P_M can be obtained by applying the Chernoff bound presented earlier in Section 2-1-6. We assume again that the all-zero code was transmitted. In comparing the received code word to the all-zero code word and to a code word of weight w_m , the probability of a decoding error, obtained from the Chernoff bound (Problem 8-22), is upper-bounded by the expression

$$P_2(w_m) \leq [4p(1-p)]^{w_m/2} \quad (8-1-89)$$

The union of these binary decisions yields the upper bound

$$P_M \leq \sum_{m=2}^M [4p(1-p)]^{w_m/2} \quad (8-1-90)$$

A simpler version of (8-1-90) is obtained if we employ d_{\min} in place of the weight distribution. That is,

$$P_M \leq (M - 1)[4p(1-p)]^{d_{\min}/2} \quad (8-1-91)$$

Of course (8-1-90) is a tighter upper bound than (8-1-91).

In Section 8-1-6, we compare the various bounds given above for a specific code, namely, the Golay (23, 12) code. In addition, we compare the error rate performance of hard-decision and soft-decision decoding.

8-1-6 Comparison of Performance between Hard-Decision and Soft-Decision Decoding

It is both interesting and instructive to compare the bounds on the error rate performance of linear block codes for soft-decision decoding and hard-decision decoding on an AWGN channel. For illustrative purposes, we shall use the Golay (23, 12) code, which has the relatively simple weight distribution given in Table 8-1-1. As stated previously, this code has a minimum distance $d_{\min} = 7$.

First we compute and compare the bounds on the error probability for hard-decision decoding. Since the Golay (23, 12) code is a perfect code, the exact error probability for hard-decision decoding is

$$\begin{aligned} P_M &= \sum_{m=4}^{23} \binom{23}{m} p^m (1-p)^{23-m} \\ &= 1 - \sum_{m=0}^3 \binom{23}{m} p^m (1-p)^{23-m} \end{aligned} \quad (8-1-92)$$

where p is the probability of a binary digit error for the binary symmetric channel. Binary (or four-phase) coherent PSK is assumed to be the modulation/demodulation technique for the transmission and reception of the binary digits contained in each code word. Thus, the appropriate expression for p is given by (8-1-73). In addition to the exact error probability given by (8-1-92), we have the lower bound given by (8-1-87) and the three upper bounds given by (8-1-88), (8-1-90), and (8-1-91).

Numerical results obtained from these bounds are compared with the exact error probability in Fig. 8-1-12. We observe that the lower bound is very loose.

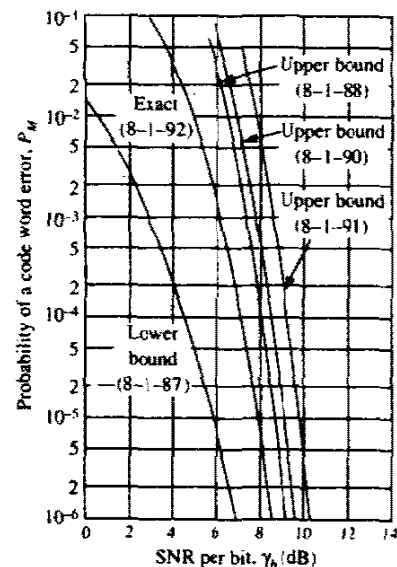


FIGURE 8-1-12 Comparison of bounds with exact error probability for hard-decision decoding of Golay (23, 12) code.

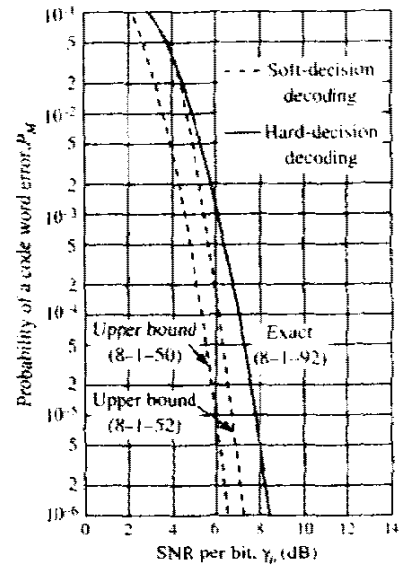


FIGURE 8-1-13 Comparison of soft-decision decoding with hard-decision decoding for the Golay (23, 12) code.

At $P_M = 10^{-5}$, the lower bound is off by approximately 2 dB from the exact error probability. At $P_M = 10^{-2}$, the difference increases to approximately 4 dB. Of the three upper bounds, the one given by (8-1-88) is the tightest; it differs by less than 1 dB from the exact error probability at $P_M = 10^{-5}$. The Chernoff bound in (8-1-90), which employs the weight distribution, is also relatively tight. Finally, the Chernoff bound that employs only the minimum distance of the code is the poorest of the three. At $P_M = 10^{-5}$, it differs from the exact error probability by approximately 2 dB. All three upper bounds are very loose for error rates above $P_M = 10^{-2}$.

It is also interesting to compare the performance between soft- and hard-decision decoding. For this comparison, we use the upper bounds on the error probability for soft-decision decoding given by (8-1-52) and the exact error probability for hard-decision decoding given by (8-1-92). Figure 8-1-13 illustrates these performance characteristics. We observe that the two bounds for soft-decision decoding differ by approximately 0.5 dB at $P_M = 10^{-6}$ and by approximately 1 dB at $P_M = 10^{-2}$. We also observe that the difference in performance between hard- and soft-decision decoding is approximately 2 dB in the range $10^{-2} < P_M < 10^{-6}$. In the range $P_M > 10^{-2}$, the curve of the error probability for hard-decision decoding crosses the curves for the bounds. This behavior indicates that the bounds for soft-decision decoding are loose when $P_M > 10^{-2}$.

The 2 dB difference between hard- and soft-decision decoding is a characteristic that applies not only to the Golay code, but is a fundamental result that applies in general to coded digital communications over the AWGN channel. This result is derived below by computing the capacity of the AWGN channel with hard- and soft-decision decoding.

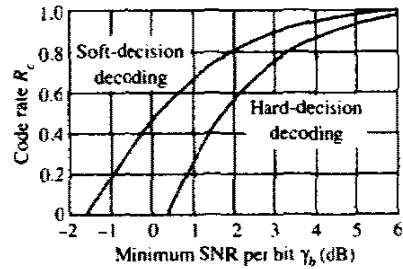


FIGURE 8-1-14 Code rate as a function of the minimum SNR per bit for soft- and hard-decision decoding.

The channel capacity of the BSC in bits per code symbol, derived in Section 7-1-2, is

$$C = 1 + p \log_2 p + (1 - p) \log_2 (1 - p) \tag{8-1-93}$$

where the probability of a bit error for binary, coherent PSK on an AWGN channel is given by (8-1-73). Suppose we use (8-1-73) for p , let $C = R_c$ in (8-1-93), and then determine the value of γ_b that satisfies this equation. The result is shown in Fig. 8-1-14 as a graph of R_c versus γ_b . For example, suppose that we are interested in using a code with rate $R_c = \frac{1}{2}$. For this code rate, note that the minimum SNR per bit required to achieve capacity with hard-decision decoding is approximately 1.6 dB.

What is the limit on the minimum SNR as the code rate approaches zero? For small values of R_c , the probability p can be approximated as

$$p \approx \frac{1}{2} - \sqrt{\gamma_b R_c / \pi} \tag{8-1-94}$$

When the expression for p is substituted into (8-1-93) and the logarithms in (8-1-93) are approximated by

$$\log_2 (1 + x) \approx (x - \frac{1}{2}x^2) / \ln 2$$

the channel capacity formula reduces to

$$C = \frac{2}{\pi \ln 2} \gamma_b R_c \tag{8-1-95}$$

Now we set $C = R_c$. Thus, in the limit as R_c approaches zero, we obtain the result

$$\gamma_b = \frac{1}{2} \pi \ln 2 \quad (0.37 \text{ dB}) \tag{8-1-96}$$

The capacity of the binary-input AWGN channel with soft-decision decoding can be computed in a similar manner. The expression for the capacity in bits per code symbol, derived in Section 7-1-2, is

$$C = \frac{1}{2} \sum_{k=0}^1 \int_{-\infty}^{\infty} p(y | k) \log_2 \frac{p(y | k)}{p(y)} dy \tag{8-1-97}$$

where $p(y | k)$, $k = 0, 1$, denote the probability density functions of the demodulator output conditioned on the transmitted bit being a 0 and a 1, respectively. For the AWGN channel, we have

$$p(y | k) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(y-m_k)^2/2\sigma^2}, \quad k = 0, 1 \quad (8-1-98)$$

where $m_0 = -\sqrt{\mathcal{E}_c}$, $m_1 = \sqrt{\mathcal{E}_c}$, $\sigma^2 = \frac{1}{2}N_0$, and $\mathcal{E}_c = R_c \mathcal{E}_b$. The unconditional probability density $p(y)$ is simply one-half of the sum of $p(y | 1)$ and $p(y | 0)$. As R_c approaches zero, the expression (8-1-97) for the channel capacity can be approximated as

$$C \approx \gamma_b R_c / \ln 2 \quad (8-1-99)$$

Again, we set $C = R_c$. Thus, as $R_c \rightarrow 0$, the minimum SNR per bit to achieve capacity is

$$\gamma_b = \ln 2 \quad (-1.6 \text{ dB}) \quad (8-1-100)$$

By using (8-1-98) in (8-1-97) and setting $C = R_c$, a numerical solution can be obtained for code rates in the range $0 \leq R_c \leq 1$. The result of this solution is also shown in Fig. 8-1-14.

From the above, we observe that in the limit as R_c approaches zero, the difference in SNR γ_b between hard- and soft-decision decoding is $\frac{1}{2}\pi$, which is approximately 2 dB. On the other hand, as R_c increases toward unity, the difference in γ_b between these two decoding techniques decreases. For example, at $R_c = 0.8$, the difference is about 1.5 dB.

The curves in Fig. 8-1-14 provide more information than just the difference in performance between soft- and hard-decision decoding. These curves also specify the minimum SNR per bit that is required for a given code rate. For example, a code rate of $R_c = 0.8$ can provide arbitrarily small error probability at an SNR per bit of 2 dB, when soft-decision decoding is used. By comparison, an uncoded binary PSK requires 9.6 dB to achieve an error probability of 10^{-5} . Hence, a 7.6 dB gain is possible by employing a rate $R_c = \frac{4}{5}$ code. Unfortunately, to achieve such a large coding gain usually implies the use of an extremely long block length code, which leads to a very complex receiver. Nevertheless, the curves in Fig. 8-1-14 provide a benchmark for comparing the coding gains achieved by practically implementable codes with the ultimate limits for either soft- or hard-decision decoding.

Instead of comparing the difference between hard- and soft-decision decoding based on the channel capacity relations, we may perform similar comparisons based on the random coding rate parameters. In Chapter 7, we demonstrated that the ensemble average probability of error for randomly selected binary code words is upper-bounded as

$$\bar{P}_e < 2^{-n(R_0 - R_c)} \quad (8-1-101)$$

where $R_c = k/n$ is the code rate and the cutoff rate R_0 represents the upper