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Generalized-α Time Integration Solutions for Hanging Chain Dynamics

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Abstract: In this paper, we study numerically the two- and three-dimensional nonlinear dynamic response of a chain hanging under its own weight. Previous authors have employed the box method, a finite-difference scheme popular in cable dynamics problems, for this purpose. The box method has significant stability problems, however, and thus is not well suited to this highly nonlinear problem. We illustrate these stability problems and propose a new time integration procedure based on the generalized-α method. The new method exhibits superior stability properties compared to the box method and other algorithms such as backward differences and trapezoidal rule. Of four time integration methods tested, the generalized-α algorithm was the only method that produced a stable solution for the three-dimensional whirling motions of a hanging chain driven by harmonic linear horizontal motion at the top.

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Introduction

The dynamics of a chain hanging under its own weight is a classic problem in mechanics. Two of the more interesting aspects of the problem are the simultaneous presence of both high- and low-tension regimes in the chain and the unstable nature of large amplitude motions. Triantafyllou and Howell (1993) and Howell and Triantafyllou (1993) considered both of these phenomena using a combination of analytic, numerical, and experimental results. They observed that the stability of the response in a harmonically driven system is strongly dependent on the frequency and amplitude of the excitation.

The numerical model that they employed was based on a finite-difference scheme known as the box method. This method was first applied to a cable dynamics problem by Ablow and Schechter (1983). Because the box method is an implicit scheme, box method solutions for the classical cable dynamics equations are singular when the tension goes to zero anywhere on the cable. Howell and Triantafyllou (1993) removed this singularity by adding bending stiffness to the governing equations, thus providing a mechanism to propagate energy in the presence of zero tension (Burgess 1993). For small values of artificial bending stiffness this modification stabilized the numerical solution with no loss of accuracy compared to experimental results.

The box method is popular because it is second-order accurate in both space and time and is relatively easy to implement. Because the box method preserves the frequency content of the solution across all frequencies, however, it has the disadvantage of relatively poor stability in its temporal discretization. In a nonlinear problem, spurious high-frequency content can cause numerical instabilities, and thus, it is desirous that a temporal integration scheme should be numerically dissipative at high frequencies. Koh et al. (1999) addressed this shortcoming of the box method by replacing the box method's temporal integration scheme with backward differences. They preserved the box method's straightforward and easy to implement spatial discretization. Backward differences have also been used by Chatjigeorgiou and Mavrakos (1999) and Chiou and Leonard (1991) in conjunction with spatial discretizations based on collocation and direct integration, respectively. The scheme is only first-order accurate, but is very stable because it has strong numerical dissipation at high frequencies.

Another temporal integration scheme that has been used in cable dynamics applications is the generalized trapezoidal rule (Sun et al. 1994). This scheme offers controllable numerical dissipation, but is second-order accurate only in its least dissipative form. Thomas (1993) compared three historically popular algorithms from the structural dynamics community, Newmark, Houbolt, and Wilson-θ, for use in mooring dynamics problems. His conclusion was that Houbolt was the best choice of the three. Other authors, however, have noted that Houbolt has an undesirable amount of low-frequency dissipation (Chung and Hulbert 1994; Hughes 1987).

Turning to the more recent structural dynamics literature, Gobat and Grosenbaugh (2001) proposed replacing the box method's temporal integration with the generalized-α method developed for the second-order structural dynamics problem by Chung and Hulbert (1993). This algorithm has the advantages of controllable numerical dissipation, second-order accuracy, and straightforward adaptation to the first-order nonlinear cable dynamics problem. Through appropriate choices of parameters, the method can also reproduce the spectral properties of several other algo-

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rithms including the box method, backward differences, and trapezoidal rule. This latter property makes it a particularly convenient choice for the type of comparative study undertaken herein.

The analyses of the box and generalized-α methods from Gobat and Grosenbaugh (2001) are summarized below. The performance of the new algorithm is studied by comparison to analytic and experimental results for the free and forced response of the hanging chain. Throughout the analyses, comparisons are also made to trapezoidal rule and backward difference solutions.

Analysis of Box Method

The governing equations for a cable or chain can be written as a system of partial differential equations of the form (Howell 1992)

$$\mathbf{M}\frac{\partial \mathbf{Y}}{\partial t}+\mathbf{K}\frac{\partial Y}{\partial s}+\mathbf{F}(\mathbf{Y},s,t)=0 \tag{1}$$

where $\mathbf{Y}$=vector of N-dependent variables, $\mathbf{M}$ and $\mathbf{K}$ =coefficient matrices, and $\mathbf{F}$=force vector. The independent variables are s, the Lagrangian coordinate measuring length along the unstretched cable, and t, time. Howell and Triantafyllou (1993) used the box method to discretize Eq. (1). In the box method the discrete equations are written using what look like traditional backward differences in both space and time, but because the discretization is applied on the half-grid points with spatial and temporal averaging of adjacent grid points, the method is second-order accurate. The result is a four-point average centered around the half-grid point.

The stability of the box method can be analyzed by considering an equivalent linear, single degree-of-freedom system in semidiscrete form. This approach separates the spatial and temporal discretizations into distinct procedures. For each of the $n-1$ spatial half-grid points between the n nodes a set of N discrete equations is assembled. Combining these $N(n-1)$ equations with N equations describing the boundary conditions yields the semidiscrete equation of motion for all of the dependent variables at all of the nodes as (Gobat and Grosenbaugh 2001)

$$\tilde{\mathbf{M}}\dot{\mathbf{Y}}+\tilde{\mathbf{K}}\mathbf{Y}+\tilde{\mathbf{F}}=0 \tag{2}$$

The tilde over the matrices signifies that these are now discretized, assembled quantities. The single degree-of-freedom, linear, homogeneous analog of Eq. (2) is

$$\dot{y}+\omega y=0 \tag{3}$$

Applying the box method's temporal discretization to Eq. (3) yields

$$\dot{y}^i+\dot{y}^{i-1}+\omega(y^i+y^{i-1})=0 \tag{4}$$

where

$$\dot{y}^i+\dot{y}^{i-1}=2\left(\frac{y^i-y^{i-1}}{\Delta t}\right) \tag{5}$$

Rearranging Eq. (5) gives the recursion relationships

$$\dot{y}^i=2\left(\frac{y^i-y^{i-1}}{\Delta t}\right)-\dot{y}^{i-1} \tag{6}$$

$$y^i=\frac{\Delta t}{2}(\dot{y}^i+\dot{y}^{i-1})+y^{i-1} \tag{7}$$

Substituting each of the recursion relationships separately into Eq. (4), we can write equations for y^i and $\dot{y}^i$ in matrix form as

$$\begin{bmatrix} y^i \\ \dot{y}^i \end{bmatrix}=\begin{bmatrix} \dfrac{2-\omega\Delta t}{2+\omega\Delta t} & 0 \\ \dfrac{-4}{2+\omega\Delta t} & -1 \end{bmatrix}\begin{bmatrix} y^{i-1} \\ \dot{y}^{i-1} \end{bmatrix} \tag{8}$$

The 2×2 matrix on the right-hand side of Eq. (8) is the amplification matrix. Spectral radius ρ of this matrix, defined as

$$\rho=\max(|\lambda_1|,|\lambda_2|) \tag{9}$$

governs the growth or decay of the solution from one time step to the next (Hughes 1987). $\lambda_{1,2}$=eigenvalues of the amplification matrix. For $\rho\leq1$, the solution will remain steady or decay and is said to be stable. For $\rho>1$, the solution will grow and is said to be unstable. For the box method,

$$\lambda_1=\frac{2-\omega\Delta t}{2+\omega\Delta t} \tag{10}$$

$$\lambda_2=-1 \tag{11}$$

and the spectral radius is unity (and the scheme is stable) for all values of ω and Δt.

In spite of this unconditional stability, however, the box method has three significant problems. The first problem is illustrated by considering the update equation for y^i written in the form

$$y^i=\left(\frac{2-\omega\Delta t}{2+\omega\Delta t}\right)y^{i-1} \tag{12}$$

As $\omega\Delta t$ goes to infinity this becomes

$$y^i=-y^{i-1} \tag{13}$$

This is the phenomenon known as Crank–Nicholson noise, whereby the high-frequency components of the solution oscillate with every time step. A second, related, problem is that the spectral radius is constant at unity. An artifact of the spatial discretization process is that at some point the high-frequency (or equivalently, high-spatial wave-number) components of the solution are not well resolved and the numerical solution is inaccurate. For this reason it is desirous to have numerical dissipation in a scheme such that the spectral radius is less than unity for increasing values of $\omega\Delta t$. The box method has no numerical dissipation. Finally, Hughes (1977) cites a problem with averaging schemes in general as applied to nonlinear problems. For the nonlinear single degree-of-freedom case, Eq. (4) can be written as

$$\dot{y}^i+\dot{y}^{i-1}+\omega^i y^i+\omega^{i-1}y^{i-1}=0 \tag{14}$$

The update equation for y^i, Eq. (12), becomes

$$y^i=\left(\frac{2-\omega^{i-1}\Delta t}{2+\omega^i\Delta t}\right)y^{i-1} \tag{15}$$

and the stability becomes conditional as parameter ω changes with time. The practice suggested by Hughes (1977) for avoiding this problem is to use an averaged value of ω, i.e.,

$$\dot{y}^i+\dot{y}^{i-1}+\left(\frac{\omega^i+\omega^{i-1}}{2}\right)(y^i+y^{i-1})=0 \tag{16}$$

Generalized-α Method

Given the stability problems associated with the box method, Gobat and Grosenbaugh (2001) proposed replacing the temporal integration with Chung and Hulbert's (1993) generalized-α

Table 1. Algorithms Included in Generalized-α Method

Algorithm	γ	α_k	α_m	1st order problem	2nd order problem
Box method	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	Ablow and Schechter (1983)	
Backward differences	1	0	0	Koh et al. (1999)	
Generalized trapezoidal	$[\frac{1}{2},1]$	0	0	Sun et al. (1994)	Newmark (1959)
Cornwell and Malkus	$\frac{1}{2}-\alpha$	α	0	Cornwell and Malkus (1992)	Hilber et al. (1977)
WBZ-α	$\frac{1}{2}+\alpha$	0	α		Wood et al. (1981)

method. The generalized-α method is a reasonably complete family of algorithms that is second-order accurate, has controllable numerical dissipation, and offers a clear approach to coefficient averaging for the nonlinear problem. Following Chung and Hulbert's development of the generalized-α method for second-order equations, semidiscrete Eq. (2) becomes

$$(1-\alpha_m)\tilde{\mathbf{M}}\dot{\mathbf{Y}}^i+\alpha_m\tilde{\mathbf{M}}\dot{\mathbf{Y}}^{i-1}+(1-\alpha_k)\tilde{\mathbf{K}}\mathbf{Y}^i+\alpha_k\tilde{\mathbf{K}}\mathbf{Y}^{i-1}$$
$$+(1-\alpha_k)\tilde{\mathbf{F}}^i+\alpha_k\tilde{\mathbf{F}}^{i-1}=0 \qquad (17)$$

The difference equation is the same as for the generalized trapezoidal rule (Hughes 1987),

$$\mathbf{Y}^i=\mathbf{Y}^{i-1}+\Delta t[(1-\gamma)\dot{\mathbf{Y}}^{i-1}+\gamma\dot{\mathbf{Y}}^i] \qquad (18)$$

The three parameter family of algorithms given by Eqs. (17) and (18) defines the generalized-α method for the first-order semidiscrete problem. The method is second-order accurate if

$$\alpha_m-\alpha_k+\gamma=\tfrac{1}{2} \qquad (19)$$

From the eigenvalues of the amplification matrix, the stability requirement is

$$\alpha_k\leqslant\tfrac{1}{2} \quad \alpha_m\leqslant\tfrac{1}{2} \quad \gamma\geqslant\tfrac{1}{2} \qquad (20)$$

Requiring second-order accuracy according to Eq. (19) and forcing the eigenvalues of the amplification matrix to be equal as $\omega\Delta t\to\infty$ to prevent bifurcation, yields formulas for α_k and α_m as a function of λ^∞ only

$$\alpha_k=\frac{\lambda^\infty}{\lambda^\infty-1} \quad \alpha_m=\frac{3\lambda^\infty+1}{2\lambda^\infty-2} \qquad (21)$$

This yields a second-order accurate algorithm in which the only parameter is the eigenvalue (or spectral radius) at infinity.

Algorithms that can be obtained through various choices of α_k, α_m, γ, and λ^∞ are listed in Table 1. Spectral radii of some of these algorithms are shown in Fig. 1. Note that taking $\lambda^\infty \in[0,1)$ as the basis for the spectral radius results in a different set of algorithms than $\lambda^\infty\in[-1,0]$. For $\rho^\infty=1$ the only option is the negative eigenvalue and this results in the box method. A nondissipative algorithm with $\lambda^\infty=+1$ cannot be achieved.

In applying the generalized-α method to the nonlinear problem we must choose the time point at which we will evaluate $\tilde{\mathbf{M}}$, $\tilde{\mathbf{K}}$, and $\tilde{\mathbf{F}}$. A natural choice, consistent with the practice suggested by Hughes (1977) for nonlinear first-order problems and exemplified by Eq. (16), is provided by the temporal averaging of terms that is already a part of the method. At time step i Eq. (17) becomes

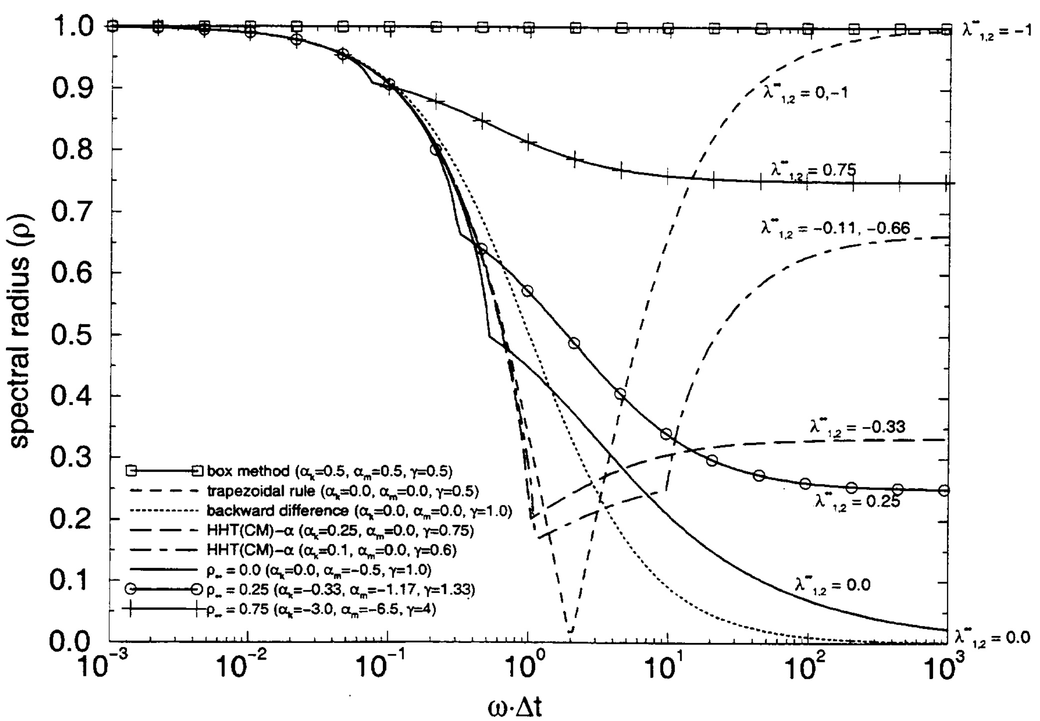

Fig. 1. Spectral radii of generalized-α family algorithms

$$(1-\alpha_m)\tilde{\mathbf{M}}^{i-\alpha_m}\dot{\mathbf{Y}}^i+\alpha_m\tilde{\mathbf{M}}^{i-\alpha_m}\dot{\mathbf{Y}}^{i-1}+(1-\alpha_k)\tilde{\mathbf{K}}^{i-\alpha_k}\mathbf{Y}^i$$
$$+\alpha_k\tilde{\mathbf{K}}^{i-\alpha_k}\mathbf{Y}^{i-1}+(1-\alpha_k)\tilde{\mathbf{F}}^i+\alpha_k\tilde{\mathbf{F}}^{i-1}=0 \quad (22)$$

where the averaged coefficient matrices are defined as

$$\tilde{\mathbf{M}}^{i-\alpha_m}=(1-\alpha_m)\tilde{\mathbf{M}}^i+\alpha_m\tilde{\mathbf{M}}^{i-1} \quad (23)$$

$$\tilde{\mathbf{K}}^{i-\alpha_k}=(1-\alpha_k)\tilde{\mathbf{K}}^i+\alpha_k\tilde{\mathbf{K}}^{i-1} \quad (24)$$

This scheme has been implemented in a computer program for two- and three-dimensional simulations of cable dynamics (Gobat and Grosenbaugh 2000). At each time step, Eq. (22) is solved using a Newton–Raphson procedure. The solution from the previous time step (or the static solution at the initial time step) serves as the initial guess in the nonlinear iterations. Because of this, the ultimate success of the solution is dependent on both the stability of the time integration and on the ability of the nonlinear solver to converge on a solution at time step i given an initial guess based on the solution at time step $i-1$. To improve convergence the program implements an adaptive time stepping scheme whereby the time step (the distance between the guess at $i-1$ and the solution at i) is reduced by factors of 10 at any spots where the solver is not successful. A practical limit of four orders of magnitude below the base-line time step is set to prevent the solution from proceeding in the face of a physical or numerical instability unrelated to the nonlinear solution procedure (e.g., Crank–Nicholson noise).

All of the numerical solutions that follow were obtained using this program. Thus, the box method, trapezoidal rule, and backward difference results, while spectrally equivalent to previous implementations, may be more stable than previous solutions because of the coefficient averaging scheme in Eq. (22). For clarity, spectrally equivalent historical names are retained in discussions of comparative algorithm performance that follow.

Application to Hanging Chain Problem

The performance of the different algorithms that can be implemented with the generalized-α family is studied by considering the free and forced response of the hanging chain shown in Fig. 2. In the free-response problem, we apply a small initial displacement to the chain and then at time $t=0$, release it. The dynamic response of the chain for $t>0$ can be calculated analytically for the small motions that result. In the forced response problem we impose a sinusoidally varying horizontal displacement to the top of the chain and analyze the forced response. This latter problem was studied both numerically and experimentally by Howell and Triantafyllou (1993).

Free Response to Initial Displacement

For small motions and an inextensible chain, the equation of motion is

$$m\frac{\partial^2 q}{\partial t^2}=\frac{\partial}{\partial s}\left[mgs\frac{\partial q}{\partial s}\right] \quad (25)$$

where m = mass per length of the chain, q = transverse displacement of the chain, g = acceleration due to gravity, and s = independent coordinate along the chain with $s=0$ at the free end. Assuming a harmonic solution of the form

$$q(s,t)=q(s)[A\cos\omega t+B\sin\omega t] \quad (26)$$

the mode shapes; $q(s)$, are (Triantafyllou et al. 1986)

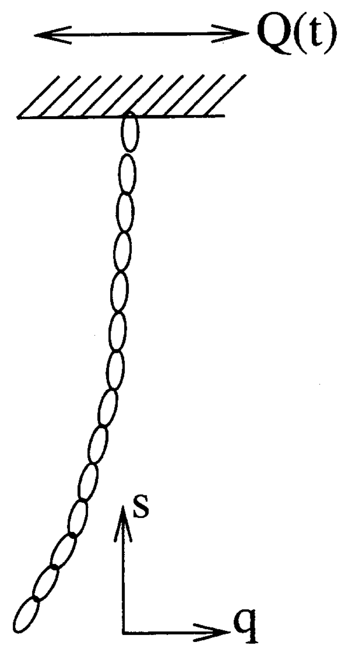

Fig. 2. Definitions for hanging chains problems

$$q(s)=c_1J_0\left(2\omega\sqrt{\frac{s}{g}}\right)+c_2Y_0\left(2\omega\sqrt{\frac{s}{g}}\right) \quad (27)$$

where J_0 and Y_0 = zero-order Bessel functions of the first and second kind, respectively. The requirement that the solution be finite at $s=0$ leads to the elimination of the Y_0 term and the requirement that $q(L)=0$ leads to the natural frequencies, ω. They are given by the roots of

$$J_0\left(2\omega\sqrt{\frac{L}{g}}\right)=0 \quad (28)$$

The complete response is given as the sum of the response in all modes:

$$q(s,t)=\sum_{n=1}^{\infty}J_0\left(2\omega_n\sqrt{\frac{s}{g}}\right)[A_n\cos\omega t+B_n\sin\omega t] \quad (29)$$

The coefficients A_n and B_n are determined from the initial displacement, $q_0(s)$, and velocity, $\dot{q}_0(s)$. Given $\dot{q}_0(s)=0$, we can immediately determine that $B_n=0$. To determine A_n we first write

$$q(s,0)=\sum_{n=1}^{\infty}A_nJ_0\left(2\omega_n\sqrt{\frac{s}{g}}\right)=q_0(s) \quad (30)$$

Multiplying both sides by $J_0(2\omega_n\sqrt{s/g})$, integrating from $s=0$ to $s=L$, and making use of the fact that

$$\int_0^L J_0\left(2\omega_n\sqrt{\frac{s}{g}}\right)J_0\left(2\omega_m\sqrt{\frac{s}{g}}\right)ds=0 \quad \text{for } n\neq m \quad (31)$$

yields the following equation for A_n:

$$A_n=\frac{\int_0^L q_0(s)J_0\left(2\omega_n\sqrt{\frac{s}{g}}\right)ds}{\int_0^L J_0^2\left(2\omega_n\sqrt{\frac{s}{g}}\right)ds} \quad (32)$$

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