DIGITAL SIGNAL PROCESSING

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11.3.1 *Definition of the Periodogram*

As an estimate of the power density spectrum let us consider the Fourier transform of the biased autocorrelation estimate $c_{xx}(m)$. That is,

$$I_N(\omega) = \sum_{m=-(N-1)}^{N-1} c_{xx}(m)e^{-j\omega m} \tag{11.24}$$

Since the Fourier transform of the real finite-length sequence $x(n)$, $0 \leq n \leq N - 1$, is

$$X(e^{j\omega}) = \sum_{n=0}^{N-1} x(n)e^{-j\omega n}$$

it can be shown that (see Problem 1 of this chapter).

$$I_N(\omega) = \frac{1}{N}|X(e^{j\omega})|^2 \tag{11.25}$$

The spectrum estimate $I_N(\omega)$ is often called the *periodogram.*

As before, it is of interest to determine the bias and variance of the periodogram as an estimate of the power spectrum. The expected value of $I_N(\omega)$ is

$$E[I_N(\omega)] = \sum_{m=-(N-1)}^{N-1} E[c_{xx}(m)]e^{-j\omega m} \tag{11.26}$$

Since we have shown that for a zero mean process

$$E[c_{xx}(m)] = \frac{N-|m|}{N}\phi_{xx}(m), \qquad |m| < N$$

then

$$E[I_N(\omega)] = \sum_{m=-(N-1)}^{N-1} \left(\frac{N-|m|}{N}\right)\phi_{xx}(m)e^{-j\omega m} \tag{11.27}$$

Thus because of the finite limits of summation and the factor $(N - |m|)/N$, $E[I_N(\omega)]$ is not equal to the Fourier transform of $\phi_{xx}(m)$, and therefore the periodogram is a biased estimate of the power spectrum, $P_{xx}(\omega)$.

Alternatively, consider the Fourier transform of the estimate $c'_{xx}(m)$; i.e.,

$$P_N(\omega) = \sum_{m=-(N-1)}^{N-1} c'_{xx}(m)e^{-j\omega m} \tag{11.28}$$

The expected value of $P_N(\omega)$ is

$$\begin{aligned} E[P_N(\omega)] &= \sum_{m=-(N-1)}^{N-1} E[c'_{xx}(m)]e^{-j\omega m} \\ &= \sum_{m=-(N-1)}^{N-1} \phi_{xx}(m)e^{-j\omega m} \end{aligned} \tag{11.29}$$

Again, because of the finite limits of summation
$P_{xx}(\omega)$, even though $c_{xx}(m)$ is an unbiased estim
We can interpret Eqs. (11.27) and (11.29
windowed autocorrelation sequences. In the cas
is the triangular window

$$w_B(m) = \begin{cases} \frac{N - |m|}{N}, & |n \\ 0, & \text{ot} \end{cases}$$

In Chapter 5 we called this the Bartlett window.
is rectangular; i.e.,

$$w_R(n) = \begin{cases} 1, & |m| < \\ 0, & \text{other} \end{cases}$$

Using the concepts introduced in Chapter 5
and (11.29) can be interpreted in the frequency

$$E[I_N(\omega)] = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_{xx}(\theta) W_B($$

and

$$E[P_N(\omega)] = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_{xx}(\theta) W_R($$

where

$$W_B(e^{j\omega}) = \frac{1}{N}\left(\frac{\sin [\omega N/}{\sin [\omega/2}\right.$$

and

$$W_R(e^{j\omega}) = \frac{\sin [\omega(2N -}{\sin [\omega/2}$$

are the Fourier transforms of the Bartlett and r
tively.

11.3.2 Variance of the Periodogram

To obtain an expression for the variance of
venient to first assume that the sequence $x(n)$,
of a real, white, zero-mean process with G
functions. The periodogram $I_N(\omega)$ can be expr

$$I_N(\omega) = \frac{1}{N} |X(e^{j\omega})|^2$$

To evaluate the covariance of $I_N(\omega)$ at two frequencies ω_1 and ω_2 we first consider

$$E[I_N(\omega_1)I_N(\omega_2)] = \frac{1}{N^2}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1}\sum_{m=0}^{N-1}\sum_{n=0}^{N-1} E[x(k)x(l)x(m)x(n)]e^{j[\omega_1(k-l)+\omega_2(m-n)]} \tag{11.36}$$

To obtain a useful result, we must simplify Eq. (11.36). In general, it is not possible to obtain a very simple result even when $x(n)$ is white, because $E[x(n)x(n+m)] = \sigma_x^2\delta(m)$ does not guarantee a simple expression for $E[x(k)x(l)x(m)x(n)]$ for all combinations of k, l, m, and n. However, in the case of a white Gaussian process, it can be shown [7] that

$$\begin{aligned} E[x(k)x(l)x(m)x(n)] = {} & E[x(k)x(l)]E[x(m)x(n)] \\ & + E[x(k)x(m)]E[x(l)x(n)] \\ & + E[x(k)x(n)]E[x(l)x(m)] \end{aligned}$$

Therefore,

$$E[x(k)x(l)x(m)x(n)] = \begin{cases} \sigma_x^4, & k = l \text{ and } m = n \\ & \text{or } k = m \text{ and } l = n \\ & \text{or } k = n \text{ and } l = m \\ 0, & \text{otherwise} \end{cases} \tag{11.37}$$

For other than Gaussian joint density functions, the result will not necessarily be so simple. However, our objective is to give a result that will lend insight into the problems of spectrum estimation rather than to give a general formula with wide validity which would be difficult to interpret. Thus, if we substitute Eq. (11.37) into Eq. (11.36), we obtain

$$E[I_N(\omega_1)I_N(\omega_2)] = \frac{\sigma_x^4}{N^2}\left\{N^2 + \sum_{m=0}^{N-1}\sum_{n=0}^{N-1} e^{j(m-n)(\omega_1+\omega_2)} + \sum_{m=0}^{N-1}\sum_{n=0}^{N-1} e^{j(n-m)(\omega_1-\omega_2)}\right\}$$

or

$$\begin{aligned} E[I_N(\omega_1)I_N(\omega_2)] = \sigma_x^4\Bigg\{1 & + \left(\frac{\sin[(\omega_1+\omega_2)N/2]}{N\sin[(\omega_1+\omega_2)/2]}\right)^2 \\ & + \left(\frac{\sin[(\omega_1-\omega_2)N/2]}{N\sin[(\omega_1-\omega_2)/2]}\right)^2\Bigg\} \end{aligned} \tag{11.38}$$

(If the signal is non-Gaussian, Eq. (11.38) contains additional terms which are proportional to $1/N$ [4, 8].) The covariance of the periodogram is

$$\operatorname{cov}[I_N(\omega_1), I_N(\omega_2)] = E[I_N(\omega_1)I_N(\omega_2)] - E[I_N(\omega_1)]E[I_N(\omega_2)] \tag{11.39}$$

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