Journal of Combinatorial Theory, Series B
Volume 16, Number 1, February 1974

CONTENTS

Volume 16, Number 2, April 19

Journal of Combinatorial Theory

Series B

ACADEMIC PRESS
New York and London

Journal of Combinatorial Theory

Series B

Published bimonthly at 37 Tempelhof, Bruges, Belgium
by Academic Press, Inc., 111 Fifth Avenue, New York, N.Y. 10003.

1974: Volumes 16–17. Price per volume: $17.00 U.S.A.; $21.00 outside U.S.A., plus postage (outside U.S.A. and Canada).

Information concerning personal subscription rates may be obtained by writing to the Publisher.

All correspondence and subscription orders should be addressed to the office of the Publishers at 111 Fifth Avenue, New York, N.Y. 10003.

Send notices of change of address to the office of the Publishers at least 6–8 weeks in advance. Please include both old and new addresses.

Application to mail at second class rates is pending at Jamaica, N.Y. 11431.

Printed in Bruges, Belgium, by the St. Catherine Press, Ltd.

JOURNAL OF COMBINATORIAL THEORY, Series B

Volume 16, Number 2, April 1974

JOURNAL OF COMBINATORIAL THEORY (B) 16, 124–133 (1974)

Construction of Quartic Graphs

S. TOIDA

Department of Systems Design, University of Waterloo, Waterloo, Ontario, Canada

Communicated by W. T. Tutte

Received May 2, 1972

It is shown that all quartic graphs can be constructed successively from a K_5 by applying two types of operations called *H*-type and *V*-type expansions.
It is also shown that the two types of operations are necessary to successively construct a regular graph of an even degree from a complete graph of the same degree.

1. INTRODUCTION

Properties of cubic graphs have been investigated by a number of people. Among them is Johnson's work [1] (see also Ore [2]) on construction and reduction of cubic graphs. According to Johnson, a cubic graph can be reduced to another cubic graph with a smaller number of vertices by an operation called *H*-reduction. Also a cubic graph can be expanded to another cubic graph with a larger number of vertices. Recently Toida [3] modified Johnson's results to find a method of constructing a planar cubic graph from another planar cubic graph with a smaller number of vertices. These works suggest the problems of how to reduce and/or construct regular graphs of a general degree.

In this paper as a first step toward the generalization the problems are solved for quartic graphs, i.e., regular graphs of degree 4. Some of the results on quartic graphs are generalized for regular graphs of even degrees.

2. PRELIMINARIES

We list symbols which are used often in this paper:

G_{2n}^m is a simple regular undirected graph of degree $2n$ with m vertices where $n \geqslant 2$.

If u and v are vertices, then uv is the edge (no parallel edges are allowed) between u and v and $\{u, v\}$ is a set of vertices.

K_m is a complete graph with m vertices.

If G is a graph and V is a set of vertices of G, then $G - V$ is the graph obtained from G by removing all the vertices of V and the edges connected to them.

3. REDUCTION OF QUARTIC GRAPHS

Given a G_{2n}^m, there are a number of ways of reducing it to a G_{2n}^k where $k < m$. Let us consider two among them:

(1) *V-type reduction.* Let v be a vertex of G_{2n}^m. Then v has $2n$ adjacent vertices. Let $v_1, v_2, \ldots, v_{2n}$ be the adjacent vertices of v. Suppose that $v_1, v_2, \ldots, v_{2n}$ can be paired such that each of the pairs is not connected by an edge in G_{2n}^m. For example $\{v_1, v_2\}, \{v_3, v_4\}, \ldots, \{v_{2i-1}, v_{2i}\}, \ldots, \{v_{2n-1}, v_{2n}\}$ are such pairs if there are no edges $v_{2i-1}v_{2i}$ in G_{2n}^m where $i = 1, 2, \ldots, n$. Then we can produce G_{2n}^{m-1} from G_{2n}^m by removing the vertex v and its edges and by adding an edge between each of the pairs of the vertices. In the example $(G_{2n}^m - \{v\}) \cup (\bigcup_{i=1}^n \{v_{2i-1}v_{2i}\})$ is a desired G_{2n}^{m-1}. We call this type of reduction a *V-type reduction.*

(2) *H-type reduction.* Let u and v be a pair of vertices of G_{2n}^m connected by an edge. Then there are $2n$ vertices connected to u and $2n$ vertices connected to v.

Let $u, v_1, v_2, \ldots, v_{2n-1}$ be the $2n$ vertices connected to v and let $v, u_1, u_2, \ldots, u_{2n-1}$ be the $2n$ vertices connected to u. Some of the u_i's may be identical to some of the v_j's, where $i, j = 1, 2, \ldots, 2n - 1$. Since a G_{2n}^m is simple, no two of u_i's (or v_j's) can be identical. Hence any u_i cannot be identical to more than one v_j.

Suppose that $u_1, u_2, \ldots, u_{2n-1}, v_1, v_2, \ldots, v_{2n-1}$ can be paired such that

(1) each pair consists of two distinct vertices,

(2) the vertices in any pair are not connected by an edge of G_{2n}^m,

(3) each of u_i's (v_i's) appears once and only once in the pairs unless it is identical to a v_j (a u_j). If a u_k is identical to a v_l then u_k is paired with two distinct vertices to form two pairs.

Suppose such a pairing is possible, then we can produce a G_{2n}^{m-2} from G_{2n}^m by removing the vertices u and v and their edges and by adding an edge between the vertices of each pair.

For example, suppose that, in a G_{2n}^{m}, v, u_1, u_2, and u_3 are adjacent to u, v_1, v_2, and v_3 are adjacent to v, u_1 is identical to v, u_1 is identical to v_1, all other vertices are distinct, edges u_2v_2 and u_3v_3 exist, and no other edges exist among u's and v's. Then clearly (u_1, u_2), (u_1, v_3), and (u_3, v_2) is a pairing which satisfies the above-mentioned (1)–(3). Hence we can produce a G_{2n}^{m-2} from G_{2n}^{m} by removing vertices u and v and their edges and by adding edges u_1u_2, u_1v_3, and u_3v_2, that is,

$$(G_{2n}^{m} - \{u, v\}) \cup \{u_1u_2, u_1v_3, u_3v_2\}$$

is a G_{2n}^{m-2}.

We call this type of reduction an *H-type reduction*.

We say G_{2n}^{m} is *H-(or V)-irreducible* if it cannot be reduced to a G_{2n}^{m-2} (or G_{2n}^{m-1}) by an *H-* (or *V*)-type reduction.

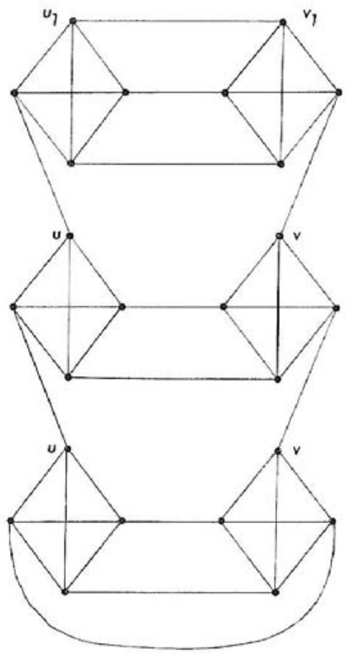

FIGURE 1

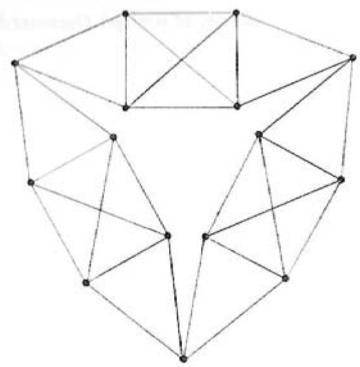

FIGURE 2

The graph of Figure 1 is *V*-irreducible and the graph of Figure 2 is *H*-irreducible. It can easily be shown that there are infinitely many *V*-irreducible and *H*-irreducible graphs among simple regular graphs of even degrees.

Thus an *H*-type or a *V*-type reduction alone cannot always be used to produce a regular graph of an even degree from a given regular graph of the same degree. A natural question then is whether one can always apply at least one of the two types of reduction to the given graph. Though we do not say anything definite on general regular graphs, we show in the following that this is the case for connected quartic graphs except K_5. That is, any connected quartic graph can be reduced to K_5 by applying *H*-type and/or *V*-type reductions.

We first investigate the cases in which a vertex of a quartic graph cannot be eliminated by a *V*-type reduction.

DEFINITION 1. A *star* consists of four vertices and three edges such that one of the vertices is connected to each of the other three vertices by an edge.

A *delta* is a complete graph with three vertices plus an isolated vertex.

In the following, a "graph" means a simple regular graph of degree 4.

DEFINITION 2. Let v be a vertex of a graph G.

If there is a subgraph in G which is a star (a delta) and the four vertices adjacent to v in G are in the subgraph we say that v *is connected to the star* (*the delta*).

LEMMA 1. *If a vertex, call it v, of a graph G cannot be removed by a V-type reduction, then v is connected to either a star or a delta.*

Proof. Suppose that v is connected to vertices w, x, y, and z and that these form neither a star nor a delta in G. Since v cannot be removed by a V-type reduction, at least one of each of the pairs of edges $\{wx, yz\}$, $\{wy, xz\}$, and (wz, xy) is in G. Without loss of generality assume that wx and wy are edges of G. Since, w, x, y, and z do not form a delta, the edge xy is not in G. Furthermore, since w, x, y, and z do not form a star either the edge wz is not in G. But one of the edges xy and wz must be in G. This is a contradiction. Thus we have the lemma.

The following theorem is one of the main results of this paper.

DEFINITION 3. A B-graph is a graph obtained by connecting two K_4's by an edge. See Figure 3.

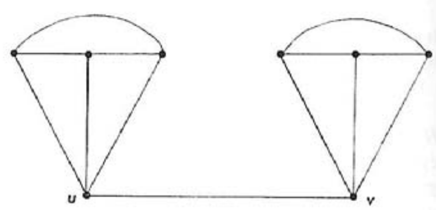

FIGURE 3

THEOREM 1. *If a connected graph G is V-irreducible, then either G is K_5 or G contains a B-graph as a subgraph.*

Proof. Let v be a vertex of G. Since v cannot be removed from G by a V-type reduction, v is connected to either a delta or a star.

Case 1. *v is connected to a delta.* Let w be the isolated vertex of the delta to which v is connected. Since G is V-irreducible, the vertex w is also connected to either a delta or a star.

(1) *w is connected to a delta.* Consider the following four cases:

(a) w is adjacent to v and to no other vertices of the delta to which v is connected.

(b) w is adjacent to v and to one of the vertices of the delta to which v is connected.

(c) w is adjacent to v and to two of the vertices of the delta to which v is connected.

(d) w is adjacent to v and to three of the vertices of the delta to which v is connected.

In (a) there is obviously a B-graph. Case (b) cannot happen. Case (d) clearly gives us a K_5 as G.

In case (c) let p be the fourth vertex adjacent to w. Since p cannot be in a star, p must be connected to a delta. If p is adjacent to one of the vertices of the delta to which v is connected, then p can be removed from G by a [illegible]-type reduction. Hence p is not adjacent to any of the vertices of the delta [illegible] w. Thus G contains a B-graph in case (c).

Hence the theorem is true in this case.

(2) *w is connected to a star.* In this case we show that G is a K_5.

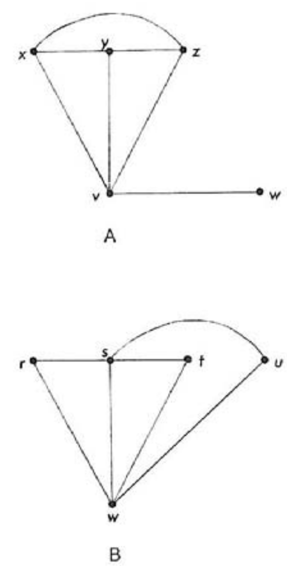

FIGURE 4

First we name the vertices of a star and a delta as in Figure 4 where v and w in Figure 4A are the v and the w of G. Since w is connected to a star, w in Figure 4A must be identical to w in Figure 4B and v must be identical to one of r, s, t, and u of Figure 4B.

DOCKET ALARM

Explore Litigation Insights

Docket Alarm provides insights to develop a more informed litigation strategy and the peace of mind of knowing you're on top of things.

Real-Time Litigation Alerts

Keep your litigation team up-to-date with **real-time alerts** and advanced team management tools built for the enterprise, all while greatly reducing PACER spend.

Our comprehensive service means we can handle Federal, State, and Administrative courts across the country.

Advanced Docket Research

With over 230 million records, Docket Alarm's cloud-native docket research platform finds what other services can't. Coverage includes Federal, State, plus PTAB, TTAB, ITC and NLRB decisions, all in one place.

Identify arguments that have been successful in the past with full text, pinpoint searching. Link to case law cited within any court document via Fastcase.

Analytics At Your Fingertips

Learn what happened the last time a particular judge, opposing counsel or company faced cases similar to yours.

Advanced out-of-the-box PTAB and TTAB analytics are always at your fingertips.

API

Docket Alarm offers a powerful API (application programming interface) to developers that want to integrate case filings into their apps.

LAW FIRMS

Build custom dashboards for your attorneys and clients with live data direct from the court.

Automate many repetitive legal tasks like conflict checks, document management, and marketing.

FINANCIAL INSTITUTIONS

Litigation and bankruptcy checks for companies and debtors.

E-DISCOVERY AND LEGAL VENDORS

Sync your system to PACER to automate legal marketing.

WHAT WILL YOU BUILD? | sales@docketalarm.com | 1-866-77-FASTCASE

fastcase® Smarter legal research.