The Fourier Transform and Its Applications

Second Edition

Ronald N. Bracewell
Lewis M. Terman Professor of Electrical Engineering
Stanford University

McGraw-Hill Book Company
New York St. Louis San Francisco Auckland Bogotá Düsseldorf
Johannesburg London Madrid Mexico Montreal New Delhi
Panama Paris São Paulo Singapore Sydney Tokyo Toronto

Th
Tran
its A

A

Practicing
tors who
Professor
formative
reference
publicati
Requirin
mathema
duces th
plains it
range of
chapters
underst
ples of
cluding
the delt
ems. Ch
applica
concen
he or s
as sign
statisti
tures in
plemen
pictori
forms;
crete F

The Fourier Transform and Its Applications

7890 DODO 83

This book was set in Scotch Roman by Bi-Comp, Incorporated.
The editors were Julienne V. Brown and Michael Gardner;
the production supervisor was Dennis J. Conroy.
The drawings were done by J & R Services, Inc.
R. R. Donnelley & Sons Company was printer and binder.

Library of Congress Cataloging in Publication Data

Bracewell, Ronald Newbold, date
The Fourier transform and its applications.

(McGraw-Hill electrical and electronic engineering series)
Includes index.
1. Fourier transformations. 2. Transformations (Mathematics) 3. Harmonic analysis. I. Title.
QA403.5.B7 1978 515′.723 77-13376
ISBN 0-07-007013-X

Joseph Fourier, 21 March 1768–16 May 1830. (*By permission of the Bibliothèque Municipale de Grenoble.*)

Th
an

New Y
Joh
Pan

The customary formulas exhibiting the reversibility of the Fourier transformation are

$$F(s) = \int_{-\infty}^{\infty} f(x)e^{-i2\pi xs}\,dx \qquad t=x$$
$$f(x) = \int_{-\infty}^{\infty} F(s)e^{i2\pi xs}\,ds. \qquad f=s$$

In this form, two successive transformations are made to yield the original function. The second transformation, however, is not exactly the same as the first, and where it is necessary to distinguish between these two sorts of Fourier transform, we shall say that $F(s)$ is the minus-i transform of $f(x)$ and that $f(x)$ is the plus-i transform of $F(s)$.

Writing the two successive transformations as a repeated integral, we obtain the usual statement of Fourier's integral theorem:

$$f(x) = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(x)e^{-i2\pi xs}\,dx\right]e^{i2\pi xs}\,ds.$$

The conditions under which this is true are given in the next section, but it must be stated at once that where $f(x)$ is discontinuous the left-hand side should be replaced by $\frac{1}{2}[f(x+) + f(x-)]$, that is, by the mean of the unequal limits of $f(x)$ as x is approached from above and below.

The factor 2π appearing in the transform formulas may be lumped with s to yield the following version (system 2):

$$F(s) = \int_{-\infty}^{\infty} f(x)e^{-ixs}\,dx$$
$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)e^{ixs}\,ds.$$

And for the sake of symmetry, authors occasionally write (system 3):

$$F(s) = \frac{1}{(2\pi)^{\frac{1}{2}}}\int_{-\infty}^{\infty} f(x)e^{-ixs}\,dx$$
$$f(x) = \frac{1}{(2\pi)^{\frac{1}{2}}}\int_{-\infty}^{\infty} F(s)e^{ixs}\,ds.$$

All three versions are in common use, but here we shall keep the 2π in the exponent (system 1). If $f(x)$ and $F(s)$ are a transform pair in system 1, then $f(x)$ and $F(s/2\pi)$ are a transform pair in system 2, and $f[x/(2\pi)^{\frac{1}{2}}]$ and $F[s/(2\pi)^{\frac{1}{2}}]$ are a transform pair in system 3. An example of a transform pair in each of the three systems follows.

System 1		*System 2*		*System 3*	
$f(x)$	$F(s)$	$f(x)$	$F(s)$	$f(x)$	$F(s)$
$e^{-\pi x^2}$	$e^{-\pi s^2}$	$e^{-\pi x^2}$	$e^{-s^2/4\pi}$	$e^{-\frac{1}{2}x^2}$	$e^{-\frac{1}{2}s^2}$

An excellent notation which may be used as an alternative to $F(s)$ is $\bar{f}(s)$. Various advantages and disadvantages are found in both notations.

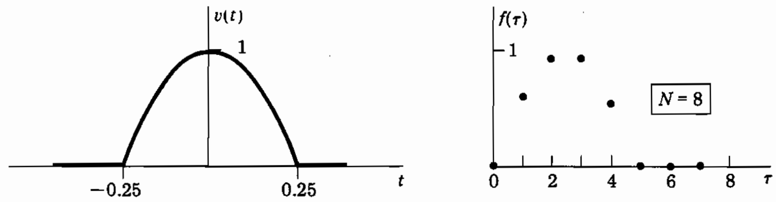

Fig. 18.2 A function of the continuous variable t and one way of representing it by eight sample values.

In what follows, $f(\tau)$ forms the point of departure. It will be noticed that no provision is made for cases where there is no starting point, as with a function such as $\exp(-t^2)$. This is in keeping with the practical character of the discrete transform, which does not contemplate data trains dating back to the indefinitely remote past. A second feature to note is that the finishing point must occur after a finite time. However, it need not come at $\tau = 5$ as in Table 18.2; one might choose to let τ run on to 15 and assign values of zero to the extra samples. This is a conscious choice that must always be made. It may be important; for example, Table 18.2 does not convey the information given in the equation preceding it—that following the half-period cosine, the voltage remains zero. The table remains silent on that point, and if it is important, the necessary number of zeros would need to be appended.

By *definition*, $f(\tau)$ possesses a discrete Fourier transform $F(\nu)$ given by

$$F(\nu) = N^{-1} \sum_{\tau=0}^{N-1} f(\tau) e^{-i2\pi(\nu/N)\tau}. \tag{1}$$

The quantity ν/N is analogous to frequency measured in cycles per sampling interval. The correspondence of symbols may be summarized as follows:

	Time	*Frequency*
Continuous case	t	f
Discrete case	τ	ν/N

The symbol ν has been chosen in the discrete case, instead of f, to emphasize that the frequency integer ν is *related* to frequency but is not the *same* as frequency f. For example, if the sampling interval is 1 second and there are eight samples ($N = 8$), then the component of frequency f will be found at $\nu = 8f$; conversely, the frequency represented by a frequency integer $\nu = 1$ will be $\frac{1}{8}$ hertz.