DIGITAL SIGNAL PROCESSING

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To Phyllis and Dorothy

$$) = \sum_{m=-(N-1)}^{N-1} c_{xx}(m)e^{-j\omega m} \quad (11.24)$$

of the real finite-length sequence $x(n)$, $0 \le n \le$

$$X(e^{j\omega}) = \sum_{n=0}^{N-1} x(n)e^{-j\omega n}$$

oblem 1 of this chapter).

$$_N(\omega) = \frac{1}{N}|X(e^{j\omega})|^2 \quad (11.25)$$

) is often called the *periodogram.*
est to determine the bias and variance of the of the power spectrum. The expected value of

$$] = \sum_{m=-(N-1)}^{N-1} E[c_{xx}(m)]e^{-j\omega m} \quad (11.26)$$

r a zero mean process

$$\frac{N-|m|}{N}\phi_{xx}(m), \qquad |m| < N$$

$$\sum_{m=-(N-1)}^{N-1}\left(\frac{N-|m|}{N}\right)\phi_{xx}(m)e^{-j\omega m} \quad (11.27)$$

nits of summation and the factor $(N-|m|)/N$, Fourier transform of $\phi_{xx}(m)$, and therefore the nate of the power spectrum, $P_{xx}(\omega)$.
e Fourier transform of the estimate $c'_{xx}(m)$; i.e.,

$$= \sum_{m=-(N-1)}^{N-1} c'_{xx}(m)e^{-j\omega m} \quad (11.28)$$

is

$$] = \sum_{m=-(N-1)}^{N-1} E[c'_{xx}(m)]e^{-j\omega m}$$
$$= \sum_{m=-(N-1)}^{N-1} \phi_{xx}(m)e^{-j\omega m} \quad (11.29)$$

windowed autocorrelation sequences. In the case of Eq. (11.27) the windo is the triangular window

$$w_B(m) = \begin{cases} \dfrac{N-|m|}{N}, & |m| < N \\ 0, & \text{otherwise} \end{cases} \quad (11.3$$

In Chapter 5 we called this the Bartlett window. For Eq. (11.29) the windo is rectangular; i.e.,

$$w_R(n) = \begin{cases} 1, & |m| < N \\ 0, & \text{otherwise} \end{cases} \quad (11.3$$

Using the concepts introduced in Chapter 5 we can see that Eqs. (11.2 and (11.29) can be interpreted in the frequency domain as the convolution

$$E[I_N(\omega)] = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_{xx}(\theta)W_B(e^{j(\omega-\theta)})\,d\theta \quad (11.3$$

and

$$E[P_N(\omega)] = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_{xx}(\theta)W_R(e^{j(\omega-\theta)})\,d\theta \quad (11.3$$

where

$$W_B(e^{j\omega}) = \frac{1}{N}\left(\frac{\sin[\omega N/2]}{\sin[\omega/2]}\right)^2 \quad (11.3$$

and

$$W_R(e^{j\omega}) = \frac{\sin[\omega(2N-1)/2]}{\sin[\omega/2]} \quad (11.3$$

are the Fourier transforms of the Bartlett and rectangular windows, respe tively.

11.3.2 *Variance of the Periodogram*

To obtain an expression for the variance of the periodogram, it is co venient to first assume that the sequence $x(n)$, $0 \le n \le N-1$, is a samp of a real, white, zero-mean process with Gaussian probability densi functions. The periodogram $I_N(\omega)$ can be expressed as

$$I_N(\omega) = \frac{1}{N}|X(e^{j\omega})|^2$$
$$= \frac{1}{N}\sum_{l=0}^{N-1}\sum_{m=0}^{N-1} x(l)x(m)e^{j\omega m}e^{-j\omega l}$$

11.3.1 Definition of the Periodogram

As an estimate of the power density spectrum let us consider the Fourier ansform of the biased autocorrelation estimate $c_{xx}(m)$. That is,

$$I_N(\omega) = \sum_{m=-(N-1)}^{N-1} c_{xx}(m)e^{-j\omega m} \tag{11.24}$$

nce the Fourier transform of the real finite-length sequence $x(n)$, $0 \leq n \leq$ $- 1$, is

$$X(e^{j\omega}) = \sum_{n=0}^{N-1} x(n)e^{-j\omega n}$$

can be shown that (see Problem 1 of this chapter).

$$I_N(\omega) = \frac{1}{N}\,|X(e^{j\omega})|^2 \tag{11.25}$$

he spectrum estimate $I_N(\omega)$ is often called the *periodogram*.

As before, it is of interest to determine the bias and variance of the eriodogram as an estimate of the power spectrum. The expected value of $_N(\omega)$ is

$$E[I_N(\omega)] = \sum_{m=-(N-1)}^{N-1} E[c_{xx}(m)]e^{-j\omega m} \tag{11.26}$$

nce we have shown that for a zero mean process

$$E[c_{xx}(m)] = \frac{N - |m|}{N}\,\phi_{xx}(m), \qquad |m| < N$$

en

$$E[I_N(\omega)] = \sum_{m=-(N-1)}^{N-1} \left(\frac{N - |m|}{N}\right)\phi_{xx}(m)e^{-j\omega m} \tag{11.27}$$

hus because of the finite limits of summation and the factor $(N - |m|)/N$, $[I_N(\omega)]$ is not equal to the Fourier transform of $\phi_{xx}(m)$, and therefore the eriodogram is a biased estimate of the power spectrum, $P_{xx}(\omega)$.

Alternatively, consider the Fourier transform of the estimate $c'_{xx}(m)$; i.e.,

$$P_N(\omega) = \sum_{m=-(N-1)}^{N-1} c'_{xx}(m)e^{-j\omega m} \tag{11.28}$$

he expected value of $P_N(\omega)$ is

$$E[P_N(\omega)] = \sum_{m=-(N-1)}^{N-1} E[c'_{xx}(m)]e^{-j\omega m}$$

(11.29)

Again, because of the finite limits of summatio $P_{xx}(\omega)$, even though $c_{xx}(m)$ is an unbiased estim

We can interpret Eqs. (11.27) and (11.29 windowed autocorrelation sequences. In the ca is the triangular window

$$w_B(m) = \begin{cases} \dfrac{N - |m|}{N}, & |n \\ 0, & \text{ot} \end{cases}$$

In Chapter 5 we called this the Bartlett window. is rectangular; i.e.,

$$w_R(n) = \begin{cases} 1, & |m| < \\ 0, & \text{other} \end{cases}$$

Using the concepts introduced in Chapter 5 and (11.29) can be interpreted in the frequency

$$E[I_N(\omega)] = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_{xx}(\theta)W_B($$

and

$$E[P_N(\omega)] = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_{xx}(\theta)W_R($$

where

$$W_B(e^{j\omega}) = \frac{1}{N}\left(\frac{\sin[\omega N/}{\sin[\omega/2}\right.$$

and

$$W_R(e^{j\omega}) = \frac{\sin[\omega(2N -}{\sin[\omega/2]}$$

are the Fourier transforms of the Bartlett and r tively.

11.3.2 Variance of the Periodogram

To obtain an expression for the variance of venient to first assume that the sequence $x(n)$, of a real, white, zero-mean process with G functions. The periodogram $I_N(\omega)$ can be expre

$$I_N(\omega) = \frac{1}{N}\,|X(e^{j\omega})|^2$$

To evaluate the covariance of $I_N(\omega)$ at two frequencies ω_1 and ω_2 we first consider

$$E[I_N(\omega_1)I_N(\omega_2)] = \frac{1}{N^2}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1}\sum_{m=0}^{N-1}\sum_{n=0}^{N-1} E[x(k)x(l)x(m)x(n)]e^{j[\omega_1(k-l)+\omega_2(m-n)]} \tag{11.36}$$

To obtain a useful result, we must simplify Eq. (11.36). In general, it is not possible to obtain a very simple result even when $x(n)$ is white, because $E[x(n)x(n+m)] = \sigma_x^2\delta(m)$ does not guarantee a simple expression for $E[x(k)x(l)x(m)x(n)]$ for all combinations of k, l, m, and n. However, in the case of a white Gaussian process, it can be shown [7] that

$$\begin{aligned} E[x(k)x(l)x(m)x(n)] = {} & E[x(k)x(l)]E[x(m)x(n)] \\ & + E[x(k)x(m)]E[x(l)x(n)] \\ & + E[x(k)x(n)]E[x(l)x(m)] \end{aligned}$$

Therefore,

$$E[x(k)x(l)x(m)x(n)] = \begin{cases} \sigma_x^4, & k = l \text{ and } m = n \\ & \text{or } k = m \text{ and } l = n \\ & \text{or } k = n \text{ and } l = m \\ 0, & \text{otherwise} \end{cases} \tag{11.37}$$

For other than Gaussian joint density functions, the result will not necessarily be so simple. However, our objective is to give a result that will lend insight into the problems of spectrum estimation rather than to give a general formula with wide validity which would be difficult to interpret. Thus, if we substitute Eq. (11.37) into Eq. (11.36), we obtain

$$E[I_N(\omega_1)I_N(\omega_2)] = \frac{\sigma_x^4}{N^2}\left\{N^2 + \sum_{m=0}^{N-1}\sum_{n=0}^{N-1} e^{j(m-n)(\omega_1+\omega_2)} + \sum_{m=0}^{N-1}\sum_{n=0}^{N-1} e^{j(n-m)(\omega_1-\omega_2)}\right\}$$

or

$$\begin{aligned} E[I_N(\omega_1)I_N(\omega_2)] = \sigma_x^4\Bigg\{1 & + \left(\frac{\sin[(\omega_1+\omega_2)N/2]}{N\sin[(\omega_1+\omega_2)/2]}\right)^2 \\ & + \left(\frac{\sin[(\omega_1-\omega_2)N/2]}{N\sin[(\omega_1-\omega_2)/2]}\right)^2\Bigg\} \end{aligned} \tag{11.38}$$

(If the signal is non-Gaussian, Eq. (11.38) contains additional terms which are proportional to $1/N$ [4, 8].) The covariance of the periodogram is

$$\text{cov}\,[I_N(\omega_1), I_N(\omega_2)] = E[I_N(\omega_1)I_N(\omega_2)] - E[I_N(\omega_1)]E[I_N(\omega_2)] \tag{11.39}$$

Since $E[I_N(\omega_1)] = E[I_N(\omega_2)] =$

cov $[I_N(\omega_1), I_N(\omega$

From Eq. (11.40) we can dra
the periodogram. The variance
frequency $\omega = \omega_1 = \omega_2$ is

var $[I_N(\omega)] =$ cov $[I_N$

Clearly, the variance of $I_N(\omega)$
infinity. Thus the periodogra
var $[I_N(\omega)]$ is of the order of σ_x^4
We also see from Eq. (11.40
$2\pi l/N$, where k and l are intege

cov $[I_N(\omega_1), I_N(\omega_2)] = \sigma_x^4\Big\{\Big(\frac{\text{s}}{N\text{ s}}$

which is equal to zero for $k \neq$
frequency by integer multiples
these uncorrelated frequency
together. It is reasonable to exp
trum should approach a consta
the signal was white. A conse
periodogram approaches a non
spectral samples with zero cova
record length becomes longer, t
gram increases. This behavior i
gram is plotted for record lengt

11.3.3 General Variance Ex

All the previous discussion
of white noise. If we consider
analysis is considerably more di
spectrum samples in this more
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shall take is heuristic; a more
Watts [5]. With the applicatio
the approximate results derive

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