“Evolution” by vertex of even-order regular graphs

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In graph theory circles, it is generally interesting to examine a structurally similar problem, but from a different point of view, to see alternate forms of graphs under certain transformations.

For example, Γ_n, the n-vertex complete graph, can be constructed from Γ_{n-1}, the complete graph on n-1 vertices, by adding a vertex and connecting it to each vertex of Γ_{n-1}. The same transformation on any other regular graph of other degree does not achieve the goal. Pal Érdős and Alfréd Rényi addressed graph evolution in [3] from another perspective.

In the present work we are only concerned with undirected, even-order regular graphs.

In the first section of this paper we define the evolutionary transformation and prove several related theorems. The second and third sections concern graphs for which the transformation cannot be applied.

Definition 1. Let $\Gamma = (P,E)$ be an n-vertex simple graph and $p_i \in P$, $p_j \in P$ be neighboring vertices, and p be a vertex which is a neighbor to neither p_i nor p_j. (P and E represent the sets of vertices and edges of the graph Γ.)

We take the EC transformation on vertex p to be the following:

(1) $\quad EC: E \rightarrow E'$

where $\quad E' = E \setminus \{(p_i p_j)\} \cup \{(pp_i), (pp_j)\}$

Herein we designate the set of all even regular graphs using as **PR** and the sets of all 2, 4, ..., k (k even) order regular graphs as $\mathbf{PR_2}$, $\mathbf{PR_4}$, ..., $\mathbf{PR_k}$. Then the following must be the case:

1. $\mathbf{PR} = \mathbf{PR_2} \cup \mathbf{PR_4} \cup \ldots \cup \mathbf{PR_k} \cup \ldots$
2. $\forall i \neq j$, (i, j even), $\mathbf{PR_i} \cap \mathbf{PR_j} = \emptyset$
3. $\forall \Gamma_{n,k} \in \mathbf{PR_k}$, $\exists!\, \mathbf{PR_k}$: $\Gamma_{n,k} \in \mathbf{PR_k}$; that is, the set **PR** has a classification.

Definition 2. Let $\Gamma_{n,k} \in \mathbf{PR_k}$ be an n-vertex, k-regular graph. Further suppose $\Gamma_{n,k} = (P,E)$ and $\Gamma_{n+1} = (P \cup \{p\}, E')$, where $p \notin P$.

We take the ET transformation on vertex p to be the k/2 EC transformation on vertex p, that is, let (p_1p_2), (p_3p_4), ..., $(p_{k-1}p_k)$ be edges in $\Gamma_{n,k}$, then

(2) $\quad ET: \Gamma_{n,k} \rightarrow \Gamma_n$

where $E' = E \setminus \{ (p_1p_2), (p_3p_4), \ldots, (p_{k-1}p_k) \} \cup \{ (pp_1),(pp_2), \ldots, (pp_k) \}$

Theorem 1. Let k be an even number. The ET transformation can be used for every $\Gamma_{n,k} \in \mathbf{PR_k}$.

PROOF. It is sufficient if we observe that in every $\Gamma_{n,k}$ there exist k/2 independent edges. For the proof we use the following theorem from G. A. Dirac [2]:

“If in a simple graph every vertex has degree of at least r (r > 1), then there exists a cycle in the graph of length at least r + 1.”

If we take K to be a cycle of length k+1 in $\Gamma_{n,k}$ (one certainly exists on the basis of the theorem cited), then using the method illustrated in figure 1 to select the edges (p_1p_2), (p_3p_4), ... , $(p_{k-1}p_k)$ we get a set of exactly k/2 unconnected edges.

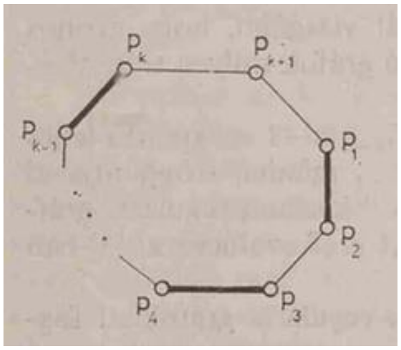

Figure 1

Definition 3. Let $\Gamma_{n,k} = (P,E) \in \mathbf{PR_k}$, $p \in P$, and we take the vertex pairs p_1p_2, p_3p_4, ..., $p_{k-1}p_k$ to be neighbors of p but pairwise not neighbors. Then we take the ET transformation to be:

(3) $ET^{-1}: \Gamma_{n,k} \rightarrow \Gamma_{n-1} = (P', E')$, $P' = P \setminus \{p\}$

$E' = E \setminus \{ (pp_1), (pp_2), \ldots, (pp_k) \cup \{ (p_1p_2),(p_3p_4), \ldots,(p_{k-1}p_k) \}$

Definition 4. Let $\Gamma_{n,k} = (P, E)$ be a graph on n vertices, $p \in P$, and q_1, q_2, ... , q_r be neighbors of p in Γ_n. Then we take the subgraph of Γ_n generated by p (designate this $G_p = (P', E')$), and it turns out this is the subgraph where $P' = \{q_1, q_2, ..., q_r\}$ and $(q_i, q_j) \in E' \Leftrightarrow (q_i, q_j) \in E$.

Definition 5. Let $\Gamma_n \in \mathbf{PR}$ and $p \in \Gamma_n$. Then the vertex p is T-characterized if the complement of the subgraph on Γ_n generated by it contains a 1-factor.

Theorem 2. The ET^{-1} transformation can be applied to a graph $\Gamma_{n,k} = (P,K) \in \mathbf{PR_k}$ if and only if there exists a T-characterized vertex $p \in \Gamma_{n,k}$.

PROOF. Necessity: From the definition of ET^{-1} it follows that there exists $p \in \Gamma_{n,k}$ from which we can select neighbors p_1p_2, p_3p_4, ... , $p_{k-1}p_k$ which are pairwise not neighbors. In the complement of the subgraph on Γ_n constructed on p_1, p_2, ... , p_k they will be neighbors, and they comprise a 1-factor, as any two vertex pairs have no common vertex. This is precisely what it means for the vertex p to be T-characterized.

Sufficiency: The proof in this direction is the exactly the reverse of the above necessity proof.

Theorem 3. For every even $k \geq 4$ there is a graph $\Gamma_{n,k} \in \mathbf{PR_k}$ which has no vertex p which is T-characterized.

Equivalent formulation: For every even $k \geq 4$ there exists a graph $\Gamma_{n,k} \in \mathbf{PR_k}$ for which no graph $\Gamma_{n-1,k} \in \mathbf{PR_k}$ can be constructed with the ET transformation.

PROOF. Let k be an arbitrary even number, $k \geq 4$. Let us examine two complete graphs on k+1 vertices $\Gamma^1_{k+1,k} = (P_1, E_2)$ and $\Gamma^2_{k+1,k} = (P_2, E_2)$. Let $(p_{i1}, p_{j1}) \in E_1$ and $(p_{i2}, p_{j2}) \in E$. Then we construct the following graph $\Gamma_{n,k} = (P , E)$

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(4) $P = P_1 \cup P_2$

(5) $E = E_1 \cup E_2 \setminus \{ (p_{i1}, p_{j1}), (p_{i2}, p_{j2}) \} \cup \{ (p_{i1}, p_{i2}), (p_{j1}, p_{j2}) \}\}$

The graph $\Gamma_{n,k}$ where for k = 4 is illustrated in figure 2.

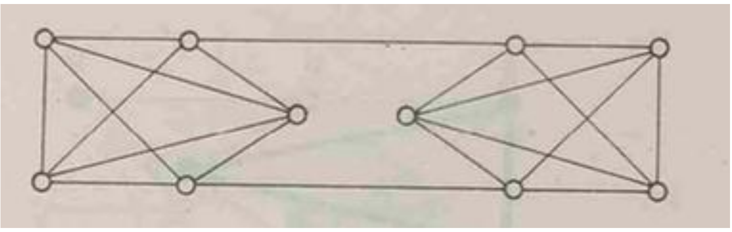

Figure 2

We will demonstrate that $\Gamma_{\mathbf{n,k}}$ constructed in this way has does not have a T-characterized vertex.

As every graph $\Gamma_{\mathbf{n,k}}$ constructed in this way is symmetric, for the proof it is enough to demonstrate for each vertex of the subgraph (for example $\Gamma^{1}_{k+1,k}$). Here we can differentiate between two cases of selected pairs of vertices:

a) p_{i1}, p_{j1}
b) any other pair

For the case in a), we take $G_1 = (A_1, B_1)$ to be the subgraph constructed on neighbors of p_{i1}. Then:

(6) $A_1 = P_1 \cup \{p_{i2}\} \setminus \{p_{i1}\}$

(7) $B_1 = E_1 \setminus \{ (p_{i2}p_r) \mid \forall p_r \in P_1 \setminus \{p_{i1}\} \}$

So the complement $\overline{G}_1 = (A'_1, B'_1)$ is therefore the following:

(8) $A'_1 = A_1$

(9) $B'_1 = E_{\mathbf{1}} \setminus \{ (p_{i2}p_s) \mid \forall p_s \in P_1 \setminus\{p_{i1}\} \}$

So $\overline{G}_1$ does not contain a 1-factor, that is, p is not T-characterized (and similarly $p_{\mathbf{i2}}, p_{\mathbf{j1}}, \ldots , p_{\mathbf{j2}}$). For the case in b), let $p \in P_{\mathbf{1}}$, $p \neq p_{\mathbf{i1}} \neq p_{\mathbf{j1}}$, and $G_{\mathbf{2}} = (A_{\mathbf{2}}, B_{\mathbf{2}})$ be the subgraph constructed on the neighbors of p:

(10) $A_{\mathbf{2}} = P_{\mathbf{1}} \setminus \{p\}$

(11) $B_{\mathbf{2}} = E_{\mathbf{2}} \setminus \{ (p_{\mathbf{i1}}p_{\mathbf{j1}}), (pp_{\mathbf{r}}) \mid \forall p_{\mathbf{r}} \in P_{\mathbf{1}} \setminus \{ p \} \}$

Then $\overline{G}_2 = (A', B')$ will be the following:

(12) $A'_{\mathbf{2}} = A_{\mathbf{2}}$

(13) $B'_{\mathbf{2}} = \{ (p_{\mathbf{i1}}p_{\mathbf{j1}}) \}$

So $\overline{G}_2$ does not contain a 1-factor, that is, no vertex in case b) is T-characterized. And so the 2 theorem is proved.

In figure 3/a-d the graphs G_1, $\overline{G}_1$, G_2, $\overline{G}_2$ are illustrated for k = 4. (The bolded edges and vertices belong to the corresponding graphs.)

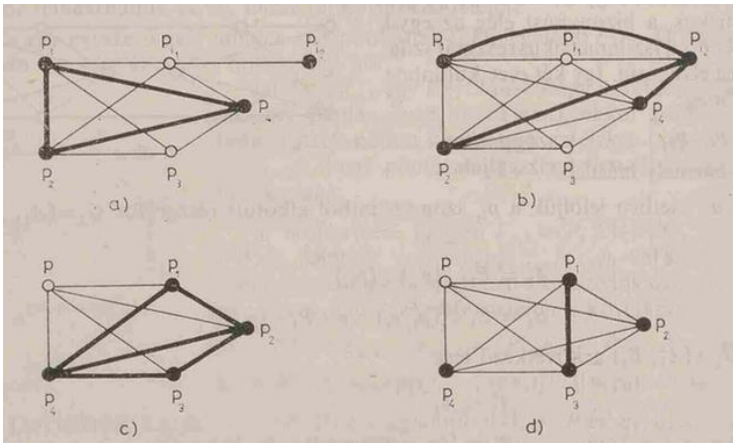

Figure 3

6. Definition. We define a Vertex Prime (VP) graph as a graph in **PR** which contains T-characterized vertex.

Then Theorem 3 can be formulated as follows: Each of the classes of graphs $\mathbf{PR_4}$, $\mathbf{PR_6}$, ... , $\mathbf{PR_k}$, ... contains a VP graph.

In the following we examine the cases under which a vertex is not T-characterized. We take advantage of the following well-known result [1].

Lemma 1. A complete graph Γ_n (where n is even) can be expanded into n-1 one-degree factors.

Theorem 4. Let $\Gamma_{n,k} \in \mathbf{PR}_k$ and $p \in \Gamma_{n,k} = (P, E)$. If p is not T-characterized, then there exist at least k-1 cycles of length 3 which contain p.

PROOF. If p is not T-characterized, then the complement of the subgraph of $\Gamma_{\mathbf{n,k}}$ generated by it (G = (A,B)) does not contain 1-factor. By Lemma 1, $\overline{G}$ contains n-1 edge-independent 1-factors, and in G from these factors an edge must exist such that in $\overline{G}$ they are not a 1-factor. So the number of edges in G is at least k-1. As it is the case in G that every edge endpoint is a neighbor of p, a 3-cycle containing p is thus constructed. And hence the theorem is proved.

For k = 4, the following illustrations in figure 4/a-h depict cases where the vertex p is not T-characterized (bold edges indicate $\overline{G}$).

Lemma 2. Let $\Gamma_{\mathbf{n}}$ be an n-vertex complete graph (n an even number) and we denote the number of all 1-factors of $\Gamma_{\mathbf{n}}$ as F. Then

(14) $$F_n = \frac{\binom{n}{2}\binom{n-2}{2}\dots\binom{2}{2}}{\binom{n}{2}!}$$

PROOF. It is easy to see that the assertion is true for the case n = 2. Suppose it also holds for any

$k \leq n - 2$, that is:

(15) $$F_k = \frac{\binom{k}{2}\binom{k-2}{2}\cdots\binom{2}{2}}{\binom{k}{2}!}$$

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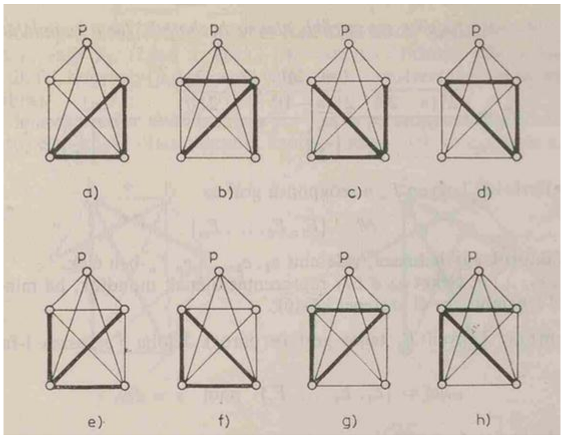

Figure 4

Then Γ_k has two vertices beyond which k+1 new vertex pairs can be selected (we don't take note of the ordering of the two points), each of which has an F_k numbered 1-factor which each share exactly one edge, so

(16) $$F_{k+2} = (k + 1) \cdot F_k$$

Recalling that

(17) $$\frac{\binom{k+2}{2}}{\frac{k+2}{2}} = \frac{2*(k+2)!}{2!\,k!(k+2)} = k + 1$$

So thus

(18) $$F_{k+2} = \frac{\binom{k+2}{2}}{\frac{k+2}{2}} * \frac{\binom{k}{2}\binom{k-2}{2}\cdots\binom{2}{2}}{\frac{k}{2}!} = \frac{\binom{k+2}{2}\binom{k}{2}\cdots\binom{2}{2}}{\frac{k+2}{2}!}$$

This concludes the proof of the theorem.

Corollary.

4. In the context of (16) we can write F_n recursively as follows:

(19) $$F_n = (n - 1) \cdot F_{n-2}$$

2. The proof of Lemma 2 shows that if e is an edge contained in an E 1-factor of Γ_n, then exactly F_{n-2} such

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